UNIT: I

Differential Equation of first order and first degree

Differential equation: An equation involving differentials or one dependent variable and its derivatives with respect to one or more independent variable is called a differential equation.

Ordinary differential equation: A differential equation is said to be ordinary, if the derivatives in the equation have reference to only a single independent variable.

Ex 1: $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \cos x,$

Ex 2: $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x.$

Partial differential equation: A differential equation is said to be partial, if the derivatives in the equation have reference to two or more independent variables.

Ex 1: $(y+z)\frac{\partial z}{\partial x} + (z+x)\frac{\partial z}{\partial y} = x+y$

Ex 2: $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4z.$

Order of differential equation: A differential equation is said to be of order n, if the n^{th} derivative is the highest derivative in that equation.

Ex: $(x^2+1)\frac{dy}{dx} + 2xy = 4x^2$

Degree of a differential equation: Let $F(x, y, y', y'', \ldots y^n) = 0$ be a differential equation of order n, if the given differential equation is a polynomial in $y^{(n)}$, then the highest degree of $y^{(n)}$ is defined as the degree of the differential equation.

Example: $a^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$

This is a polynomial equation in $\frac{dy}{dx}$. The highest degree of $\frac{dy}{dx}$ is two. Hence the degree of the above differential equation is 2.

(i) Solution of a differential equation: A relation between the dependent and independent variables when substituted in the differential equation reduces it to an identity, is called a solution or integral or primitive of the differential equation.

(ii) General solution of a differential equation: Let $f(x, y, y^{(1)}, y^{(2)}, \ldots y^{(n)}) = 0$ be a differential equation of order n. If $\phi(x, y, c_1, c_2, \ldots, c_n) = 0$ where $c_1, c_2, \ldots, c_n$ are n independent arbitrary constants, is a solution of the given differential equations, then it is called the general solution of the given differential equation.

(iii) Particular solution of a differential equation: The solution obtained by giving particular values to arbitrary constants in the general solution of the differential equation $f(x, y, y^{(1)}, \ldots y^{(n)}) = 0$ is called a particular solution of the given differential equation.

(iv) Singular solution of a differential equation: An equation $\psi(x, y) = 0$ is called singular solution of the differential equation $f(x, y, y^{(1)}, \ldots, y^{(n)}) = 0$ if

(i) $\psi(x, y) = 0$ is a solution of the given differential equation.

(ii) $\psi(x, y) = 0$ does not contain arbitrary constants and

(iii) $\psi(x, y) = 0$ is not obtained by giving particular values to arbitrary constants in the general solution.

Linear differential equations in y :

An equation of the form $\frac{dy}{dx}+Py=Q$ where P and Q are constants or functions of x defined over an interval I alone is called a linear differential equation of first order in y.

If $Q=0$ for all x in I, then the corresponding equation $\frac{dy}{dx}+Py=0$ is called a homogeneous linear differential equation of first order.

If $Q\neq 0$ for all x in I, then $\frac{dy}{dx}+Py=Q$ is called a non-homogeneous linear equation of first order.

Working rule to solve linear equation $\frac{dy}{dx}+Py=Q$ where $P=f(x)$ and $Q=g(x)$:

1. First reduce the given equation to the standard form and then identify P and Q.
2. Find $\int P\,dx$ and then $I.F = e^{\int P\,dx}$
3. Then obtain general solution by using $y(I.F)=\int Q(I.F)\,dx+C$

Problem: Solve $(x^2+1)\frac{dy}{dx}+4xy=\frac{1}{x^2+1}$

Solution: Given equation is $(x^2+1)\frac{dy}{dx}+4xy=\frac{1}{x^2+1} \longrightarrow (1)$

divide equation (1) with x^2+1

$$\frac{dy}{dx}+\frac{4x}{(x^2+1)}\cdot y=\frac{1}{(x^2+1)^2}$$

which is a standard form

where $P=\frac{4x}{x^2+1}$; $Q=\frac{1}{(x^2+1)^2}$

Integrating factor $e^{\int P\,dx}=e^{\int \frac{4x}{x^2+1}dx}$

$$= e^{\int \frac{4x}{x^2+1} dx}$$

$$= e^{2\int \frac{2x}{x^2+1} dx}$$

$$= e^{2\log x^2+1}$$

$$= e^{\log(x^2+1)^2}$$

$$= (x^2+1)^2$$

Therefore, The general solution is $y(I.f) = \int Q(I.f)dx + c$

$$y(x^2+1)^2 = \int \frac{1}{(x^2+1)^2}(x^2+1)^2 dx + c$$

$$= \int 1dx + c$$

$$y(x^2+1)^2 = x + c$$

Problem : Solve $x\frac{dy}{dx} + 2y - x^2\log x = 0$

Solution : Given equation is $x\frac{dy}{dx} + 2y - x^2\log x = 0 \rightarrow (1)$

divide the given equation with x to reduce it to standard form

$$\frac{dy}{dx} + \frac{2}{x}y = x\log x \longrightarrow (2)$$

where $P = \frac{2}{x}$; $Q = x\log x$

Integrating factor, $e^{\int Pdx} = e^{\int \frac{2}{x}dx}$

$$= e^{2\int \frac{1}{x}dx}$$

$$= e^{2\log x}$$

$$= e^{\log x^2}$$

$$= x^2$$

The general solution is $y(I.f) = \int Q(I.f)dx + c$

$$y(x^2) = \int x\log x(x^2)\, dx + c$$

$$x^2y = \int (x^3\log x)dx + c$$

$$= \log x\int x^3dx - \int \frac{d}{dx}(\log x)\int x^3 dx + c$$

$$= \log x \frac{x^4}{4} - \int \frac{1}{x}\left(\frac{x^4}{4}\right) dx + c$$

$$= \log x \frac{x^4}{4} - \frac{1}{4}\int x^3 \, dx + c$$

$$= \log x \frac{x^4}{4} - \frac{1}{4}\frac{x^4}{4} + c$$

$$x^2 y = \frac{x^4}{4}\log x - \frac{x^4}{16} + c$$

Problem: Solve $x\cos x \frac{dy}{dx} + (x\sin x + \cos x)y = 1$

Solution: Given equation is $x\cos x \frac{dy}{dx} + (x\sin x + \cos x)y = 1 \rightarrow (1)$

dividing the given equation by $x\cos x$, we get

$$\frac{dy}{dx} + \frac{x\sin x + \cos x}{x\cos x} y = \frac{1}{x\cos x}$$

$$\frac{dy}{dx} + \left(\tan x + \frac{1}{x}\right)y = \frac{1}{x\cos x}$$

which is a standard form

Here $P = \tan x + \frac{1}{x}$; $Q = \frac{1}{x\cos x}$

Integrating factor $e^{\int P dx}$

$$= e^{\int (\tan x + \frac{1}{x}) dx}$$

$$= e^{\int \tan x \, dx + \int \frac{1}{x} dx}$$

$$= e^{\log \sec x + \log x}$$

$$= e^{\log \sec x \cdot x}$$

$$I.f = x \sec x$$

$\therefore$ The general solution is $y(I.f) = \int Q(I.f)\, dx + c$

$$y(x\sec x) = \int \frac{1}{x\cos x}(x\sec x)\, dx + c$$

$$= \int (\sec x \cdot \sec x)\, dx + c$$

$$= \int \sec^2 x \, dx + c$$

$$y(x\sec x) = \tan x + c$$

Problem: $x(x-1)\frac{dy}{dx}-(x-2)y=x^3(2x-1)$

Solution: Given equation is $x(x-1)\frac{dy}{dx}-(x-2)y=x^3(2x-1) \rightarrow ①$

dividing the given equation with $x(x-1)$, we get

$$\frac{dy}{dx}-\frac{x-2}{x(x-1)}y=\frac{x^3(2x-1)}{x(x-1)}$$

$$\frac{dy}{dx}-\frac{x-2}{x(x-1)}y=\frac{x^2(2x-1)}{x-1}$$

where $P=-\left(\frac{x-2}{x(x-1)}\right)$; $Q=\frac{x^2(2x-1)}{x-1}$

Integrating factor $e^{\int P\,dx}$

$$= e^{\int -\left(\frac{x-2}{x(x-1)}\right)dx}$$

$$= e^{\int\left(\frac{1}{x-1}-\frac{2}{x}\right)dx}$$

$$= e^{\int \frac{1}{x-1}dx-\int \frac{2}{x}dx}$$

$$= e^{\log(x-1)-2\log x}$$

$$= e^{\log(x-1)+\log x^{-2}}$$

$$= e^{\log (x-1).x^{-2}}$$

$$= (x-1)\,x^{-2}$$

$$I.f = \frac{x-1}{x^2}$$

$\therefore$ The general solution is $y(I.f)=\int Q(I.f)\,dx+c$

$$y\left(\frac{x-1}{x^2}\right)=\int\left(\frac{x^2(2x-1)}{x-1}\cdot\frac{x-1}{x^2}\right)dx+c$$

$$=\int(2x-1)\,dx+c$$

$$=2\int x\,dx-\int 1\,dx+c$$

$$=2\frac{x^2}{2}-x+c$$

$$y\left(\frac{x-1}{x^2}\right)=x^2-x+c$$

Problem: Solve $\frac{dy}{dx} + \frac{1}{(1-x)\sqrt{x}} = 1-\sqrt{x}$

Solution: Given equation is $\frac{dy}{dx} + \frac{1}{(1-x)\sqrt{x}} = 1-\sqrt{x}$

This is a linear differential equation in y

Where $P = \frac{1}{(1-x)\sqrt{x}}$; $Q = 1-\sqrt{x}$

Integrating factor $= e^{\int P dx}$

$$I.f = e^{\int\left(\frac{1}{(1-x)\sqrt{x}}\right) dx}$$

Put $\sqrt{x} = t$

$$\frac{1}{2\sqrt{x}} dx = dt$$

$$\frac{1}{\sqrt{x}} dx = 2\,dt$$

and $\sqrt{x} = t$

S.O.B.S

$$(\sqrt{x})^2 = t^2$$

$$x = t^2$$

$$I.f = e^{\int \frac{1}{1-t^2} 2dt}$$

$$= e^{2\int \frac{1}{1-t^2} dt}$$

$$= e^{2 \cdot \frac{1}{2} \log \frac{1-t}{1+t}}$$

$$= e^{\log \frac{1+t}{1-t}}$$

$$= \frac{1+t}{1-t} = \frac{1+\sqrt{x}}{1-\sqrt{x}}$$

$$I.f = \frac{1+\sqrt{x}}{1-\sqrt{x}}$$

$\therefore$ The general Solution is

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = \int (1-\sqrt{x})\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)dx + c$$

$$= \int (1+\sqrt{x})\,dx + c$$

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = x + \frac{2}{3}x^{3/2} + c$$

Problem: Solve $y\,dx - x\,dy + \log x\,dx = 0$

Solution: Given Eq'n is $y\,dx - x\,dy + \log x\,dx = 0 \rightarrow (1)$

dividing Equation (1) with dx we get

$$y - x\frac{dy}{dx} + \log x = 0$$

$$x\frac{dy}{dx} - y = \log x$$

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{\log x}{x}$$

where $P = -1/x$; $Q = \frac{\log x}{x}$

$$I.f = e^{\int P\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

$$\therefore I.f = \frac{1}{x}$$

The general Solution is $y\left(\frac{1}{x}\right) = \int\left(\frac{\log x}{x}\cdot\frac{1}{x}\right)dx + c$

$$y\left(\frac{1}{x}\right) = \int\left(\log x\cdot\frac{1}{x^2}\right)dx + c$$

$$= \log x\int\frac{1}{x^2}dx - \int\frac{d}{dx}(\log x)\int\frac{1}{x^2}dx + c$$

$$= \log x\left(-\frac{1}{x}\right) - \int\frac{1}{x}(-1/x)\,dx + c$$

$$= \frac{-\log x}{x} + \int\frac{1}{x^2}dx + c$$

$$y\left(\frac{1}{x}\right) = -\frac{\log x}{x} - \frac{1}{x} + c$$

$\therefore$ The general Solution is $y = cx - (1+\log x)$

Problem: Solve $(1+x)\frac{dy}{dx} - xy = (1-x)$

Solution: Given equation is $(1+x)\frac{dy}{dx} - xy = (1-x) \rightarrow (1)$

dividing equation (1) with $1+x$

$$\frac{dy}{dx} - \frac{x}{1+x}y = \frac{1-x}{1+x}$$

This is a linear differential equation in y

Here $P = \frac{-x}{1+x}$; $Q = \frac{1-x}{1+x}$

Integrating factor $= e^{\int P dx}$

$$= e^{\int \frac{-x}{1+x} dx}$$

$$= e^{-\int \left(\frac{1+x-1}{1+x}\right) dx}$$

$$= e^{-\int \frac{1+x}{1+x}dx - \int \frac{1}{1+x} dx}$$

$$= e^{-\left[\int 1 dx - \int \frac{1}{1+x} dx\right]}$$

$$= e^{-\left[x - \log(1+x)\right]}$$

$$= e^{-x + \log(1+x)}$$

$$= e^{-x} e^{\log(1+x)}$$

$$= e^{-x}(1+x)$$

The general solution is $y(I.f) = \int Q(I.f)\, dx + C$

$$y(e^{-x}(1+x)) = \int\left(\frac{1-x}{1+x} e^{-x}(1+x)\right) dx + C$$

$$= \int (1-x) e^{-x} dx + C$$

$$= \int (e^{-x} - xe^{-x}) dx + C$$

$$= \int e^{-x}dx - \int xe^{-x} dx + C$$

$= -e^{-x} - \left(x\int e^{-x}dx - \int \frac{d}{dx}(x) \int e^{-x}dx\right) + c$

$= e^{-x} - \left(x(-e^{-x}) - \int 1\,(-e^{-x})dx\right) + c$

$= e^{-x} + xe^{-x} + \int e^{-x}dx + c$

$= e^{-x} + xe^{-x} - e^{-x} + c$

$= xe^{-x} + c$

$\therefore$ The general solution is

$y(e^{-x}(1+x)) = xe^{-x} + c$

Problem: Solve $x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2$

Solution: Given Eq'n is $x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2 \rightarrow (1)$

dividing Eq'n (1) with $x(x-1)$

$$\frac{dy}{dx} - \frac{1}{x(x-1)}y = \frac{x^2(x-1)^2}{x(x-1)}$$

$$\frac{dy}{dx} - \frac{1}{x(x-1)}y = x(x-1)$$

This is a linear differential Eq'n in y

where $P = -\frac{1}{x(x-1)}$; $Q = x(x-1)$

Integrating factor $= e^{\int P dx}$

$= e^{\int \frac{-1}{x(x-1)}dx}$

$= e^{\int \left(\frac{1}{x} - \frac{1}{x-1}\right)dx}$

$= e^{\int \frac{1}{x}dx - \int \frac{1}{x-1}dx}$

$= e^{\log x - \log(x-1)}$

$$= e^{\log \frac{x}{x-1}}$$

$$I.F = \frac{x}{x-1}$$

The general solution is $y(I.F) = \int Q(I.F)\,dx + c$

$$y\left(\frac{x}{x-1}\right) = \int x(x-1)\frac{x}{x-1}\,dx + c$$

$$y\left(\frac{x}{x-1}\right) = \int x^2\,dx + c$$

$$y\left(\frac{x}{x-1}\right) = \frac{x^3}{3} + c$$

<u>Problem:</u> solve $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$

<u>Solution:</u> Given Equation is $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2} \rightarrow (1)$

dividing the given Equation with $(1-x^2)$ we ge

$$\frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x\sqrt{1-x^2}}{1-x^2}$$

where $P = \frac{2x}{1-x^2}$; $Q = \frac{x\sqrt{1-x^2}}{1-x^2}$

$$I.f = e^{\int P dx} = e^{\int \frac{2x}{1-x^2}dx}$$

$$= e^{-\int -\frac{2x}{1-x^2}dx}$$

$$= e^{-\log(1-x^2)}$$

$$= e^{\log(1-x^2)^{-1}}$$

$$= (1-x^2)^{-1}$$

$$= \frac{1}{1-x^2}$$

The general solution is $y\left(\frac{1}{1-x^2}\right) = \int \frac{x\sqrt{1-x^2}}{1-x^2}\left(\frac{1}{1-x^2}\right)dx + c$

$$= \int \frac{x\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-x^2}}\,\frac{1}{(1-x^2)}\,dx + c$$

$$= \int \frac{x}{(1-x^2)^{1/2}} \cdot \frac{1}{1-x^2}\, dx + c$$

$$= \int \frac{x}{(1-x^2)^{3/2}}\, dx + c$$

$$= \int (1-x^2)^{-3/2} \cdot x\, dx + c$$

$$= -\frac{1}{2} \int (1-x^2)^{-3/2} (-2x)\, dx + c$$

$$= -\frac{1}{2} \frac{(1-x^2)^{-3/2+1}}{-\frac{3}{2}+1} + c$$

$$= -\frac{1}{2} \frac{(1-x^2)^{-1/2}}{-1/2} + c$$

$$= (1-x^2)^{-1/2} + c$$

$$y\left(\frac{1}{1-x^2}\right) = \frac{1}{(1-x^2)^{1/2}} + c$$

$\therefore$ The general solution is

$$y\left(\frac{1}{1-x^2}\right) = \frac{1}{(1-x^2)^{1/2}} + c$$

Problem: Solve $\frac{dy}{dx} + 2x \cdot y = e^{-x^2}$

Solution: Given equation is $\frac{dy}{dx} + 2x \cdot y = e^{-x^2} \rightarrow (1)$

This is a linear differential equation in y

where $P = 2x$; $Q = e^{-x^2}$

Integrating factor $= e^{\int P dx}$

$$= e^{\int 2x\, dx}$$

$$= e^{2\frac{x^2}{2}}$$

$$I.f = e^{x^2}$$

The general solution is

$$y(e^{x^2}) = \int e^{-x^2}(e^{x^2})\,dx + c$$

$$= \int e^{-x^2+x^2}\,dx + c$$

$$= \int e^{0}\,dx + c$$

$$= \int 1\,dx + c$$

$$\therefore y(e^{x^2}) = x + c$$

Problem: Solve $\cos^2 x \frac{dy}{dx} + y = \tan x$

Solution: Given Equation is $\cos^2 x \frac{dy}{dx} + y = \tan x \rightarrow (1)$

equation (1) is dividing with $\cos^2 x$; we get

$$\frac{dy}{dx} + \frac{1}{\cos^2 x} y = \frac{\tan x}{\cos^2 x}$$

This is a linear differential equation in y

Where $P = \frac{1}{\cos^2 x}$; $Q = \frac{\tan x}{\cos^2 x}$

$$I.f = e^{\int P dx} = e^{\int \frac{1}{\cos^2 x}} = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

The general solution is $y(I.f) = \int Q(I.f)\,dx + c$

$$y(e^{\tan x}) = \int \left(\frac{\tan x}{\cos^2 x} e^{\tan x}\right) dx + c$$

$$= \int (\tan x \sec^2 x\, e^{\tan x})\,dx + c$$

put $t = \tan x$

$dt = \sec^2 x\, dx$

$$y(e^t) = \int (t e^t)\,dt + c$$

$$= t\int e^t dt - \int \frac{d}{dt}(t) \int e^t dt + c$$

$$= te^t - e^t + c$$

$$y(e^t) = e^t(t-1) + c$$

$\therefore$ The general solution is

$$y(e^{\tan x}) = e^{\tan x}(\tan x - 1) + c$$

<u>Problem</u>: obtained the equation of the curve satisfying the differential equation $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ and passing through the origin?

<u>Solution</u>: Given equation is $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0 \rightarrow (1)$

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$$

This is a linear differential equation in y

Here $P = \frac{2x}{1+x^2}$; $Q = \frac{4x^2}{1+x^2}$

$$I.f = e^{\int P dx} = e^{\int \frac{2}{1+x^2}dx} = e^{\log(1+x^2)} = (1+x^2)$$

The general solution is

$$y(1+x^2) = \int\left(\frac{4x^2}{1+x^2}\, 1+x^2\right)dx$$

$$= \int 4x^2 dx + c$$

$$y(1+x^2) = 4\frac{x^3}{3} + c$$

The curve passes through the origin $(0,0)$

$$0 = 4(0) + c$$

$$\therefore c = 0$$

The required differential equation of the curve is

$$y(1+x^2) = 4\frac{x^3}{3} + 0$$

$$\therefore y(1+x^2) = \frac{4x^3}{3}$$

Linear differential Equations in x: An Equation of the form $\frac{dx}{dy}+Px=Q$ where P and Q are constants (or) functions of y alone is called a linear differential equation of first order in x.

If $Q=0$ for all y in I, then the corresponding Equation $\frac{dx}{dy}+Px=0$ it is called homogeneous equation.

If $Q\neq 0$ then the equation $\frac{dx}{dy}+Px=Q$ is called non-homogeneous equation.

Working rule to Solve linear equation $\frac{dx}{dy}+Px=Q$ where $P=f(x)$ and $Q=g(x)$

Problem: Solve $(1+y^2)dx = \tan^{-1}y - x\,dy$

Solution: Given Equation is $(1+y^2)dx = \tan^{-1}y - x\,dy \longrightarrow (1)$

dividing by $(1+y^2)$ to reduce this to standard form

$$\frac{dx}{dy}+\frac{1}{1+y^2}x=\frac{\tan^{-1}y}{1+y^2}$$

Where $P=\frac{1}{1+y^2}$; $Q=\frac{\tan^{-1}y}{1+y^2}$

$$\text{Integrating factor} = e^{\int P\,dy}$$

$$= e^{\int\frac{1}{1+y^2}dy}$$

$$\text{I.f} = e^{\tan^{-1}y}$$

The general solution $x(\text{I.f})=\int Q(\text{I.f})\,dy+c$

$$x\left(e^{\tan^{-1}y}\right)=\int\frac{\tan^{-1}y}{1+y^2}\left(e^{\tan^{-1}y}\right)dy+c$$

Put $\tan^{-1}y=t$

$$\frac{1}{1+y^2}dy = dt$$

$$x(e^t) = \int te^t dt + c$$

$$= t\int e^t dt - \int \frac{d}{dt}(t)\int e^t dt + c$$

$$= te^t - e^t + c$$

$$xe^t = e^t(t-1) + c$$

$\therefore$ The required general solution is

$$xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$$

Problem : Solve $(1+x+xy^2)\frac{dy}{dx} + (y+y^3) = 0$

Solution : Given equation is $(1+x+xy^2)\frac{dy}{dx} + (y+y^3) = 0$

which can be written as $y(1+y^2)\frac{dx}{dy} + 1 + x(1+y^2) = 0$

$$\Rightarrow \frac{dx}{dy} + \frac{x(1+y^2)}{y(1+y^2)} = \frac{-1}{y(1+y^2)}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{y}x = \frac{-1}{y(1+y^2)}$$

which is a linear equation in x.

$$I.f = e^{\int P\,dy}$$

$$= e^{\int \frac{1}{y}dy}$$

$$= e^{\log y}$$

$$I.f = y$$

The general solution is $x(y) = \int \frac{-1}{y(1+y^2)}\, y\, dy + c$

$$= \int \frac{-1}{1+y^2}dy + c$$

$$xy = -\tan^{-1}y + c$$

Problem: Solve $(x+2y^3)\frac{dy}{dx}=y$

Solution: Given equation is $(x+2y^3)\frac{dy}{dx}=y$

which can be written as $y\frac{dx}{dy}=x+2y^3$

$\Rightarrow \frac{dx}{dy}-\frac{1}{y}x=2y^2$

This is a linear differential equation in x.

Where $P=-\frac{1}{y}$ and $Q=2y^2$

$$I.f = e^{\int P dy}$$
$$= e^{-\int \frac{1}{y} dy}$$
$$= e^{-\log y}$$
$$= e^{\log y^{-1}}$$
$$= y^{-1}$$
$$= \frac{1}{y}$$

The general solution is $x(I.f)=\int Q(I.f)dy+c$

$$x\left(\frac{1}{y}\right)=\int\left(2y^2\frac{1}{y}\right)dy+c$$
$$=\int 2y\, dy+c$$
$$=2\frac{y^2}{2}+c$$
$$x\left(\frac{1}{y}\right)=y^2+c$$

$\therefore$ The required general solution is $x=y^3+cy$

problem: Solve $(x+y+1)\frac{dy}{dx}=1$

Solution: Given equation is $(x+y+1)\frac{dy}{dx}=1$

$\Rightarrow \frac{dx}{dy}=x+y+1 \Rightarrow \frac{dx}{dy}-x=(y+1)$

This is a linear differential equation in x

Where $P = -1$; $Q = y+1$

$$I.f = e^{\int P dy}$$

$$= e^{\int -1 dy}$$

$$= e^{-y}$$

The general solution is $x(I.f) = \int Q(I.f)\,dy + c$

$$x(e^{-y}) = \int[(y+1)e^{-y}]dy + c$$

$$= \int(ye^{-y} + e^{-y})dy + c$$

$$= \int ye^{-y}dy + \int e^{-y}dy + c$$

$$= y\int e^{-y}dy - \int \frac{d}{dy}(y)\int e^{-y}dy + \int e^{-y}dy + c$$

$$= y(-e^{-y}) - \int -e^{-y}dy - e^{-y} + c$$

$$= -ye^{-y} + \int e^{-y}dy - e^{-y} + c$$

$$= -ye^{-y} - e^{-y} - e^{-y} + c$$

$$xe^{-y} = -ye^{-y} - 2e^{-y} + c$$

$$\Rightarrow x + y + 2 = ce^{y}$$

$\therefore$ The required general solution is $x + y + 2 = ce^{y}$

Equations Reducible to linear form:

Bernoulli's Equation: An equation of the form $\frac{dy}{dx} + Py = Qy^n$ where P and Q are real numbers or functions of x alone and n is real number such that $n \neq 0$ and $n \neq 1$, is called a Bernoulli's differential equation.

We can solve Bernoulli's equation by reducing it to linear differential equation in y as follows

Given equation is $\frac{dy}{dx} + Py = Qy^n$

Multiplying by y^{-n}, we get $y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q$

Let $y^{1-n} = u \Rightarrow (1-n)y^{-n}\frac{dy}{dx} = \frac{du}{dx}$

$$\Rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$$

$$\Rightarrow \frac{1}{1-n}\frac{dy}{dx} + Pu = Q$$

$$\Rightarrow \frac{du}{dx} + (1-n)Pu = (1-n)Q$$

This is a linear differential equation of first order in u and x.

I.F $= \exp[\int(1-n)P\,dx] = 1-n\int Q\cdot\exp[\int(1-n)P\,dx]\,dx + c$

Substitution of $u = y^{1-n}$, we get the general solution

Problem: Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x\sin x,\ x>0$

Solution: Given equation is $\frac{dy}{dx} + \frac{y}{x} = y^2 x\sin x \rightarrow (1)$

which is Bernoulli's equation.

divided both sides with y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{y}{x}\left(\frac{1}{y^2}\right) = x\sin x$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = x\sin x \rightarrow (2)$$

Take $\frac{1}{y} = u$

differentiating with respect to x

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{1}{y^2}\frac{dy}{dx} = -\frac{du}{dx}$$

from ② $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\left(\frac{1}{y}\right) = x\sin x$

$$-\frac{du}{dx} + \frac{1}{x}u = x\sin x$$

$$\frac{du}{dx} - \frac{1}{x}u = -x\sin x$$

which is linear equation in u

where $P = -1/x$, $Q = -x\sin x$

$$I.f = e^{\int P\,dx}$$

$$= e^{-\int \frac{1}{x}dx}$$

$$= e^{-\log x}$$

$$= e^{\log x^{-1}}$$

$$= x^{-1}$$

$$= \frac{1}{x}$$

The general solution is $u(I.f) = \int Q(I.f)\,dx + c$

$$u\left(\frac{1}{x}\right) = \int\left(-x\sin x \cdot \frac{1}{x}\right)dx + c$$

$$u\left(\frac{1}{x}\right) = -\int \sin x\,dx + c$$

$$u\left(\frac{1}{x}\right) = -(-\cos x) + c$$

$$\frac{1}{xy} = \cos x + c$$

$$\Rightarrow xy\cos x + cxy = 1$$

∴ The required general solution is $xy\cos x + cxy = 1$.

Problem: Solve $x\frac{dy}{dx} + y = y^2\log x$

Solution: Given equation is $x\frac{dy}{dx} + y = y^2\log x \longrightarrow ①$

dividing x on both sides

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2 \log x}{x}$$

This is a Bernoulli's equation.

dividing y^2 on both sides

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{y}{x}\frac{1}{y^2} = \frac{\log x}{x}$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\frac{1}{y} = \frac{\log x}{x}$$

Take $\frac{1}{y} = u$

differentiating with respect to x

$$\frac{-1}{y^2}\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{-du}{dx}$$

from ①

$$-\frac{du}{dx} + \frac{1}{x}u = \frac{\log x}{x}$$

$$\frac{du}{dx} - \frac{1}{x}u = -\frac{\log x}{x}$$

which is a linear equation in u

where $P = -\frac{1}{x}$; $Q = -\frac{\log x}{x}$

$$I.f = e^{\int P\,dx}$$

$$= e^{-\int \frac{1}{x}dx}$$

$$= e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

The general solution is $u(I.f) = \int Q(I.f)\,dx + c$

$$u\left(\frac{1}{x}\right) = \int -\frac{\log x}{x}\,\frac{1}{x}\,dx + c$$

$$= -\int \left(\log x \cdot \frac{1}{x^2}\right)dx + c$$

$$= -\left(\log x \int \frac{1}{x^2} - \int \left(\frac{d}{dx}(\log x) \int \frac{1}{x^2}\,dx\right) dx + c\right.$$

$$= -\left(\log x \left(-\frac{1}{x}\right) - \int \frac{1}{x}\left(-\frac{1}{x}\right) dx\right) + c$$

$$= -\left(\log x \left(-\frac{1}{x}\right) + \int \frac{1}{x^2}\,dx\right) + c$$

$$= \frac{\log x}{x} + \frac{1}{x} + c$$

$$\frac{1}{y}\left(\frac{1}{x}\right) = \frac{1}{x}(\log x + 1) + c$$

$$\frac{1}{y} = \log x + 1 + c$$

$\therefore$ The required general solution is $\frac{1}{y} = \log x + 1 + c$

<u>Problem</u>: Solve $\frac{dy}{dx} + \frac{xy}{1-x^2} = x\sqrt{y}$

<u>Solution</u>: Given Equation is $\frac{dy}{dx} + \frac{xy}{1-x^2} = x\sqrt{y} \longrightarrow (1)$

dividing equation with $\sqrt{y}$ on both sides

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} + \frac{xy}{(1-x^2)\sqrt{y}} = x$$

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} + \frac{x(\sqrt{y}\sqrt{y})}{(1-x^2)\sqrt{y}} = x$$

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} + \frac{x\sqrt{y}}{(1-x^2)} = x \longrightarrow (2)$$

Take $u = \sqrt{y}$

$$\frac{du}{dx} = \frac{1}{2\sqrt{y}}\frac{dy}{dx}$$

$$2\frac{du}{dx} = \frac{1}{\sqrt{y}}\frac{dy}{dx}$$

From (2)

$$2\frac{du}{dx} + \frac{x}{1-x^2}u = x$$

$$\frac{du}{dx} + \frac{x}{2(1-x^2)}u = \frac{x}{2}$$

Where $P = \frac{x}{2(1-x^2)}$; $Q = \frac{x}{2}$

$$I.f = e^{\int P dx}$$

$$= e^{\frac{1}{2}\int \frac{x}{1-x^2} dx} \quad = e^{\frac{1}{2}\int -\frac{1}{2}\left(\frac{-2x}{1-x^2}\right) dx}$$

$$= e^{-\frac{1}{4}\log(1-x^2)}$$

$$= e^{\log(1-x^2)^{-1/4}}$$

$$= (1-x^2)^{-1/4}$$

The general solution is $v(I.f) = \int Q(I.f)\,dx + c$

$$v\left(\frac{1}{(1-x^2)^{1/4}}\right) = \int \frac{x}{2}\left(\frac{1}{(1-x^2)^{1/4}}\right) dx + c$$

$$= \frac{1}{2}\int x(1-x^2)^{-1/4} dx + c$$

$$= \frac{1}{2}\int -\frac{1}{2}(-2x)(1-x^2)^{-1/4} dx + c$$

$$= -\frac{1}{4}\left(\frac{(1-x^2)^{-1/4+1}}{-\frac{1}{4}+1}\right) + c$$

$$= -\frac{1}{4}\frac{(1-x^2)^{3/4}}{3/4} + c$$

$$= -\frac{1}{4}(1-x^2)^{3/4}\cdot\frac{4}{3} + c$$

$$= \frac{-(1-x^2)^{3/4}}{3} + c$$

$\therefore$ The required general solution is

$$v\left(\frac{1}{(1-x^2)^{1/4}}\right) = \frac{-(1-x^2)^{3/4}}{3} + c$$

Problem: Solve $\frac{dy}{dx} = 2y\tan x + y^2 \tan^2 x$

Solution: Given equation is $\frac{dy}{dx} = 2y\tan x + y^2\tan^2 x \rightarrow (1)$

$$\Rightarrow \frac{dy}{dx} - 2y\tan x = y^2\tan^2 x$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}2\tan x = \tan^2 x \rightarrow (2)$$

Take $u = -\frac{1}{y}$

$$\frac{du}{dx} = -1/y^2 \frac{dy}{dx}$$

from (2) we get

$$\Rightarrow \frac{du}{dx} + u(2\tan x) = \tan^2 x$$

Where $P = 2\tan x$; $Q = \tan^2 x$

$$I.f = e^{\int P dx} = e^{2\int \tan x dx} = e^{2\log \sec x} = e^{\log \sec^2 x} = \sec^2 x.$$

The general solution is $u(I.f) = \int Q(I.f)dx + C$

$$u(\sec^2 x) = \int(\tan^2 x \cdot \sec^2 x)dx + C$$

$$= \frac{\tan^{2+1} x}{2+1} + C$$

$$-\frac{1}{y}(\sec^2 x) = \frac{\tan^3 x}{3} + C$$

The required general solution is $-\frac{1}{y}\sec^2 x = \frac{\tan^3 x}{3} + C$

Problem: Solve $\frac{dy}{dx} + \frac{y}{x} = xy^2$

Solution: Given equation is $\frac{dy}{dx} + \frac{y}{x} = xy^2 \rightarrow (1)$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x \rightarrow (2)$$

which is a bernouli's equation

Take $u = \frac{1}{y}$

diff. w. r. to x

$$\frac{du}{dx} = -1/y^2 \frac{dy}{dx}$$

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$$\frac{1}{y^2}\frac{dy}{dx} = \frac{-du}{dx}$$

from ②

$$\frac{du}{dx} - \frac{1}{x}u = -x$$

where $P = -\frac{1}{x}$; $Q = -x$

$$I.f \; e^{\int P dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

The general solution $u(I.f) = \int Q(I.f)\,dx + c$

$$u\left(\frac{1}{x}\right) = \int -x \left(\frac{1}{x}\right) dx + c$$

$$u\left(\frac{1}{x}\right) = \int -1\, dx + c$$

$$\frac{1}{xy} = -x + c$$

∴ The required general solution is

$$\frac{1}{xy} = -x + c$$

__Problem__: Solve $\frac{dy}{dx} + \frac{y}{x} = x^2y^6$

__Solution__: Given equation is $\frac{dy}{dx} + \frac{y}{x} = x^2y^6 \longrightarrow ①$

dividing equation ① with y^6 we get

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{y^5}\cdot\frac{1}{x} = x^2 \longrightarrow ②$$

which is a bernouli's Equation

Take $u = \frac{1}{y^5}$

diff. w.r. to x

$$\frac{du}{dx} = \frac{-5}{y^6}\frac{dy}{dx}$$

$$\frac{1}{5}\frac{du}{dx} = -\frac{1}{y^6}\frac{dy}{dx}$$

from ②

$$\Rightarrow -\frac{1}{5}\frac{du}{dx} + \frac{1}{x}u = x^2$$

$$\Rightarrow \frac{du}{dx} - \frac{5}{x}u = -5x^2$$

This is a linear diff. Eq'n in u

$\Rightarrow$ Take $P = -\frac{5}{x}$; $Q = -5x^2$

$$I.f = e^{\int P dx} = e^{-5\int \frac{1}{x}dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5} = \frac{1}{x^5}$$

The general solution equation is $u(I.f) = \int Q(I.f)\,dx + c$

$$\frac{1}{y^5}\left(\frac{1}{x^5}\right) = \int -5x^2\left(\frac{1}{x^5}\right)dx + c$$

$$= -5\int \frac{1}{x^3}dx + c$$

$$= -5\int x^{-3}dx + c$$

$$= -5\,\frac{x^{-3+1}}{-3+1} + c$$

$$\frac{1}{x^5y^5} = -5\,\frac{x^{-2}}{-2} + c$$

$\therefore$ The required general solution is

$$\frac{1}{x^5y^5} = \frac{5}{2}\frac{1}{x^2} + c$$

<u>Bernouli's Equation in x</u> :

<u>Problem</u>: Solve $\frac{dy}{dx}(x^2y^3 + xy) = 1$

<u>Solution</u>: Given equation is $\frac{dy}{dx}(x^2y^3 + xy) = 1$

$$\Rightarrow x^2y^3 + xy = \frac{dx}{dy}$$

$$\Rightarrow x^2y^3 = \frac{dx}{dy} - xy$$

$$\Rightarrow \frac{dx}{dy} - xy = x^2y^3$$

$$\Rightarrow \frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}\cdot y = y^3$$

Which is a Bernoulli's equation.

Take $u = \frac{1}{x}$

diff. w.r.to y

$$\frac{du}{dy} = -\frac{1}{x^2}\frac{dx}{dy}$$

$$-\frac{du}{dy} - u \cdot y = y^3$$

$$\frac{du}{dy} + u \cdot y = -y^3$$

Where $P = y$; $Q = -y^3$

$$I.f = e^{\int P dy} = e^{\int y dy} = e^{y^2/2}$$

The general solution is $u(I.f) = \int Q(I.f)\,dy + c$

$$\frac{1}{x}(e^{y^2/2}) = \int(-y^3 e^{y^2/2})\,dy + c$$

Take $y^2 = t$

$2y\,dy = dt$

$y\,dy = \frac{1}{2}dt$

$$= -\int y^2 \cdot y e^{y^2/2}\,dy + c$$

$$= -\int t \cdot e^{t/2}\frac{dt}{2} + c$$

$$= -\frac{1}{2}\int(t e^{t/2})\,dt + c$$

$$= -\frac{1}{2}\left[t\int e^{t/2}dt - \int \frac{d}{dt}(t)\int e^{t/2}dt\right] + c$$

$$= -\frac{1}{2}\left[t\frac{e^{t/2}}{1/2} - \int 1\,\frac{e^{t/2}}{1/2}dt\right] + c$$

$$= -\frac{1}{2}\left[2te^{t/2} - 2\int e^{t/2}dt\right] + c$$

$$= -\frac{1}{2}\left[2te^{t/2} - 2\frac{e^{t/2}}{1/2}\right] + c$$

$$= -\frac{1}{2}\left(2te^{t/2} - 4\left(e^{t/2}\right)\right) + c$$

$$= -\frac{2}{2}\left(te^{t/2} - 2e^{t/2}\right) + c$$

$$= -\left(te^{t/2} - 2e^{t/2}\right) + c$$

$$= 2e^{t/2} - te^{t/2} + c$$

$$\frac{1}{x}\left(e^{y^2/2}\right) = e^{t/2}(2-t) + c$$

$\therefore$ The required general solution is

$$\frac{1}{x}\left(e^{y^2/2}\right) = e^{y^2/2}(2-y^2) + c$$

Exact differential Equations:

Let $M(x,y)dx + N(x,y)dy = 0$ be a first order and first degree differential equation where M, N are real valued functions defined for some real x, y on some rectangle $R: |x-x_0| \le a, |y-y_0| \le b$. Then the equation $Mdx + Ndy = 0$ is said to be an exact differential equation. if there exists a function $f(x,y)$ having continuous first partial derivatives in R such that $d[f(x,y)] = M(x,y)dx + N(x,y)dy$

$$\Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

Working rule for Solving an Exact differential equation:

1. compare the given equation with $Mdx + Ndy = 0$ and find out M and N. Then find out $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$

2. Given equation will be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

3. Integrate M partially with respect to x, treating y as constant. Denote this by $\int M dx$.

4. Integrate only those terms of N, which do not contain x, with respect to y.

5. Equate the sum of the results obtained from (3) and (4) to a constant to obtain the required solution.

$\therefore$ The general solution of the given exact differential equation is $\int M dx + \int (\text{terms of } N \text{ not involving } x) dy = c$

<u>Problem:</u> Solve $(e^y+1)\cos x\, dx + e^y \sin x\, dy = 0$

<u>Solution:</u> Given equation is $(e^y+1)\cos x\, dx + e^y \sin x\, dy = 0 \rightarrow ①$

equation ① is the form $M dx + N dy = 0$

where $M = (e^y+1)\cos x$ and $N = e^y \sin x$

Now $\frac{\partial M}{\partial y} = e^y \cos x$ $\qquad \frac{\partial N}{\partial x} = e^y \cos x$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

equation ① is an exact equation.

The general solution is $\int M dx + \int (\text{terms of } N \text{ involving } x) dy = c$

$\int (e^y+1)\cos x\, dx + \int 0\, dy = c$

$e^y+1 \int \cos x\, dx = c$

$(e^y+1)\int \cos x\, dx = c$

$(e^y+1)\sin x = c$

$\therefore$ The required general solution is $(e^y+1)\sin x = c$.

<u>Problem:</u> Solve $(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$

<u>Solution:</u> Given equation is $(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$
$\rightarrow ①$

Equation ① is the form $M dx + N dy = 0$

Where $M = 2xy + y - \tan y$ and $N = x^2 - x\tan^2 y + \sec^2 y$

$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y$ $\qquad \frac{\partial N}{\partial x} = 2x - \tan^2 y$

$= 2x - \tan^2 y$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The given differential equation is an Exact.

The general Solution is $\int (2xy+y-\tan y)dx + \int \sec^2 y\, dy = c$

$2y\int x\,dx + y\int dx - \tan y\int dx + \int \sec^2 y\, dy = c$

$2y\frac{x^2}{2} + yx - \tan y \cdot x + \tan y = c$

$x^2y + xy - x\tan y + \tan y = c$

$x^2y + (y-\tan y)x + \tan y = c$

$\therefore$ The required general Solution is

$x^2y + (y-\tan y)x + \tan y = c$

Problem: Show that the equation $(y^2e^{xy^2}+4x^3)dx + (2xye^{xy^2}-3y^2)dy = 0$ is an exact and hence Solve it.

Solution: Given Equation is $(y^2e^{xy^2}+4x^3)dx + (2xye^{xy^2}-3y^2)dy = 0 \rightarrow (1)$

Equation (1) is the form $Mdx + Ndy = 0$

Where $M = y^2e^{xy^2} + 4x^3$ and $N = 2xye^{xy^2} - 3y^2$

$\frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2(e^{xy^2})(2xy)$; $\frac{\partial N}{\partial x} = 2ye^{xy^2} + 2xye^{xy^2}(y^2)$

$\frac{\partial M}{\partial y} = 2ye^{xy^2} + 2xy^3e^{xy^2}$; $\frac{\partial N}{\partial x} = 2ye^{xy^2} + 2xy^3e^{xy^2}$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

which is an Exact differential equation.

The general Solution is $\int (y^2e^{xy^2}+4x^3)dx + \int -3y^2dy = C$

$y^2\int e^{xy^2}dx + 4\int x^3dx - 3\int y^2dy = C$

$y^2\frac{e^{xy^2}}{y^2} + 4\frac{x^4}{4} - 3\frac{y^3}{3} = c$

$\Rightarrow e^{xy^2} + x^4 - y^3 = C$

Therefore, the required general Solution is

$e^{xy^2} + x^4 - y^3 = C$

Problem: Solve $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+(x+\log x-x\sin y)dy=0$

Solution: Give equation is $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+(x+\log x-x\sin y)dy=0 \rightarrow (1)$

Equation (1) is the form $Mdx+Ndy=0$

Where $M=y\left(1+\frac{1}{x}\right)+\cos y$ and $N=x+\log x-x\sin y$

$$\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y \qquad \frac{\partial N}{\partial x}=1+\frac{1}{x}-\sin y$$

$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ is an exact differential equation.

The general solution is $\int\left(y\left(1+\frac{1}{x}\right)+\cos y\right)dx+\int 0\,dy=c$

$\Rightarrow y(x+\log x)+x\cos y=c.$

Problem: Solve $(x^2-2xy-y^2)dx-(x+y)^2dy=0$

Solution: Given equation is $(x^2-2xy-y^2)dx-(x+y)^2dy=0 \rightarrow (1)$

Where $M=x^2-2xy-y^2$ and $N=-(x+y)^2=-x^2-2xy-y^2$

$$\frac{\partial M}{\partial y}=-2x-2y \qquad \frac{\partial N}{\partial x}=-2x-2y$$

$$=-2(x+y \qquad =-2(x+y$$

$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

The Equation (1) is an exact diff. equation.

The general solution is $\int(x^2-2xy-y^2)dx+\int -y^2dy=c$

$$\int x^2dx-2y\int x\,dx-y^2\int dx-\int y^2dy=c$$

$$\frac{x^3}{3}-2y\frac{x^2}{2}-y^2x-\frac{y^3}{3}=c$$

$$x^3-3x^2y-3xy^2-y^3=3c$$

Problem: Solve $(1+e^{x/y})dx+e^{x/y}(1-x/y)dy=0$

Solution: Given equation is $(1+e^{x/y})dx+e^{x/y}(1-x/y)dy=0 \rightarrow (1)$

equation (1) is the form $Mdx+Ndy=0$

Where $M=1+e^{x/y}$ and $N=e^{x/y}(1-x/y)$

$$\frac{\partial M}{\partial y} = e^{x/y}\frac{\partial}{\partial y}(x/y) \;; \qquad \frac{\partial N}{\partial x} = e^{x/y}\frac{\partial}{\partial x}(x/y) - \frac{\partial}{\partial x}\left(e^{x/y}\cdot x/y\right)$$

$$= e^{x/y}x\left(-\frac{1}{y^2}\right) \;; \qquad = e^{x/y}\frac{1}{y} - \left(e^{x/y}\frac{\partial}{\partial x}(x/y) - \frac{x}{y}\frac{\partial}{\partial x}(e^{x/y})\right)$$

$$\frac{\partial M}{\partial y} = -e^{x/y}\frac{x}{y^2} \;; \qquad = \frac{e^{x/y}}{y} - \frac{e^{x/y}}{y} - \frac{x}{y}e^{x/y}\frac{1}{y}$$

$$\frac{\partial N}{\partial x} = -e^{x/y}\frac{x}{y^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Equation ① is an exact equation.

$\therefore$ The general solution is $\int (1+e^{x/y})\,dx = c$

$$\Rightarrow \int 1\,dx + \int e^{x/y}\,dx = c$$

$$\Rightarrow x + \frac{e^{x/y}}{\frac{1}{y}} = c$$

$$\Rightarrow x + ye^{x/y} = c$$

<u>Problem</u>: Solve $(xe^{xy}+2y)\frac{dy}{dx} + ye^{xy} = 0$

<u>Solution</u>: Given equation is $(xe^{xy}+2y)\frac{dy}{dx} + ye^{xy} = 0 \rightarrow ①$

$$(xe^{xy}+2y)\,dy + ye^{xy}\,dx = 0$$

$$ye^{xy}\,dx + (xe^{xy}+2y)\,dy = 0$$

which is the form of $M\,dx + N\,dy = 0$

where $M = ye^{xy}$ and $N = xe^{xy}+2y$

$$\frac{\partial M}{\partial y} = y\frac{\partial}{\partial y}(e^{xy}) + e^{xy}\frac{\partial}{\partial y}(y) \;; \qquad \frac{\partial N}{\partial x} = x\frac{\partial}{\partial x}e^{xy} + e^{xy}\frac{\partial}{\partial x}(x)$$

$$\frac{\partial M}{\partial y} = ye^{xy}x + e^{xy} \;; \qquad \frac{\partial N}{\partial x} = xe^{xy}y + e^{xy}$$

$$\frac{\partial M}{\partial y} = e^{xy}(xy+1) \;; \qquad \frac{\partial N}{\partial x} = e^{xy}(xy+1)$$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is an exact equation.

$\therefore$ The general solution is $\int y e^{xy} dx + \int 2y dy = c$

$$y \int e^{xy} dx + 2\int y\, dy = c$$

$$y \frac{e^{xy}}{y} + 2\frac{y^2}{2} = c$$

$$\therefore e^{xy} + y^2 = c$$

<u>Problem:</u> Show that the equation $x dx + y dy = \frac{a^2(x dy - y dx)}{x^2+y^2}$ is exact and hence solve it.

<u>Solution:</u> Given equation is $x dx + y dy = \frac{a^2(x dy - y dx)}{x^2+y^2}$

$$x dx + y dy = \frac{a^2 x dy - a^2 y dx}{x^2+y^2}$$

$$x dx + y dy = \frac{a^2 x\, dy}{x^2+y^2} - \frac{a^2 y}{x^2+y^2} dx$$

$$x dx + y dy - \frac{a^2 x}{x^2+y^2} dy + \frac{a^2 y}{x^2+y^2} dx$$

$$\left(x + \frac{a^2 y}{x^2+y^2}\right) dx + \left(y - \frac{a^2 x}{x^2+y^2}\right) dy = 0$$

Where $M = x + \frac{a^2 y}{x^2+y^2}$; $N = y - \frac{a^2 x}{x^2+y^2}$

$$\frac{\partial M}{\partial y} = a^2 \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= a^2 \left[\frac{x^2+y^2 \frac{\partial}{\partial y}(y) - y \frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}\right]$$

$$= a^2 \left[\frac{x^2+y^2 - y(2y)}{(x^2+y^2)^2}\right]$$

$$= a^2 \left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]$$

$$\frac{\partial M}{\partial y} = a^2 \left[\frac{x^2-y^2}{(x^2+y^2)^2}\right]$$

Similarly $\frac{\partial N}{\partial x} = a^2\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right]$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The given equation is an exact

The general solution is $\int\left(x+\frac{a^2y}{x^2+y^2}\right)dx + \int y\,dy = c$

$\int x\,dx + a^2y\int\frac{1}{x^2+y^2}dx + \int y\,dy = c$

$\frac{x^2}{2} + a^2y\left(\frac{1}{y}\tan^{-1}(x/y)\right) + \frac{y^2}{2} = c$

$\Rightarrow x^2 + 2a^2\tan^{-1}(x/y) + y^2 = 2c$

$\therefore$ The required general Solution of eq'n ① is

$x^2+2a^2\tan^{-1}(x/y)+y^2=2c$

Problem: Solve $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$ and show that this differential equation represents a family conics.

Solution: Given equation is $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$

$(hx+by+f)dy + (ax+hy+g)dx = 0$

$(ax+hy+g)dx + (hx+by+f)dy = 0 \longrightarrow ①$

equation ① is the form $Mdx+Ndy=0$

Where $M=ax+hy+g$ and $N=hx+by+f$

$\frac{\partial M}{\partial y} = h$ $\qquad$ $\frac{\partial N}{\partial x} = h$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Equation ① is an exact diff. equation.

$\therefore$ The general solution is

$\int(ax+hy+g)dx + \int(by+f)dy = c$

$a\frac{x^2}{2} + hxy + gx + b\frac{y^2}{2} + fy = c$

$ax^2+2hxy+2gx+by^2+2fy=2c$

Equations reducible to exact form:

Integrating factors: Let $M(x,y)dx + N(x,y)dy = 0$ be not an exact differential equation. $Mdx+Ndy=0$ can be made exact by multiplying it with a suitable function $\mu(x,y) \neq 0$ then $\mu(x,y)$ is called an integrating of $Mdx+Ndy=0$

Example: Let $ydx - xdy = 0 \rightarrow ①$ Where $M = y, N = -x$

Then $\frac{\partial M}{\partial y} = 1$; $\frac{\partial N}{\partial x} = -1 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ is not an exact equation.

Multiplying (1) with $\frac{1}{x^2}$, we get $(y/x^2)dx - (1/x)dy = 0$

where $M = \frac{y}{x^2}$; $N = \frac{-1}{x}$ —— ②

Since $\frac{\partial M}{\partial y} = \frac{1}{x^2}$; $\frac{\partial N}{\partial x} = \frac{1}{x^2}$

equation ② is an exact equation.

Hence $\frac{1}{x^2}$ is an integrating factor of $\frac{ydx - xdy}{xy}$ and

$d\left[\operatorname{Tan}^{-1}(x/y)\right] = \frac{ydx - xdy}{x^2+y^2}$, the functions $\frac{1}{y^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}$ are also integrating factors of $ydx - xdy = 0$.

Methods to find integrating factors of $Mdx+Ndy=0$

Methods to find integrating factor (I.F)

Method 1: By inspection: An integrating factor can be found by inspection of given equation $Mdx+Ndy=0$ can be found by inspection of the given as explained below. By rearranging the terms of the given equation or by dividing with a suitable function x and y, the equation thus obtained will contain several parts integrable easily. In this connection the following exact differentials will be found useful.

(i) $d(xy) = xdy + ydx$

(ii) $d[\log(xy)] = \frac{xdy + ydx}{xy}$

(iii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iv) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

(v) $d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$

(vi) $d\left(\frac{x^2}{y}\right) = \frac{2xydx - y^2dy}{y^2}$

(vii) $d\left(\frac{y^2}{x^2}\right) = \frac{2x^2y\,dy - 2xy^2\,dx}{x^4}$ (xi) $d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$

(viii) $d\left(\frac{x^2}{y^2}\right) = \frac{2y^2x\,dx - 2yx^2\,dy}{y^4}$ (xii) $d\left(\frac{e^y}{x}\right) = \frac{xe^y dy - e^y dx}{x^2}$

(ix) $d\left(\mathrm{Tan}^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2+y^2}$ (xiii) $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{xy}$

(x) $d\left(\mathrm{Tan}^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{x^2+y^2}$ (xiv) $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{xy}$

<u>Problem</u>: Solve $x\,dy - y\,dx = xy^2\,dx$

<u>Solution</u>: Given equation is $x\,dy - y\,dx = xy^2\,dx \rightarrow (1)$

dividing (1) by $y^2 \Rightarrow \frac{x\,dy - y\,dx}{y^2} = x\,dx$

$\Rightarrow x\,dx + \frac{y\,dx - x\,dy}{y^2} = 0$

$\Rightarrow x\,dx + d(x/y) = 0$

Integrating $\frac{x^2}{2} + \frac{x}{y} = c$

$\therefore$ The general solution (1) is $\frac{x^2}{2} + \frac{x}{y} = c$

<u>Problem</u>: Solve $(1+xy)x\,dy + (1-yx)y\,dx = 0$

<u>Solution</u>: Given equation is $(1+xy)x\,dy + (1-yx)y\,dx = 0 \rightarrow (1)$

$\Rightarrow x\,dy + x^2y\,dy + y\,dx - y^2x\,dx = 0$

$\Rightarrow x\,dy + y\,dx + (x\,dy - y\,dx)xy = 0 \longrightarrow (2)$

Multiplying (2) with $\frac{1}{x^2y^2} \Rightarrow \frac{x\,dy + y\,dx}{x^2y^2} + \frac{x\,dy - y\,dx}{xy} = 0$

$\Rightarrow \frac{d(xy)}{x^2y^2} + \frac{1}{y}dy - \frac{1}{x}dx = 0$

Integrating: $\int \frac{d(xy)}{x^2y^2} + \int \frac{1}{y}dy - \int \frac{1}{x}dx = c$

$-\frac{1}{xy} + \log y - \log x = c.$

$\therefore$ The general solution of (1) is $xy\log\left(\frac{y}{x}\right) - 1 = cxy$

<u>Problem</u>: Solve $x\,dx + y\,dy + \frac{x\,dx - y\,dx}{x^2+y^2} = 0$

<u>Solution</u>: Given equation is $x\,dx + y\,dy + \frac{x\,dx - y\,dy}{x^2+y^2} = 0$

$\Rightarrow x\,dx + y\,dy + \frac{(x\,dy - y\,dx)/x^2}{1 + (y^2/x^2)} = 0 \longrightarrow (1)$

$$\Rightarrow x\,dx + y\,dy + \frac{d(y/x)}{1+(y/x)^2} = 0$$

$$\Rightarrow d\left(\frac{x^2+y^2}{2}\right) + \frac{d(y/x)}{1+(y/x)^2} = 0$$

Integrating, we get $\frac{x^2+y^2}{2} + \tan^{-1}\left(\frac{y}{x}\right) = c$

$\therefore$ The general Solution of ① is $(x^2+y^2) + 2\tan^{-1}(y/x) = 2c$

Problem: Solve $y\,dx - x\,dy + \log x\,dx = 0$

Solution: Given Equation is $y\,dx - x\,dy + \log x\,dx = 0 \rightarrow$ ①

$$\Rightarrow \log x\,dx - (x\,dy - y\,dx) = 0$$

Multiplying with $\frac{1}{x^2} \Rightarrow \frac{1}{x^2}\log x\,dx - \frac{(x\,dy - y\,dx)}{x^2} = 0$

$$\Rightarrow \frac{1}{x^2}\log x\,dx - d\left(\frac{y}{x}\right) = 0$$

Integrating $\int \frac{1}{x^2}\log x\,dx - \int d\left(\frac{y}{x}\right) = c$

$$\Rightarrow -\frac{1}{x}\log x - \int\left(-\frac{1}{x}\right)\cdot\frac{1}{x}\,dx - \frac{y}{x} = c$$

$\therefore$ The general solution is $cx + y + (1+\log x) = 0.$

Problem: Solve $x\,dy = [y + x\cos^2(y/x)]\,dx \rightarrow$ ①

Solution: Given equation is $x\,dy = [y + x\cos^2(y/x)]\,dx$

$$\Rightarrow x\,dy - y\,dx = x\cos^2(y/x)\,dx$$

dividing with x^2

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = \frac{1}{x}\cos^2\left(\frac{y}{x}\right)dx$$

$$\Rightarrow \sec^2 y/x \cdot \frac{x\,dy - y\,dx}{x^2} = \frac{1}{x}\,dx$$

$$\Rightarrow \sec^2\left(\frac{y}{x}\right) d(y/x) = \frac{1}{x}\,dx$$

Integrating $\int \sec^2(y/x)\,d(y/x) = \int (1/x)\,dx + c$

$$\Rightarrow \tan(y/x) = \log|x| + c.$$

$\therefore$ The general solution is $\tan(y/x) = \log|x| + c.$

Problem: Solve $(x^2+y^2+x)dx-(2x^2+2y^2-y)dy=0$

Solution: Given equation is $(x^2+y^2+x)dx-(2x^2+2y^2-y)dy=0 \rightarrow (1)$

$\Rightarrow (x^2+y^2)(dx-2dy)+xdx+ydy=0$

$\Rightarrow dx-2dy+\dfrac{xdx+ydy}{x^2+y^2}=0$

$\Rightarrow 2dx-4dy+\dfrac{2xdx+2ydy}{x^2+y^2}=0$

$\Rightarrow 2dx-4dy+d\log(x^2+y^2)=0$

$\Rightarrow 2\int dx-4\int dy+\int d\log(x^2+y^2)=c$

$\Rightarrow 2x-4y+\log(x^2+y^2)=c$

$\therefore$ The general solution is $2x-4y+\log(x^2+y^2)=c$.

Problem: Solve $(x^2+y^2-2y)dy=2xdx$.

Solution: Given equation is $(x^2+y^2-2y)dy=2xdx \rightarrow (1)$

$\Rightarrow (x^2+y^2)dy=d(x^2+y^2)$

$\Rightarrow dy=\dfrac{d(x^2+y^2)}{x^2+y^2}$

$\Rightarrow \int dy=\int\dfrac{d(x^2+y^2)}{x^2+y^2}+c$

$y=\log(x^2+y^2)+c$

$\therefore$ The general solution is $y=\log(x^2+y^2)+c$.

Problem: Solve $ydx-xdy+(1+x^2)dx+x^2\sin y\,dy=0$

Solution: Given equation is $ydx-xdy+(1+x^2)dx+x^2\sin y\,dy=0 \rightarrow (1)$

dividing (1) by $x^2 \Rightarrow \dfrac{ydx-xdy}{x^2}+\left(\dfrac{1}{x^2}+1\right)dx+\sin y\,dy=0$

$\Rightarrow -\dfrac{xdy-ydx}{x^2}+\left(\dfrac{1}{x^2}+1\right)dx+\sin y\,dy=0$

$\Rightarrow -d(y/x)+\left(\dfrac{1}{x^2}+1\right)dx+\sin y\,dy=0$

Integrating $-\int d(y/x)+\int\left(\dfrac{1}{x^2}+1\right)dx+\int\sin y\,dy=c$

$\Rightarrow -\left(\dfrac{y}{x}\right)-\left(\dfrac{1}{x}\right)+x-\cos y=c$

$\therefore$ The general solution is $x^2-y-1-x\cos y=cx$

Problem: Solve $y(2x^2+e^x)dx-(e^x+y^3)dy=0$

Solution: Given equation is $y(2x^2+e^x)dx-(e^x+y^3)dy=0 \rightarrow (1)$

$\Rightarrow 2x^2y\,dx+ye^x dx-e^x dy-y^3dy=0$

dividing by $y^2 \Rightarrow 2x^2dx+\left(\frac{ye^x dx-e^x dy}{y^2}\right)-y\,dy=0$

$\Rightarrow 2x^2dx+d\left(\frac{e^x}{y}\right)-y\,dy=0$

Integrating: $2\int x^2dx+\int d\left(\frac{e^x}{y}\right)-\int y\,dy=c$

$$\Rightarrow \frac{2x^3}{3}+\frac{e^x}{y}-\frac{y^2}{2}=c$$

$\therefore$ The general solution is $\frac{2x^3}{3}+\frac{e^x}{y}-\frac{y^2}{2}=c$

Problem: Solve $(y-xy^2)dx-(x+x^2y)dy=0$

Solution: Given equation is $(y-xy^2)dx-(x+x^2y)dy=0 \rightarrow (1)$

$(y\,dx-x\,dy)-xy(y\,dx+x\,dy)=0 \rightarrow (2)$

dividing (2) by $xy \Rightarrow \left(\frac{dx}{x}-\frac{dy}{y}\right)-(y\,dx+x\,dy)=0$

$\Rightarrow \frac{dx}{x}-\frac{dy}{y}-d(xy)=0$

Integrating: $\int\frac{dx}{x}-\int\frac{dy}{y}-\int d(xy)=c$

$\Rightarrow \log x-\log y-xy=c$

$\therefore$ The general solution is $\log(x/y)-xy=c$

Problem: Solve $x\,dy-y\,dx=a(x^2+y^2)dy$

Solution: Given equation is $x\,dy-y\,dx=a(x^2+y^2)dy \rightarrow (1)$

equation (1) can be written as $\frac{x\,dy-y\,dx}{x^2+y^2}=a\,dy$

$\Rightarrow d\left(\tan^{-1}\frac{y}{x}\right)=a\,dy$

Integrating: $\int d\left(\tan^{-1}\frac{y}{x}\right)=a\int dy \Rightarrow \tan^{-1}(y/x)=ay+c$

$\therefore$ The general solution is $\tan^{-1}(y/x)=ay+c$

Problem: Solve $y\,dx-x\,dy=3x^2e^{x^3}y^2dx$

Solution: Given equation is $y\,dx-x\,dy=3x^2e^{x^3}y^2dx \rightarrow (1)$

$\Rightarrow \frac{y\,dx-x\,dy}{y^2}=3x^2e^{x^3}dx \Rightarrow d(x/y)=3x^2e^{x^3}dx$

$\Rightarrow \int d(x/y)=\int 3x^2e^{x^3}dx+c \Rightarrow x=ye^{x^3}+cy$

$\therefore$ The general solution is $x=ye^{x^3}+cy$

Problem: Solve $(1+xy)y\,dx + (1-xy)x\,dy = 0$

Solution: Given equation is $(1+xy)y\,dx + (1-xy)x\,dy \rightarrow ①$

$y\,dx + xy^2\,dx + x\,dy - x^2y\,dy = 0$

$\Rightarrow y\,dx + x\,dy + xy^2\,dx - x^2y\,dy = 0$

$y\,dx + x\,dy + xy(y\,dx - x\,dy) = 0$

divide $(xy)^2$ in Eq^n ②

$$\frac{y\,dx + x\,dy}{(xy)^2} + \frac{y\,dx - x\,dy}{xy} = 0$$

$$\frac{d(xy)}{(xy)^2} + d(\log(x/y)) = 0$$

Integrating: $\int \frac{1}{(xy)^2}\,d(xy) + \int d[\log(x/y)] = c$

$$-\frac{1}{xy} + \log(x/y) = c$$

Method II: To find an integrating factor of $M\,dx + N\,dy = 0$

Working rule to Solve $M\,dx + N\,dy = 0$

1. General equation is $M\,dx + N\,dy = 0 \rightarrow ①$. observe $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\hookrightarrow ①$ is not exact.

2. observe M and N are homogeneous functions of same order

3. find $Mx + Ny$ and observe it $\neq 0$. Then $\frac{1}{Mx+Ny}$ is an Integrating factor of (i)

4. Multiply (i) with I.F to transform it into an Exact equation of ① $M_1\,dx + N_1\,dy = 0 \rightarrow$ (ii)

5. Solve (ii) to get the general solution of (i)

Problem: Solve $x^2y\,dx-(x^3+y^3)dy=0$

Solution: Given equation is $x^2y\,dx-(x^3+y^3)dy=0 \rightarrow (1)$

Comparing (1) with $M\,dx+N\,dy=0$

$\Rightarrow M=x^2y \quad ; \quad N=-(x^3+y^3)$

$\frac{\partial M}{\partial y}=x^2 \quad ; \quad \frac{\partial N}{\partial x}=-3x^2$

$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \Rightarrow$ (1) is not Exact Equation.

But (1) is homogeneous equation in x and y.

Now $Mx+Ny=x^3y-x^3y-y^4=-y^4\neq 0$

$\therefore$ Integrating factor $I.F=\frac{1}{Mx+Ny}=-\frac{1}{y^4}$

Multiplying (1) by $(-1/y^4) \Rightarrow -\frac{x^2}{y^3}dx+\left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy=0 \rightarrow (2)$

(2) is in the form $M_1dx+N_1dy=0$

Where $M_1=-\frac{x^2}{y^3}$ and $N_1=\frac{x^3}{y^4}+\frac{1}{y}$

Since $\frac{\partial M_1}{\partial y}=\frac{3x^2}{y^4}$ and $\frac{\partial N_1}{\partial x}=\frac{3x^2}{y^4}$

$\Rightarrow \frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$, equation (2) is an Exact Equation.

The general solution is $\int M_1dx+\int(\text{terms } N_1 \text{ not involving } x)dy=c$

$\int -\frac{x^2}{y^3}dx+\int\frac{1}{y}dy=c$

$\Rightarrow -\frac{1}{y^3}\int x^2dx+\int\frac{1}{y}dy=c$

$\frac{-x^3}{3y^3}+\log y=c$

$\therefore$ The general Solution is $\frac{-x^3}{3y^3}+\log y=c$

Problem: Solve $y^2dx+(x^2-xy-y^2)dy=0$

Solution: Given equation is $y^2dx+(x^2-xy-y^2)dy=0 \rightarrow (1)$

Comparing (1) with $M\,dx+N\,dy=0$

$\Rightarrow M=y^2 \quad ; \quad N=x^2-xy-y^2$

$$\frac{\partial M}{\partial y} = 2y \; ; \; \frac{\partial N}{\partial x} = 2x - y$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation ① is not exact.

But ① is homogeneous equation in x and y

$$Mx + Ny = xy^2 + x^2y - xy^2 - y^3 = y(x^2 - y^2) \neq 0$$

$$\text{I.F} = \frac{1}{Mx+Ny} = \frac{1}{y(x^2-y^2)}$$

Multiplying ① by $\frac{1}{y(x^2-y^2)} \Rightarrow \frac{y}{x^2-y^2}\,dx + \frac{x^2-xy-y^2}{y(x^2-y^2)}\,dy = 0$ —— ②

Equation ② is in the form $M_1 dx + N_1 dy = 0$

Where $M_1 = \frac{y}{x^2-y^2}$ and $N = \frac{x^2-xy-y^2}{y(x^2-y^2)} = \frac{1}{y} - \frac{x}{x^2-y^2}$

$$\frac{\partial M_1}{\partial y} = \frac{(x^2-y^2)\frac{\partial}{\partial y}(y) - y\frac{\partial}{\partial y}(x^2-y^2)}{(x^2-y^2)^2} \; ; \; \frac{\partial N_1}{\partial x} = -\left[\frac{(x^2-y^2)\frac{\partial}{\partial x}(x) - x\frac{\partial}{\partial x}(x^2-y^2)}{(x^2-y^2)^2}\right]$$

$$= \frac{x^2-y^2(1) - y(-2y)}{(x^2-y^2)^2} \; ; \; \frac{\partial N_1}{\partial x} = \frac{x^2-y^2-x(2x)}{(x^2-y^2)^2}$$

$$= \frac{x^2-y^2+2y^2}{(x^2-y^2)^2} = \frac{x^2+y^2}{(x^2-y^2)^2} \; ; \; \frac{\partial N_1}{\partial x} = \frac{-(-x^2-y^2)}{(x^2-y^2)^2} = \frac{x^2+y^2}{(x^2-y^2)^2}$$

$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$; equation ② is an exact equation.

The general solution is $\int M_1 dx + \int(\text{terms of } N_1 \text{ not involving } x)\,dy = c$

$$\int \frac{y}{x^2-y^2}\,dx + \int \frac{1}{y}\,dy = c$$

$$y\int \frac{1}{x^2-y^2}\,dx + \int \frac{1}{y}\,dy = c$$

$$y\left(\frac{1}{2y}\log\frac{x-y}{x+y}\right) + \log y = \log c$$

$$\frac{1}{2}\log\frac{x-y}{x+y} + \log y = \log c$$

$$\Rightarrow \log y\sqrt{\frac{x-y}{x+y}} = \log c$$

$$\Rightarrow y\sqrt{\frac{x-y}{x+y}} = c$$

$$\Rightarrow y^2(x-y) = c^2(x+y)$$

$\therefore$ The general Solution is $y^2(x-y) = c^2(x+y)$

Problem: Solve $y^2+x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

Solution: Given Equation is $y^2+x^2\frac{dy}{dx} = xy\frac{dy}{dx} \rightarrow (1)$

$$\Rightarrow y^2dx + (x^2-xy)dy = 0 \rightarrow (2)$$

comparing (2) with $Mdx+Ndy=0$

$\Rightarrow M = y^2$; $N = x^2-xy$

$\frac{\partial M}{\partial y} = 2y$; $\frac{\partial N}{\partial x} = 2x-y$; equation (2) is not exact

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

But (1) is a homogeneous Equation in x and y.

Integrating factor $\frac{1}{Mx+Ny} = \frac{1}{xy^2+x^2y-xy^2} = \frac{1}{x^2y} \neq 0$

Multiplying Equation (2) with $\frac{1}{x^2y}$, we get

$$\frac{y^2}{x^2y}dx + \frac{x^2-xy}{x^2y}dy = 0$$

$$\Rightarrow \frac{y}{x^2}dx + \left(\frac{1}{y}-\frac{1}{x}\right)dy = 0 \rightarrow (3)$$

Equation (3) is in the form $M_1dx+N_1dy=0$

where $M_1 = \frac{y}{x^2}$ and $N_1 = \frac{1}{y}-\frac{1}{x}$

$\frac{\partial M_1}{\partial y} = \frac{1}{x^2}$; $\frac{\partial N_1}{\partial x} = \frac{1}{x^2}$

Equation (3) is an Exact Equation.

The general Solution is $\int\frac{y}{x^2}dx + \int\frac{1}{y}dy = c$

$$-y\int\frac{1}{x^2}dx + \int\frac{1}{y}dy = c$$

$\Rightarrow -(y/x) + \log y = -\log c$

$\therefore$ The general solution is $\log c + \log y = y/x$

$\Rightarrow y/x = \log(cy)$

Problem: Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Solution: Given equation is $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0 \rightarrow (1)$

Comparing the equation (1) in form $Mdx + Ndy = 0$

$\Rightarrow M = x^2y - 2xy^2$ and $N = -(x^3 - 3x^2y)$

$\frac{\partial M}{\partial y} = x^2 - 4xy$ $\qquad \frac{\partial N}{\partial x} = -3x^2 + 6xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$; equation (1) is not an exact.

$Mx + Ny = x^3y - 2x^2y^2 + 3x^2y^2 - x^3y = x^2y^2 \neq 0$

$\frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$ is an Integrating factor.

Multiplying equation (1) with $\frac{1}{x^2y^2}$

$$\frac{x^2y - 2xy^2}{x^2y^2}dx - \frac{(x^3 - 3x^2y)}{x^2y^2}dy = 0$$

$$\frac{1}{y}dx - \frac{2}{x}dx - \frac{x}{y^2}dy + \frac{3}{y}dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(-\frac{x}{y^2} + \frac{3}{y}\right)dy = 0 \rightarrow (2)$$

Equation (2) in the form $M_1dx + N_1dy = 0$

Where $M_1 = \frac{1}{y} - \frac{2}{x}$ and $N_1 = -\frac{x}{y^2} + \frac{3}{y}$

$\frac{\partial M_1}{\partial y} = -\frac{1}{y^2}$ $\qquad \frac{\partial N_1}{\partial x} = \frac{-1}{y^2}$

$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$; equation (2) is an Exact.

The general Solution is $\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}dy = c$

$\frac{x}{y} - 2\log x + 3\log y = c.$

Method (iii): To find an integrating factor of $Mdx+Ndy=0$

Working rule to solve $Mdx+Ndy=0$

1. General Solution is $Mdx+Ndy=0 \rightarrow (i)$ observe $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\Rightarrow$ (i) is not Exact.

2. observe (i) is of the form $y\,f(xy)dx + x\,g(xy)dy=0$

3. find $Mx-Ny$ and observe it $\neq 0$. Then $\frac{1}{Mx-Ny}$ is an I.F of (i)

4. Multiply (i) with I.F to transform it into an Exact Equation of (i) $M_1dx+N_1dy=0 \rightarrow$ (ii)

5. Solve (ii) to get the general Solution of (i)

Problem: Solve $y(xy+2x^2y^2)dx + x(xy-x^2y^2)dy=0$

Solution: Given equation is $y(xy+2x^2y^2)dx+x(xy-x^2y^2)dy=0 \rightarrow (1)$

comparing (1) with $Mdx+Ndy=0$

$\Rightarrow M = y(xy+2x^2y^2)$ and $N = x(xy-x^2y^2)$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(xy^2+2x^2y^3)$ $\qquad$ $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2y-x^3y^2)$

$= 2xy+6x^2y^2$ $\qquad$ $= 2xy-6x^2y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ is not an Exact

$Mx-Ny = (xy^2+2x^2y^3)x - (x^2y-x^3y^2)y$

$= x^2y^2+2x^3y^3-x^2y^2+x^3y^3$

$= 3x^3y^3 \neq 0$

$\therefore$ I.F $= \frac{1}{Mx-Ny} = \frac{1}{3x^3y^3}$

Multiplying (1) by $\frac{1}{3x^3y^3}$

$$\Rightarrow \frac{xy^2+2x^2y^3}{3x^2y^3}dx + \frac{x^2y-x^3y^2}{3x^3y^3}dy=0 \rightarrow (2)$$

which is of the form $M_1dx+N_1dy=0$

Where $M_1 = \frac{xy^2 + 2x^2y}{3x^3y^3} = \frac{1}{3x^2y} + \frac{2}{3x}$

$$\frac{\partial M_1}{\partial y} = \frac{-1}{3x^2y^2}$$

$$N_1 = \frac{x^2y - x^3y^2}{3x^3y^3} = \frac{1}{3xy^2} - \frac{1}{3y}$$

$$\frac{\partial N_1}{\partial x} = \frac{-1}{3x^2y^2}$$

$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$; equation (2) is an exact

$\therefore$ The general solution is

$$\Rightarrow \int \left(\frac{1}{3x^2y} + \frac{2}{3x}\right) dx + \int -\frac{1}{3y}\, dy = c$$

$$\Rightarrow \frac{1}{3} \int \frac{1}{x^2y}\, dx + \frac{2}{3}\int \frac{1}{x}\, dx - \frac{1}{3}\int \frac{1}{y}\, dy = c$$

$$\Rightarrow \frac{-1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

$$\Rightarrow \frac{1}{3}\left(-\frac{1}{xy} + 2\log x - \log y\right) = c$$

$$\Rightarrow \frac{-1}{xy} + 2\log x - \log y = 3c$$

$\therefore$ The required general solution is

$$-\frac{1}{xy} + 2\log x - \log y = 3c$$

Problem: Solve $(xy \sin xy + \cos xy) y\, dx + (xy \sin xy - \cos xy) x\, dy = 0$

Solution: Given equation is

$(xy \sin xy + \cos xy) y\, dx + (xy \sin xy - \cos xy) x\, dy = 0 \longrightarrow (1)$

Comparing equation (1) with $M dx + N dy = 0$

Where $M = xy^2 \sin xy + \cos xy \cdot y$ and

$N = x^2y \sin xy - \cos xy \cdot x$

$$\frac{\partial M}{\partial y} = (x^2y^2+1)\cos xy + xy\sin xy \; ; \; \frac{\partial N}{\partial x} = 3xy\sin xy + (x^2y^2-1)\cos xy$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$; equation ① is not Exact

$$I.F = \frac{1}{Mx - Ny}$$

$$\Rightarrow Mx - Ny = x^2y^2\sin xy + xy\cos xy - x^2y^2\sin xy + xy\cos xy$$

$$= 2xy\cos xy \neq 0$$

$$I.f = \frac{1}{Mx-Ny} = \frac{1}{2xy\cos xy}$$

Multiplying ① with $\frac{1}{2xy\cos xy}$

$$\Rightarrow \frac{1}{2}\left(y\tan xy + \frac{1}{x}\right)dx + \frac{1}{2}\left(x\tan xy - \frac{1}{y}\right)dy = 0 \rightarrow ②$$

Equation ② is the form $M_1 dx + N_1 dy = 0$

When $M_1 = \frac{1}{2}\left(y\tan xy + \frac{1}{x}\right)$; $N_1 = \frac{1}{2}\left(x\tan xy - \frac{1}{y}\right)$

$$\frac{\partial M_1}{\partial y} = \frac{1}{2}(xy\sec^2 xy + \tan xy) ; \frac{\partial N_1}{\partial x} = \frac{1}{2}(xy\sec^2 xy + \tan xy)$$

$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N}{\partial x}$; equation ② is an Exact.

The general solution is

$$\int \frac{1}{2}\left(y\tan xy + \frac{1}{x}\right)dx + \int \frac{1}{2}\left(-\frac{1}{y}\right)dy = c$$

$$\frac{1}{2}\int y\tan xy\, dx + \frac{1}{2}\int \frac{1}{x}dx - \frac{1}{2}\int \frac{1}{y}dy = c$$

$$\Rightarrow \frac{1}{2}\, y\, \frac{\log|\sec(xy)|}{y} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\Rightarrow \frac{1}{2}\left(\frac{y\log|\sec(xy)}{y} + \log x - \log y\right) = c$$

$$\Rightarrow \log|\sec(xy)| + \log x - \log y = 2c$$

$\therefore$ The required general solution is

$$\log|\sec(xy)| + \log x - \log y = 2c$$

Problem: Solve $(x^3y^3+x^2y^2+xy+1)y\,dx+(x^3y^3-x^2y^2-xy+1)x\,dy$

Solution: Given equation is

$$(x^3y^3+x^2y^2+xy+1)y\,dx+(x^3y^3-x^2y^2-xy+1)x\,dy=0 \rightarrow (1)$$

equation (1) comparing with $M\,dx+N\,dy=0$

$M=x^3y^4+x^2y^3+xy^2+y$; $N=x^4y^3-x^3y^2-x^2y+x$

$\frac{\partial M}{\partial y}=4y^3x^3+3x^2y^2+2xy+1$ $\quad \frac{\partial N}{\partial x}=4x^3y^3-3x^2y^2-2xy+1$

$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$; equation (1) is not exact.

$$\begin{aligned} Mx-Ny &= (x^3y^4+x^2y^3+xy^2+y)x-(x^4y^3-x^3y^2-x^2y+x)y \\ &= x^4y^4+x^3y^3+x^2y^2+xy-x^4y^4+x^3y^3+x^2y^2-xy \\ &= 2x^3y^3+2x^2y^2 \\ &= 2x^2y^2(xy+1)\neq 0 \end{aligned}$$

$$\text{I.f}=\frac{1}{Mx-Ny}=\frac{1}{2x^2y^2(xy+1)} \longrightarrow (*)$$

multiplying (1) with $\frac{1}{2x^2y^2(xy+1)}$

$$\Rightarrow \frac{x^2y^2+1}{2x^2y}dx+\frac{(xy-1)^2}{2xy^2}\text{ and } \longrightarrow (2)$$

equation (2) comparing $M_1dx+N_1dy=0$

When $M_1=\frac{x^2y^2+1}{2x^2y}$; $N_1=\frac{(xy-1)^2}{2xy^2}$

$M_1=\frac{y}{2}+\frac{1}{2x^2y}$ $\quad N_1=\frac{x}{2}+\frac{1}{2xy^2}-\frac{1}{y}$

$\frac{\partial M_1}{\partial y}=\frac{1}{2}-\frac{1}{2x^2y^2}$ $\quad \frac{\partial N_1}{\partial x}=\frac{1}{2}-\frac{1}{2x^2y^2}$

$\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$; equation (2) is an exact.

$\therefore$ The general solution is

$$\int\left(\frac{y}{2}+\frac{1}{2x^2y}\right)dx+\int -\frac{1}{y}\,dy = c$$

$$\Rightarrow \frac{xy}{2}+\frac{1}{2y}\left(-\frac{1}{x}\right)-\log y = c$$

$$\Rightarrow xy-\frac{1}{xy}-\log y^2 = 2c.$$

Problem: solve $y[x^2y^2+xy+1]\,dx + x[x^2y^2-xy+1]\,dy = 0$.

Solution: Given equation is

$$y[x^2y^2+xy+1]\,dx + x[x^2y^2-xy+1]\,dy = 0 \longrightarrow (1)$$

Equation (1) compare with $M\,dx+N\,dy=0$

$$\Rightarrow M = x^2y^3+xy^2+y \quad ; \quad N = x^3y^2-x^2y+x$$

$$\frac{\partial M}{\partial y} = 3x^2y^2+2xy+1 \quad ; \quad \frac{\partial N}{\partial x} = 3x^2y^2-2xy+1$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$; equation (1) is not exact.

$$Mx-Ny = (x^2y^3+xy^2+y)x-(x^3y^2-x^2y+x)y$$

$$= x^3y^3+x^2y^2+xy-x^3y^3+x^2y^2-xy$$

$$= 2x^2y^2 \neq 0$$

$$\text{I.F} = \frac{1}{Mx-Ny} = \frac{1}{2x^2y^2}$$

Multiplying Equation (1) with $\frac{1}{2x^2y^2}$

$$\Rightarrow \left(\frac{y}{2}+\frac{1}{2x}+\frac{1}{2x^2y}\right)dx+\left(\frac{x}{2}+\frac{1}{2y}+\frac{1}{2xy^2}\right)dy = 0 \longrightarrow (2)$$

Comparing (2) with $M_1dx+N_1dy=0$

Where $M_1 = \frac{y}{2}+\frac{1}{2x}+\frac{1}{2x^2y}$; $N_1 = \frac{x}{2}+\frac{1}{2y}+\frac{1}{2xy^2}$

$$\frac{\partial M_1}{\partial y} = \frac{1}{2}-\frac{1}{2x^2y^2} \quad ; \quad \frac{\partial N}{\partial x} = \frac{1}{2}-\frac{1}{2x^2y^2}$$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$; Equation ② is an exact.

The general solution is

$$\int\left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \int\frac{1}{2y}\,dy = c$$

$$\frac{1}{2}\left(xy + \log x - \frac{1}{xy} + \log y\right) = c$$

$$\therefore\ xy + \log x - \frac{1}{xy} + \log y = 2c$$

<u>Problem</u>: Solve $y(1+xy)dx + x(1-xy)\,dy = 0$

<u>Solution</u>: Given equation is $y(1+xy)dx + x(1-xy)dy = 0 \to ①$

Compare equation ① with $Mdx + Ndy = 0$

$\Rightarrow M = y + xy^2$; $N = x - x^2y$

$\frac{\partial M}{\partial y} = 1 + 2xy$; $\frac{\partial N}{\partial x} = 1 - 2xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$; Equation ① is not exact.

Now $Mx - Ny = (y + xy^2)x - (x - x^2y)y$

$= xy + x^2y^2 - xy + x^2y^2$

$= 2x^2y^2$

$$I.f = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$$

multiplying equation ① with $\frac{1}{2x^2y^2}$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0 \longrightarrow ②$$

Comparing equation ② with $M_1dx + N_1dy = 0$

When $M_1 = \frac{1}{2x^2y} + \frac{1}{2x}$; $N_1 = \frac{1}{2xy^2} - \frac{1}{2y}$

$\frac{\partial M_1}{\partial y} = \frac{-1}{2x^2y^2}$; $\frac{\partial N_1}{\partial x} = \frac{-1}{2x^2y^2}$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$; equation ② is an Exact.

$\therefore$ The general Solution is

$$\int \left(\frac{1}{2x^2 y} + \frac{1}{2x}\right) dx + \int -\frac{1}{2y}\, dy = c$$

$$\frac{1}{2}\left(-\frac{1}{xy} + \log x - \log y\right) = c$$

$$-\frac{1}{xy} + \log x - \log y = 2c.$$

Therefore, the required general Solution is

$$\frac{-1}{xy} + \log x - \log y = 2c.$$

Method IV : To find an integrating factor of $Mdx + Ndy = 0$

Working rule to solve $Mdx + Ndy = 0$:

1. General Equation is $Mdx + Ndy = 0 \rightarrow$ ① observe $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ is not Exact

2. find $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ and observe it as a function of x alone $= f(x)$ or real constant k.

3. Then $e^{\int f(x)\,dx}$ or $e^{\int k\,dx}$ is an I.F of (i)

4. Multiply (i) with I.F to transform it into an exact Equation of (i) $M_1 dx + N_1 dy = 0 \rightarrow$ (ii)

5. Solve (ii) to get the general Solution of (i)

Problem: Solve $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right) dx + \frac{1}{4}(x + xy^2)\, dy = 0$

Solution: Given equation is $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right) dx + \frac{1}{4}(x + xy^2)\, dy = 0$ $\rightarrow$ ①

Equation ① compare with $Mdx + Ndy = 0$

where $M = y + \frac{y^3}{3} + \frac{x^2}{2}$ and $N = \frac{1}{4}(x + xy^2)$

Where $M = y + \frac{y^3}{3} + \frac{x^2}{2}$; $N = \frac{1}{4}(x + xy^2)$

$\frac{\partial M}{\partial y} = 1 + \frac{3y^2}{3}$ $\qquad \frac{\partial N}{\partial x} = \frac{1}{4}(1 + y^2)$

$\frac{\partial M}{\partial y} = 1 + y^2$ $\qquad \frac{\partial N}{\partial x} = \frac{1}{4}(1 + y^2)$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$; Equation ① is not Exact.

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\frac{1}{4}(x + xy^2)}\left[1 + y^2 - \frac{1}{4}(1 + y^2)\right]$

$$= \frac{4}{x(1+y^2)} \cdot \frac{3}{4}(1 + y^2) = \frac{3}{x} = f(x)$$

I.F $= e^{\int f(x)\,dx} = e^{3\int \frac{1}{x}dx} = e^{3\log x} = e^{\log x^3} = x^3$

Multiplying Equation ① with x^3

$\Rightarrow \left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)x^3\,dx + \frac{1}{4}(x + xy^2)x^3\,dy = 0.$

Where $M_1 = \left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)x^3$; $N_1 = \frac{1}{4}(x + xy^2)x^3$

$= x^3y + \frac{x^3y^3}{3} + \frac{x^5}{2}$; $N_1 = \frac{1}{4}(x^4 + x^4y^2)$

$\frac{\partial M_1}{\partial y} = x^3 + \frac{3x^3y^2}{3}$ $\qquad \frac{\partial N_1}{\partial x} = \frac{1}{4}(4x^3 + 4x^3y^2$

$= x^3 + x^3y^2$ $\qquad = x^3 + x^3y^2$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation ② is an Exact.

$\therefore$ The general solution is $\int\left(x^3y + \frac{x^3y^3}{3} + \frac{x^5}{2}\right)dx + \int 0\,dy = c$

$\frac{x^4y}{4} + \frac{x^4y^3}{12} + \frac{x^6}{12} = C$

$3x^4y + x^4y^3 + x^6 = C$

Problem: Solve $(x^2+y^2+2x)\,dx + 2y\,dy = 0$

Solution: Given equation is $(x^2+y^2+2x)\,dx + 2y\,dy = 0 \rightarrow (1)$

equation (1) compare with $M\,dx + N\,dy$

$\Rightarrow M = x^2+y^2+2x \quad ; \quad N = 2y$

$\frac{\partial M}{\partial y} = 2y \quad ; \quad \frac{\partial N}{\partial x} = 0$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$; equation (1) is not an exact.

Now $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{2y}(2y) = 1$

I.F $= e^{\int f(x)\,dx} = e^{\int 1\,dx} = e^x$

Multiplying equation (1) with e^x

$(x^2+y^2+2x)e^x\,dx + 2ye^x\,dy = 0$

Where $M_1 = (x^2+y^2+2x)e^x \quad ; \quad N_1 = 2ye^x$

$\frac{\partial M_1}{\partial y} = 2ye^x \quad ; \quad \frac{\partial N}{\partial x} = 2ye^x$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$; equation (2) is an exact.

The general solution $\int (x^2e^x + y^2e^x + 2xe^x)\,dx = c$

$x^2\int e^x dx - \int \frac{d}{dx}(x^2)\int e^x dx + y^2\int e^x dx + 2x\int e^x dx - \int \frac{d}{dx}(x)\int e^x dx = c$

$x^2e^x - \int 2xe^x dx + y^2e^x + 2(xe^x - e^x)$

$x^2e^x - 2\left(x\int e^x dx - \int \frac{d}{dx}(x)\int e^x dx\right) + y^2e^x + 2(xe^x - e^x) = c$

$x^2e^x - 2\left(xe^x - \int e^x dx\right) + y^2e^x + 2(xe^x - e^x) = c$

$x^2e^x - 2(xe^x - e^x) + y^2e^x + 2(xe^x - e^x) = c$

$x^2e^x + y^2e^x = c$

$(x^2+y^2)e^x = c$

$\therefore$ The general Solution is $(x^2+y^2)e^x = c$.

Problem: solve $(x^3-2y^2)dx+2xy\,dy=0$

Solution: Given equation is $(x^3-2y^2)dx+2xy\,dy=0 \rightarrow (1)$

Comparing Equation (1) with $Mdx+N\,dy=0$

$M=x^3-2y^2 \qquad N=2xy$

$\frac{\partial M}{\partial y}=-4y \qquad \frac{\partial N}{\partial x}=2y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$; equation (1) is not an exact.

Now $\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]=\frac{1}{2xy}\left[-4y-2y\right]=\frac{1}{2xy}\left[-6y\right]=\frac{-3}{x}$

I.f $= e^{\int f(x)dx} = e^{-3\int \frac{1}{x}dx} = e^{-3\log x} = e^{\log x^{-3}} = x^{-3} = \frac{1}{x^3}$

Multiplying equation (1) with $\frac{1}{x^3}$ we get.

$$\frac{-2y^2}{x^3}dx+\frac{2y}{x^2}dy=0 \rightarrow (2)$$

When $M_1=-\frac{2y^2}{x^3}$; $N_1'=\frac{2y}{x^2}$

$\frac{\partial M_1}{\partial y}=\frac{-4y}{x^3}$; $\frac{\partial N_1'}{\partial x}=\frac{-4y}{x^3}$

$\frac{\partial M_1}{\partial y}=\frac{\partial N_1'}{\partial x}$; equation (2) is an exact.

$\therefore$ The general solution is

$$\int \frac{-2y^2}{x^3}dx=c$$

$$-2y^2\int x^{-3}dx=c$$

$$-2y^2\left(-\frac{1}{2x^2}\right)=c$$

$$\frac{y^2}{x^2}=c$$

$\therefore$ The required general Solution $=\frac{y^2}{x^2}=c$.

Problem: Solve $2xy\,dy - (x^2+y^2+1)\,dx = 0$

Solution: Given equation is $2xy\,dy - (x^2+y^2+1)\,dx = 0 \rightarrow (1)$

Equation (1) is compare with $M\,dx + N\,dy = 0$

Where $M = -(x^2+y^2+1)$; $N = 2xy$

$$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2y$$

Now $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{2xy}\left[-2y-2y\right] = \frac{1}{2xy}\left[-4y\right] = -\frac{2}{x}$

$I.f = e^{\int f(x)\,dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$

Multiplying equation (1) with $\frac{1}{x^2}$; we get

$$\Rightarrow \left(\frac{2y}{x}\right) dy - \left[1+\frac{y^2}{x^2}+\frac{1}{x^2}\right] dx = 0$$

Consider $M_1 = -\left[1+\frac{y^2}{x^2}+\frac{1}{x^2}\right]$; $N_1 = \frac{2y}{x}$

$$\frac{\partial M_1}{\partial y} = -\frac{2y}{x^2} \quad ; \quad \frac{\partial N_1}{\partial x} = -\frac{2y}{x^2}$$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$; Equation (2) is an Exact

$\therefore$ The general solution is

$$\int -\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right) dx = c$$

$$\Rightarrow -\left[\int 1\,dx + y^2\int \frac{1}{x^2}\,dx + \int \frac{1}{x^2}\,dx\right] = c$$

$$\Rightarrow -\left[x - \frac{y^2}{x} + \frac{1}{x}\right] = c$$

$$\Rightarrow -x + \frac{y^2}{x^2} + \frac{1}{x} = c.$$

Method V : To find an integrating factor of $Mdx+Ndy=0$

Working rule to solve $Mdx+Ndy=0$:

1. General equation is $Mdx+Ndy=0 \rightarrow$ (i). observe $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ is not exact

2. find $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ and observe it as a function of y alone $g(y)$

3. Then $e^{\int g(y)dy}$ is an I.F of (i)

4. Multiply (i) with I.F to transform it into an exact equation of (i) $M_1dx+N_1dy=0 \rightarrow$ (ii)

5. Solve (ii) to get the general solution.

Problem : Solve $(xy^2-x^2)dx+(3x^2y^2+x^2y-2x^3+y^2)dy=0$

Solution: Given equation is $(xy^2-x^2)dx+(3x^2y^2+x^2y-2x^3+y^2)dy=0$

Where $M=xy^2-x^2$; $N=3x^2y^2+x^2y-2x^3+y^2$

$$\frac{\partial M}{\partial y}=2xy \quad ; \quad \frac{\partial N}{\partial x}=6xy^2+2xy-6x^2$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow$ ① is not an exact equation.

But
$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = \frac{1}{xy^2-x^2}(6xy^2+2xy-6x^2-2xy)$$
$$= \frac{1}{xy^2-x^2}6(xy^2-x^2) = 6 = g(y)$$

$$I.f = e^{\int g(y)dy} = e^{\int 6dy} = e^{6y}$$

Multiplying equation ① with e^{6y}

$$\Rightarrow (xy^2-x^2)e^{6y}dx+(3x^2y^2+x^2y-2x^3+y^2)e^{6y}dy=0$$

Where $M_1 = xy^2e^{6y}-x^2e^{6y}$; $N_1=(3x^2y^2+x^2y-2x^3+y^2)e^{6y}$

$$\frac{\partial M_1}{\partial y} = x\left(y^2\frac{\partial}{\partial y}e^{6y}+e^{6y}\frac{\partial}{\partial y}(y^2)\right)-x^2\frac{\partial}{\partial y}(e^{6y})$$
$$= x(y^2e^{6y}.6+e^{6y}2y)-x^2e^{6y}.6$$
$$= x(6y^2e^{6y}+e^{6y}2y-6x^2e^{6y})$$

$$\frac{\partial N_1}{\partial x} = 3y^2e^{6y}\frac{\partial}{\partial x}(x^2) + 2xye^{6y} - 6x^2e^{6y}$$
$$= x(6y^2e^{6y} + e^{6y}2y) - 6x^2e^{6y}$$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$; equation ② is an exact.

∴ The general solution is

$$\int(xy^2e^{6y} - x^2e^{6y})\,dx + \int y^2e^{6y}$$
$$y^2e^{6y}\int x\,dx - e^{6y}\int x^2dx + y^2\int e^{6y}dy - \int\frac{d}{dy}(y^2)\int e^{6y}dy = c$$
$$y^2e^{6y}\frac{x^2}{2} - e^{6y}\frac{x^3}{3} + y^2\frac{e^{6y}}{6} - \int 2y\frac{e^{6y}}{6}\,dy = c$$
$$y^2e^{6y}\frac{x^2}{2} - e^{6y}\frac{x^3}{3} + y^2\frac{e^{6y}}{6} - \frac{2}{6}\left[y\int e^{6y}dy - \int\frac{d}{dy}(y)\int e^{6y}dy\right] = c$$
$$y^2e^{6y}\frac{x^2}{2} - e^{6y}\frac{x^3}{3} + y^2\frac{e^{6y}}{6} - \frac{2}{6}\left[y\frac{e^{6y}}{6} - \int\frac{e^{6y}}{6}\right] = c$$
$$y^2e^{6y}\frac{x^2}{2} - e^{6y}\frac{x^3}{3} + y^2\frac{e^{6y}}{6} - \frac{1}{3}\left[y\frac{e^{6y}}{6} - \frac{1}{6}\frac{e^{6y}}{6}\right] = c.$$
$$\Rightarrow \frac{x^2y^2e^{6y}}{2} - \frac{x^3e^{6y}}{3} + \frac{e^{6y}y^2}{6} - \frac{1}{3}\left[\frac{ye^{6y}}{6} - \frac{e^{6y}}{36}\right] = c$$

∴ The required general solution is

$$\Rightarrow \frac{x^2y^2e^{6y}}{2} - \frac{x^3e^{6y}}{3} + \frac{e^{6y}y^2}{6} - \frac{1}{3}\left[\frac{ye^{6y}}{6} - \frac{e^{6y}}{36}\right] = c.$$

Problem: Solve $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$

Solution: Given equation is $(y^4+2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$ → ①

Compare ① with $Mdx + Ndy = 0$

Where $M = y^4 + 2y$; $N = xy^3 + 2y^4 - 4x$

$\frac{\partial M}{\partial y} = 4y^3 + 2$; $\frac{\partial N}{\partial x} = y^3 - 4$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$; Equation ① is not exact.

Now $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{y^4+2y}\left[y^3 - 4 - 4y^3 - 2\right]$

$$= \frac{1}{y(y^3+2)}\left[-3(y^2+2)\right]$$

$$= \frac{-3}{y}$$

$$\text{I.F} = e^{-3\int \frac{1}{y}dy} = e^{-3\log y} = e^{\log y^{-3}} = y^{-3} = \frac{1}{y^3}$$

Multiply with $\frac{1}{y^3}$ in Equation (1)

$$\Rightarrow \left(y+\frac{2}{y^2}\right)dx + \left(x+2y-\frac{4x}{y^3}\right)dy = 0 \rightarrow (2)$$

$$M_1 = y+\frac{2}{y^2} \quad ; \quad N_1 = x+2y-\frac{4x}{y^3}$$

$$\frac{\partial M_1}{\partial y} = 1-\frac{4}{y^3} \qquad \frac{\partial N_1}{\partial x} = 1-\frac{4}{y^3}$$

$\frac{\partial M_1}{\partial y} = \frac{\partial N}{\partial x}$; equation (2) is an exact.

The general solution is

$$\int\left(y+\frac{2}{y^2}\right)dx + \int 2y\,dy = c$$

$$y\int 1dx + \frac{2}{y^2}\int dx + 2\int y\,dy = c$$

$$yx + \frac{2}{y^2}x + \frac{2y^2}{2} = c$$

$$\Rightarrow xy + \frac{2x}{y^2} + y^2 = c$$

∴ The required general solution of (1) is

$$xy + \frac{2x}{y^2} + y^2 = c.$$

Problem: Solve $(xy^3+y)dx + 2(x^2y^2+x+y^4)dy = 0$

Solution: Given equation is $(xy^3+y)dx + 2(x^2y^2+x+y^4)dy = 0 \rightarrow (1)$

Compare (1) with $Mdx + Ndy = 0$

Where $M = xy^3+y$; $N = 2x^2y^2+2x+2y^4$

$\frac{\partial M}{\partial y} = 3xy^2+1$; $\frac{\partial N}{\partial x} = 4xy^2+2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$; equation (1) is not exact

Now $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{xy^3+y}\left[4xy^2+2-3xy^2-1\right]$

$= \frac{1}{y(xy^2+1)}\left[xy^2+1\right] = \frac{1}{y}$

I.F $= e^{\int g(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$

Multiplying equation (1) with y; we get

$\Rightarrow (xy^4+y^2)dx + (2x^2y^3+2xy+2y^5)dy = 0 \rightarrow (2)$

Where $M_1 = xy^4+y^2$ $N_1 = 2x^2y^3+2xy+2y^5$

$\frac{\partial M_1}{\partial y} = 4xy^3+2y$ $\frac{\partial N_1}{\partial x} = 4xy^3+2y$

$\frac{\partial M_1}{\partial y} = \frac{\partial N}{\partial x}$; equation is an exact.

$\therefore$ The general solution is

$\int (xy^4+y^2)dx + \int 2y^5 dy = c$

$y^4\int x\,dx + y^2\int dx + 2\int y^5 dy = c$

$\frac{x^2y^4}{2} + xy^2 + \frac{2y^6}{6} = c$

$3x^2y^4 + 6xy^2 + 2y^6 = 6c$

$\therefore$ The required general solution of (1) is

$3x^2y^4 + 6xy^2 + 2y^6 = 6c.$

Problem: Solve $(2x^2y - 3y^2)dx + (2x^3 - 12xy + \log y)dy = 0$

Solution: Given equation is $(2x^2y - 3y^2)dx + (2x^3 - 12xy + \log y)dy = 0 \rightarrow (1)$

Compare (1) with $Mdx + Ndy = 0$

$M = 2x^2y - 3y^2$; $N = 2x^3 - 12xy + \log y$

$\frac{\partial M}{\partial y} = 2x^2 - 6y$; $\frac{\partial N}{\partial x} = 6x^2 - 12y$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$; Equation (1) is not Exact.

Now $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{2x^2y - 3y^2}\left[6x^2 - 12y - 2x^2 + 6y\right]$

$= \frac{1}{y(2x^2 - 3y)}\left[2(2x^2 - 3y)\right] = \frac{2}{y}$

$I.f = e^{\int g(y)\,dy} = e^{2\int \frac{1}{y}dy} = e^{2\log y} = e^{\log y^2} = y^2$

Multiplying equation (1) with y^2

$\Rightarrow (2x^2y^3 - 3y^4)dx + (2x^3y^2 - 12xy^3 + y^2\log y)dy = 0 \rightarrow (2)$

Where $M_1 = 2x^2y^3 - 3y^4$ $N_1 = 2x^3y^2 - 12xy^3 + y^2\log y$

$\frac{\partial M_1}{\partial y} = 6x^2y^2 - 12y^3$ $\frac{\partial N_1}{\partial x} = 6x^2y^2 - 12y^3$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$; equation (2) is an Exact.

The general Solution is

$\int (2x^2y^3 - 3y^4)dx + \int y^2 \log y\,dy = c$

$2y^3\int x^2 dx - 3y^4\int dx + \int y^2 \log y\,dy = c$

$\frac{2x^3y^3}{3} - 3xy^4 + \log y\left(\int y^2 dy - \int \frac{d}{dy}(\log y)\int y^2 dy\right)dy = c$

$\frac{2x^3y^3}{3} - 3xy^4 + \log y\,\frac{y^3}{3} - \frac{y^3}{9} = c$

Therefore; the general Solution is

$6x^3y^3 - 27xy^4 + 3y^3\log y = C$

Unit - II

Differential Equations of First order but not of the first degree :

Definition:- An equation of the form $f(x,y,p)=0$, where p is not of first degree, is called a differential equation of first order and not of first degree.

An equation of the form

$p^n + P_1(x,y)p^{n-1} + \cdots + P_{n-1}(x,y)p + P_n(x,y) = 0$ is called the general first order equation of degree n.

Equations solvable for p:-

(1) solve $p^2-5p+6=0$

Solution:- Given that $p^2-5p+6=0$ — (1)

$$p^2-2p-3p+6=0$$
$$p(p-2)-3(p-2)=0$$
$$(p-2)(p-3)=0$$

$p-2=0$ — (2) and $p-3=0$ — (3)

Solving Equation (2)

$$\frac{dy}{dx}-2=0$$
$$dy-2dx=0$$

Integrating on both sides

$$\int dy - 2\int dx = c$$
$$y-2x=c$$
$$y-2x-c=0$$

Solving Equation (3)

$$\frac{dy}{dx}-3=0$$
$$dy-3dx=0$$

Integrating on both sides

$$\int dy - 3\int dx = c$$
$$y-3x=c$$
$$y-3x-c=0$$

$\therefore$ The General solution of (1) is $(y-2x-c)(y-3x-c)=0$

(2) **Problem:** $x + yp^2 = (1+xy)p$

Solution: Given that $x+yp^2 = (1+xy)p$

$$x + yp^2 = p + xyp$$
$$\Rightarrow x + yp^2 - p - xyp = 0$$
$$\Rightarrow yp^2 - p - xyp + x = 0$$
$$\Rightarrow p(yp-1) - x(yp-1) = 0$$
$$\Rightarrow (yp-1)(p-x) = 0$$

$yp - 1 = 0$ — (2) and $p - x = 0$ — (3)

Solving Equation (2)

$$y\left(\frac{dy}{dx}\right) - 1 = 0$$
$$\Rightarrow y\,dy - dx = 0$$
$$\Rightarrow \int y\,dy - \int dx = c$$
$$\Rightarrow y^2/2 - x = c$$
$$\Rightarrow y^2 - 2x - 2c = 0$$

Solving Equation (3)

$$\frac{dy}{dx} - x = 0$$
$$\Rightarrow dy - x\,dx = 0$$
$$\Rightarrow \int dy - \int x\,dx = c$$
$$\Rightarrow y - x^2/2 - c = 0$$
$$\Rightarrow 2y - x^2 - 2c = 0$$

$\therefore$ The general solution of (1) is

$$(y^2 - 2x - 2c)(2y - x^2 - 2c) = 0$$

Problem: $4y^2p^2 + 2xy(3x+1)p + 3x^3 = 0$

Solution: Given that $4y^2p^2 + 2xy(3x+1)p + 3x^3 = 0$ — (1)

$$4y^2p^2 + 6x^2yp + 2xyp + 3x^3 = 0$$
$$4y^2p^2 + 2xyp + 6x^2yp + 3x^3 = 0$$
$$2yp(2yp+x) + 3x^2(2yp+x) = 0$$
$$(2yp+x)(2yp+3x^2) = 0$$

$2yp + 3x^2 = 0$ — (2) and $2yp + x = 0$ — (3)

Solving Eq^n (2)

Solving Eq^n (3)

$$2y\left(\frac{dy}{dx}\right)+3x^2=0$$
$$\Rightarrow 2y\,dy+3x^2dx=0$$
$$\Rightarrow 2\int y\,dy+3\int x^2dx=c$$
$$\Rightarrow 2(y^2/2)+3(x^3/3)=c$$
$$\Rightarrow x^3+y^2-c=0$$

$$2yp+x=0$$
$$\Rightarrow 2y\left(\frac{dy}{dx}\right)+x=0$$
$$\Rightarrow 2y\,dy+x\,dx=0$$
$$\Rightarrow 2\int y\,dy+\int x\,dx=c$$
$$\Rightarrow 2(y^2/2)+x^2/2=c$$
$$\Rightarrow x^2+2y^2-2c=0$$

$\therefore$ The General Solution of ① is

$$(x^3+y^2-c)(x^2+2y^2-2c)=0$$

<u>Problem:-</u> Solve $x^2p^2+xyp-6y^2=0$

<u>Solution:</u> Given that $x^2p^2+xyp-6y^2=0$ — ①

$$x^2p^2+3xyp-2xyp-6y^2=0$$
$$xp(xp+3y)-2y(xp+3y)=0$$
$$(xp-2y)(xp+3y)=0$$

$xp-2y=0$ — ②

Solving Equation ②

$$x\left(\frac{dy}{dx}\right)-2y=0$$
$$\Rightarrow \frac{1}{2y}dy=\frac{1}{x}dx$$
$$\Rightarrow \frac{1}{2}\int\frac{1}{y}dy=\int\frac{1}{x}dx$$
$$\Rightarrow \frac{1}{2}\log y=\log x+\log c$$
$$\Rightarrow \log y^{1/2}=\log(xc)$$
$$\Rightarrow y^{1/2}-cx=0$$

$xp+3y=0$ — ③

Solving Equation ③

$$x\frac{dy}{dx}+3y=0$$
$$\Rightarrow \frac{1}{3y}dy=-1/x\,dx$$
$$\Rightarrow \frac{1}{3}\int\frac{1}{y}dy=-\int\frac{1}{x}dx$$
$$\Rightarrow \frac{1}{3}\log y=-\log x+\log c$$
$$\Rightarrow \log y^{1/3}=\log(c/x)$$
$$\Rightarrow xy^{1/3}-c=0$$

$\therefore$ The General solution of ① is

$$(y^{1/2}-cx)(xy^{1/3}-c)=0$$

Problem:- solve $p^2 - 7p + 10 = 0$

Solution:- Given that $p^2 - 7p + 10 = 0$ —(1)

$\Rightarrow p^2 - 2p - 5p + 10 = 0$

$\Rightarrow p(p-2) - 5(p-2) = 0$

$\Rightarrow (p-2)(p-5) = 0$

$p - 2 = 0$ —(2) and $p - 5 = 0$ —(3)

Solving Equation (2)

$\frac{dy}{dx} - 2 = 0$

$\Rightarrow dy - 2dx = 0$

$\Rightarrow \int dy - 2\int dx = c$

$\Rightarrow y - 2x - c = 0$

Solving Equation (3)

$\frac{dy}{dx} - 5 = 0$

$\Rightarrow dy - 5dx = 0$

$\Rightarrow \int dy - 5\int dx = c$

$\Rightarrow y - 5x - c = 0$

$\therefore$ The General solution of (1) is

$(y - 2x - c)(y - 5x - c) = 0$

Problem:- solve $p^2x^2 = y^2$

Solution:- Given that $p^2x^2 = y^2$ —(1)

$\Rightarrow p^2 = \frac{y^2}{x^2}$

$\Rightarrow p = \pm \frac{y}{x}$

$p = \frac{y}{x}$ —(2) and $p = -\frac{y}{x}$ —(3)

$\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \frac{1}{y}dy = \frac{1}{x}dx$

$\Rightarrow \int \frac{1}{y}dy = \int \frac{1}{x}dx + \log c$

$\Rightarrow \log y = \log x + \log c$

$\Rightarrow y - cx = 0$

$\frac{dy}{dx} = -\frac{y}{x}$

$\Rightarrow 1/y\,dy = -1/x\,dx$

$\Rightarrow \int \frac{1}{y}dy = -\int \frac{1}{x}dx + \log c$

$\Rightarrow \log y = -\log x + \log c$

$\Rightarrow y = c/x$

$\Rightarrow xy - c = 0$

$\therefore$ The General solution of (1) is $(y - cx)(xy - c) = 0$

...

Problem: solve $xy(p^2+1) = (x^2+y^2)p$

Solution:- Given that $xy(p^2+1) = (x^2+y^2)p$

$$xyp^2 + xy = x^2p + y^2p$$

$$\Rightarrow xyp^2 - x^2p - y^2p + xy = 0$$

$$\Rightarrow xp(yp-x) - y(yp-x) = 0$$

$$\Rightarrow (xp-y)(yp-x) = 0$$

$xp - y = 0$ — ②

Solving Equation ②

$$x\left(\frac{dy}{dx}\right) - y = 0$$

$$\Rightarrow x\,dy - y\,dx = 0$$

$$\Rightarrow \frac{1}{y}dy = \frac{1}{x}dx$$

$$\Rightarrow \int \frac{1}{y}dy = \int \frac{1}{x}$$

$$\Rightarrow \log y = \log x + \log c$$

$$\Rightarrow \log y = \log xc$$

$$\Rightarrow y - cx = 0$$

$yp - x = 0$ — ③

Solving Equation ③

$$y\left(\frac{dy}{dx}\right) - x = 0$$

$$\Rightarrow y\,dy - x\,dx = 0$$

$$\Rightarrow \int y\,dy - \int x\,dx = c$$

$$\Rightarrow y^2/2 - x^2/2 = c$$

$$\Rightarrow x^2 - y^2 - 2c = 0$$

$$\Rightarrow x^2 - y^2 - c = 0$$

$\therefore$ The General solution of ① is $(y-cx)(x^2-y^2-c) = 0$

Problem:- solve $p^3 + (2x-y^2)p^2 = 2xy^2p$

Solution:- Given that $p^3 + (2x-y^2)p^2 = 2xy^2p$ — ①

$$\Rightarrow p(p^2 + (2x-y^2)p - 2xy^2) = 0$$

$$\Rightarrow p(p^2 + 2xp - y^2p - 2xy^2) = 0$$

$$\Rightarrow p(p(p+2x) - y^2(p+2x)) = 0$$

$$\Rightarrow p[(p+2x)(p-y^2)] = 0$$

$p = 0$ — ②

$$\frac{dy}{dx} = 0$$

$p + 2x = 0$ — ③

$$\frac{dy}{dx} + 2x = 0$$

$p - y^2 = 0$ — ④

$$\frac{dy}{dx} - y^2 = 0$$

$\Rightarrow dy = 0$

$\Rightarrow \int dy = c$

$\Rightarrow y = c$

$\Rightarrow y - c = 0$

$\Rightarrow dy + 2x\,dx = 0$

$\Rightarrow \int dy + 2\int x\,dx = c$

$\Rightarrow y + 2(x^2/2) = c$

$\Rightarrow y + x^2 - c = 0$

$\Rightarrow dy = y^2 dx$

$\Rightarrow \frac{1}{y^2} dy = dx$

$\Rightarrow \int \frac{1}{y^2} dy = \int dx + c$

$\Rightarrow -1/y = x + c$

$\Rightarrow xy + cy + 1 = 0$

$\therefore$ The General solution of ① is

$(y-c)(y+x^2-c)(xy+cy+1) = 0$

Problem: Solve $xyp^2 + (x^2+xy+y^2)p + (x^2+xy) = 0$

Solution: Given that $xyp^2 + (x^2+xy+y^2)p + (x^2+xy) = 0$ — ①

$\Rightarrow xyp^2 + x^2p + xyp + y^2p + x^2 + xy = 0$

$\Rightarrow xyp^2 + x^2p + xyp + x^2 + y^2p + xp = 0$

$\Rightarrow xp(yp+x) + x(yp+x) + y(yp+x) = 0$

$\Rightarrow (yp+x)(xp+x+y) = 0$

$yp + x = 0$ — ②

$\Rightarrow y\left(\frac{dy}{dx}\right) + x = 0$

$\Rightarrow y\,dy + x\,dx = 0$

$\Rightarrow \int y + \int x = c$

$\Rightarrow y^2/2 + x^2/2 = c$

$\Rightarrow x^2 + y^2 - 2c = 0$

$\Rightarrow x^2 + y^2 - c = 0$

$xp + x + y = 0$ — ③

$\Rightarrow x\left(\frac{dy}{dx}\right) + x + y = 0$

$\Rightarrow x\,dy + y\,dx + x\,dx = 0$

$\Rightarrow d(xy) + x\,dx = 0$

$\Rightarrow \int d(xy) + \int x\,dx = c$

$\Rightarrow xy + x^2/2 = c$

$\Rightarrow 2xy + x^2 - c = 0$

$\therefore$ The General solution of ① is

$(x^2+y^2-c)(2xy+x^2-c) = 0$

Problem:- Solve $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$

Solution:- Given that $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$ —①

$$\Rightarrow xyp^2 - 2y^2p + 3x^2p - 6xy = 0$$

$$\Rightarrow xyp^2 + 3x^2p - 2y^2p - 6xy = 0$$

$$\Rightarrow xp(yp+3x) - 2y(yp+3x) = 0$$

$$\Rightarrow (xp-2y)(yp+3x) = 0$$

$xp - 2y = 0$ —② and $yp + 3x = 0$ —③

$x\left(\frac{dy}{dx}\right) - 2y = 0$

$\Rightarrow \frac{1}{2y}dy = \frac{1}{x}dx$

$\Rightarrow \frac{1}{2}\int \frac{1}{y}dy = \int \frac{1}{x}dx + \log c$

$\Rightarrow \frac{1}{2}\log y = \log x + \log c$

$\Rightarrow \log y^{1/2} = \log(xc)$

$\Rightarrow y^{1/2} - cx = 0$

$\Rightarrow y\left(\frac{dy}{dx}\right) + 3x = 0$

$\Rightarrow y\,dy + 3x\,dx = 0$

$\Rightarrow \int y\,dy + 3\int x\,dx = c$

$\Rightarrow \frac{y^2}{2} + 3\left(\frac{x^2}{2}\right) = c$

$\Rightarrow 3x^2 + y^2 - c = 0$

$\therefore$ The General Solution of ① is

$$(y^{1/2} - cx)(3x^2 + y^2 - c) = 0$$

Problem:- Solve $4xp^2 = (3x-a)^2$

Solution:- Given that $4xp^2 = (3x-a)^2$

$$\Rightarrow p^2 = \frac{(3x-a)^2}{4x}$$

$$\Rightarrow p = \pm \frac{(3x-a)}{2\sqrt{x}}$$

$$p = \frac{3x-a}{2\sqrt{x}} \quad \text{—②}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x-a}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2}(x^{1/2}) - \frac{a}{2}(x)^{-1/2}$$

$$\Rightarrow \int dy = \frac{3}{2}\int x^{1/2}dx - \frac{a}{2}\int x^{-1/2}dx + c$$

$$\Rightarrow y = \frac{3}{2}\left(\frac{x^{3/2}}{3/2}\right) - \frac{a}{2}\left(\frac{x^{1/2}}{1/2}\right) + c$$

$$\Rightarrow y = x^{3/2} - ax^{1/2} + c$$

$$\Rightarrow x^{3/2} - y - ax^{1/2} + c = 0$$

$$p = \frac{-(3x-a)}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-3x}{2\sqrt{x}} + \frac{a}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{3}{2}(x^{1/2}) + \frac{a}{2}(x)^{-1/2}$$

$$\Rightarrow \int dy = -\frac{3}{2}\int x^{1/2}dx + \frac{a}{2}\int x^{-1/2}dx + c$$

$$\Rightarrow y = -\frac{3}{2}\left(\frac{x^{3/2}}{3/2}\right) + \frac{a}{2}\left(\frac{x^{1/2}}{1/2}\right) + c$$

$$\Rightarrow y = -x^{3/2} + ax^{1/2} + c$$

$$\Rightarrow -x^{3/2} - y + ax^{1/2} + c = 0$$

$\therefore$ The General solution of ① is

$$(x^{3/2} - y - ax^{1/2} + c)(-x^{3/2} - y + ax^{1/2} + c) = 0$$

Problem:- solve $x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) - x^2y^2 - x^4 + y^2 = 0$

Solution:- Given that $x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) - x^2y^2 - x^4 + y^2 = 0$ —①

$$x^2p^2 - 2xyp - x^2y^2 - x^4 + y^2 = 0$$

$$p = \frac{2xy \pm \sqrt{(-2xy)^2 - 4(x^2)(-x^2y^2 - x^4 + y^2)}}{2x^2}$$

$$p = \frac{2xy \pm \sqrt{4x^4y^2 + 4x^6}}{2x^2}$$

$$p = \frac{2xy \pm 2x^2\sqrt{y^2 + x^2}}{2x^2}$$

$$P = \frac{y \pm x\sqrt{y^2+x^2}}{x}$$

$$\frac{dy}{dx} = \frac{y \pm x\sqrt{y^2+x^2}}{x} \quad \text{--- (2)}$$

put $y = vx \Rightarrow v = \frac{y}{x}$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

put the values of y and $\frac{dy}{dx}$ in Equation (2)

$$v + x\frac{dv}{dx} = \frac{vx \pm x\sqrt{(vx)^2+x^2}}{x}$$

$$v + x\frac{dv}{dx} = v \pm x\sqrt{v^2+1}$$

$$x\frac{dv}{dx} = \pm x\sqrt{v^2+1}$$

$$\frac{dv}{dx} = \pm\sqrt{v^2+1}$$

$$\frac{1}{\sqrt{v^2+1}}\, dv = \pm\, dx$$

$$\frac{1}{\sqrt{v^2+1}}\, dv = dx \quad ; \quad \frac{1}{\sqrt{v^2+1}}\, dv = -dx$$

$$\Rightarrow \int \frac{1}{\sqrt{v^2+1}}\, dv = \int dx + c \qquad \Rightarrow \int \frac{1}{\sqrt{v^2+1}}\, dv = -\int dx + c$$

$$\Rightarrow \sinh^{-1}(v) = x + c \qquad \Rightarrow \sinh^{-1}(v) = -x + c$$

$$\Rightarrow \sinh^{-1}\left(\frac{y}{x}\right) = x + c \qquad \Rightarrow \sinh^{-1}\left(\frac{y}{x}\right) = -x + c$$

$$\Rightarrow y - x\sinh^{-1}(x+c) = 0 \qquad \Rightarrow y - x\sinh^{-1}(-x+c) = 0$$

$\therefore$ The General solution of (1) is

$$(y - x\sinh^{-1}(x+c))(y - x\sinh^{-1}(-x+c)) = 0$$

Problem:- solve $p^2 + 2py\cot x = y^2$

Solution:- Given that $p^2 + 2py\cot x = y^2$

$$\Rightarrow p^2 + 2py\cot x + y^2\cot^2 x = y^2 + y^2\cot^2 x$$

$$\Rightarrow (p + y\cot x)^2 = y^2\operatorname{cosec}^2 x$$

$$\Rightarrow p + y\cot x = \pm y\operatorname{cosec} x.$$

$p + y\cot x = y\operatorname{cosec} x$ and $p + y\cot x = -y\operatorname{cosec} x$

$$\Rightarrow p = -y(\cot x - \operatorname{cosec} x)$$

$$\Rightarrow \frac{dy}{dx} = y(\operatorname{cosec} x - \cot x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1-\cos x}{\sin x}\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2\sin^2 x/2}{2\sin x/2\cos x/2}\right]$$

$$\Rightarrow \frac{dy}{dx} = y\operatorname{Tan} x/2$$

$$\Rightarrow \frac{1}{y}dy = \operatorname{Tan} x/2\, dx$$

$$\Rightarrow \int \frac{1}{y}dy = \int \operatorname{Tan} x/2\, dx + \log c$$

$$\Rightarrow \log y = \frac{\log \sec x/2}{1/2} + \log c$$

$$\Rightarrow \log y = 2\log \sec x/2 + \log c$$

$$\Rightarrow \log y = \log(c\cdot \sec^2 x/2)$$

$$\Rightarrow y - c\sec^2 x/2 = 0$$

$$\Rightarrow p = -y\operatorname{cosec} x - y\cot x$$

$$\Rightarrow p = -y(\operatorname{cosec} x + \cot x)$$

$$\Rightarrow p = -y\left[\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right]$$

$$\Rightarrow \frac{dy}{dx} = -y\left[\frac{1+\cos x}{\sin x}\right]$$

$$\Rightarrow \frac{dy}{dx} = -y\left[\frac{2\cos^2 x/2}{2\sin x/2\cdot\cos x/2}\right]$$

$$\Rightarrow \frac{dy}{dx} = -y\cot x/2$$

$$\Rightarrow \frac{1}{y}dy = -\cot x/2\, dx$$

$$\Rightarrow \int \frac{1}{y}dy = -\int \cot x/2\, dx + \log c$$

$$\Rightarrow \log y = -\frac{\log \sin x/2}{1/2} + \log c$$

$$\Rightarrow \log y = -\log \sin^2 x/2 + \log c$$

$$\Rightarrow \log y = \log\left(\frac{c}{\sin^2 x/2}\right)$$

$$\Rightarrow y - \frac{c}{\sin^2 x/2} = 0$$

$\therefore$ The General solution of (1) is

$$\left(y - c\sec^2 x/2\right)\left(y - \frac{c}{\sin^2 x/2}\right) = 0$$

Problem: solve $x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + 2y^2 - x^2 = 0$

Solution: Given that $x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + 2y^2 - x^2 = 0$ — ①

$$\Rightarrow x^2p^2 - 2xyp + 2y^2 - x^2 = 0$$

$$\Rightarrow p = \frac{2xy \pm \sqrt{4x^2y^2 - 4x^2(2y^2 - x^2)}}{2x^2}$$

$$\Rightarrow p = \frac{2xy \pm 2x\sqrt{x^2 - y^2}}{2x^2}$$

$$\Rightarrow p = \frac{y \pm \sqrt{x^2 - y^2}}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \pm \sqrt{x^2 - y^2}}{x} \quad \text{— ②}$$

put $y = vx \Rightarrow y/x = v$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

put the values of y and $\frac{dy}{dx}$ in Equation ②

$$v + x\frac{dv}{dx} = \frac{vx \pm \sqrt{x^2 - v^2x^2}}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v \pm \sqrt{1 - v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \pm\sqrt{1 - v^2}$$

$$\Rightarrow \frac{1}{\sqrt{1 - v^2}}\,dv = \pm\frac{1}{x}\,dx.$$

$\frac{1}{\sqrt{1 - v^2}}\,dv = \frac{1}{x}\,dx$ and $\frac{1}{\sqrt{1 - v^2}}\,dv = -\frac{1}{x}\,dx$

$\Rightarrow \int \frac{1}{\sqrt{1-v^2}}\,dv = \int \frac{1}{x}\,dx + \log c$, $\int \frac{1}{\sqrt{1-v^2}}\,dv = -\int \frac{1}{x}\,dx + c$

$\Rightarrow \sin^{-1} v = \log x + \log c$, $\Rightarrow \sin^{-1} v = -\log x + \log c$

$\Rightarrow \sin^{-1}\left(\frac{y}{x}\right) - \log xc = 0$, $\Rightarrow \sin^{-1}\left(\frac{y}{x}\right) - \log\left(\frac{c}{x}\right) = 0$

$\therefore$ The General solution of (1) is

$$\left(\sin^{-1}\left(\frac{y}{x}\right) - \log xc\right)\left(\sin^{-1}\left(\frac{y}{x}\right) - \log\left(\frac{c}{x}\right)\right) = 0$$

Differential Equations Solvable for x:-

Let $f(x,y,p) = 0$ — (1) be the given differential equation

If the equation (1) cannot be split up into rational and linear factors and (1) is of first degree in x, then (1) can be solved for x.

Equation (1) can be expressed in the form $x = F(y,p)$ — (2)

Differentiating Equation (2) w.r. to y given an equation of the form $\frac{1}{p} = g\left(y, p, \frac{dp}{dy}\right)$ — (3)

Since (3) is an equation in two variables p and y, it can be solved.

$\therefore$ The solution of (3) is $\phi(y,p,c) = 0$

Eliminating p from (1) and (4),

general solution of (1) is $\psi(x,y,c) = 0$

Note:- 1. If it is possible to eliminate p, then the values of x and y in terms of p in the form $x = f_1(p,c)$ and $y = f_2(p,c)$ together give the general solution.

2. This method is specially useful for equations y being absent.

Problem: solve $y^2 \log y = xpy + p^2$

Solution: Given that $y^2 \log y = xpy + p^2$ — (1)

since x is first degree in (1), It can be solved for x

$$x = \frac{y \log y}{p} - \frac{p}{y} \quad \text{— (2)}$$

Differentiating (2) w.r.t. y

$$\frac{dx}{dy} = \frac{p\left[y\left(\frac{1}{y}\right) + \log y\right] - y \log y \frac{dp}{dy}}{p^2} - \frac{\left[y\frac{dp}{dy} - p\right]}{y^2}$$

$$\Rightarrow \frac{1}{p} = (1+\log y)\frac{1}{p} - \frac{y \log y}{p^2}\frac{dp}{dy} - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

$$\Rightarrow \frac{1}{p} = \frac{1}{p} + \frac{1}{p}\log y + \frac{p}{y^2} - \left[\frac{y \log y}{p^2} + \frac{1}{y}\right]\frac{dp}{dy} = 0$$

$$\Rightarrow \frac{p}{y}\left(\frac{y \log y}{p^2} + \frac{1}{y}\right) - \left(\frac{y \log y}{p^2} + \frac{1}{y}\right)\frac{dp}{dy} = 0$$

$$\Rightarrow \left(\frac{y \log y}{p^2} + \frac{1}{y}\right)\left(\frac{p}{y} - \frac{dp}{dy}\right) = 0$$

$$\Rightarrow \frac{y \log y}{p^2} + \frac{1}{y} = 0 \text{ — (3)}, \quad \frac{p}{y} - \frac{dp}{dy} = 0 \text{ — (4)}$$

(3) is discarded as it gives singular solution

Solving (4), $\frac{dp}{dy} = \frac{p}{y} \Rightarrow \frac{dp}{p} = \frac{dy}{y}$

$$\Rightarrow \int \frac{dp}{p} = \int \frac{dy}{y} + \log c$$

$$\Rightarrow \log p = \log y + \log c$$

$\Rightarrow p = cy$ —⑤

Eliminating p from ① and ⑤

$$y^2 \log y = cxy^2 + c^2y^2$$

$$\Rightarrow \log y = cx + c^2$$

$\therefore$ The general solution of ① is $\log y = cx + c^2$

Problem: solve $p^2y + 2px = y$

Solution: Given that $p^2y + 2px = y$ —①

Since x is of first degree in ①, it can be solved for x.

$$x = \frac{y}{2p} - \frac{py}{2} \quad \text{—②}$$

Differentiating ② w.r.t. y

$$\Rightarrow \frac{dx}{dy} = \frac{1}{2}\left[\frac{p - y\frac{dp}{dy}}{p^2}\right] - \frac{1}{2}\left(p + y\frac{dp}{dy}\right)$$

$$\Rightarrow \frac{1}{p} = \frac{1}{2}\left[\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - p - y\frac{dp}{dy}\right)$$

$$\Rightarrow 2 = p\left[\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - p - y\frac{dp}{dy}\right]$$

$$\Rightarrow 2 = 1 - \frac{y}{p}\frac{dp}{dy} - p^2 - yp\frac{dp}{dy}$$

$$\Rightarrow 1 + p^2 + \frac{y}{p}\frac{dp}{dy}(1+p^2) = 0$$

$$\Rightarrow (1+p^2)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

$\Rightarrow 1 + p^2 = 0$ —③ and $1 + \frac{y}{p}\frac{dp}{dy} = 0$ —④

③ is discarded as it gives a singular solution.

solving ④ $\Rightarrow 1+\frac{y}{p}\frac{dp}{dy}=0$

$$\Rightarrow \frac{1}{p}dp = -\frac{1}{y}dy$$

$$\Rightarrow \int \frac{1}{p}dp = -\int \frac{1}{y}dy + \log c$$

$$\Rightarrow \log p = -\log y + \log c$$

$$\Rightarrow p = \frac{c}{y}$$

Eliminating 'p' from ① and ⑤

$$① \Rightarrow \left(\frac{c}{y}\right)^2 y + 2\left(c/y\right)x = y$$

$$\Rightarrow c^2 + 2cx = y^2$$

∴ The General solution of ① is $y^2 = c^2 + 2cx$.

Problem: solve $xp^3 = a+bp$

Solution: Given that $xp^3 = a+bp$ —①

solving ① for x, we have $x = \frac{a}{p^3} + \frac{b}{p^2}$ —②

Differentiating ② w.r.t y

$$\Rightarrow \frac{dx}{dy} = -\frac{3a}{p^4}\frac{dp}{dy} - \frac{2b}{p^3}\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = -\frac{1}{p}\left(\frac{3a}{p^3} + \frac{2b}{p^2}\right)\frac{dp}{dy}$$

$$\Rightarrow dy = -\left(\frac{3a}{p^3} + \frac{2b}{p^2}\right)dp$$

$$\Rightarrow \int dy = -3a\int \frac{1}{p^3}dp - 2b\int \frac{1}{p^2}dp + c$$

$$\Rightarrow y = -3a\left(\frac{p^{-2}}{-2}\right) - 2b\left(\frac{p^{-1}}{-1}\right) + c$$

$$\Rightarrow y = \frac{3a}{2p^2} + \frac{2b}{p} + c \quad \text{--- (3)}$$

It is not possible to eliminate p from (1) and (3)

$\therefore$ General solution of (1) is $x = \frac{a}{p^3} + \frac{b}{p^2}$ and

$$y = \frac{3a}{p^2} + \frac{2b}{p} + c$$

Problem:- solve $x = y + p^2$

Solution:- Given that $x = y + p^2$ --- (1)

Differentiating (1) w.r.t y

$$\frac{dx}{dy} = 1 + 2p\frac{dp}{dy}$$

$$\frac{1}{p} = 1 + 2p\frac{dp}{dy}$$

$$\Rightarrow \frac{dp}{dy} = \frac{1-p}{2p^2}$$

$$\Rightarrow dy = \frac{-2p^2}{p-1}\,dp$$

$$\Rightarrow \int dy = -2\int \frac{p^2}{p-1}\,dp + c$$

$$\Rightarrow y = -2\int\left(p+1+\frac{1}{p-1}\right)dp + c$$

$$\Rightarrow y = -2\left[\int p\,dp + \int dp + \int \frac{1}{p-1}\,dp\right] + c$$

$$\Rightarrow y = -2\left[\frac{p^2}{2} + p + \log(p-1)\right] + c \quad \text{--- (2)}$$

substituting the values of y from (2) in (1),

we get $x = c - 2\left[p + \log(p-1)\right]$ --- (3)

which shows that it is not possible to eliminate p from ① and ②

$\therefore$ The general solution of ① is

$x = c - 2[p + \log(p-1)],\ y = -p^2 - 2p - 2\log(p-1) + c$

Problem:- solve $2px = 2\tan y + p^3\cos^2 y$

Solution:- Given equation is $2px = 2\tan y + p^3\cos^2 y$ — ①

since x is of first degree in ①, it can be solved for x

$$\therefore x = \frac{\tan y}{p} + \frac{p^2\cos^2 y}{2} \quad \text{— ②}$$

Differentiating ② w.r.t y

$$\frac{dx}{dy} = \frac{p\sec^2 y - \tan y \frac{dp}{dy}}{p^2} + \frac{1}{2}\left[p^2(-2\cos y\sin y) + \cos^2 y\, 2p\frac{dp}{dy}\right]$$

$$\Rightarrow \frac{1}{p} = \frac{1}{p}\sec^2 y - \frac{1}{p^2}\tan y\frac{dp}{dy} - p^2\sin y\cos y + p\cos^2 y\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = \frac{1+\tan^2 y}{p} - p^2\sin y\cos y - \frac{1}{p^2}\tan y\frac{dp}{dy} + p\cos^2 y\frac{dp}{dy}$$

$$\Rightarrow \left[\frac{1}{p}\tan^2 y - p^2\sin y\cos y\right] + \left[p\cos^2 y - \frac{\tan y}{p^2}\right]\frac{dp}{dy} = 0$$

$$\Rightarrow -p\tan y\left[p\cos^2 y - \frac{\tan y}{p^2}\right] + \left[p\cos^2 y - \frac{\tan y}{p}\right]\frac{dp}{dy} = 0$$

$$\Rightarrow \left[p\cos^2 y - \frac{\tan y}{p^2}\right]\left[\frac{dp}{dy} - p\tan y\right] = 0$$

$$\Rightarrow p\cos^2 y - \frac{\tan y}{p^2} = 0 \quad \text{— ③}, \quad \frac{dp}{dy} - p\tan y = 0 \quad \text{— ④}$$

③ is rejected as it gives a singular solution solving (4), $\frac{dP}{dy} = p\tan y$

$\Rightarrow \frac{1}{p}dp = \tan y\, dy$

$\Rightarrow \int \frac{1}{p}dp = \int \tan y\, dy + \log c$

$\Rightarrow \log p = \log\sec y + \log c$

$\Rightarrow \log p = \log(c\sec y)$

$\Rightarrow p = c\sec y$ —— (5)

Eliminating p from (1) and (5)

$\Rightarrow 2cxy = 2\tan y + c^3\sec^3 y\cos^2 y$

$\Rightarrow 2cx = 2\sin y + c^3$

$\therefore$ The general solution of (1) is $2cx = 2\sin y + c^3$.

<u>Differential Equations solvable for</u> y:-

Let $f(x,y,p)=0$ —— (1) be the given differential equation

If (1) cannot be resolved into two rational and linear factors and (1) is of first degree in y, then it can be solved for y

(1) can be expressed in the form $y = f(x,p)$ —— (2)

Differentitation of (2) w.r.t x gives an equation of the form $p = g\left(x, p, \frac{dP}{dx}\right)$ —— (3)

Since (3) is an equation in two variables p and x, it can be solved. The solution of (1) is $\varphi(x,y,c)=0$

Note: * If it is not possible to eliminate p from ① and ④, the general solution of ① is given in the form $\phi(x,p,c)=0$, $f(x,y,p)=0$ (or) $x=f_1(p,c)$, $y=f_2(p,c)$

This is regarded as parametric form of the required solution where p is regarded as parameter.

* This method is specially useful for equations in which x is absent.

Problem: solve $y+px=p^2x^4$

solution:- Given that $y+px=p^2x^4$ — ①

But ① is of first degree in y and hence ① can be solved for y.

$$y=p^2x^4-px \quad \text{—} ②$$

Differentiating ② w.r.t x, we get

$$\frac{dy}{dx}=\left[p^2(4x^3)+x^4(2p)\frac{dp}{dx}\right]-\left(p+x\frac{dp}{dx}\right)$$

$$\Rightarrow p+p+x\frac{dp}{dx}-2px^4\frac{dp}{dx}-4x^3p^2=0$$

$$\Rightarrow 2p-4x^3p^2+x(1-2px^3)\frac{dp}{dx}=0$$

$$\Rightarrow 2p(1-2px^3)+(1-2px^3)x\frac{dp}{dx}=0$$

$$\Rightarrow (1-2px^3)\left(2p+x\frac{dp}{dx}\right)=0$$

$$\Rightarrow 1-2px^3=0 \quad \text{—} ③ \quad \text{and} \quad 2p+x\frac{dp}{dx}=0 \quad \text{—} ④$$

③ is discarded as it gives a singular solution.

Solving (4) $x\frac{dp}{dx}+2p=0$

$$\Rightarrow \frac{dp}{p}+\frac{2}{x}dx=0$$

$$\Rightarrow \int\frac{1}{p}dp+2\int\frac{1}{x}dx=\log c$$

$$\Rightarrow \log p+2\log x=\log c$$

$$\Rightarrow \log p+\log x^2=\log c$$

$$\Rightarrow \log(px^2)=\log c$$

$$\Rightarrow px^2=c \Rightarrow p=\frac{c}{x^2}$$

Eliminating p from (1) and (5), the general solution of (1) is

$$y+\left(\frac{c}{x}\right)=\left(\frac{c^2}{x^4}\right)x^4$$

$$\Rightarrow y+\left(\frac{c}{x}\right)=c^2$$

<u>Problem</u>:- solve $y=2xp+x^2p^4$

<u>Solution</u>: Given that $y=2xp+x^2p^4$ —— (1)

Differentiating (1) w.r.t x, we get

$$\frac{dy}{dx}=2x\frac{dp}{dx}+2p+x^2(4x^3)\frac{dp}{dx}+p^4(2x)$$

$$\Rightarrow p=2p+2xp^4+2x(1+2xp^3)\frac{dp}{dx}$$

$$\Rightarrow p+2xp^4+2x(1+2xp^3)\frac{dp}{dx}=0$$

$$\Rightarrow p(1+2xp^3)+2x(1+2xp^3)\frac{dp}{dx}=0$$

$$\Rightarrow (1+2xp^3)\left(p+2x\frac{dp}{dx}\right)=0$$

$\Rightarrow 1+2xp^3=0 \quad — (2) \qquad , \qquad p+2x\frac{dp}{dx}=0 \quad — (3)$

(2) is discarded as it gives a singular solution.

Solving (3). $p+2x\frac{dp}{dx}=0$.

$$\Rightarrow \frac{2}{p}dp+\frac{dx}{x}=0$$

$$\Rightarrow 2\int\frac{1}{p}dp+\int\frac{1}{x}dx=\log c$$

$$\Rightarrow 2\log p+\log x=\log c$$

$$\Rightarrow \log p^2+\log x=\log c$$

$$\Rightarrow \log(xp^2)=\log c$$

$$\Rightarrow p^2x=c$$

$$\Rightarrow p^2=\frac{c}{x} \quad — (4)$$

eliminating p from (1) and (4)

$$\Rightarrow (y-x^2p^4)^2=4x^2p^2$$

$\therefore$ The general solution of (1) is

$$(y-(x^2c^2/x^2))^2=4x^2(c/x)$$

$$\Rightarrow (y-c^2)^2=4cx.$$

Problem:- solve $y=xp^2+p$

Solution:- Given that $y=xp^2+p \quad — (1)$

Differentiating (1) w.r.t x

$$\Rightarrow \frac{dy}{dx}=p^2+2xp\frac{dp}{dx}+\frac{dp}{dx}$$

$$\Rightarrow p = p^2 + 2xp\frac{dp}{dx} + \frac{dp}{dx}$$

$$\Rightarrow (p^2 - p)\frac{dx}{dp} + 2xp + 1 = 0$$

$$\Rightarrow \frac{dx}{dp} + \frac{2}{p-1}x = -\frac{1}{p(p-1)} \quad \text{—}(2)$$

(2) is a linear equation in x, where

$$P = \frac{2}{p-1}, \quad Q = \frac{-1}{p(p-1)}$$

$$I.F = e^{\int P\,dp}$$

$$= e^{\int \frac{2}{p-1}dp}$$

$$= e^{2\log(p-1)} = (p-1)^2$$

$\therefore$ The general solution of (2) is

$$x(I.F) = \int Q(I.F)\,dp + c$$

$$\Rightarrow x(p-1)^2 = \int \frac{-1}{p(p-1)}(p-1)^2\,dp + c$$

$$\Rightarrow x(p-1)^2 = -\int\left(\frac{p-1}{p}\right)dp + c$$

$$\Rightarrow x(p-1)^2 = -\int dp + \int \frac{1}{p}\,dp + c$$

$$\Rightarrow x(p-1)^2 = -p + \log p + c$$

$$\Rightarrow x(p-1)^2 = c - p + \log p \quad \text{—}(3)$$

It is not possible to eliminating p from (1) and (3)

$\therefore$ The General solution of (1) is given by

$x = (c - p + \log p)(p-1)^{-2}$ and $y = xp^2 + p$

Problem: Solve $y = 2px - p^2$

Solution: Given equation is $y = 2px - p^2$ —— ①

Differentiating ① w.r.t x

$$\Rightarrow p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\Rightarrow 2(p-x)\frac{dp}{dx} = p$$

$$\Rightarrow p\frac{dx}{dp} + 2x = 2p$$

$$\Rightarrow \frac{dx}{dp} + \frac{2x}{p} = 2 \quad \text{—— ②}$$

② is a linear equation in x.

$$I.F = e^{\int P\,dp}$$

$$= e^{\int \frac{2}{p}dp}$$

$$= e^{2\log p}$$

$$= e^{\log p^2}$$

$$I.F = p^2$$

∴ The general solution of ② is

$$x(I.F) = \int Q\,(I.F)\,dp + c$$

$$\Rightarrow xp^2 = \int 2p^2\,dp + c$$

$$\Rightarrow xp^2 = \frac{2p^3}{3} + c \Rightarrow 3xp^2 = 2p^3 + 3c$$

$$\Rightarrow x = \frac{2}{3}p + \frac{c}{p^2} \quad \text{—— ③}$$

It is not possible to eliminate p from ① and ③

$\therefore$ The general solution of ① is given by two equations

$$x = \left(\frac{2p}{3}\right) + \left(\frac{c}{p^2}\right) \quad \text{and} \quad y = 2p\left(\frac{2p}{3} + \frac{c}{p^2}\right) - p^2 = \frac{p^2}{3} + \frac{2c}{p}$$

<u>Problem</u>:- Solve $e^y = p^3 + p$

<u>Solution</u>:- Given equation is $e^y = p^3 + p$ —①

Differentiating ① w.r.t x

$$\Rightarrow \quad e^y \frac{dy}{dx} = 3p^2 \frac{dp}{dx} + \frac{dp}{dx}$$

$$\Rightarrow \quad pe^y = (3p^2+1)\frac{dp}{dx}$$

$$\Rightarrow \quad p(p^3+p) = (3p^2+1)\frac{dp}{dx}$$

$$\Rightarrow \quad dx = \frac{3p^2+1}{p^2(p^2+1)}\, dp$$

$$\Rightarrow \quad dx = \frac{2p^2+p^2+1}{p^2(p^2+1)}\, dp$$

$$\Rightarrow \quad dx = \left(\frac{2}{p^2+1} + \frac{1}{p^2}\right) dp$$

$$\Rightarrow \int dx = 2\int \frac{1}{p^2+1}\, dp + \int \frac{1}{p^2}\, dp + c$$

$$\Rightarrow \quad x = 2\tan^{-1}p - \frac{1}{p} + c \quad —③$$

It is not possible to eliminate p from ① and ③

$\therefore$ The General solution of ① is given by

$$e^y = p^3 + p \quad \text{and} \quad x = 2\tan^{-1}p - \frac{1}{p} + c.$$

Clairauts Equation :-

The differential equation of the form $y = xp + f(p)$ is called clairauts equation. This equation is solved by considering it as $y = f(x, p)$, solvable for y type.

Solution of clairauts Equation :-

Let the given equation be $y = xp + f(p)$ — (1)

Differentiating (1) w.r.t 'x'

$$\Rightarrow \frac{dy}{dx} = x\frac{dp}{dx} + p + f'(p)\frac{dp}{dx}$$

$$\Rightarrow p = [x + f'(p)]\frac{dp}{dx} + p$$

$$\Rightarrow [x + f'(p)]\frac{dp}{dx} = 0$$

$$\Rightarrow \frac{dp}{dx} = 0 \text{ — (2)} \quad \text{and} \quad x + f'(p) = 0 \text{ — (3)}$$

Solving (2), $\frac{dp}{dx} = 0$

$$\Rightarrow dp = 0$$

$$\Rightarrow \int dp = c$$

$$\Rightarrow p = c$$ — (4), where c is a real number

Eliminating p from (1) and (4) $y = cx + f(c)$

$\therefore$ The general solution of clairauts equation (1) is

$y = cx + f(c)$.

Problem: solve $p = \log(px - y)$

Solution: Given equation is $p = \log(px - y)$ — ①

$$\Rightarrow px - y = e^{p}$$

$$\Rightarrow y = px - e^{p}$$

which is in clairauts form

$\therefore$ The general solution of ① is $y = x - e^{c}$

where 'c' is any real number.

Problem: solve $(y - xp)(p-1) = p$

Solution: Given equation is $(y - xp)(p-1) = p$ — ①

$$\Rightarrow y(p-1) - xp(p-1) = p$$

$$\Rightarrow y(p-1) = xp(p-1) + p$$

$$\Rightarrow y = xp + \frac{p}{p-1}$$

which is in clairauts form

$\therefore$ The general solution of ① is $y = xc + \frac{c}{c-1}$

where 'c' is any real number.

Problem: solve $p = \tan(xp - y)$

Solution: Given Equation is $p = \tan(xp - y)$ — ①

$$\Rightarrow xp - y = \tan^{-1} p$$

$$\Rightarrow y = xp - \tan^{-1} p$$

which is in clairauts form

$\therefore$ The General solution of ① is $y = xc - \tan^{-1} c$

where c is any real number.

Problem: solve $\sin px \cos y = \cos px \sin y + p$

Solution: Given equation is $\sin px \cos y = \cos px \sin y + p$ — ①

$\Rightarrow \sin px \cos y - \cos px \sin y = p$

$\Rightarrow \sin(px - y) = p$

$\Rightarrow px - y = \sin^{-1}(p)$

$\Rightarrow y = px - \sin^{-1}(p)$

which is in clairauts form

$\therefore$ The general solution of ① is $y = cx - \sin^{-1}(c)$,

c being an arbitrary real number

Problem: solve $y^2 - 2pxy + p^2(x^2 - 1) = m^2$

Solution: Given equation is $y^2 - 2pxy + p^2(x^2-1) = m^2$ — ①

$\Rightarrow y^2 - 2pxy + p^2x^2 = p^2 + m^2$

$\Rightarrow (y - px)^2 = p^2 + m^2$

$\Rightarrow y - px = \sqrt{p^2 + m^2}$

$\Rightarrow y = xp + \sqrt{p^2 + m^2}$

which is in clairauts form

$\therefore$ General solution of ① is $y = xc + \sqrt{c^2 + m^2}$

Problem: solve $(py + x)(px - y) = 2p$

Solution: Given Equation is $(py+x)(px-y) = 2p$ — ①

Put $x^2 = X$ and $y^2 = Y$

$\Rightarrow 2x\,dx = dX$ $\quad 2y\,dy = dY$

$\Rightarrow P = \frac{dY}{dX} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}p \Rightarrow p = \frac{x}{y}P$ — ②

1 and 2 $\Rightarrow \left(\frac{x}{y}p\cdot y+x\right)\left(\frac{x}{y}p\cdot x-y\right)=2\left(\frac{x}{y}\right)p$

$\Rightarrow x(p+1)(x^2p-y^2)=2xp$

$\Rightarrow (p+1)(x^2p-y^2)=2p$

$\Rightarrow (P+1)(XP-Y)=2P$

$\Rightarrow PX-Y=\frac{2P}{P+1}$

$\Rightarrow Y=PX-\frac{2P}{P+1}$ — 3 is clairauts equation

The general solution of 3 is $Y=CX-\frac{2C}{C+1}$

$\therefore$ The general solution of 1 is $y^2=cx^2-\frac{2c}{c+1}$

Problem:- solve $x^2(y-px)=p^2y$

Solution:- Given equation is $x^2(y-px)=p^2y$ — 1

put $x^2=X$ and $y^2=Y$

$\Rightarrow 2xdx=dX$ and $2ydy=dY$

$\Rightarrow P=\frac{dY}{dX}=\frac{2ydy}{2xdx}=\frac{y}{x}\frac{dy}{dx}=\frac{y}{x}p$

$\Rightarrow p=\frac{Px}{y}$ — 2

1 and 2 $\Rightarrow X\left(y-x.\frac{Px}{y}\right)=y\frac{P^2x^2}{y^2}$

$\Rightarrow X(y^2-x^2P)=x^2P^2$

$\Rightarrow X(Y-XP)=XP^2$

$\Rightarrow y - xp = p^2$

$\Rightarrow y = xp + p^2$ —③

③ is clairaut's Equation

$\therefore$ General solution of ③ is $y = cx + c^2$

$\therefore$ The general solution of ① is $y^2 = cx^2 + c^2$

Problem:- solve $(px - y)(py + x) = h^2 p$

solution:- Given Equation is $(px - y)(py + x) = h^2 p$ —①

put $x^2 = X$ and $y^2 = Y$

$2x\,dx = dX$ and $2y\,dy = dY$

$$\frac{dY}{dX} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}p$$

$\Rightarrow P = \frac{y}{x}p \Rightarrow p = \frac{Px}{y}$ —②

① and ② $\Rightarrow \left(\frac{Px}{y}\cdot x - y\right)\left(\frac{Px}{y}\cdot y + x\right) = h^2\left(\frac{Px}{y}\right)$

$\Rightarrow (Px^2 - y^2)(Px + x) = h^2 Px$

$\Rightarrow (Px^2 - y^2)(P+1) = h^2 P$

$\Rightarrow Px^2 - y^2 = \frac{h^2 P}{P+1}$

$\Rightarrow PX - Y = \frac{h^2 P}{P+1}$

$\Rightarrow Y = PX - \frac{h^2 P}{P+1}$ —③

(3) is clairauts Equation

$\therefore$ General solution of (3) is $Y = XC - \frac{h^2c}{c+1}$

$\therefore$ General solution of (1) is $y^2 = cx^2 - \frac{h^2c}{c+1}$

<u>Problem</u>:- solve $y = 2px + y^2p^3$

<u>Solution</u>: Given Equation is $y = 2px + y^2p^3$ — (1)

Put $2x = X$ and $y^2 = Y$

$2dx = dX \qquad 2ydy = dY$

$$P = \frac{dY}{dX} = \frac{2ydy}{2dx} = y\frac{dy}{dx} = yp$$

$$\Rightarrow p = \frac{P}{y} \quad \text{— (2)}$$

(1) and (2) $\quad y = 2\left(\frac{P}{y}\right)x + y^2\left(\frac{P}{y}\right)^3$

$$y^2 = 2xP + P^3$$

$$Y = 2xP + P^3$$

$$Y = XP + P^3 \quad \text{— (3)}$$

(3) is clairauts Equation

General solution of (3) is $Y = XC + C^3$

$\therefore$ General solution of (1) is $y^2 = 2xc + c^3$

<u>Problem</u>:- Reduce the equation $xyp^2 - (x^2+y^2+1)p + xy = 0$ to clairauts form by using the transformations $u = x^2, v = y^2$ and find its singular solution.

Solution: Given Equation is

$$xyp^2 - (x^2+y^2-1)p + xy = 0 \quad — (1)$$

put $u = x^2,\ v = y^2$

$du = 2x\,dx,\ dv = 2y\,dy$

$$P = \frac{dv}{du} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}\,p$$

$$\Rightarrow p = \frac{xP}{y} \quad — (2)$$

(1) and (2)

$$\Rightarrow xy\left(\frac{xP}{y}\right)^2 - (x^2+y^2-1)\left(\frac{xP}{y}\right) + xy = 0$$

$$\Rightarrow x^3P^2 - (x^2+y^2-1)(xP) + y^2 = 0$$

$$\Rightarrow x^2P^2 - (u+v-1)P + y^2 = 0$$

$$\Rightarrow uP^2 - uP - vP + P + v = 0$$

$$\Rightarrow uP(P-1) - v(P-1) + P = 0$$

$$\Rightarrow (P-1)(uP-v) + P = 0$$

$$\Rightarrow uP - v + \frac{P}{P-1} = 0$$

$$\Rightarrow v = uP + \frac{P}{P-1} \quad — (3)$$

(3) is Clairaut's Equation

Differentiating (3) w.r.t 'u'

$$\Rightarrow \frac{dv}{du} = u\frac{dP}{du} + P + \frac{(P-1)\frac{dP}{du} - P\frac{dP}{du}}{(P-1)^2}$$

$$\Rightarrow p = u\frac{dp}{du} + p - \frac{1}{(p-1)^2}\frac{dp}{du} = 0$$

$$\Rightarrow u\frac{dp}{du} - \frac{1}{(p-1)^2}\frac{dp}{du} = 0$$

$$\Rightarrow \frac{dp}{du}\left[u - \frac{1}{(p-1)^2}\right] = 0$$

$\frac{dp}{du} = 0$ — (4) and $u - \frac{1}{(p-1)^2} = 0$ — (5)

$\Rightarrow dp = 0$

$\Rightarrow \int dp = c$

$\Rightarrow p = c$

$\Rightarrow u = \frac{1}{(p-1)^2}$

$\Rightarrow (p-1)^2 = \frac{1}{u}$

$\Rightarrow p = 1 \pm \frac{1}{\sqrt{u}}$ — (6)

(3) and (6) $v = up + \frac{p}{p-1}$

$$\Rightarrow v = u\left(1+\frac{1}{\sqrt{u}}\right) + \frac{1+\frac{1}{\sqrt{u}}}{1+\frac{1}{\sqrt{u}}-1}$$

$$\Rightarrow v = u + \sqrt{u} + \sqrt{u} + 1$$

$$\Rightarrow v = u + 2\sqrt{u} + 1$$

$\therefore$ General solution of (1) is $y^2 = x^2 + 2x + 1$

$y^2 = (x+1)^2$

UNIT - III

Higher Order Linear Differential equations - I

Linear differential equations with Constant Coefficients:

Definition:- An equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1\frac{dy}{dx} + a_0 y = Q(x) \quad (1)$$

where $a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n$ are Linear Constants (or) real constants and $Q(x)$ is continuous function of x defined on interval is called a linear differential equation of order 'n' with constant coefficients.

Solution of homogenious Linear differential equation of order n with constant coefficients:

Consider a solution homogenious linear differential equation of order 'n' with constant coefficient

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1\frac{dy}{dx} + a_0 y = Q(x) \quad (1)$$

where $a_0, a_1, a_2, \ldots, a_n$

$$a_n m^n e^{mx} + a_{n-1} m^{n-1} e^{mx} + a_{n-2} m^{n-2} e^{mx} + \cdots + a_1 m e^{mx} + a_0 e^{mx} = 0 \quad (2)$$

$$[a_n m^n + a_{n-1} m^{n-1} + a_{n-2} m^{n-2} + \cdots + a_1 m + a_0] e^{mx} = 0.$$

The equation (2) is called characteristic equation or Auxiliary equation. Since equation (2) is an algebraic equation in n of degree n by the fundamental theorem of Algebraic equation (2) has not greater than equation (1).

The roots of the Auxiliary equation (2) may be

1. Real and distinct
2. Real and Repeated
3. Conjugate Complex Numbers.
4. Repeated Conjugate Complex Numbers.

Case(i):

All the roots are real and distinct. Suppose $m_1, m_2, \dots, m_n$ are roots of the equation ① and the roots are distinct then $y = e^{m_1 x}, e^{m_2 x}, \dots, e^{m_n x}$ are solutions of ①.

The general solution of equation of ① is

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$ where $c_1, c_2, \dots, c_n$ are real constants.

Case(ii):

All the roots are real and some are repeated. Suppose one root is repeated K times. The remaining roots $m_k, m_{k+1}, m_{k+2}, \dots, m_n$ are real and distinct then the General solution of equation ① is

$$y = [c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1}] e^{mx} + c_{k+1} e^{m_{k+1}} + c_{k+2} e^{m_{k+2}} + c_n e^{m_n x}.$$

Case(iii):

A Pair of conjugate complex roots and remaining real and distinct. Suppose two roots are Pair of conjugate complex numbers $(a \pm ib)$ and the remaining roots are $m_3, m_4, \dots, m_n$ are real and distinct.

Then the General solution of ① is

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}.$$

Case (iv):

A Pair of conjugate complex roots are repeated and the remaining roots are repeated. Suppose a Pair of conjugate complex roots repeated twice and remaining roots $m_5, m_6, \dots, m_n$ are real and distinct.

Then the general solution of ① is

$$y = e^{ax}[(c_1 + c_2 x)\cos bx + (c_3 + c_4 x)\sin bx] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}.$$

Linear differential equations of order n:

An equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(y) = \theta(x)$$

where $a_0, a_1, a_2, \dots, a_{n-1}$ and θ are continuous real function in x defined as interval I is called an linear differential equations of order 'n' over the interval I

<u>Note:</u>

A linear differential equations of order 'n' can be written as $\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_n(x)y = Q(x)$.

Here $P_1, P_2, \ldots, P_n$ and Q are continuous equations (or) functions of x defined on an interval I.

<u>Differential operators:</u>

Let the differential operator $\frac{d}{dx}$ be denoted as 'D' and the differential operation $\frac{d^2}{dx^2}, \frac{d^2}{dx^3}, \frac{d^2}{dx^4}, \ldots, \frac{d^n}{dx^n}$. we denoted by $D^2, D^3, D^4, \ldots, D^n$.

when applied on the function $y(x)$

$Dy = \frac{dy}{dx}$, $D^2y = \frac{d^2y}{dx^2}, \ldots, D^n y = \frac{d^n y}{dx^n}$.

Know the $D^n + D_1 D^{n+1} + \cdots + D_n$ is in D it is called a differential operator of order 'n' and it is denoted by $f(D)$.

$$\therefore f(D) = D^n + P_1 D^{n-1} + \cdots + P_n.$$

Thus $f(D)y = (D^n + P_1 D^{n-1} + \cdots + P_n)y$.

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n y = Q(x)$$

as $f(D)y = Q(x)$

which is called the operator.

<u>Note:</u>

1) If $Q \neq 0$ for some x in I then the equation $f(D)y = Q$ is called linear and non-homogeneous equation.

2) If $Q = 0$ for all x in I then the equation $f(D)y = 0$ is called Linear and homogeneous equation.

Problems:

Case -(i):

1) Solve $\frac{d^2y}{dx^2} - a^2y = 0$ where $a \neq 0$

Sol: Given that $\frac{d^2y}{dx^2} - a^2y = 0$ — ①

$$\Rightarrow D^2y - a^2y = 0$$

$$\Rightarrow (D^2 - a^2)y = 0$$

The operator form $f(D)y = 0$

where $f(D) = D^2 - a^2$.

The Auxiliary equation is $f(m) = 0$

$$\Rightarrow m^2 - a^2 = 0$$

$$\Rightarrow m^2 = a^2$$

$$\Rightarrow m = \pm a$$

$$\Rightarrow m = a, -a$$

∴ General Solution of equation ① is

$$y = c_1e^{m_1x} + c_2e^{m_2x}$$

$$y = c_1e^{ax} + c_2e^{-ax}$$

2) Solve $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$

Sol: Given that $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$ — ①

The operator form is $D^3y + 6D^2y + 11Dy + 6y = 0$

$$\Rightarrow (D^3 + 6D^2 + 11D + 6)y = 0.$$

Where $f(D) = D^3 + 6D^2 + 11D + 6$

$$\begin{array}{r|rrrr}
-1 & 1 & 6 & 11 & 6 \\
 & 0 & -1 & -5 & -6 \\
\hline
-2 & 1 & 5 & 6 & \boxed{0} \\
 & 0 & -2 & -6 & \\
\hline
-3 & 1 & 3 & \boxed{0} & \\
 & 0 & -3 & & \\
\hline
 & 1 & \boxed{0} & &
\end{array}$$

$m = -1, -2, -3.$

$\therefore$ General Solution of equation ① is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$$

3) Solve $(D^3 - 2D^2 - 5D + 6)y = 0$, where $x = 0$, $y = 1$, $y' = -7$, $y'' = -1$

Sol: Given that $(D^3 - 2D^2 - 5D + 6)y = 0$ —①

where $f(D) = D^3 - 2D^2 - 5D + 6$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & 0 & 1 & -1 & -6 \\ \hline -2 & 1 & -1 & -6 & \boxed{0} \\ & 0 & -2 & 6 & \\ \hline 3 & 1 & -3 & \boxed{0} & \\ & 0 & 3 & & \\ \hline & 1 & \boxed{0} & & \end{array}$$

$\therefore$ General Solution of equation ① is

$$m = 1, -2, 3$$

$$y = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{3x} \quad \text{—②}$$

At $x = 0$, $y = 1$

from ②,

$$1 = c_1 e^{0} + c_2 e^{-2(0)} + c_3 e^{3(0)}$$

$$1 = c_1 + c_2 + c_3$$

$$c_1 + c_2 + c_3 = 1 \quad \text{—③}$$

From ②,

$$y' = c_1 e^{x} + c_2 (e^{-2x}(-2)) + c_3 e^{3x}(3)$$

$$y' = c_1 e^{x} - 2c_2 e^{-2x} + 3c_3 e^{3x}$$

$$y'' = c_1 e^{x} - 2c_2 e^{-2x}(-2) + 3c_3 e^{3x}(3)$$

$$y'' = c_1 e^{x} + 4c_2 e^{-2x} + 9c_3 e^{3x}$$

At $x = 0$, $y' = -7$

$$y' = c_1 e^{x} - 2c_2 e^{-2x} + 3c_3 e^{3x}$$

$$-7 = c_1 e^{0} - 2c_2 e^{-2(0)} + 3c_3 e^{3(0)}$$

$$-7 = c_1 - 2c_2 + 3c_3 \quad \text{—④}$$

At $x=0$, $y''=-1$

$$y'' = c_1e^{x} + 4c_2e^{-2x} + 9c_3e^{3x}$$

$$-1 = c_1e^{0} + 4c_2e^{-2(0)} + 9c_3e^{3(0)}$$

$$-1 = c_1 + 4c_2 + 9c_3 \quad \text{--- (5)}$$

(3) - (4) $\Rightarrow$

$$\begin{array}{r} c_1 + c_2 + c_3 = 1 \\ c_1 - 2c_2 + 3c_3 = -7 \\ - \quad + \quad - \qquad + \\ \hline 3c_2 - 2c_3 = 8 \end{array} \quad \text{--- (6)}$$

(4) - (5) $\Rightarrow$

$$\begin{array}{r} c_1 - 2c_2 + 3c_3 = -7 \\ c_1 + 4c_2 + 9c_3 = -1 \\ - \quad - \quad - \qquad + \\ \hline -6c_2 - 6c_3 = -6 \end{array}$$

$$\Rightarrow c_2 + c_3 = 1 \quad \text{--- (7)}$$

Adding

2 x (7) and (6) $\Rightarrow$

$$\begin{array}{r} 2c_2 + 2c_3 = 2 \\ 3c_2 - 2c_3 = 8 \\ \hline 5c_2 \qquad = 10 \end{array}$$

$$\Rightarrow c_2 = 2$$

From (7),

$$c_2 + c_3 = 1$$

$$\Rightarrow 2 + c_3 = 1$$

$$\Rightarrow c_3 = -1$$

From (3),

$$1 = c_1 + c_2 + c_3$$

$$1 = c_1 + 2 + (-1)$$

$$1 = c_1 + 1$$

$$c_1 = 0$$

$\therefore c_1 = 0,\ c_2 = 0,\ c_3 = 1$

$\therefore$ The General Solution is

$$y = c_1e^{mx} + c_2e^{m_2x} + c_3e^{m_3x}$$

$$= 0e^{m_1x} + 2e^{m_2x} + (-1)e^{m_3x}$$

$$y = 2e^{m_2x} - e^{m_3x}$$

4) Solve $(D^3-2D^2-3D)y=0$ when $\frac{d}{dx}=D$

Sol: Given that $(D^3-2D^2-3D)y=0$

where $f(D)=D^3-2D^2-3D$ — ①

Auxiliary equation ① is $f(m)=0$

$$\Rightarrow m^3-2m^2-3m=0$$
$$\Rightarrow m(m^2-2m-3)=0$$
$$\Rightarrow m=0,\ m^2-2m-3=0$$
$$\Rightarrow m^2-3m+m-3=0$$
$$\Rightarrow m(m-3)+1(m-3)=0$$
$$\Rightarrow (m-3)(m+1)=0$$
$$\Rightarrow m=3,-1$$

$m=0,-1,3$

General Solution of equation ① is

$$y=c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}$$
$$=c_1e^{0x}+c_2e^{-x}+c_3e^{3x}$$
$$=c_1+c_2e^{-x}+c_3e^{3x}$$

5) Solve $(D^4-2D^3-13D^2+38D-24)y=0$

Sol: Given that $(D^4-2D^3-13D^2+38D-24)y=0$ — ①

which is of the form $f(D)y=0$

where $f(D)=D^4-2D^3-13D^2+38D-24$

-4	1	-2	-13	38	-24
	0	-4	24	-44	24
3	1	-6	11	-6	$\lfloor 0$
	0	3	-9	6	
2	1	-3	2	$\lfloor 0$	
	0	2	-2		
1	1	-1	$\lfloor 0$		
	0	1			
	1	$\lfloor 0$			

$m=1,2,3,-4$

General Solution of equation ① is

$$y=c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}+c_4e^{m_4x}$$
$$=c_1e^{x}+c_2e^{2x}+c_3e^{3x}+c_4e^{-4x}.$$

Case -(ii) :

1) Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

Sol: Given that $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

Operator form is $D^3y - 3Dy + 2y = 0$

$\Rightarrow (D^3 - 3D + 2)y = 0$

where $f(D) = D^3 - 3D + 2$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline -2 & 1 & 1 & -2 & \boxed{0} \\ & 0 & -2 & 2 & \\ \hline 1 & 1 & -1 & \boxed{0} & \\ & 0 & 1 & & \\ \hline & 1 & \boxed{0} & & \end{array}$$

$m = 1, 1, -2$

General Solution of ① is

$y = c_1 e^{m_1 x} + (c_2 + c_3 x) e^{m_2 x}$

$= c_1 e^{-2x} + (c_2 + c_3 x) e^{x}$

2) Solve $(D^3 + 6D^2 + 12D + 8)y = 0$.

Sol: Given that $(D^3 + 6D^2 + 12D + 8)y = 0$ — ①

where $f(D) = D^3 + 6D^2 + 12D + 8$

$$\begin{array}{r|rrrr} -2 & 1 & 6 & 12 & 8 \\ & 0 & -2 & -8 & -8 \\ \hline -2 & 1 & 4 & 4 & \boxed{0} \\ & 0 & -2 & -4 & \\ \hline -2 & 1 & 2 & \boxed{0} & \\ & 0 & -2 & & \\ \hline & 1 & \boxed{0} & & \end{array}$$

$m = -2, -2, -2$

General solution of equation ① is

$y = (c_1 + c_2 x + c_3 x^2) e^{-2x}$.

3) Solve $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0$.

Sol: Given that $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0$.

Operator form is $D^3y - D^2y - 8Dy + 12y = 0$.

$\Rightarrow (D^3 - D^2 - 8D + 12)y = 0$

where $f(D) = D^3 - D^2 - 8D + 12$

$$\begin{array}{r|rrrr}
2 & 1 & -1 & -8 & 12 \\
 & 0 & 2 & 2 & -12 \\
\hline
2 & 1 & 1 & -6 & \boxed{0} \\
 & 0 & 2 & 6 & \\
\hline
-3 & 1 & 3 & \boxed{0} & \\
 & 0 & -3 & & \\
\hline
 & 1 & \boxed{0} & &
\end{array}$$

$m = 2, 2, -3$

General Solution of equation ① is

$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_2 x}$

$= (c_1 + c_2 x)e^{2x} + c_3 e^{-3x}$

2

4) Solve $(D^4 - 4D^3 + 6D^2 - 4D + 1)y = 0$

Sol: Given that $(D^4 - 4D^3 + 6D^2 - 4D + 1)y = 0$ —①

where $f(D) = D^4 - 4D^3 + 6D^2 - 4D + 1$

$$\begin{array}{r|rrrrr}
1 & 1 & -4 & 6 & -4 & 1 \\
 & 0 & 1 & -3 & 3 & -1 \\
\hline
1 & 1 & -3 & 3 & -1 & \boxed{0} \\
 & 0 & 1 & -2 & 1 & \\
\hline
1 & 1 & -2 & 1 & \boxed{0} & \\
 & 0 & 1 & -1 & & \\
\hline
1 & 1 & -1 & \boxed{0} & & \\
 & 0 & 1 & & & \\
\hline
 & 1 & \boxed{0} & & &
\end{array}$$

$m = 1, 1, 1, 1$

General Solution of the equation ① is

$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)e^{mx}$

$= (c_1 + c_2 x + c_3 x^2 + c_4 x^3)e^{x}$.

2

Case - (iii) :

1) Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$.

Sol: Given that $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Operator form is $D^2y + Dy + y = 0$

$\Rightarrow (D^2+D+1)y = 0$ —(1)

where $f(D) = D^2+D+1$

Auxiliary equation is $f(m) = 0$

$\Rightarrow m^2+m+1 = 0$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Here $a=1$, $b=1$, $c=1$

$$m = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{i^2 3}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2}$$

$$m = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

General solution of (1) is

$$y = e^{ax}(C_1 \cos bx + C_2 \sin bx)$$

$$= e^{-\frac{1}{2}x}\left(C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x\right).$$

2) Solve $(D^3+1)y = 0$.

Sol: Given that $(D^3+1)y = 0$ —(1)

where $f(D) = D^3+1$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & 0 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

$\Rightarrow m^2 - m + 1 = 0$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Here $a = 1,\ b = -1,\ c = 1$

$$m = \frac{1 \pm \sqrt{1-4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1 \pm \sqrt{i^2 3}}{2}$$

$$= \frac{1 \pm i\sqrt{3}}{2}$$

General Solution of equation ① is

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

$$= e^{x}\left(c_1 \cos\left(\tfrac{\sqrt{3}}{2}\right)x + c_2 \sin\left(\tfrac{\sqrt{3}}{2}\right)x\right)$$

3) Solve $(D^4 - 2D^3 + 2D^2 - 2D + D)y = 0$

Sol:- Given equation is $(D^4 - 2D^3 + 2D^2 - 2D + 1)y = 0$ — ①

The operator form is $(D^4 - 2D^3 + 2D^2 - 2D + 1)y = 0$

where $f(D) = D^4 - 2D^3 + 2D^2 - 2D + 1 = 0$

The Auxiliary equation is $f(m) = 0$

$$\Rightarrow m^4 - 2m^3 + 2m^2 - 2m + 1 = 0$$

1	1	-2	2	-2	1
	0	1	-1	1	-1
1	1	-1	1	-1	$\boxed{0}$
	0	1	0	1	
	1	0	1	$\boxed{0}$	

$$m^2 + 1 = 0$$

$$\Rightarrow m^2 = -1$$

$$\Rightarrow m^2 = i^2$$

$$\Rightarrow m = \pm i$$

$$\Rightarrow m = 1, 1, \pm i$$

The general Solution of equation ① is

$$y = (C_1 + C_2 x) e^{m_1 x} + e^{m_2 x} (\cos bx + \sin bx)$$

$$= (C_1 + C_2 x) e^{x} + e^{0} (\cos x + \sin x)$$

$$= (C_1 + C_2 x) e^{x} + \cos x + \sin x .$$

2

4) Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$.

Sol:. Given that $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$ —①

operator form is $(D^2y - 4Dy + 5y) = 0$

$$\Rightarrow (D^2 - 4D + 5)y = 0$$

where $f(D) = D^2 - 4D + 5$

The Auxiliary equation of ① is $f(m) = 0$

$$\Rightarrow m^2 - 4m + 5 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $a = 1,\ b = -4,\ c = 5$

$$m = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2}$$

$$= \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{4 \pm \sqrt{-4}}{2}$$

$$= \frac{4 \pm \sqrt{i^2 2^2}}{2}$$

$$m = 2 \pm i$$

General Solution of equation ① is

$$y = e^{ax} (C_1 \cos bx + C_2 \sin bx)$$

$$= e^{2x} (C_1 \cos 2x + C_2 \sin x)$$

2

Case - (iv):

1) Solve $(D^4+8D^2+16)y=0$.

Sol:- Given that $(D^4+8D^2+16)y=0$

Which is of the form $f(D)y=0$

Where $f(D)=D^4+8D^2+16$

The Auxiliary equation is $f(m)=0$

$$\Rightarrow m^4+8m^2+16=0$$
$$\Rightarrow (m^2)^2+2(m^2)(4)+4^2=0$$
$$\Rightarrow (m^2+4)^2=0$$
$$\Rightarrow (m^2+4)(m^2+4)=0$$
$$\Rightarrow m^2+4=0,\ m^2+4=0$$
$$\Rightarrow m^2=-4,\ m^2=-4$$
$$\Rightarrow m=\pm 2i,\ m=\pm 2i$$

General Solution of equation (1) is

$$y=e^{ax}((c_1+c_2x)\cos bx+(c_3+c_4x)\sin bx)$$
$$=e^{0x}((c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x)$$
$$=(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x$$

2) Solve $((D-1)^2(D^2+1)^2)y=0$

Sol:- Given that $((D-1)^2(D^2+1)^2)y=0$

Which is of the form $f(D)y=0$

Where $f(D)=(D-1)^2(D^2+1)^2$

Auxiliary equation is $f(m)=0$

$$\Rightarrow (m-1)^2(m^2+1)^2=0$$
$$\Rightarrow (m-1)^2=0,\ (m^2+1)^2=0$$
$$\Rightarrow (m-1)(m-1)=0,\ (m^2+1)(m^2+1)=0$$
$$\Rightarrow m-1=0, m-1=0,\ m^2+1=0,\ m^2+1=0$$
$$\Rightarrow m=1,\ m=1,\ m^2=-1,\ m^2=-1$$
$$m=\pm i,\ m=\pm i$$

General Solution is $y=e^{mx}(c_1+c_2x)+e^{ax}[(c_3+c_4x)\cos bx+(c_5+c_6x)\sin bx]$

$$\Rightarrow y=e^{x}(c_1+c_2x)+[(c_3+c_4x)\cos x+(c_5+c_6x)\sin x]$$

Solution of non-homogeneous linear differential equations with Constant coefficient by means of Polynomial operator:

Let $f(D)y = \Theta$ be a non-homogeneous linear differential equation with constant coefficients.

If $y_c = y$ is the general solution of $f(D)y = \Theta$ and $y_p = y$ is the particular integral (or) particular solution of $f(D)y = \Theta$

$\therefore$ Then $y = y_c + y_p$ is the general solution of $f(D)y = \Theta$, where y_c is the complimentary function of $f(D)y = \Theta$.

Inverse Operator:

The operator D inverse is called the inverse of differential operator that is, if Θ is any function of 'x' defined on an inverse of Θ.
interval I. Then $D^{-1}\Theta$ (or) $\frac{1}{D}\Theta$ is called the

Note:-

If Θ is any function of x defined on n interval and α is any constant then the particular value of

1) $\frac{1}{D-\alpha}\Theta = e^{\alpha x}\int \Theta e^{-\alpha x}dx$

2) $\frac{1}{D+\alpha}\Theta = \frac{1}{D-(-\alpha)}\Theta = e^{-\alpha x}\int \Theta . e^{\alpha x}dx$

3) $\int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

4) $\int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$.

Problems:

1) Find the particular values of

a) $\frac{1}{D}x^2$ b) $\frac{1}{D^2}e^{4x}$ c) $\frac{1}{D^3}\cos x$.

Sol:- a) $\frac{1}{D}x^2$

$= \int x^2 dx$

$= \frac{x^3}{3} + c$

b) $\frac{1}{D^2} e^{4x}$

$= \frac{1}{D}\left[\frac{1}{D} e^{4x}\right]$

$= \frac{1}{D}\left[\int e^{4x} dx\right]$

$= \frac{1}{D}\left[\frac{e^{4x}}{4}\right]$

$= \frac{1}{4}\left[\frac{1}{D} e^{4x}\right]$

$= \frac{1}{4} \frac{e^{4x}}{4}$

$= \frac{e^{4x}}{16} + c$

c) $\frac{1}{D^3} \cos x$

$= \frac{1}{D} \cdot \frac{1}{D}\left[\frac{1}{D} \cos x\right]$

$= \frac{1}{D^2}\left[\int \cos x\right] dx$

$= \frac{1}{D}\left[\frac{1}{D} \sin x\right]$

$= \frac{1}{D}(-\cos x)$

$= \int -\cos x \, dx$

$= -\sin x + c.$

2) Find the Particular values of

a) $\frac{1}{D+1} x$ b) $\frac{1}{D-2} e^{2x}$ c) $\frac{1}{D+3} \cos x$

Sol:- a) $\frac{1}{D+1} x$

$= e^{-x} \int x e^{x} dx$

$= e^{-x}\left[x \int e^{x} dx - \int \frac{d}{dx}(x) \int e^{x} dx\right] dx + c$

$= e^{-x}\left[x e^{x} - \int e^{x} dx + c\right]$

$= e^{-x}\left[x e^{x} - e^{x}\right]$

$= x e^{x} \cdot e^{-x} - e^{x} \cdot e^{-x}$

$= x - 1$

b) $\frac{1}{D-2} e^{2x}$

$= e^{2x}\int e^{2x} \cdot e^{-2x} dx$

$= e^{2x}\int 1 dx$

$= e^{2x} \cdot x$

$= x e^{2x}$

c) $\frac{1}{D+3} \cos x$

$= e^{-3x} \int \cos x \, e^{3x} dx$

$= e^{-3x} \left[\int e^{3x} \cos x \, dx \right]$

$= e^{-3x} \left[\frac{e^{3x}}{9+1} (3\cos x + \sin x) \right]$

$= e^{-3x} \left[\frac{e^{3x}}{10} (3\cos x + \sin x) \right]$

$= \frac{1}{10} \left[3\cos x + e^{-3x} \sin x \right].$

3) Find the particular value of

a) $\frac{1}{(D-2)(D-3)} e^{2x}$ b) $\frac{1}{(D+1)(D-1)} x$

Sol:- a) Given that $\frac{1}{(D-2)(D-3)} e^{2x}$

$= \frac{1}{D-2} \left[\frac{1}{D-3} e^{2x} \right]$

$= \frac{1}{D-2} \left[e^{3x} \int e^{-3x} e^{2x} dx \right]$

$= \frac{1}{D-2} \left[e^{3x} \int e^{-x} dx \right]$

$= \frac{1}{D-2} \left[e^{3x} (-e^{-x}) \right]$

$= -\left[\frac{1}{D-2} e^{2x} \right]$

$= -\left[e^{2x} \int e^{-2x} e^{2x} dx \right]$

$= -\left[e^{2x} \int 1 dx \right]$

$= -e^{2x} \cdot x$

b) Given that $\frac{1}{(D+1)(D-1)}x$

$$= \frac{1}{D+1}\left[\frac{1}{D-1}x\right]$$

$$= \frac{1}{D+1}\left[e^{(-1)x}\int e^{x}\cdot x\,dx\right]$$

$$= \frac{1}{D+1}\left[e^{-x}\int x\cdot e^{x}dx\right]$$

$$= \frac{1}{D+1}\left[e^{x}\left[x\int e^{-x}dx - \int\left[\frac{d}{dx}(x)\int e^{-x}dx\right]dx\right]\right]$$

$$= \frac{1}{D+1}\left[e^{x}x(-e^{-x}) - \int 1(-e^{-x})dx\right]$$

$$= \frac{1}{D+1}\left[e^{x}(-xe^{-x}) + \int e^{-x}dx\right]$$

$$= \frac{1}{D+1}\left[e^{x}(-xe^{-x} - e^{-x})\right]$$

$$= \frac{1}{D+1}\left[e^{x}\cdot e^{-x}(-x-1)\right]$$

$$= \frac{1}{D+1}\left[e^{x-x}(-x-1)\right]$$

$$= \frac{1}{D+1}\left[e^{0}(-x-1)\right]$$

$$= \frac{1}{D-(-1)}(-x-1)$$

$$= e^{-x}\int(-x-1)e^{x}dx$$

$$= e^{-x}\left[\int -xe^{x}dx - \int e^{x}dx\right]$$

$$= e^{-x}\left[-\int xe^{x}dx - \int e^{x}dx\right]$$

$$= e^{-x}\left[-e^{x}(x-1) - e^{x}\right]$$

$$= e^{-x}\left[e^{x}(-(x-1)-1)\right]$$

$$= e^{-x}e^{x}(-x+1-1)$$

$$= e^{-x+x}(-x)$$

$$= e^{0}(-x)$$

$$= -x$$

Particular integral of $f(D)y = \theta_1$ when $\frac{1}{f(D)}$ is expressed has a partial fractions:

Let $f(D) = (D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)$

$$y_p = \frac{1}{f(D)}\theta_1 = \frac{1}{(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)}\theta_1$$

$$= \left[\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \cdots + \frac{A_n}{D-\alpha_n}\right]\theta_1$$

$$= A_1 e^{\alpha_1 x}\int \theta_1 e^{-\alpha_1 x}dx + A_2 e^{\alpha_2 x}\int \theta_1 e^{-\alpha_2 x}dx + \cdots + A_n e^{\alpha_n x}\int \theta_1 e^{-\alpha_n x}dx.$$

1) Solve $(D^2+a^2)y = \sec ax$

Sol: Given that $(D^2+a^2)y = \sec ax$

which is of the form $f(D)y = \theta_1$

where $f(D) = D^2+a^2$, $\theta_1 = \sec ax$

Auxiliary equation is $f(m) = 0$

$$\Rightarrow m^2 + a^2 = 0$$
$$\Rightarrow m^2 = -a^2$$
$$\Rightarrow m = \pm ai$$

$$y_c = e^{ax}(C_1 \cos bx + C_2 \sin bx)$$
$$= e^{0x}(C_1 \cos ax + C_2 \sin ax)$$
$$= C_1 \cos ax + C_2 \sin ax$$

$$y_p = \frac{1}{f(D)}\theta_1$$
$$= \frac{1}{D^2+a^2}\sec ax$$
$$= \frac{1}{(D+ai)(D-ai)}\sec ax$$
$$= \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right]\sec ax$$
$$= \frac{1}{2ai}\left[\frac{1}{D-ai}\sec ax - \frac{1}{D+ai}\sec ax\right]$$

Take $\frac{1}{D-ai}\sec ax$

$$= e^{a_i x} \int e^{-a_i x} \sec ax \, dx$$

$$= e^{a_i x} \int \left[\frac{\cos ax - i \sin ax}{\cos ax} \right] dx$$

$$= e^{a_i x} \left[\int 1 \, dx - i \int \tan ax \, dx \right]$$

$$= e^{a_i x} \left[x - \frac{i \log|\sec ax|}{a} \right]$$

$$= e^{a_i x} \left[x + \frac{i}{a} \log \cos ax \right]$$

Take $\frac{1}{D+a_i} \sec ax$

$$= e^{-a_i x} \int e^{a_i x} \sec ax \, dx$$

$$= e^{-a_i x} \int \left[\frac{\cos ax + i \sin ax}{\cos ax} \right] dx$$

$$= e^{-a_i x} \left[\int 1 \, dx + i \int \tan ax \, dx \right]$$

$$= e^{-a_i x} \left[x + \frac{i \log|\sec ax|}{a} \right]$$

$$= e^{-a_i x} \left[x - \frac{i}{a} \log \cos ax \right]$$

$$y_p = \frac{1}{2a_i} \left[\frac{1}{D-a_i} \sec ax - \frac{1}{D+a_i} \sec ax \right]$$

$$= \frac{1}{2a_i} \left[e^{a_i x} \left(x + \frac{i}{a} \log \cos ax \right) - e^{-a_i x} \left(x - \frac{i}{a} \log \cos ax \right) \right]$$

$$= \frac{1}{2a_i} e^{a_i x} (x) + \frac{1}{2a_i} e^{a_i x} \frac{i}{a} \log \cos ax - \frac{1}{2a_i} e^{-a_i x} (x) + \frac{1}{2a_i} e^{-a_i x} \frac{i}{a} \log \cos ax$$

$$= \frac{x}{a} \left(\frac{e^{a_i x} - e^{-a_i x}}{2i} \right) + \left[\frac{e^{a_i x} + e^{-a_i x}}{2} \right] \frac{1}{a_i} \cdot \frac{i}{a} \log \cos ax$$

$$y_p = \frac{x}{a} \sin ax + \cos ax \cdot \frac{1}{a^2} \log \cos ax$$

$\therefore$ General Solution of equation (1) is $y = y_c + y_p$

$$y = C_1 \cos ax + C_2 \sin ax + \frac{x}{a} \sin ax + \cos ax \cdot \frac{1}{a^2} \log \cos ax.$$

2

Method - I :

When $Q(x) = be^{ax}$

In this case $f(D)y = Q(x)$

becomes $f(D)y = be^{ax}$ —— ①

Let y_c is the complementary function of ①

$$y_p = \frac{1}{f(D)} be^{ax}$$

$$= \frac{1}{f(a)} be^{ax} \text{ when } f(a) \neq 0$$

The general solution of equation ① is

$$y = y_c + y_p$$

Note :

The zero case is $\frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$

Problems:

1) Solve $(D^2 - 5D + 6)y = e^{4x}$

Sol: Given that $(D^2 - 5D + 6)y = e^{4x}$

where $f(D) = D^2 - 5D + 6$, $Q = e^{4x}$

Auxiliary equation is $f(m) = 0$

$$\Rightarrow m^2 - 5m + 6 = 0$$

$$\Rightarrow m^2 - 3m - 2m + 6 = 0$$

$$\Rightarrow m(m-3) - 2(m-3) = 0$$

$$\Rightarrow (m-2)(m-3) = 0$$

$$\Rightarrow m = 2, 3.$$

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$= c_1 e^{2x} + c_2 e^{3x}$$

$$y_p = \frac{1}{f(D)} Q$$

$$= \frac{1}{D^2 - 5D + 6} e^{4x}$$

$$= \frac{1}{4^2 - 5(4) + 6} e^{4x}$$

$$= \frac{1}{16 - 20 + 6} e^{4x}$$

$$y_p = \frac{e^{4x}}{2}$$

General Solution of equation ① is $y = y_c + y_p$

$$\Rightarrow y = c_1 e^{3x} + c_2 e^{2x} + \frac{e^{4x}}{2}$$

2) Solve $(D^2 - 2D - 3)y = 5$

Sol:- Given that $(D^2 - 2D - 3)y = 5$

where $f(D) = D^2 - 2D - 3$, $Q = 5$

Auxiliary form $f(m) = 0$

$$\Rightarrow m^2 - 2m - 3 = 0$$
$$\Rightarrow m^2 - 3m + m - 3 = 0$$
$$\Rightarrow m(m-3) + 1(m-3) = 0$$
$$\Rightarrow (m-3)(m+1) = 0$$
$$\Rightarrow m = 3, -1$$

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$
$$= c_1 e^{3x} + c_2 e^{-x}$$

$$y_p = \frac{1}{f(D)} Q$$
$$= \frac{1}{D^2 - 2D - 3} 5$$
$$= \frac{1}{D^2 - 2D - 3} 5e^{0x}$$
$$= \frac{1}{0 - 2(0) - 3} 5e^{0x}$$
$$y_p = -\frac{5}{3}$$

$\therefore$ General Solution of equation ① is $y = y_c + y_p$.

$$\therefore y = c_1 e^{3x} + c_2 e^{-x} + \left(-\frac{5}{3}\right).$$

3) Solve $(D^3 - 7D + 6)y = e^{2x}$

Sol:- Given that $(D^3 - 7D + 6)y = e^{2x}$ — ①

where $f(D) = D^3 - 7D + 6$, $Q = e^{2x}$

Auxiliary form $f(m) = 0$

$\Rightarrow m^3 - 7m + 6 = 0$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -7 & 6 \\ & 0 & 1 & 1 & -6 \\ \hline 2 & 1 & 1 & -6 & \underline{0} \\ & 0 & 2 & 6 & \\ \hline -3 & 1 & 3 & \underline{0} & \\ & 0 & -3 & & \\ \hline & 1 & \underline{0} & & \end{array}$$

$m = 1, 2, -3$

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$

$= c_1 e^{x} + c_2 e^{2x} + c_3 e^{-3x}$

$y_p = \frac{1}{f(D)} \theta_1$

$= \frac{1}{D^3 - 7D + 6} e^{2x}$

$= \frac{1}{(D-1)(D-2)(D+3)} e^{2x}$

$= \frac{1}{(D-1)(D+3)} \left[\frac{1}{D-2} e^{2x}\right]$

$= \frac{1}{(2-1)(2+3)} \left[\frac{x}{1!} e^{2x}\right]$

$y_p = \frac{1}{5} x e^{2x}$

$\therefore$ General Solution of equation ① is $y = y_c + y_p$.

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{-3x} + \frac{1}{5} x e^{2x}$$

4) Solve $(D^3 - 4D^2) y = 5$

Sol:- Given that $(D^3 - 4D^2) y = 5$ — ①

where $f(D) = D^3 - 4D^2$, $\theta = 5$

Auxiliary form $f(m) = 0$

$\Rightarrow m^3 - 4m^2 = 0$

$$
\begin{array}{r|rrrr}
4 & 1 & -4 & 0 & 0 \\
 & 0 & 4 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & \boxed{0} \\
 & 0 & 0 & 0 & \\
\hline
0 & 1 & 0 & \boxed{0} & \\
 & 0 & 0 & & \\
\hline
 & 1 & \boxed{0} & &
\end{array}
$$

$m = 4, 0, 0$

$$y_c = c_1 e^{4x} + (c_2 + c_3 x) e^{0x}$$

$$= c_1 e^{4x} + (c_2 + c_3 x)$$

$$y_p = \frac{1}{f(D)} \theta_1$$

$$= \frac{1}{D^3 - 4D^2} 5$$

$$= \frac{1}{D^3 - 4D^2} 5e^{0x}$$

$$= \frac{1}{D^2(D-4)} 5e^{0x}$$

$$= \frac{5}{D-4} \left[\frac{1}{D^2} e^{0x}\right]$$

$$= \frac{5}{0-4} \left[\frac{1}{D} \left[\frac{1}{D} e^{0x}\right]\right]$$

$$= \frac{5}{-4} \left[\frac{1}{D} \left[\int 1 dx\right]\right]$$

$$= \frac{-5}{4} \left[\frac{1}{D} x\right]$$

$$= \frac{-5}{4} \left[\int x dx\right]$$

$$= \frac{-5}{4} \left(\frac{x^2}{2}\right)$$

$$= -\frac{5x^2}{8}$$

$\therefore$ General Solution of equation (1) is $y = y_c + y_p$

$$y = c_1 e^{4x} + (c_2 + c_3 x) - \frac{5x^2}{8}$$

2

5) Solve $(D^2-5D+6)y = e^x$

Sol:- Given that $(D^2-5D+6)y = e^x$ — ①

where $f(D) = D^2-5D+6$, $\theta = e^x$

Auxiliary form $f(m) = 0$

$$\Rightarrow m^2-5m+6=0$$

$$\Rightarrow m^2-3m-2m+6=0$$

$$\Rightarrow m(m-3)-2(m-3)=0$$

$$\Rightarrow (m-3)(m-2)=0$$

$$\Rightarrow m = 3, 2$$

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$= c_1 e^{3x} + c_2 e^{2x}$$

$$y_p = \frac{1}{f(D)}\theta$$

$$= \frac{1}{D^2-5D+6} e^x$$

$$= \frac{1}{1-5+6} e^x$$

$$= \frac{1}{2} e^x$$

General solution of equation ① is $y = y_c + y_p$

$$\Rightarrow y = c_1 e^{3x} + c_2 e^{2x} + \frac{e^x}{2}.$$

6) Solve $(D^3-5D^2+8D-4)y = e^{2x}$

Sol:- Given that $(D^3-5D^2+8D-4)y = e^{2x}$

where $f(D) = D^3-5D^2+8D-4$, $\theta = e^{2x}$

Auxiliary form, $f(m) = 0$

$$\Rightarrow m^3-5m^2+8m-4=0$$

1	1	-5	8	-4
	0	1	-4	4
2	1	-4	4	0
	0	2	-4	
2	1	-2	0	
	0	2		
	1	0		

$m = 1, 2, 2$

$y_c = c_1 e^{x} + (c_2 + c_3 x) e^{2x}$

$y_p = \frac{1}{f(D)} Q_1$

$= \frac{1}{D^3 - 5D^2 + 8D - 4} e^{2x}$

$= \frac{1}{(D-1)(D-2)(D-2)} e^{2x}$

$= \frac{1}{\alpha - 1}\left[\frac{1}{(D-2)^2} e^{2x}\right]$

$= \frac{1}{1}\left[\frac{x^2}{2!} e^{2x}\right]$

General Solution of equation ① is $y = y_c + y_p$

$$y = c_1 e^{x} + (c_2 + c_3 x) e^{2x} + \frac{x^2}{2!} e^{2x}.$$

7) Solve $(D^2 - 2D + 5)y = e^{-x}$ 2

Sol:- Given that $(D^2 - 2D + 5)y = e^{-x}$ — ①

Where $f(D) = D^2 - 2D + 5$, $Q_1 = e^{-x}$

Auxiliary form $f(m) = 0$

$\Rightarrow m^2 - 2m + 5 = 0$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here $a = 1$, $b = -2$, $c = 5$

$= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$

$= \frac{2 \pm \sqrt{4 - 20}}{2}$

$= \frac{2 \pm \sqrt{-16}}{2}$

$= \frac{2 \pm \sqrt{i^2 16}}{2}$

$= \frac{2 \pm i4}{2}$

$= 1 \pm 2i$

$y_c = e^{ax}(C_1 \cos bx + C_2 \sin bx)$

$= e^{x}(C_1 \cos 2x + C_2 \sin 2x)$

$y_p = \frac{1}{f(D)} Q_1$

$= \frac{1}{D^2 - 2D + 5} e^{-x}$

$= \frac{1}{(-1)^2 - 2(-1) + 5} e^{-x}$

$= \frac{1}{1+2+5} e^{-x}$

$= \frac{1}{8} e^{-x}$

General solution of equation ① is $y = y_c + y_p$

$y = e^{x}(C_1 \cos 2x + C_2 \sin 2x) + \frac{1}{8} e^{-x}$

8) Solve $(D^2 - 3D + 2)y = \cosh x$

Sol: Given that $(D^2 - 3D + 2)y = \cosh x$ — ①

which is of the form $f(D)y = Q_1$

where $f(D) = D^2 - 3D + 2$, $Q_1 = \cosh x$

Auxiliary equation is $f(m) = 0$

$\Rightarrow m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - 2m + m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-2)(m-1) = 0$

$\Rightarrow m = 2, 1$

$y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$= C_1 e^{2x} + C_2 e^{x}$

$y_p = \frac{1}{f(D)} Q_1$

$= \frac{1}{D^2 - 3D + 2} \cosh x$

$$= \frac{1}{D^2-3D+2}\left[\frac{e^x+e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-3D+2}e^x + \frac{1}{D^2-3D+2}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(D-2)(D-1)}e^x + \frac{1}{1+3+2}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(1-2)}\left[\frac{1}{(D-1)}e^x\right] + \frac{1}{6}e^{-x}\right]$$

$$= \frac{1}{2}\left[-1\left(\frac{x!}{1!}e^x\right) + \frac{1}{6}e^{-x}\right]$$

$$y_p = -\frac{1}{2}xe^x + \frac{1}{12}e^{-x}$$

General solution of equation ① is $y = y_c + y_p$

$$y = c_1e^{2x} + c_2e^x - \frac{1}{2}xe^x + \frac{1}{12}e^{-x}.$$

Method - II :-

When $Q(x) = b\sin ax$ (or) $b\cos ax$

In this case $f(D)y = Q(x)$ becomes $f(D)y = b\cos ax$ (or) $b\sin ax$ —①

Let y_c be the complementary function of equation ①

$f(D)$ is a polynomial in D^2.

i.e., $f(D) = f(D)^2$

$$y_p = \frac{1}{f(D)} b\sin ax \text{ (or) } b\cos ax$$

$$= \frac{1}{f(-a^2)} b\sin ax \text{ (or) } b\cos ax \text{ if } f(-a^2) \neq 0$$

$\therefore$ The General Solution is $y = y_c + y_p$

Note :-

* $\frac{1}{D^2+b^2}\sin bx = \frac{-x}{2b}\cos bx$

* $\frac{1}{D^2+b^2}\cos bx = \frac{x}{2b}\sin ax$

Problems :-

1) Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = \sin 2x$

Sol:- Given that $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = \sin 2x$

The operator form is $D^2y - Dy - 2y = \sin 2x$ — ①

$$(D^2 - D - 2)y = \sin 2x$$

where $f(D) = D^2 - D - 2$

Auxiliary equation is $f(m) = 0$

$$\Rightarrow m^2 - m - 2 = 0$$

$$\Rightarrow m^2 - 2m + m - 2 = 0$$

$$\Rightarrow m(m-2) + 1(m-2) = 0$$

$$\Rightarrow (m-2)(m+1) = 0$$

$$\Rightarrow m = 2, -1$$

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$= c_1 e^{2x} + c_2 e^{-x}$$

$$y_p = \frac{1}{f(D)} Q$$

$$= \frac{1}{D^2 - D - 2} \sin 2x$$

$$= \frac{1}{-4 - D - 2} \sin 2x$$

$$= \frac{1}{-6 - D} \sin 2x$$

$$= \frac{1}{-(6+D)} \sin 2x$$

$$y_p = \frac{-1}{(6+D)} \times \frac{6-D}{6-D} \sin 2x$$

$$= -\frac{(6-D)}{36 - D^2} \sin 2x$$

$$= \frac{D-6}{36-D^2} \sin 2x .$$

$$= \frac{D-6}{36-(-4)} \sin 2x$$

$$= \frac{D(\sin 2x) - 6\sin 2x}{40}$$

$$= \frac{\cos 2x(2) - 6\sin 2x}{40}$$

$\therefore$ General solution of equation ① is $y = y_c + y_p$

$$y = c_1 e^{2x} + c_2 e^{-x} + \frac{2\cos 2x - 6\sin 2x}{40}$$

2) Solve $(D^2+9)y = \cos^3 x$

Sol:- Given that $(D^2+9)y = \cos^3 x$ — ①

where $f(D) = D^2+9$

Auxiliary form is $f(m) = 0$

$$\Rightarrow m^2+9=0$$

$$\Rightarrow m^2 = -9$$

$$\Rightarrow m = \pm 3i$$

$$y_c = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

$$y_c = c_1 \cos 3x + c_2 \sin 3x$$

$$y_p = \frac{1}{f(D)} \theta$$

$$= \frac{1}{D^2+9} \cos^3 x$$

$$= \frac{1}{D^2+9}\left[\frac{\cos 3x + 3\cos x}{4}\right]$$

$$= \frac{1}{4}\left[\frac{1}{D^2+9}\cos 3x + \frac{1}{D^2+9} 3\cos x\right]$$

$$= \frac{1}{4}\left[\frac{1}{D^2+3^2}\cos 3x + \frac{3}{-1+9}\cos x\right]$$

$$= \frac{1}{4}\left[\frac{x}{2(3)}\sin 3x + \frac{3}{8}\cos x\right]$$

$$= \frac{x}{24}\sin 3x + \frac{3}{32}\cos x$$

∴ General Solution of equation ① is

$$y = y_c + y_p$$

$$y = C_1\cos 3x + C_2\sin 3x + \frac{x}{24}\sin 3x + \frac{3}{32}\cos x$$

2

3) Solve $(D^2 - 4D + 3)y = \sin 3x \cdot \cos 2x$

Sol: Given that $(D^2 - 4D + 3)y = \sin 3x \cdot \cos 2x$ — ①

Where $f(D) = D^2 - 4D + 3$

Auxiliary equation is $f(m) = 0$

$$\Rightarrow m^2 - 4m + 3 = 0$$

$$\Rightarrow m^2 - 3m - m + 3 = 0$$

$$\Rightarrow m(m-3) - 1(m-3) = 0$$

$$\Rightarrow (m-3)(m-1) = 0$$

$$\Rightarrow m = 3, 1$$

$$y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$\Rightarrow y_c = C_1 e^{3x} + C_2 e^{x}$$

$$y_p = \frac{1}{f(D)} Q$$

$$= \frac{1}{D^2 - 4D + 3}\sin 3x \cdot \cos 2x$$

$$= \frac{1}{D^2 - 4D + 3}\left[\frac{1}{2}\{\sin(3x+2x) + \sin(3x-2x)\}\right]$$

$$= \frac{1}{D^2 - 4D + 3}\left[\frac{1}{2}(\sin 5x + \sin x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 4D + 3}\sin 5x + \frac{1}{D^2 - 4D + 3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-25 - 4D + 3}\sin 5x + \frac{1}{-1 - 4D + 3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-(4D + 22)}\sin 5x + \frac{1}{2 - 4D}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{-1}{4D + 22} \times \frac{4D - 22}{4D - 22}\sin 5x + \frac{1}{2 - 4D} \times \frac{2 + 4D}{2 + 4D}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{22 - 4D}{16D^2 - (22)^2}\sin 5x + \frac{2 + 4D}{4 - 16D^2}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{22-4D}{16(-5)-484}\sin 5x + \frac{2+4D}{4-16(-1)}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{(22-4D)}{-884}\sin 5x + \frac{(2+4D)}{20}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{22(\sin 5x)-4D(\sin 5x)}{-884} + \frac{2\sin x + 4D\sin x}{20}\right]$$

$$= \frac{1}{2}\left[\frac{22\sin 5x - 4(5)\cos 5x}{-884} + \frac{2\sin x + 4\cos x}{20}\right]$$

$\therefore$ General solution of equation ① is

$$y = y_c + y_p$$

$$y = c_1e^{3x} + c_2e^{x} + \frac{1}{2}\left[\frac{22\sin 5x - 20\cos 5x}{-884} + \frac{2\sin x + 4\cos x}{20}\right]$$

4) Solve $(D^2-3D+2)y = \cos 3x \cdot \cos 2x$

Sol:- Given that $(D^2-3D+2)y = \cos 3x \cdot \cos 2x$ — ①

Where $f(D) = D^2-3D+2$

Auxiliary form $f(m) = 0$

$$\Rightarrow m^2-3m+2=0$$

$$\Rightarrow m^2-2m-m+2=0$$

$$\Rightarrow m(m-2)-(m-2)=0$$

$$\Rightarrow (m-2)(m-1)=0$$

$$\Rightarrow m = 2, 1$$

$$y_c = c_1e^{m_1x} + c_2e^{m_2x}$$

$$= c_1e^{2x} + c_2e^{x}$$

$$y_p = \frac{1}{f(D)}\,Q$$

$$= \frac{1}{D^2-3D+2}\cos 3x \cos 2x$$

$$= \frac{1}{D^2-3D+2}\left[\frac{1}{2}\{\cos(3x+2x) + \cos(3x-2x)\}\right]$$

$$= \frac{1}{D^2-3D+2}\left[\frac{1}{2}(\cos 5x + \cos x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-3D+2}\cos 5x + \frac{1}{D^2-3D+2}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-25-3D+2}\cos 5x + \frac{1}{-1-3D+2}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-23-3D}\cos 5x + \frac{1}{1-3D}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-(23+3D)}\cos 5x + \frac{1}{1-3D}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-(23+3D)}\times\frac{23-3D}{23-3D}\cos 5x + \frac{1}{1-3D}\times\frac{1+3D}{1+3D}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{-(3D-23)}{-(23+3D)(23-3D)}\cos 5x + \frac{1+3D}{(1-3D)(1-3D)}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{3D-23}{(23)^2-3^2D^2}\cos 5x + \frac{1+3D}{1^2-(3D)^2}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{3D-23}{529-(-25)9}\cos 5x + \frac{1+3D}{1-9D^2}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{3D-23}{529+225}\cos 5x + \frac{1+3D}{1-9(-1)}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{3D-23}{529+225}\cos 5x + \frac{1+3D}{1+9}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{3D(\cos 5x)-23\cos 5x}{754} + \frac{\cos x+3D(\cos x)}{1+9}\right]$$

$$= \frac{1}{2}\left[\frac{3(-5\sin 5x)-23\cos 5x}{754} + \frac{\cos x+3(-\sin x)}{10}\right]$$

$$= \frac{1}{2}\left[\frac{-15\sin 5x-23\cos 5x}{754} + \frac{\cos x-3\sin x}{10}\right]$$

$$y_p = \frac{1}{2}\left[\frac{-(15\sin x+23\cos 5x)}{754} + \frac{\cos x-3\sin x}{10}\right]$$

$\therefore$ General Solution of equation ① is $y = y_c + y_p$

$$y = c_1e^{2x} + c_2e^{x} + \frac{1}{2}\left[\frac{-(15\sin x+23\cos 5x)}{754} + \frac{\cos x-3\sin x}{10}\right]$$

6) Solve $(D^4+3D^2-4)y = \cos^2 x - \cosh x$

Sol:- Given that $(D^4+3D^2-4)y = \cos^2 x - \cosh x$ — ①

where $f(D) = D^4+3D^2-4$

Auxiliary equation is $f(m)=0$

$$\Rightarrow m^4+3m^2-4=0$$

$$\begin{array}{r|rrrrr}
1 & 1 & 0 & 3 & 0 & -4 \\
 & 0 & 1 & 1 & 4 & 4 \\
\hline
-1 & 1 & 1 & 4 & 4 & \boxed{0} \\
 & 0 & -1 & 0 & -4 & \\
\hline
 & 1 & 0 & 4 & \boxed{0} &
\end{array}$$

$$m^2+4=0$$

$$\Rightarrow m^2=-4$$

$$\Rightarrow m^2 = \pm 2i$$

$$y_c = c_1e^{x}+c_2e^{-x}+c_3\cos 2x + c_4 \sin 2x .$$

$$y_p = \frac{1}{f(D)}\,\theta(x)$$

$$= \frac{1}{D^4+3D^2-4}(\cos^2 x - \cosh x)$$

$$= \frac{1}{D^4+3D^2-4}\left[\frac{1+\cos 2x}{2} - \left(\frac{e^{x}+e^{-x}}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^4+3D^2-4}e^{0x} + \frac{1}{D^4+3D^2-4}\cos 2x - \frac{1}{D^2+3D^2-4}e^{x} - \frac{1}{D^4 3D^2-4}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{0+3(0)-4} + \frac{1}{(D+1)(D-1)(D^2+4)}\cos 2x - \frac{1}{(D+1)(D-1)(D^2+4)}e^{x} - \frac{1}{(D+1)(D-1)(D^2+4)}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{4} + \frac{1}{(D^2-1)(D^2+4)}\cos 2x - \frac{1}{(1+1)(1+4)}\left(\frac{1}{D-1}e^{x}\right) - \frac{1}{(-1-1)(1+4)}\left(\frac{1}{D+1}e^{-x}\right)\right]$$

$= \frac{1}{2}\left[-\frac{1}{4} + \frac{1}{-4-1}\left(\frac{1}{D^2+2^2}\cos 2x\right) - \frac{1}{2(5)}\left(\frac{x}{1!}e^{x}\right) - \frac{1}{(-2)(5)}\frac{x}{1!}e^{-x}\right]$

$= \frac{1}{2}\left[-\frac{1}{4} - \frac{1}{5}\left(\frac{x}{4}\sin 2x\right) - \frac{1}{10}xe^{x} + \frac{1}{10}xe^{-x}\right]$

$\therefore$ General solution of equation ① is $y = y_c + y_p$

$y = c_1e^{x} + c_2e^{-x} + c_3\cos 2x + c_4\sin 2x + \frac{1}{2}\left[-\frac{1}{4} - \frac{1}{5}\left(\frac{x}{4}\sin 2x\right) - \frac{1}{10}xe^{x} + \frac{1}{10}xe^{-x}\right]$

7) Solve $\frac{d^3y}{dx^3} + a^2\frac{dy}{dx} = \sin ax$

Sol: Given that $\frac{d^3y}{dx^3} + a^2\frac{dy}{dx} = \sin ax$ — ①

Operator form is $D^3y + a^2Dy = \sin ax$

$\Rightarrow (D^3 + a^2D)y = \sin ax$

Where $f(D) = D^3 + a^2D$

Auxiliary equation of $f(m) = 0$

$\Rightarrow m^3 + a^2m = 0$

$\Rightarrow m(m^2 + a^2) = 0$

$\Rightarrow m = 0,\ m^2 + a^2 = 0$

$m^2 = -a^2$

$m = \pm ai$

$y_c = c_1 + c_2\cos ax + c_3\sin ax$

$y_p = \frac{1}{f(D)}Q$

$= \frac{1}{D^3 + a^2D}\sin ax$

$= \frac{1}{D(D^2 + a^2)}\sin ax$

$= \frac{1}{D^2 + a^2}\left[\frac{1}{D}\sin ax\right]$

$= \frac{1}{D^2 + a^2}\left[\frac{-\cos ax}{a}\right]$

$= \frac{-1}{a}\left[\frac{1}{D^2 + a^2}\cos ax\right]$

$$= -\frac{1}{a}\left[\frac{x}{2a}\sin ax\right]$$

$\therefore$ The general solution of equation ① is $y = y_c + y_p$

$$y = C_1 + C_2\cos ax + C_3\sin ax - \left[\frac{1}{a}\cdot\frac{x}{2a}\sin ax\right]$$

8) Solve $(D^3+4D)y = \sin 2x$

Sol:- Given that $(D^3+4D)y = \sin 2x$ __ ①

where $f(D) = D^3+4D$

The Auxiliary equation is $f(m)=0$

$$\Rightarrow m^3+4m = 0$$

$$\Rightarrow m(m^2+4) = 0$$

$$\Rightarrow m=0,\ m^2=-4$$

$$m = \pm 2i$$

$$m = 0, \pm 2i$$

$$y_c = C_1 + C_2\cos 2x + C_3\sin 2x$$

$$y_p = \frac{1}{f(D)}Q$$

$$= \frac{1}{D^3+4D}\sin 2x$$

$$= \frac{1}{D(D^2+4)}\sin 2x$$

$$= \frac{1}{D^2+4}\left[\frac{1}{D}\sin 2x\right]$$

$$= \frac{1}{D^2+4}\left[\int \sin 2x\right]$$

$$= \frac{1}{D^2+4}\left[-\frac{\cos 2x}{2}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{D^2+2^2}\cos 2x\right]$$

$$= -\frac{1}{2}\left[\frac{x}{2(2)}\sin 2x\right]$$

$$y_p = -\frac{x}{8}\sin 2x$$

$\therefore$ The general solution of equation ① is $y = y_c + y_p$

$$y = C_1 + C_2\cos 2x + C_3\sin 2x - \frac{x}{8}\sin 2x$$

9) Solve $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x + \cos 2x$

Sol:- Given equation is $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x + \cos 2x$ — ①

Operator form is $D^2y + 4y = e^x + \sin 2x + \cos 2x$

$\Rightarrow (D^2+4)y = e^x + \sin 2x + \cos 2x$

where $f(D) = D^2+4$

The auxiliary form is $f(m) = 0$

$\Rightarrow m^2 + 4 = 0$

$\Rightarrow m^2 = -4$

$\Rightarrow m = \pm 2i$

$y_c = (c_1 \cos 2x + c_2 \sin 2x)e^{0x}$

$= c_1 \cos 2x + c_2 \sin 2x$

$y_p = \frac{1}{f(D)} Q$

$= \frac{1}{D^2+4}(e^x + \sin 2x + \cos 2x)$

$= \frac{1}{D^2+4} e^x + \frac{1}{D^2+4} \sin 2x + \frac{1}{D^2+4} \cos 2x$

$= \frac{1}{1+4} e^x + \left[\frac{-x}{2(2)} \cos 2x\right] + \frac{x}{2(2)} \sin 2x$

$= \frac{1}{5} e^x - \frac{x}{4} \cos 2x + \frac{x}{4} \sin 2x$

$\therefore$ The general solution of equation ① is $y = y_c + y_p$

$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5} e^x - \frac{x}{4} \cos 2x + \frac{x}{4} \sin 2x$

10) Solve $(D^2+4)y = 2\cos^2 x$

Sol:- Given that $(D^2+4)y = 2\cos^2 x$ — ①

where $f(D) = D^2+4$

Auxiliary form $f(m) = 0$

$\Rightarrow m^2 + 4 = 0$

$\Rightarrow m^2 = -4$

$\Rightarrow m^2 = \pm 2i$

$y_c = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

$= e^0 (c_1 \cos 2x + c_2 \sin 2x)$

$= c_1 \cos 2x + c_2 \sin 2x$

$$y_p = \frac{1}{f(D)} Q_1$$

$$= \frac{1}{D^2+4} 2\cos^2 x$$

$$= \frac{2}{D^2+4}\left[\frac{1+\cos 2x}{2}\right]$$

$$= \left[\frac{1}{D^2+4} + \frac{1}{D^2+4}\cos 2x\right]$$

$$= \left[\frac{1}{D^2+4} e^{0x} + \frac{1}{D^2+2^2}\cos 2x\right]$$

$$= \frac{1}{0+4} + \frac{x}{(2)(2)} \sin 2x$$

$$= \frac{1}{4} + \frac{x}{4}\sin 2x .$$

$=$

UNIT-4

Higher order Linear Differential Equations-II

Method- When $Q(x) = bx^k$ and $f(D) = D - a_0$, $a_0 \neq 0$.

In this case the equation $f(D)y = Q(x)$ becomes $(D-a_0)y = bx^k \rightarrow (1)$

Let $y = y_c$ be the complementary function of eqn (1).

particular Integral of equation (1) is

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D-a_0} bx^k$$

$$= \frac{-b}{a_0\left(1-\frac{D}{a_0}\right)} x^k$$

$$= \frac{-b}{a_0}\left(1-\frac{D}{a_0}\right)^{-1} x^k$$

$$= \frac{-b}{a_0}\left[1+\frac{D}{a_0}+\frac{D^2}{a_0^2}+\dots+\frac{D^k}{a_0^k}\right]x^k$$

Therefore the General solution of equation (1) is $y = y_c + y_p$

Note: If $k=0$ then $y_p = \frac{-b}{a_0}$

* $[(1+x)]^{-1} = 1-x+x^2-x^3+\dots\dots$
* $[(1-x)]^{-1} = 1+x+x^2+x^3+\dots\dots$
* $[(1+x)]^{-2} = 1-2x+3x^2-4x^3+\dots\dots$
* $[(1+x)]^{-2} = 1+2x+3x^2+4x^3+\dots\dots$

problems:

1. Solve $(D^2-4D+4)y = x^3$

Sol: Given equation is $(D^2-4D+4)y = x^3 \rightarrow (1)$ Where $f(D) = D^2-4D+4$

The Auxilary equation is $f(m) = 0$

$$\Rightarrow m^2-4m+4=0$$

$$\Rightarrow m^2-2m-2m+4=0$$

$$\Rightarrow m(m-2)-2(m-2)=0$$

$$\Rightarrow (m-2)(m-2)=0$$

$$\Rightarrow m=2,2$$

$\therefore$ Complementary function of equation (1) is $y_c = (C_1 + C_2 x) e^{2x}$

$$y_P = \frac{1}{f(D)} Q(x) = \frac{1}{D^2 - 4D + 4} x^3$$

$$= \frac{1}{(D-2)^2} x^3 = \frac{1}{\left[-2\left(1 - \frac{D}{2}\right)\right]^2} x^3 = \frac{1}{4} (1 - D/2)^{-2} x^3$$

$$= \frac{1}{4}\left[1 + 2\left(\frac{D}{2}\right) + 3\left(\frac{D}{2}\right)^2 + 4\left(\frac{D}{2}\right)^3 + \cdots\right] x^3$$

$$= \frac{1}{4}\left[x^3 + D(x^3) + \frac{3}{4} D^2 x^3 + \frac{4}{8} D^3 (x^3) + \cdots\right]$$

x^3

$D \to 3x^2$

$D^2 \to 6x$

$D^3 \to 6$

$$= \frac{1}{4}\left[x^3 + 3x^2 + \frac{3}{4}(6x) + \frac{4}{8}(6)\right]$$

$$= \frac{1}{4}\left[x^3 + 3x^2 + \frac{9x}{2} + 3\right]$$

$$= \frac{1}{4}\left[\frac{2x^3 + 6x^2 + 9x + 3}{2}\right]$$

$$y_P = \frac{1}{8}(2x^3 + 6x^2 + 9x + 3)$$

Therefore the general solution of eqn (1) is $y = y_c + y_P$

$$\Rightarrow y = (C_1 + C_2 x) e^{2x} + \frac{1}{8}(2x^3 + 6x^2 + 9x + 3).$$

2: Solve $(D^2 - 3D + 2) y = 2x^2$

Sol: Given equation is $(D^2 - 3D + 2) y = 2x^2 \to (1)$ Where $f(D) = D^2 - 3D + 2$

The Auxilary equation is $f(m) = 0$

$$\Rightarrow m^2 - 3m + 2 = 0$$

$$\Rightarrow m^2 - 2m - m + 2 = 0$$

$$\Rightarrow m(m-2) - 1(m-2) = 0$$

$$\Rightarrow (m-1)(m-2) = 0$$

$$\Rightarrow m = 1, 2$$

$\therefore$ Complementary function of equation (1) is $y_c = C_1 e^{x} + C_2 e^{2x}$.

$$y_P = \frac{1}{f(D)} Q(x) = \frac{1}{D^2 - 3D + 2} 2x^2$$

$$= \frac{1}{2\left[1 + \left(\frac{D^2 - 3D}{2}\right)\right]} 2x^2$$

$$= \left[1 + \left(\frac{D^2 - 3D}{2}\right)\right]^{-1} x^2$$

$$= \left[1 - \left(\frac{D^2 - 3D}{2}\right) + \left(\frac{D^2 - 3D}{2}\right)^2 + \cdots\right] x^2$$

ii, $\frac{1}{D^4+D^2} 4\sin x$

$D^2 \to -a^2 \to -1$

$$\frac{1}{D^2(1+D^2)} 4\sin x = \frac{1}{1+D^2}\left[\frac{1}{D^2} 4\sin x\right]$$

$$= \frac{1}{1+D^2}[-4\sin x]$$

$$= -4\left[\frac{1}{D^2+1^2}\sin x\right]$$

$$= -4\left[\frac{-x}{2(1)}\cos(1x)\right]$$

$$= \frac{4x\cos x}{2}$$

$$= 2x\cos x \to \text{ii,}$$

$\left[\frac{1}{D^2} 4\sin x = 4\left[\frac{1}{D}\int \sin x\,dx\right]\right.$
$= 4\left[\frac{1}{D}[-\cos x]\right]$
$= 4\left[\int -\cos x\,dx\right]$
$= -4\sin x.\Big]$

$\left[\because \frac{1}{D^2+b^2}\cos bx = \frac{+x}{2b}\sin bx\right.$
$\left.\frac{1}{D^2+b^2}\sin bx = \frac{-x}{2b}\cos bx\right]$

iii, $\frac{1}{D^4+D^2} 2\cos x = \frac{1}{D^2(1+D^2)} 2\cos x$

$$= \frac{1}{1+D^2}\left[\frac{1}{D^2} 2\cos x\right]$$

$$= \frac{1}{1+D^2}[-2\cos x]$$

$$= -2\cdot\frac{1}{D^2+1^2}\cos x$$

$$= -2\cdot\frac{x}{2(1)}\sin(1)x = -x\sin x \to \text{iii,}$$

$\left[\frac{1}{D^2} 2\cos x = 2\left[\frac{1}{D}\int \cos x\,dx\right]\right.$
$= 2\left[\frac{1}{D}\sin x\right]$
$= 2\left[\int \sin x\,dx\right]$
$= -2\cos x\Big]$

from ii, iii, and iv, in equation (2) we get

$y_P = \frac{1}{4}[x^4 - 12x^2 + 24] + 2x\cos x + x\sin x.$

Therefore the General solution of equation (1) is $y = y_c + y_p$

$y = (c_1 + c_2x) + (c_3\cos x + c_4\sin x) + \frac{1}{4}(x^4 - 12x^2 + 24) + 2x\cos x + x\sin x.$

4: Solve $(D^2 - 2D + 4)y = 8(x^2 + e^{2x} + \sin 2x)$

Sol: Given equation is $(D^2 - 2D + 4)y = 8(x^2 + e^{2x} + \sin 2x) \to (1)$ Where $f(D) = D^2 - 2D + 4$

The Auxiliary equation is $f(m) = 0$.

$\Rightarrow m^2 - 2m + 4 = 0$

$\Rightarrow m = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$

$\therefore$ complementary function of equation (1) is $y_c = e^x[c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x]$

$y_P = \frac{1}{f(D)} Q(x) = \frac{1}{D^2 - 2D + 4}[8x^2 + 8e^{2x} + 8\sin 2x] \to (2)$

$= \left[1 - \frac{1}{2}(D^2-3D) + \frac{1}{4}(D^4+9D^2-6D^3) - \dots\right] x^2$ $\quad x^2$

$= \left[x^2 - \frac{1}{2}(D^2-3D)x^2 + \frac{1}{4}(D^4+9D^2-6D^3)x^2 - \dots\right]$ $\quad D \to 2x$, $D^2 \to 2$

$= \left[x^2 - \frac{1}{2}(D^2(x^2) - 3D(x^2)) + \frac{1}{4}(D^4(x^2)+9D^2(x^2)-6D^3(x^2)) - \dots\right]$

$= \left[x^2 - \frac{1}{2}(2-3(2x)) + \frac{1}{4}(9(2))\right]$

$= \left[x^2 - 1 + \frac{6x}{2} + \frac{9}{2}\right]$

$= x^2 - 1 + 3x + 9/2 = x^2 + 3x + 7/2$

Therefore the General solution of equation (1) is $y = y_c + y_p$

$y = c_1 e^{x} + c_2 e^{2x} + x^2 + 3x + 7/2.$

3: Solve $(D^4+D^2)y = 3x^2 + 4(\sin x) - 2\cos x$

Sol: Given equation is $(D^4+D^2)y = 3x^2 + 4\sin x - 2\cos x \rightarrow (1)$ where $f(D) = D^4+D^2$

The Auxiliary equation is $f(m) = 0$

$\Rightarrow m^4 + m^2 = 0$

$\Rightarrow m^2(m^2+1) = 0$

$\Rightarrow m^2 = 0 \,;\, m^2 = -1 = i^2$

$\Rightarrow m = 0, 0 \,;\, m = \pm i$

complementary function of equation (1) is $y_c = (c_1 + c_2 x)e^{0x} + [c_3\cos x + c_4 \sin x]e^{0x}$

$= c_1 + c_2 x + c_3 \cos x + c_4 \sin x.$

$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^4+D^2}\left[3x^2 + 4\sin x - 2\cos x\right]$

$= 3 \cdot \frac{1}{D^4+D^2} x^2 + 4 \cdot \frac{1}{D^4+D^2} \sin x - 2 \cdot \frac{1}{D^4+D^2} \cos x \rightarrow (2)$

i, $\frac{1}{D^4+D^2} 3x^2 = \frac{1}{D^2[1+D^2]} 3x^2 = \frac{1}{[1+D^2]}\left[\frac{1}{D^2}(3x^2)\right] = \frac{1}{[1+D^2]}\left[\frac{x^4}{4}\right]$

$= \frac{1}{4}\left[(1+D^2)^{-1}\right]x^4$ $\quad x^4$

$= \frac{1}{4}\left[1 - D^2 + (D^2)^2 - (D^2)^3 + (D^2)^4 - \dots\right]x^4$ $\quad D \to 4x^3$, $D^2 \to 12x^2$

$= \frac{1}{4}\left[x^4 - D^2(x^4) + D^4(x^4) - D^6(x^4) + D^8(x^4) - \dots\right]$ $\quad D^3 \to 24x$, $D^4 \to 24$

$= \frac{1}{4}\left[x^4 - (12x^2) + 24\right]$

$= \frac{1}{4}\left[x^4 - 12x^2 + 24\right] \rightarrow$ i,

$$y_p = 8\left[\frac{1}{D^2-2D+4}x^2 + \frac{1}{D^2-2D+4}e^{2x} + \frac{1}{D^2-2D+4}\sin 2x\right] \rightarrow (2)$$

i, $\frac{1}{D^2-2D+4}x^2 = \frac{1}{4}\left[\frac{1}{\left[1+\left(\frac{D^2-2D}{4}\right)\right]}\right]x^2 = \frac{1}{4}\left[1+\left(\frac{D^2-2D}{4}\right)\right]^{-1}x^2$

$$= \frac{1}{4}\left[1-\left(\frac{D^2-2D}{4}\right)+\left(\frac{D^2-2D}{4}\right)^2 + \dots\right]x^2$$

$$= \frac{1}{4}\left[1-\frac{D^2}{4}+\frac{2D}{4}+\frac{(D^4+4D^2-4D^3)}{16} - \dots\right]x^2$$

$$= \frac{1}{4}\left[1-\frac{D^2}{4}+\frac{2D}{4}+\frac{4D^2}{16}\right]x^2$$

x^2
$D \rightarrow 2x$
$D^2 \rightarrow 2$

$$= \frac{1}{4}\left[x^2 - \frac{D^2(x^2)}{4} + \frac{2D(x^2)}{4} + \frac{D^2(x^2)}{4}\right]$$

$$= \frac{1}{4}\left[x^2 - \frac{2}{4} + \frac{2(2x)}{4} + \frac{2}{4}\right]$$

$$= \frac{1}{4}\left[x^2 - \frac{1}{2} + \frac{4x}{4} + \frac{1}{2}\right]$$

$$= \frac{1}{4}\left[x^2+x\right] \rightarrow \text{i,}$$

ii, $\frac{1}{D^2-2D+4}e^{2x}$

put $D = a = 2$

$$\frac{1}{D^2-2D+4}e^{2x} = \frac{1}{4-4+4}e^{2x} = \frac{1}{4}e^{2x} \rightarrow \text{ii,}$$

iii, $\frac{1}{D^2-2D+4}\sin 2x =$

$D^2 \rightarrow -a^2 \rightarrow -4$

$$= \frac{1}{-4-2D+4}\sin 2x = \frac{1}{-2D}\sin 2x.$$

$$= \frac{-1}{2}\left[\frac{1}{D}\sin 2x\right]$$

$$= \frac{-1}{2}\left[\int \sin 2x . dx\right]$$

$$= \frac{-1}{2}\left[\frac{-\cos 2x}{2}\right] = \frac{\cos 2x}{4} \rightarrow \text{iii,}$$

from ii, iii, iv, in equation (2) we get

$$y_p = 8\left[\frac{1}{4}(x^2+x)+\frac{1}{4}e^{2x}+\frac{\cos 2x}{4}\right] = 8\left[\frac{x^2+x}{4}+\frac{e^{2x}}{4}+\frac{\cos 2x}{4}\right]$$

Therefore the General Solution of equation (1) is $y = y_c + y_p$

$$y = (C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x)e^{x} + 8\left[\frac{x^2+x}{4}+\frac{e^{2x}}{4}+\frac{\cos 2x}{4}\right]$$

5: Solve $(D^3+2D^2+D)y = e^{2x}+(x^2+x)$

Sol: Given equation is $(D^3+2D^2+D)y = e^{2x}+(x^2+x) \rightarrow (1)$ where $f(D) = D^3+2D^2+D$

The Auxiliary equation is $f(m)=0$

$\Rightarrow m^3+2m^2+m=0$

$\Rightarrow m(m^2+2m+1)=0$

$\Rightarrow m=0,\ m^2+2m+1=0$

$\Rightarrow m=0,\ m^2+m+m+1=0$

$,\ m(m+1)+1(m+1)=0$

$,\ (m+1)(m+1)=0$

$\Rightarrow m=0,\ m=-1,-1$

$\therefore$ complementary function of equation (1) is $y_c=c_1e^{0x}+(c_2+c_3x)e^{-x}$

$= c_1+(c_2+c_3x)e^{-x}$

$$y_p=\frac{1}{f(D)}\phi(x)=\frac{1}{D^3+2D^2+D}e^{2x}+(x^2+x)$$

$$=\frac{1}{D^3+2D^2+D}[\cancel{x^2}e^{2x}]+\frac{1}{D^3+2D^2+D}[x^2+x]\rightarrow(2)$$

i, $\frac{1}{D^3+2D^2+D}e^{2x}$; put $D=a=2$

$$=\frac{1}{8+8+2}e^{2x}=\frac{1}{18}e^{2x}$$

ii, $\frac{1}{D^3+2D^2+D}\cancel{e^{2x}}[x^2+x]=\frac{1}{D(D^2+2D+1)}(x^2+x)$

$$=\frac{1}{(D+1)^2}\left[\frac{1}{D}(x^2+x)\right]$$

$$=\frac{1}{(1+D)^2}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]$$

$$=[1+D]^{-2}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]$$ $\because (1+x)^{-2}=1-2x+3x^2-4x^3+\cdots$

$$=[1-2D+3D^2-4D^3+\cdots]\left[\frac{x^3}{3}+\frac{x^2}{2}\right]$$

x^3: $D\to3x^2$, $D^2\to6x$, $D^3\to6$

x^2: $D\to2x$, $D^2\to2$

$$=\left[\frac{x^3}{3}+\frac{x^2}{2}\right]-2\left[\frac{3x^2}{3}+\frac{2x}{2}\right]+3\left[\frac{6x}{2}+\frac{2}{2}\right]-4\left[\frac{6}{3}+0\right]$$

$$=\frac{x^3}{3}+\frac{x^2}{2}-2x^2-2x+6x+3-8$$

$$=\frac{x^3}{3}+\frac{x^2}{2}-2x^2+4x-5$$

$$=\frac{x^3}{3}-\frac{3x^2}{2}+4x-5\rightarrow\text{ii,}$$

from (i) (ii) in equation (2) we get $y_p=\frac{1}{18}e^{2x}+\frac{x^3}{3}-\frac{3x^2}{2}+4x-5$

Therefore the general solution of eqn (1) is $y=y_c+y_p$

$y=c_1+(c_2+c_3x)e^{-x}+\frac{1}{18}e^{2x}+\frac{x^3}{3}-\frac{3x^2}{2}+4x-5.$

6: Solve $(D^2-2D+3)y = \cos x + x^2$

Sol: Given equation is $(D^2-2D+3)y = \cos x + x^2 \rightarrow (1)$ where $f(D) = D^2-2D+3$

The Auxiliary eqn is $f(m)=0$

$\Rightarrow m^2-2m+3=0$

$$m = \frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm 2i\sqrt{2}}{2} = 1 \pm i\sqrt{2}$$

$\therefore$ complementary function of equation (1) is $y_c = e^x[c_1 \cos x\sqrt{2} + c_2 \sin x\sqrt{2}]$

$$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^2-2D+3} \cos x + x^2$$

$$= \frac{1}{D^2-2D+3} \cos x + \frac{1}{D^2-2D+3} x^2 \rightarrow (2)$$

i, $\frac{1}{D^2-2D+3} \cos x$; $D^2 \rightarrow -a^2 \rightarrow -1$

$$= \frac{1}{-1-2D+3} \cos x = \frac{1}{2-2D} \cos x$$

$$= \frac{2+2D}{(2-2D)(2+2D)} \cos x$$

$$= \frac{2+2D}{4-4D^2} \cos x$$

$D^2 \rightarrow -a^2 \rightarrow -1$

$$= \frac{2+2D}{4-4(-1)} \cos x$$

$$= \frac{2+2D}{4+4} \cos x$$

$$= \frac{2+2D}{8} \cos x$$

$$= \frac{1}{8}[2\cos x + 2D(\cos x)] = \frac{1}{4}[\cos x - \sin x] \rightarrow \text{i,}$$

ii, $$\frac{1}{D^2-2D+3} x^2 = \frac{1}{3\left[1+\left(\frac{D^2-2D}{3}\right)\right]} x^2$$

$$= \frac{1}{3}\left[1+\left(\frac{D^2-2D}{3}\right)\right]^{-1} x^2$$

x^2
$D \rightarrow 2x$
$D^2 \rightarrow 2$

$$= \frac{1}{3}\left[1 - \left(\frac{D^2-2D}{3}\right) + \left(\frac{D^2-2D}{3}\right)^2 - \cdots\cdots\right] x^2$$

$$= \frac{1}{3}\left[1 - \frac{D^2}{3} + \frac{2D}{3} + \frac{(D^4+4D^2-4D^3)}{9} - \cdots\cdots\right] x^2$$

$$= \frac{1}{3}\left[x^2 - \frac{D^2(x^2)}{3} + \frac{2D(x^2)}{3} + \frac{4D^2(x^2)}{9}\right]$$

$$= \frac{1}{3}\left[x^2 - \frac{2}{3} + \frac{4x}{3} + 8/9\right] = \frac{x^2}{3} + \frac{4x}{9} + \frac{2}{27} \rightarrow \text{ii,}$$

Therefore the General Solution of equation (1) is $y = y_c + y_p$

$$y = [C_1 \cos x\sqrt{2} + C_2 \sin x\sqrt{2}]e^{x} + \frac{1}{4}(\cos x - \sin x) + x^2/3 + 4x/9 + 2/27$$

Method: When $Q(x) = e^{ax}.V$, Where V is a function of x.

* Let the given equation be $f(D)y = e^{ax}.V$
* To find particular Integral $= \frac{1}{f(D)} e^{ax}.V$, shift e^{ax} outside and after replacing D by $D+a$, operate V by $\frac{1}{f(D+a)}$
* $\therefore$ $P.I = \frac{1}{f(D)} e^{ax}.V = e^{ax} \frac{1}{f(D+a)} V.$

7: Solve $(D^2 - 2D + 1)y = x^2 e^{3x}$

Sol: Given equation is $(D^2-2D+1)y = x^2 e^{3x} \rightarrow (1)$ Where $f(D) = D^2 - 2D + 1$

The Auxilary equation is $f(m) = 0$

$\Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow m^2 - m - m + 1 = 0$

$\Rightarrow m(m-1) - 1(m-1) = 0$

$\Rightarrow (m-1)(m-1) = 0$

$\Rightarrow m = 1, 1$

$\therefore$ complementary function of equation (1) is $y_c = (C_1 + C_2 x)e^{x}$

$$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^2 - 2D + 1} x^2 e^{3x} = \frac{1}{(D-1)^2} x^2 e^{3x}$$

put $D = D + a = D + 3$

$$= e^{3x} \frac{1}{(D+3-1)^2} x^2$$

$$= e^{3x} \frac{1}{(D+2)^2} x^2$$

$$= e^{3x} \frac{1}{[2(1+D/2)]^2} x^2$$

$$= \frac{e^{3x}}{4} [1 + D/2]^{-2} x^2$$

$$= \frac{e^{3x}}{4} [1 - 2(D/2) + 3(D/2)^2] x^2$$

$$= \frac{e^{3x}}{4} [x^2 - D(x^2) + \frac{3}{4} D^2(x^2)]$$

x^2

$D \rightarrow 2x$

$D^2 \rightarrow 2$

$$y_p = \frac{e^{3x}}{4} [x^2 - 2x + \frac{3}{4}(2)] = \frac{e^{3x}}{4} [x^2 - 2x + 3/2]$$

Therefore the General solution of eqn (1) is $y = y_c + y_p$

$$y = (C_1 + C_2 x)e^{x} + \frac{e^{3x}}{4} [x^2 - 2x + 3/2]$$

8: Solve $(D^2-6D+13)y = 8e^{3x}\sin 2x$

Sol: Given equation is $(D^2-6D+13)y = 8e^{3x}\sin 2x \rightarrow (1)$ where $f(D) = D^2-6D+13$

The Auxiliary equation is $f(m)=0$.

$\Rightarrow m^2-6m+13=0$

$m = \frac{6 \pm \sqrt{36-52}}{2}$

$= \frac{6 \pm 4i}{2}$

$= 3 \pm 2i$

complementary function of equation (1) is $y_c = e^{3x}(C_1\cos 2x + C_2\sin 2x)$

$y_P = \frac{1}{f(D)} Q(x) = \frac{1}{D^2-6D+13} 8e^{3x}\sin 2x$

put $D = D+a = D+3$

$= 8e^{3x} \frac{1}{(D+3)^2-6(D+3)+13} \sin 2x$

$= 8e^{3x} \frac{1}{D^2+9+6D-6D-18+13} \sin 2x$

$= 8e^{3x} \frac{1}{D^2+4} \sin 2x$

$= 8e^{3x} \frac{1}{D^2+2^2} \sin 2x$

$= 8e^{3x} \left[\frac{-x}{2(2)} \cos 2x\right]$ $\left[\because \frac{1}{D^2+b^2}\sin bx = \frac{-x}{2b}\cos bx\right]$

$= -2xe^{3x}\cos 2x$

Therefore the general solution of equation (1) is $y = y_c + y_p$

$y = e^{3x}[C_1\cos 2x + C_2\sin 2x] - 2xe^{3x}\cos 2x.$

9: Solve $(D^2-4D+1)y = e^{2x}\cdot\cos^2 x$.

Sol: Given equation is $(D^2-4D+1)y = e^{2x}\cos^2 x \rightarrow (1)$ where $f(D) = D^2-4D+1$

The Auxiliary equation is $f(m)=0$

$\rightarrow m^2-4m+1=0$

$\rightarrow m = \frac{4 \pm \sqrt{16-4}}{2}$

$= \frac{4 \pm \sqrt{12}}{2}$

$= 2 \pm \sqrt{3}$

Complementary function of eq (1) is $y_c = e^{2x}[C_1\cosh(x\sqrt{3}) + C_2\sinh(x\sqrt{3})]$

$$y_p = \frac{1}{f(D)} e^{ax}.V$$

$$= \frac{1}{D^2-4D+1} e^{2x}.\cos^2 x$$

$$= \frac{1}{D^2-4D+1}\left[e^{2x}.\left(\frac{1+\cos 2x}{2}\right)\right] = \frac{1}{D^2-4D+1}\left[\frac{e^{2x}}{2}+\frac{e^{2x}\cos 2x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+1} e^{2x} + \frac{1}{D^2-4D+1} \cos 2x . e^{2x}\right] \rightarrow (2)$$

i, $\frac{1}{D^2-4D+1} e^{2x}$; put $D=a=2$

$$= \frac{1}{4-8+1} e^{2x} = \frac{1}{-3} e^{2x} \rightarrow \text{i,}$$

ii, $\frac{1}{D^2-4D+1} e^{2x}\cos 2x.$

put $D = D+a = D+2$

$$= e^{2x} \frac{1}{(D+2)^2-4(D+2)+1} \cos 2x$$

$$= e^{2x} \frac{1}{D^2+4+4D-4D-8+1} \cos 2x$$

$$= e^{2x} \frac{1}{D^2-3} \cos 2x$$

put $D^2 \rightarrow -a^2 \rightarrow -4$

$$= e^{2x} \frac{1}{-4-3} \cos 2x = -e^{2x}.\frac{1}{7}\cos 2x \rightarrow \text{ii,}$$

From i, and ii, in equation (2) we have $y_p = \frac{1}{2}\left[\frac{-e^{2x}}{3} - \frac{e^{2x}\cos 2x}{7}\right]$

Therefore the General Solution of equation (1) is $y = y_c + y_p$

$$y = \left[C_1\cosh(x\sqrt{3}) + C_2\sinh(x\sqrt{3})\right]e^{2x} - \frac{e^{2x}}{6} - \frac{e^{2x}\cos 2x}{14}$$

10: Solve $(D^2+2)y = x^2 e^{3x} + e^x \cos 2x$

Sol: Given equation is $(D^2+2)y = x^2e^{3x} + e^x\cos 2x \rightarrow (1)$ where $f(D) = D^2+2$

The Auxillary equation is $f(m)=0$

$\Rightarrow m^2+2=0$

$\Rightarrow m^2 = 2i^2$

$\Rightarrow m = \pm\sqrt{2}i$

complementary function of equation (1) is $y_c = C_1\cos\sqrt{2}x + C_2\sin\sqrt{2}x$

$$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^2+2} x^2e^{3x} + e^x\cos 2x$$

$$y_p = \frac{1}{D^2+2} x^2e^{3x} + \frac{1}{D^2+2} e^x\cos 2x \rightarrow (2)$$

....

i, $\frac{1}{D^2+2} x^2 e^{3x}$ put $D = D+a = D+3$

$= e^{3x} \frac{1}{(D+3)^2+2} x^2$

$= e^{3x} \frac{1}{D^2+9+6D+2} x^2$

$= e^{3x} \frac{1}{D^2+6D+11} x^2$

$= e^{3x} \frac{1}{11\left[1+\left(\frac{D^2+6D}{11}\right)\right]} x^2$

$= \frac{e^{3x}}{11}\left[1+\left(\frac{D^2+6D}{11}\right)\right]^{-1} x^2$

$= \frac{e^{3x}}{11}\left[1-\left(\frac{D^2+6D}{11}\right)+\left(\frac{D^2+6D}{11}\right)^2 \cdots\right] x^2$

$= \frac{e^{3x}}{11}\left[1-\frac{D^2}{11}-\frac{6D}{11}+\frac{(D^4+36D^2+12D^3)}{121} - \cdots\right] x^2$

x^2
$D \to 2x$
$D^2 \to 2$

$= \frac{e^{3x}}{11}\left[1-\frac{D^2}{11}-\frac{6D}{11}+\frac{36D^2}{121}\right] x^2$

$= \frac{e^{3x}}{11}\left[x^2-\frac{D^2(x^2)}{11}-\frac{6D(x^2)}{11}+\frac{36D^2(x^2)}{11}\right]$

$= \frac{e^{3x}}{11}\left[x^2-\frac{2}{11}-\frac{12x}{11}+\frac{72}{121}\right]$

$= \frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right] \to$ i,

ii, $\frac{1}{D^2+2} e^{x} \cos 2x$

put $D = D+a = D+1$

$= e^{x} \frac{1}{(D+1)^2+2} \cos 2x$

$= e^{x} \frac{1}{D^2+1+2D+2} \cos 2x$

$= e^{x} \frac{1}{D^2+2D+3} \cos 2x$

put $D^2 \to -a^2 \to -4$

$= e^{x} \frac{1}{-4+2D+3} \cos 2x$

$= e^{x} \frac{1}{2D-1} \cos 2x$

$= e^{x} \frac{2D+1}{(2D-1)(2D+1)} \cos 2x$

$$= e^{x}\frac{(2D+1)}{4D^2-1}\cos 2x$$

$$= e^{x}\frac{2D+1}{4(-4)-1}\cos 2x \text{ , put } D^2 \to -a^2 \to -4$$

$$= e^{x}\frac{2D+1}{-17}\cos 2x$$

$$= e^{x}\left[\frac{2D(\cos 2x)+\cos 2x}{-17}\right]$$

$$= e^{x}\left[\frac{2(-2\sin 2x)+\cos 2x}{-17}\right]$$

$$= e^{x}\left[\frac{-4\sin 2x+\cos 2x}{-17}\right] \to \text{(ii)}$$

From i, and ii, in equation (2) we get $y_p = \frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]+e^{x}\left[\frac{-4\sin 2x+\cos 2x}{-17}\right]$

Therefore the general solution of equation is $y = y_c + y_p$

$$y = (c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x) + \frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]+e^{x}\left[\frac{-4\sin 2x+\cos 2x}{-17}\right]$$

1: Solve $(D^2-4)y = x\sinh x$

Sol: Given equation is $(D^2-4)y = x\sinh x \to (1)$ where $f(D) = D^2-4$

The Auxilary equation is $f(m)=0$

$$\Rightarrow m^2-4=0$$

$$\Rightarrow m^2=4$$

$$\Rightarrow m=\pm 2$$

complementary function of equation (1) is $y_c = c_1e^{2x}+c_2e^{-2x}$

$$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{D^2-4}x\sinh x$$

$$= \frac{1}{D^2-4}\left[x\left(\frac{e^{x}-e^{-x}}{2}\right)\right]$$

$$= \frac{1}{D^2-4}\left[\frac{xe^{x}-xe^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}xe^{x}-\frac{1}{D^2-4}xe^{-x}\right] \to (2)$$

i, $\frac{1}{D^2-2}xe^{x}$ put $D = D+a = D+1$

$$= e^{x}\frac{1}{(D+1)^2-4}x = e^{x}\frac{1}{D^2+1+2D-4}x = e^{x}\frac{1}{D^2+2D+3}x$$

$$= e^{x}\frac{1}{D^2+2D-3}x$$

$$= e^{x}\frac{1}{-3\left[1-\left(\frac{D^2+2D}{3}\right)\right]}x$$

$$= \frac{-e^{x}}{3}\left[1-\left(\frac{D^2+2D}{3}\right)\right]^{-1} x$$

$$= \frac{-e^{x}}{3}\left[1+\left(\frac{D^2+2D}{3}\right)+\cdots\right] x$$

$$= \frac{-e^{x}}{3}\left[1+\frac{D^2}{3}+\frac{2D}{3}\right] x$$

x
$D \to 1$

$$= \frac{-e^{x}}{3}\left[x+\frac{D^2(x)}{3}+\frac{2D(x)}{3}\right]$$

$$= \frac{-e^{x}}{3}\left[x+2/3\right] \to \text{(i)}$$

(ii) $\frac{1}{D^2-4} x e^{-x}$ put $D = D+a = D-1$

$$= e^{-x} \frac{1}{(D-1)^2-4} x$$

$$= e^{-x} \frac{1}{D^2+1-2D-4} x$$

$$= e^{-x} \frac{1}{D^2-2D-3} x$$

$$= e^{-x} \frac{1}{-3\left[1-\left(\frac{D^2-2D}{3}\right)\right]} x$$

$$= \frac{e^{-x}}{-3}\left[1-\left(\frac{D^2-2D}{3}\right)\right]^{-1} x.$$

$$= \frac{e^{-x}}{-3}\left[1+\left(\frac{D^2-2D}{3}\right)+\cdots\right] x$$

$$= \frac{e^{-x}}{-3}\left[1+\frac{D^2}{3}-\frac{2D}{3}\right] x$$

x
$D \to 1$

$$= \frac{e^{-x}}{-3}\left[x+\frac{D^2}{3}-\frac{2D(x)}{3}\right]$$

$$= \frac{e^{-x}}{-3}\left[x+0-2/3\right] = \frac{e^{-x}}{-3}\left[x-2/3\right] \to \text{(ii)}$$

From (i) and (ii) $y_p = \frac{1}{2}\left[\frac{-e^{x}}{3}(x+2/3)-\left(\frac{e^{-x}}{-3}(x-2/3)\right)\right]$

$$= \frac{1}{2}\left[\frac{-xe^{x}}{3}-\frac{2e^{x}}{9}+\frac{e^{-x}x}{3}-\frac{2e^{-x}}{9}\right]$$

$$= \frac{-xe^{x}}{3\cdot 2}-\frac{2e^{x}}{9\cdot 2}+\frac{xe^{-x}}{3\cdot 2}-\frac{2e^{-x}}{9\cdot 2}$$

$$= -\frac{x}{3}\left[\frac{e^{x}-e^{-x}}{2}\right]-\frac{2}{9}\left[\frac{e^{x}+e^{-x}}{2}\right]$$

$$= \frac{-x}{3}\sinh x-\frac{2}{9}\cosh x.$$

Therefore the General solution of equation (1) is $y = y_c + y_p$

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x.$$

12. Solve $(D^2-4D+3)y = 2xe^{3x}+3e^{x}\cos 2x$

Sol: Given equation is $(D^2-4D+3)y = 2xe^{3x}+3e^{x}\cos 2x \rightarrow (1)$ Where $f(D)=D^2-4D+3$

The auxilary equation is $f(m)=0$.

$$\Rightarrow m^2-4m+3=0$$
$$\Rightarrow m^2-m-3m+3=0$$
$$\Rightarrow m(m-1)-3(m-1)=0$$
$$\Rightarrow (m-1)(m-3)=0$$
$$\Rightarrow m=1,3$$

$\therefore$ complementary function of equation (1) is $y_c = C_1e^{x}+C_2e^{3x}$

$$y_P = \frac{1}{f(D)}Q(x) = \frac{1}{D^2-4D+3}\left[2xe^{3x}+3e^{x}\cos 2x\right]$$
$$= \left[\frac{1}{D^2-4D+3}2xe^{3x}+\frac{1}{D^2-4D+3}3e^{x}\cos 2x\right] \rightarrow (2)$$

i, $\frac{1}{D^2-4D+3}2xe^{3x}$ put $D=D+a=D+3$

$$= 2e^{3x}\frac{1}{(D+3)^2-4(D+3)+3}x$$
$$= 2e^{3x}\frac{1}{D^2+9+6D-4D-12+3}x$$
$$= 2e^{3x}\frac{1}{D^2+2D+12-12}x$$
$$= 2e^{3x}\frac{1}{D^2+2D}x$$
$$= 2e^{3x}\frac{1}{2D(1+D/2)}x$$
$$= e^{3x}\left[1+D/2\right]^{-1}\left[\frac{1}{D}x\right]$$
$$= e^{3x}\left[1+D/2\right]^{-1}\left[x^2/2\right]$$
$$= \frac{e^{3x}}{2}\left[1+D/2\right]^{-1}x^2$$
$$= \frac{e^{3x}}{2}\left[1-D/2+D^2/4-\cdots\right]x^2$$

x^2
$D\rightarrow 2x$
$D^2\rightarrow 2$

$$= \frac{e^{3x}}{2}\left[x^2-\frac{D(x^2)}{2}+\frac{D^2(x^2)}{4}\right]$$
$$= \frac{e^{3x}}{2}\left[x^2-\frac{2x}{2}+\frac{2}{4}\right] = \frac{e^{3x}}{2}\left[x^2-x+1/2\right] \rightarrow \text{i,}$$

ii, $\frac{1}{D^2-4D+3}3e^{x}\cos 2x$ put $D=D+a=D+1$

$$= 3e^{x}\frac{1}{(D+1)^2-4(D+1)+3}\cos 2x$$

$$= 3e^x \frac{1}{D^2+1+2D-4D-4+3} \cos 2x$$

$$= 3e^x \frac{1}{D^2-2D} \cos 2x$$

$$D^2 \to -a^2 \to -4$$

$$= 3e^x \frac{1}{-4-2D} \cos 2x$$

$$= 3e^x \frac{1}{-2[2+D]} \cos 2x$$

$$= \frac{3e^x}{-2}\left[\frac{1}{D+2} \cos 2x\right]$$

$$= \frac{-3e^x}{2} \frac{D-2}{(D+2)(D-2)} \cos 2x$$

$$= \frac{-3e^x}{2}\left[\frac{D-2}{D^2-4} \cos 2x\right]$$

$$= \frac{-3e^x}{2}\left[\frac{D-2}{-4-4} \cos 2x\right]$$

$$= \frac{-3e^x}{2}\left[\frac{D-2}{-8} \cos 2x\right]$$

$$= \frac{3e^x}{16}\left[(D-2) \cos 2x\right]$$

$$= \frac{3e^x}{16}\left[(\cos 2x)D - 2\cos 2x\right]$$

$$= \frac{3e^x}{16}\left[-2\sin 2x - 2\cos 2x\right]$$

$$= \frac{-3e^x}{8}\left[\sin 2x + \cos 2x\right] \to \text{(ii)}$$

from (i), (ii) in equation (2) we get $y_p = \frac{e^{3x}}{2}(x^2 - x + 1/2) + \frac{3e^x}{8}(\sin 2x + \cos 2x)$

Therefore the General Solution of equation (1) is $y = y_c + y_p$

$$y = c_1 e^x + c_2 e^{3x} + \frac{e^{3x}}{2}(x^2 - x + 1/2) - \frac{3}{8}e^x(\sin 2x + \cos 2x)$$

13: Solve $(D^2-2D+4)y = e^x \sin(x/2)$

Sol: Given equation is $(D^2-2D+4)y = e^x \sin(x/2) \to (1)$ where $f(D) = D^2-2D+4$

The Auxilary equation is $f(m) = 0$

$$\Rightarrow m^2 - 2m + 4 = 0$$

$$\Rightarrow m = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm i\sqrt{3}$$

$$y_c = (c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)e^x.$$

....

$$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{D^2-2D+4} \cdot e^x \sin(x/2)$$

put $D = D+a = D+1$

$$= e^x \frac{1}{(D+1)^2-2(D+1)+4} \sin x/2$$

$$= e^x \frac{1}{D^2+1+2D-2D-2+4} \sin x/2$$

$$= e^x \frac{1}{D^2+3} \sin x/2$$

put $D^2 \to -a^2 \to -1/4$

$$= e^x \frac{1}{-1/4+3} \sin(x/2)$$

$$= e^x \frac{1}{11/4} \sin x/2$$

$$y_p = e^x \frac{4}{11} \sin x/2$$

Therefore the General Solution of equation (1) is $y = y_c + y_p$

$$y = e^x[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x] + e^x \cdot \frac{4}{11} \cdot \sin x/2.$$

14: Solve $(D^2+1)y = e^{-x} + \cos x + x^3 + e^x \cos x$

Sol: Given equation is $(D^2+1)y = e^{-x} + \cos x + x^3 + e^x \cos x \to (1)$ Where $f(D) = D^2+1$

The Auxilary equation is $f(m) = 0 \Rightarrow m^2+1=0$

$\Rightarrow m^2 = i^2$

$\Rightarrow m = \pm i$

complementary function of equation (1) is $y_c = c_1\cos x + c_2 \sin x$

$$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^2+1} e^{-x} + \cos x + x^3 + e^x \cos x$$

$$= \left[\frac{1}{D^2+1} e^{-x} + \frac{1}{D^2+1}\cos x + \frac{1}{D^2+1} x^3 + \frac{1}{D^2+1} e^x \cos x\right] \to (2)$$

i, $\frac{1}{D^2+1} e^{-x}$ put $D = a = -1$

$$= \frac{1}{(-1)^2+1} e^{-x} = \frac{1}{2} e^{-x} \to \text{i,}$$

ii, $\frac{1}{D^2+1} \cos x$

put $D^2 \to -a^2 \to -1$

$$= \frac{1}{D^2+1^2} \cos x = \frac{x}{2(1)} \sin x$$

$$= \frac{x}{2} \sin x \to \text{ii,}$$

(iii) $\frac{1}{D^2+1}x^3 = \frac{1}{[1+D^2]}x^3$

$= [1+D^2]^{-1}x^3$

$= [1-(D^2)+(D^2)^2-(D^2)^3+\dots]x^3$

$= [1-D^2+D^4-D^6+\dots]x^3$ $\quad x^3$

$D \to 3x^2$

$= [x^3-D^2(x^3)]$ $\quad D^2 \to 6x$

$= [x^3-6x] \to$ (iii) $\quad D^3 \to 6$

(iv) $\frac{1}{D^2+1}e^x \cos x$

put $D = D+a = D+1$

$= e^x \frac{1}{(D+1)^2+1}\cos x = e^x \frac{1}{D^2+1+2D+1}\cos x$

$= e^x \frac{1}{D^2+2D+2}\cos x$

put $D^2 \to -a^2 \to -1$

$= e^x \frac{1}{-1+2D+2}\cos x$

$= e^x \frac{1}{2D+1}\cos x$

$= e^x \left[\frac{2D-1}{(2D+1)(2D-1)}\right]\cos x$

$= e^x \left[\frac{2D-1}{4D^2-1}\right]\cos x$

$= e^x \left[\frac{2D-1}{-4-1}\right]\cos x$

$= e^x \left[\frac{2D(\cos x)-1(\cos x)}{-5}\right]$

$= e^x \left[\frac{2(-\sin x)-\cos x}{-5}\right]$

$= e^x \left[\frac{-(2\sin x+\cos x)}{-5}\right] = \frac{e^x}{5}[2\sin x+\cos x] \to$ (iv)

from (i), (ii), (iii) & (iv) in equation (2) we get

$y_p = \frac{1}{2}e^{-x} + \frac{x\sin x}{2} + [x^3-6x] + \frac{e^x}{5}[2\sin x+\cos x]$

Therefore the General solution of equation (1) is $y = y_c + y_p$

$y = C_1\cos x + C_2\sin x + \frac{e^{-x}}{2} + \frac{x\sin x}{2} + [x^3-6x] + \frac{e^x}{5}[2\sin x+\cos x]$

Method: When $Q(x) = xV$, where V is a function of x

$$y_p = \frac{1}{f(D)} xV$$
$$= x \frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} V$$

15: Solve $(D^2+4)y = x\sin x$

Sol: Given equation is $(D^2+4)y = x\sin x \rightarrow (1)$ Where $f(D) = D^2+4$.

The Auxilary equation is $f(m)=0$

$$\Rightarrow m^2+4=0$$
$$\Rightarrow m^2 = 4i^2$$
$$\Rightarrow m = \pm 2i$$

Complementary function of equation (1) is $y_c = C_1\cos 2x + C_2 \sin 2x$

$$y_p = \frac{1}{f(D)} xV$$
$$= \frac{1}{D^2+4} x\sin x$$

$\left[f(D) = D^2+4 \right.$
$\left. f'(D) = 2D+0 \right]$

$$= x\frac{1}{D^2+4}\sin x - \frac{2D}{(D^2+4)^2}\sin x$$

$= \cancel{x}$ put $D^2 \rightarrow -a^2 \rightarrow -1$

$$= x \frac{1}{-1+4}\sin x - \frac{2D}{(-1+4)^2}\sin x$$
$$= \frac{x}{3}\sin x - \frac{2D}{9}\sin x$$
$$y_p = \frac{x\sin x}{3} - \frac{2\cos x}{9}$$

Therefore the General solution of equation (1) is $y = y_c + y_p$

$$y = C_1\cos 2x + C_2 \sin 2x + \frac{x\sin x}{3} - \frac{2\cos x}{9}$$

16: Solve $(D^2-4)y = x\sin x$.

Sol: Given equation is $(D^2-4)y = x\sin x \rightarrow (1)$ Where $f(D) = D^2-4$

The Auxilary equation is $f(m)=0$

$$\Rightarrow m^2-4=0 \Rightarrow m^2=4 \Rightarrow m = \pm 2$$

Complementary function of equation (1) is $y_c = C_1 e^{2x} + C_2 e^{-2x}$

$$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^2-4} x\sin x$$
$$= x\frac{1}{D^2-4}\sin x - \frac{2D}{(D^2-4)^2}\sin x$$

$\left[f(D) = D^2-4 \right.$
$\left. f'(D) = 2D \right]$

put $D^2 \to -a^2 \to -1$

$$= x \frac{1}{-1-4} \sin x - \frac{2D}{(-1-4)^2} \sin x$$

$$= \frac{x \sin x}{-5} - \frac{2D(\sin x)}{25}$$

$$y_p = \frac{x \sin x}{5} - \frac{2(\cos x)}{25}$$

Therefore the general solution of equation (1) is $y = y_c + y_p$

$$y = C_1 e^{2x} + C_2 e^{-2x} - \frac{x \sin x}{5} - \frac{2 \cos x}{25}$$

14: Solve $(D^2+4)y = x\cos 2x$

Sol: Given equation is $(D^2+4)y = x\cos 2x \longrightarrow (1)$ where $f(D) = D^2+4$

The auxiliary equation is $f(m)=0$

$\Rightarrow m^2+4=0$

$\Rightarrow m^2 = 4i^2$

$\Rightarrow m = \pm 2i$

$\therefore$ Complementary function of equation (1) is $y_c = C_1 \cos 2x + C_2 \sin 2x$

$$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^2+4} x \cos 2x$$

$$= x \frac{1}{D^2+4} \cos 2x - \frac{2D}{(D^2+4)^2} \cos 2x$$

$$= x \frac{1}{D^2+2^2} \cos 2x - \frac{2D}{D^2+4}\left[\frac{1}{D^2+2^2} \cos 2x\right]$$

$$= x \cdot \frac{x}{2(2)} \sin 2x - \frac{2D}{D^2+4}\left[\frac{x}{2(2)} \sin 2x\right]$$

$$= \frac{x^2}{4} \sin 2x - \frac{2D}{D^2+4}\left[\frac{x}{4} \sin 2x\right]$$

$$= \frac{x^2}{4} \sin 2x - \frac{1}{2(D^2+4)}\left[D(x \sin 2x)\right]$$

$$= \frac{x^2 \sin 2x}{4} - \frac{1}{2(D^2+4)}\left[x(2\cos 2x) + \sin 2x (1)\right]$$

$$= \frac{x^2 \sin 2x}{4} - \frac{2x \cos 2x}{2(D^2+4)} - \frac{\sin 2x}{2(D^2+4)}$$

$$= \frac{x^2 \sin 2x}{4} - y_p - \frac{1}{2}\left[\frac{1}{D^2+2^2} \sin 2x\right]$$

$$= \frac{x^2 \sin 2x}{4} - y_p - \frac{1}{2}\left[\frac{-x}{2(2)} \cos 2x\right]$$

$$y_p = \frac{x^2 \sin 2x}{4} - y_p + \frac{1}{2}\frac{x\cos 2x}{4}$$

$$2y_p = \frac{x^2 \sin 2x}{4} + \frac{x\cos 2x}{8}$$

$$y_p = \frac{x^2 \sin 2x}{8} + \frac{x\cos 2x}{16}$$

Therefore the General solution of equation (1) is $y = y_c + y_p$

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{x^2 \sin 2x}{8} + \frac{x\cos 2x}{16}$$

18: Solve $(D^2+3D+2)y = xe^x \sin x$

Sol: Given equation is $(D^2+3D+2)y = xe^x \sin x \rightarrow (1)$ Where $f(D) = D^2+3D+2$

The auxiliary equation is $f(m) = 0$

$$\Rightarrow m^2+3m+2=0$$
$$\Rightarrow m^2+2m+m+2=0$$
$$\Rightarrow m(m+2)+1(m+2)=0$$
$$\Rightarrow (m+1)(m+2)=0$$
$$\Rightarrow m = -1, -2$$

Complementary function of equation (1) is $y_c = c_1 e^{-x} + c_2 e^{-2x}$

$$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{D^2+3D+2} xe^x \sin x = \frac{1}{D^2+3D+2} x(e^x \sin x)$$

put $D = D+a = D+1$

$$= e^x \frac{1}{(D+1)^2+3(D+1)+2} x \sin x$$

$$= e^x \frac{1}{D^2+1+2D+3D+3+2} x \sin x$$

$$= e^x \frac{1}{D^2+5D+6} x \sin x$$

$\left[f(D) = D^2+5D+6 \quad f'(D) = 2D+5 \right]$

$$= e^x \left[x \cdot \frac{1}{D^2+5D+6} \sin x - \frac{2D+5}{(D^2+5D+6)^2} \sin x \right]$$

$$= e^x \left[x \frac{1}{-1+5D+6} \sin x - \frac{2D+5}{(-1+5D+6)^2} \sin x \right] \quad D^2 \rightarrow -a^2 \rightarrow -1$$

$$= e^x \left[x \cdot \frac{1}{5D+5} \sin x - \frac{2D+5}{(5D+5)^2} \sin x \right]$$

$$= e^x \left[x \cdot \frac{(D-1)}{5(D-1)(D+1)} \sin x - \frac{2D+5}{[5(D+1)]^2} \sin x \right]$$

$$= e^x \left[x \cdot \frac{(D-1)}{5(D^2-1)} \sin x - \frac{2D+5}{[5(D^2+2D+1)]} \sin x \right]$$

$D^2 \rightarrow -a^2 \rightarrow -1$

$$= e^{x}\left[x.\frac{D-1}{-10}\sin x - \frac{2D+5}{25(2D)}\sin x\right]$$

$$= e^{x}\left[\frac{x}{-10}\{D(\sin x)-\sin x\} - \frac{1}{25}\left\{\frac{2D}{2D}+\frac{5}{2D}\right\}\sin x\right]$$

$$= e^{x}\left[\frac{-x}{10}(\cos x-\sin x) - \frac{1}{25}\left\{1+\frac{5}{2}\cdot\frac{1}{D}\right\}\sin x\right]$$

$$= e^{x}\left[\frac{-x}{10}(\cos x-\sin x) - \frac{1}{25}\left\{\sin x+\frac{5}{2}\cdot\frac{1}{D}\sin x\right\}\right]$$

$$= e^{x}\left[\frac{-x}{10}(\cos x-\sin x) - \left\{\frac{\sin x}{25}+\frac{5(-\cos x)}{50}\right\}\right]$$

$$= e^{x}\left[\frac{-x}{10}(\cos x-\sin x) - \frac{\sin x}{25}+\frac{5}{50}\cos x\right]$$

$$= e^{x}\left[\frac{x}{10}(\sin x-\cos x) - \frac{1}{25}\sin x+\frac{5}{50}\cos x\right]$$

$$= e^{x}\left[\frac{x}{10}(\sin x-\cos x) - \left(\frac{2\sin x-5\cos x}{50}\right)\right]$$

$$y_p = e^{x}\left[\frac{x}{10}(\sin x-\cos x) - \frac{1}{25}\sin x \div \frac{1}{10}\cos x\right]$$

$\therefore$ The General solution of equation (1) is $y = y_c + y_p$

$$y = c_1e^{-x}+c_2e^{-2x}+e^{x}\left[\frac{x}{10}(\sin x-\cos x) - \frac{1}{25}\sin x - \frac{1}{10}\cos x\right]$$

P: Solve $(D^2+1)y = x^2\sin 2x$

Sol: Given equation is $(D^2+1)y = x^2\sin 2x \rightarrow (1)$ Where $f(D) = D^2+1$

The Auxilary equation is $f(m)=0$

$\Rightarrow m^2+1=0$

$\Rightarrow m^2 = i^2$

$\Rightarrow m = \pm i$

complementary function of equation (1) is $y_c = C_1\cos x + C_2\sin x$

$$y_p = \frac{1}{f(D)}Q(x) = \frac{1}{D^2+1}x^2\sin 2x$$

$e^{ix} = \cos x + i\sin x$, $\cos x \to$ R.P, $\sin x \to$ I.P

$e^{i2x} = \cos 2x + i\sin 2x$, $\cos 2x \to$ R.P, $\sin 2x \to$ I.P

$$= \text{I.P of } \frac{1}{D^2+1}x^2e^{i2x}$$

$= $ put $D = D+a = D+2i$

$$= \text{I.P of } e^{i2x}\frac{1}{(D+2i)^2+1}x^2$$

$$= \text{I.P of } e^{i2x}\frac{1}{D^2+4i^2+4iD+1}x^2$$

$$= \text{I.P of } e^{i2x}\frac{1}{D^2-4+4iD+1}x^2$$

$= I.P \text{ of } e^{i2x} \dfrac{1}{D^2+4iD-3} x^2$

$= I.P \text{ of } e^{i2x} \dfrac{1}{-3\left[1-\left(\frac{D^2+4iD}{3}\right)\right]} x^2$

$= I.P \text{ of } e^{i2x} \dfrac{1}{(-3)} \left[1-\left(\frac{D^2+4iD}{3}\right)\right]^{-1} x^2$

$= (-1/3)\, I.P \text{ of } e^{2ix}\left[1+\left(\frac{D^2+4iD}{3}\right)+\left(\frac{D^2+4iD}{3}\right)^2+\dots\right]x^2$

$= (-1/3)\, I.P \text{ of } e^{2ix}\left[1+1/3\,(D^2+4iD)+1/9\,(D^4-16D^2+8iD^3)+\dots\right]x^2$

$= \left(\frac{-1}{3}\right) I.P \text{ of } e^{i2x}\left[x^2+\frac{1}{3}(D^2+4iD)x^2-\frac{16}{9}(D^2x^2)\right]$ $\quad x^2,\ D \to 2x,\ D^2 \to 2$

$= (-1/3)\, I.P \text{ of } e^{i2x}\left[x^2+\frac{1}{3}(2+4i(2x))-\frac{16}{9}(2)\right]$

$= (-1/3)\, I.P \text{ of } e^{i2x}\left[x^2+2/3+\frac{8ix}{3}-32/9\right]$

$= (-1/3)\, I.P \text{ of } (\cos 2x + i\sin 2x)\left[(x^2-26/9)+i(8/3)x\right]$

$= (-1/3)\, I.P \text{ of } \left[(x^2-26/9)\cos 2x + i(8/3)x\cos 2x + i(x^2-26/9)\sin 2x - (8/3)x\sin 2x\right]$

$y_p = \frac{-1}{3}\left[\frac{8}{3}x\cos 2x + (x^2-26/9)\sin 2x\right]$

Therefore the General solution of equation (1) is $y = y_c + y_p$

$$y = c_1\cos x + c_2\sin x - \frac{1}{3}\left[(8/3)x\cos 2x + (x^2-26/9)\sin 2x\right]$$

20: Solve $(D^2-4D+4)y = 8x^2 e^{2x}\sin 2x$

Sol: Given equation is $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x \longrightarrow (1)$ Where $f(D) = D^2-4D+4$.

The Auxiliary equation is $f(m)=0$

$\Rightarrow m^2-4m+4=0$

$\Rightarrow m^2-2m-2m+4=0$

$\Rightarrow m(m-2)-2(m-2)=0$

$\Rightarrow (m-2)(m-2)=0$

$\Rightarrow m=2,2$

complementary function of equation (1) is $y_c = (c_1+c_2x)e^{2x}$

$y_p = \dfrac{1}{f(D)}\phi(x) = \dfrac{1}{D^2-4D+4} 8x^2e^{2x}\sin 2x$

$= 8\dfrac{1}{D^2-4D+4} e^{2x}(x^2\sin 2x)$

put $D = D+a = D+2$

$$= 8e^{2x} \frac{1}{(D+2-2)^2} x^2 \sin 2x$$

$$= 8e^{2x} \frac{1}{D^2} x^2 \sin 2x$$

$$= 8e^{2x} \left[\text{I.P of } \frac{1}{D^2} x^2 e^{(2i)x} \right]$$

put $D = D + a = 2i \quad D + 2i$

$$= 8e^{2x} \left[\text{I.P of } e^{2ix} \frac{1}{(D+2i)^2} x^2 \right]$$

$$= 8e^{2x} \left[\text{I.P of } e^{2ix} \frac{1}{\cancel{D^2+4i^2+4iD}} \frac{1}{\left[2i\left[1+\frac{D}{2i}\right]\right]^2} x^2 \right]$$

$$= 8e^{2x} \left[\text{I.P of } e^{2ix} \frac{1}{-4\left[1+\frac{D}{2i}\right]^2} x^2 \right]$$

$$= -2e^{2x} \left[\text{I.P of } e^{2ix} \left[1+\frac{D}{2i}\right]^{-2} x^2 \right] \qquad \left[\frac{D}{2i} \times \frac{i}{i} = \frac{Di}{-2} \right]$$

$$= -2e^{2x} \left[\text{I.P of } e^{2ix} \left[1-\frac{Di}{2}\right]^{-2} x^2 \right]$$

$$= -2e^{2x} \left[\text{I.P of } e^{2ix} \left\{ 1 + 2\left(\frac{Di}{2}\right) + 3\left(\frac{Di}{2}\right)^2 + \cdots \right\} x^2 \right]$$

$$= -2e^{2x} \left[\text{I.P of } e^{2ix} \left\{ 1 + Di + 3\left(\frac{-D^2}{4}\right) \right\} x^2 \right]$$

$$= -2e^{2x} \left[\text{I.P of } e^{2ix} \left[x^2 + i(2x) - \frac{3(2)}{4} \right] \right]$$

$$= -2e^{2x} \left[\text{I.P of } e^{2ix} \left[(x^2 - 3/2) + i(2x) \right] \right]$$

$$= -2e^{2x} \left[\text{I.P of } (\cos 2x + i \sin 2x)\left((x^2-3/2) + i(2x)\right) \right]$$

$$= -2e^{2x} \left[\text{I.P of } \left[\cos 2x (x^2 - 3/2) + i(2x)\cos 2x + i \sin 2x (x^2 - 3/2) + i^2 \sin 2x (2x) \right] \right]$$

$$= -2e^{2x} \left[\text{I.P of } \left[\cos 2x (x^2 - 3/2) + i(2x)\cos 2x + i \sin 2x (x^2 - 3/2) - \sin 2x (2x) \right] \right]$$

$$y_p = -2e^{2x} \left[(x^2 - 3/2) \sin 2x + (2x) \cos 2x \right]$$

There the General solution of equation (1) $y = y_c + y_p$

$$y = (c_1 + c_2 x) e^{2x} - 2e^{2x} \left[(x^2 - 3/2) \sin 2x + (2x) \cos 2x \right]$$

21: Solve $(D^4 + 2D^2 + 1) y = x^2 \cos x$

Sol: Given Equation is $(D^4 + 2D^2 + 1) y = x^2 \cos x \rightarrow (1)$ Where $f(D) = D^4 + 2D^2 + 1$

The Auxilary equation is $f(m) = 0 \Rightarrow m^4 + 2m^2 + 1 = 0$

$$\Rightarrow (m^2+1)^2 = 0$$

$$\Rightarrow m^2 + 1 = 0, \; m^2 + 1 = 0$$

$$\Rightarrow m = \pm i, \pm i$$

complementary function of equation (1) is $y_c = [(c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x]$

$$y_P = \frac{1}{f(D)}\phi(x) = \frac{1}{D^4+2D^2+1}x^2\cos x$$

$$= R.P \text{ of } \frac{1}{D^4+2D^2+1}x^2 e^{ix} \quad ; \text{ put } D=D+a=D+i$$

$$= R.P \text{ of } \frac{1}{(D^2+1)^2}x^2 e^{ix} \quad ; \text{put } D=D+a=D+i$$

$$= R.P \text{ of } e^{ix}\frac{1}{((D+i)^2+1)^2}x^2 = R.P \text{ of } e^{ix}\frac{1}{(D^2+2Di)^2}x^2$$

$$= R.P \text{ of } e^{ix}\frac{1}{\left[2iD\left[1+\frac{D^2}{2iD}\right]\right]^2}x^2 = R.P \text{ of } \frac{e^{ix}}{4i^2D^2}\left[1+\frac{D^2}{2iD}\right]^{-2}x^2$$

$$= R.P \text{ of } \frac{e^{ix}}{-4}\left[1+\frac{D^2}{2iD}\right]\left[\frac{1}{D^2}x^2\right] = R.P \text{ of } \frac{e^{ix}}{-4}\left[1+\frac{D^2}{2iD}\right]^{-2}\left[\frac{x^4}{12}\right]$$

$$= \frac{-1}{48} R.P \text{ of } e^{ix}\left[1+\frac{D^2}{2iD}\right]^{-2}x^4 \qquad \left[\frac{D^2}{2iD} = \frac{D^2}{2iD}\times\frac{i}{i} = \frac{-D^2 i}{-2D}\right]$$

$$= \frac{-1}{48} R.P \text{ of } e^{ix}\left[1-\frac{D^2 i}{2D}\right]^{-2}x^4 \qquad \left[\frac{1}{D^2}x^2 = \frac{1}{D}\int x^2dx = \frac{1}{3}\int x^3 dx = \frac{x^4}{12}\right]$$

$$= \frac{-1}{48} R.P \; e^{ix}\left[1-\frac{Di}{2}\right]^{-2}x^4$$

$$= \frac{-1}{48}\left[R.P \text{ of } e^{ix}\left[1+2\left(\frac{Di}{2}\right)+3\left(\frac{Di}{2}\right)^2+4\left(\frac{Di}{2}\right)^3+5\left(\frac{Di}{2}\right)^4+\ldots\right]x^4\right]$$

$$= \frac{-1}{48}\left[R.P \text{ of } e^{ix}\left[x^4+iD(x^4)+\frac{3i^2}{4}D^2(x^4)+\frac{4i^3}{8}D^3(x^4)+\frac{5i^4D^4(x^4)}{16}\right]\right]$$

$$= \frac{-1}{48}\left[R.P \text{ of } e^{ix}\left[x^4+i(4x^3)-\frac{3(12x^2)}{4}-\frac{i(24x)}{2}+\frac{5(24)}{16}\right]\right]$$

$$= \frac{-1}{48}\left[R.P \text{ of } e^{ix}\left[(x^4-9x^2+15/2)+i\left(4x^3-\frac{24x}{2}\right)\right]\right]$$

$$= \frac{-1}{48}\left[R.P \text{ of } (\cos x+i\sin x)\left[(x^4-9x^2+15/2)+i(4x^3-12x)\right]\right]$$

$$= \frac{-1}{48}\left[R.P \text{ of } \left[\cos x(x^4-9x^2+15/2)+i\cos x(4x^3-12x)+i\sin x(x^4-9x^2+15/2) + i^2\sin x(4x^3-12x)\right]\right]$$

$$= \frac{-1}{48}\left[R.P \text{ of } \left[\cos x(x^4-9x^2+15/2)+i\cos x(4x^3-12x)+i\sin x(x^4-9x^2+15/2) - \sin x(4x^3-12x)\right]\right]$$

$$y_P = \frac{-1}{48}\left[\cos x(x^4-9x^2+15/2)-\sin x(4x^3-12x)\right]$$

Therefore the General Solution of equation (1) is $y = y_c + y_P$

$$y = [(C_1+C_2x)\cos x+(C_3+C_4x)\sin x] - \frac{1}{48}\left[\cos x(x^4-9x^2+15/2)-\sin x(4x^3-12x)\right]$$

Method of undetermined Coefficients:

The general solution of the differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = Q(x) \quad \text{---(A)}$$

where $a_n \neq 0$ and $Q(x) \neq 0$ is $y = y_c + y_p$ ---(B)

where y_c is a complementary function and y_p is the Particular integral

Working Rule 1: If no term of $Q(x)$ in (A) is same as a term in y_c. Then y_p will be a linear combination of the terms in $Q(x)$ and all its linearly independent derivatives.

Problem: Solve $(D^2 - 2D)y = e^x \sin x$ by the method of undetermined Coefficients

Solution: $(D^2-2D)y = e^x \sin x$ ---(1) where $f(D) = D^2 - 2D$

The Auxiliary equation is $f(m) = 0 \Rightarrow m^2 - 2m = 0 \Rightarrow m(m-2) = 0 \Rightarrow m = 0, 2$

$\therefore y_c = c_1 + c_2 e^{2x}$

Since $Q(x) = e^x \sin x$ has no term common with y_c, The particular integral y_p will be a linear Combination of $Q(x)$ and all its linearly independent derivatives.

then $y_p(x) = Ae^x \sin x + Be^x \cos x$ where A and B are to be determined.

$y_p'(x) = Ae^x \sin x + Ae^x \cos x + Be^x \cos x - Be^x \sin x$

$= (A-B)e^x \sin x + (A+B)e^x \cos x$

$y_p''(x) = (A-B)e^x \sin x + (A-B)e^x \cos x + (A+B)e^x \cos x - (A+B)e^x \sin x$

$= -2Be^x \sin x + 2Ae^x \cos x$

Substituting the values of y_p, y_p', y_p'' in (1), we get

$-2Be^x \sin x + 2Ae^x \cos x - 2(A-B)e^x \sin x - 2(A+B)e^x \cos x = e^x \sin x$

$\Rightarrow -2Ae^x \sin x - 2Be^x \cos x = e^x \sin x$

Equating the Coefficients of like terms on both sides

$-2A = 1 \Rightarrow A = -\frac{1}{2}$, $-2B = 0 \Rightarrow B = 0$

$\therefore y_p = -\frac{1}{2}e^x \sin x$

Hence the general solution of (1) is $y = y_c + y_p$

$\Rightarrow y = c_1 + c_2 e^{2x} - \frac{1}{2}e^x \sin x$

Problem: Solve $(D^2+4D+4)y = 4x^2 + 6e^x$

Solution: given equation is $(D^2+4D+4)y = 4x^2 + 6e^x$

The auxiliary equation is $f(m) = 0 \Rightarrow m^2 + 4m + 4 = 0$

$\Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2$

$\therefore y_c = (c_1 + c_2 x)e^{-2x}$

Since $Q(x) = 4x^2+6e^x$ has no term common with $y_c(x)$, $y_p(x)$ will be a linear combination of $Q(x)$ and all its linearly independent derivatives neglecting the constant coefficients.

$\therefore y_p = Ax^2+Bx+C+De^x$, where A, B, C, D are to be determined.

$y_p' = 2Ax+B+De^x$, $y_p'' = 2A+De^x$

Substituting the values of y_p, y_p' and y_p'' in (1), we get

$2A+De^x+4(2Ax+B+De^x)+4(Ax^2+Bx+C+De^x) = 4x^2+6e^x$

$\Rightarrow 4Ax^2+(8A+4B)x+(2A+4B+4C)+9De^x = 4x^2+6e^x$

Equating the coefficients of like terms on both sides

$4A = 4$, $8A+4B=0$, $2A+4B+4C=0$, $9D=6$

Solving, $A=1$, $B=-2$, $C=3/2$, $D=2/3$

$\therefore y_p = x^2-2x+\frac{2}{3}e^x+\frac{3}{2}$

$\therefore$ the general solution of (1) is $y = y_c+y_p$

$\Rightarrow y = (c_1+c_2x)e^{-2x}+x^2-2x+\frac{2}{3}e^x+\frac{3}{2}$

Problem: Solve $(D^2+2D+5)y = 12e^x-34\sin 2x$

Solution: given equation is $(D^2+2D+5)y = 12e^x-34\sin 2x$ — (1)

The auxiliary equation is $m^2+2m+5=0$

$\Rightarrow m = \frac{-2\pm\sqrt{4-20}}{2} = \frac{-2\pm 4i}{2} = -1\pm 2i$

$\therefore y_c = e^{-x}(c_1\cos 2x+c_2\sin 2x)$

Since $Q(x) = 12e^x-34\sin 2x$ has no term common with y_c.
Hence y_p will be a linear combination of $Q(x)$ and all its linearly independent derivatives neglecting the constant coefficients.

$\therefore y_p = Ae^x+B\sin 2x+C\cos 2x$, where A, B, C are to be determined

$y_p' = Ae^x+2B\cos 2x-2C\sin 2x$

$y_p'' = Ae^x-4B\sin 2x-4C\cos 2x$

Substituting the values of y_p, y_p' and y_p'' in (1)

$Ae^x-4B\sin 2x-4C\cos 2x+2(Ae^x+2B\cos 2x-2C\sin 2x)$
$+5(Ae^x+B\sin 2x+C\cos 2x) = 12e^x-34\sin 2x$

Equating the coefficients of like terms on both sides

$A+2A+5A = 12 \Rightarrow 8A = 12$

$-4B-4C+5B = -34 \Rightarrow B-4C = -34$

$-4C+4B+5C = 0 \Rightarrow 4B+C = 0$

Solving, $A = 3/2$, $B=-2$, $C=8$

Hence $y_p = \frac{3}{2}e^{x} - 2\sin x + 8\cos 2x$

The general solution of (1) is $y = y_c + y_p$

$$= e^{-x}(c_1\cos 2x + c_2\sin 2x) + \frac{3}{2}e^{x} - 2\sin x + 8\cos 2x$$

Problem: Solve $(D^2+3D+2)y = \sin x$

Solution: given equation is $(D^2+3D+2)y = \sin x$ — (1)

The auxiliary equation is $f(m)=0 \Rightarrow m^2+3m+2=0$

$\Rightarrow m = -2, -1$

$\therefore y_c = c_1 e^{-2x} + c_2 e^{-x}$

Since $Q(x) = \sin x$ has no term common with y_c. Hence y_p will be a linear combination of $Q(x)$ and all its linearly independent derivatives neglecting the constant coefficients.

$\therefore y_p = A\sin x + B\cos x$, where A, B are to be determined.

$y_p' = A\cos x - B\sin x$, $y_p'' = -A\sin x - B\cos x$.

Substituting in (1) we get

$-A\sin x - B\cos x + 3(A\cos x - B\sin x) + 2(A\sin x + B\cos x) = \sin x$

$\Rightarrow (A-3B)\sin x + (3A+B)\cos x = \sin x$

Equating the coefficients of like terms on both sides,

$A - 3B = 1$, $3A+B = 0$

Solving, $A = 1/10$, $B = -3/10$

$\therefore y_p = \frac{1}{10}(\sin x - 3\cos x)$

$\therefore$ The general solution is $y = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{10}(\sin x - 3\cos x)$

Working Rule 2:

If $Q(x)$ in (A) contains a term which is x^k times a term $f(x)$ of y_c where k is zero or a positive integer, the particular integral of (A) will be a linear combination of $x^{k+1}f(x)$ and all its linearly independent derivatives. Also if in addition $Q(x)$ contains terms which correspond to working rule 1, then proper terms required by this must also be included in y_p

Problems: Solve $(D^2-3D+2)y = 2x^2+3e^{2x}$

Solution: Given equation is $(D^2-3D+2)y = 2x^2+3e^{2x}$

The auxiliary equation is $m^2-3m+2=0$

$\Rightarrow (m-1)(m-2)=0 \Rightarrow m = 1, 2$

$\therefore y_c = c_1 e^{x} + c_2 e^{2x}$.

$Q(x) = 2x^2 + 3e^{2x}$

$Q(x)$ contains e^{2x} which is x^0 times of the same term in y_c

For this term, y_p must contain a linear combination of $x^{0+1}e^{2x}$ and its linear derivatives.

Also $Q(x)$ has the term x^2 which is not present in y_c.

$\therefore$ By Rule 1, y_p must include a linear combination of it and all its linearly independent derivatives ignoring the constant coefficients we can neglect e^{2x} as it already appeared in y_c.

$$\therefore y_p = Ax^2 + Bx + C + Dxe^{2x}$$

$$y_p' = 2Ax + B + 2Dxe^{2x} + De^{2x}$$

$$y_p'' = 2A + 4Dxe^{2x} + 4De^{2x}.$$

Substituting these values in (1), we get

$$(2A + 4Dxe^{2x} + 4De^{2x}) - 3(2Ax + B + 2Dxe^{2x} + De^{2x}) + 2(Ax^2 + Bx + C + Dxe^{2x}) = 2x^2 + 3e^{2x}.$$

$$\Rightarrow 2Ax^2 + (2B - 6A)x + (2A - 3B + 2C) + De^{2x} = 2x^2 + 3e^{2x}$$

Equating the coefficients of like terms on both sides

$$2A = 2,\ 2B - 6A = 0,\ 2A - 3B + 2C = 0,\ D = 3.$$

Solving $A = 1,\ B = 3,\ C = 7/2,\ D = 3$

$$\therefore y_p = x^2 + 3x + 7/2 + 3xe^{2x}$$

$\therefore$ the general solution is $y = y_c + y_p = c_1e^x + c_2e^{2x} + x^2 + 3x + \frac{7}{2} + 3xe^{2x}$

Problem: Solve $(D^2 - 3D + 2)y = xe^{2x} + \sin x$

Solution: given equation is $(D^2 - 3D + 2)y = xe^{2x} + \sin x$

the auxiliary equation is $m^2 - 3m + 2 = 0 \Rightarrow (m-2)(m-1) = 0$

$$\Rightarrow m = 1, 2$$

$$\therefore y_c = c_1e^x + c_2e^{2x}$$

$$Q(x) = xe^{2x} + \sin x$$

Comparing this with y_c, we observe that $Q(x)$ contains xe^{2x} which is x times the same term in y_c. For this term y_p must contain a linear combination of $x^{1+1}e^{2x} = x^2e^{2x}$ and all its linearly independent derivatives. Also $Q(x)$ contains the term $\sin x$. For this we include a linear combination of it and its derivatives.

$$\therefore y_p = Ax^2e^{2x} + Bxe^{2x} + C\sin x + D\cos x.$$

$$y_p' = 2Ax^2e^{2x} + 2Axe^{x} + 2Bxe^{2x} + Be^{2x} + C\cos x - D\sin x$$

$$y_p'' = 4Axe^{2x} + 4Ax^2e^{2x} + 2Ae^{2x} + 4Axe^{2x} + 2Be^{2x} + 4Bxe^{2x} + 2Be^{2x} - C\sin x - D\cos x$$

Substituting the values of y_p, y_p', y_p'' in (1), we get

$$2Axe^{2x} + (2A+B)e^{x} + (C+3D)\sin x + (D-3C)\cos x = xe^{2x} + \sin x$$

Equating the coefficients of like terms on both sides

$2A = 1$, $2A + B = 0$, $C + 3D = 1$, $D - 3C = 0$.

Solving $A = 1/2$, $B = -1$, $C = 1/10$. $D = 3/10$.

$$\therefore y_p = \tfrac{1}{2}x^2 e^{2x} - xe^{2x} + \frac{1}{10}\sin x + \frac{3}{10}\cos x$$

$\therefore$ The general solution is $y = y_c + y_p$

$$\Rightarrow y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2}x^2 e^{2x} - xe^{2x} + \frac{1}{10}\sin x + \frac{3}{10}\cos x.$$

Problem: Solve $(D^2 + D)y = 2\cos x + x$

Solution: Auxiliary equation is $f(m) = 0 \Rightarrow m^2 + m = 0 \Rightarrow m = 0, -1$

$$\therefore y_c = C_1 + C_2 e^{-2}$$

$$y_p = Ax^2 + Bx + C\cos x + D\sin x$$

$$y_p' = 2Ax + B - C\sin x + D\cos x$$

$$y_p'' = 2A - C\cos x - D\sin x$$

Substituting y_p, y_p', y_p'' in the given equation

$$(2A + B) + 2Ax + (D - C)\cos x - (C + D)\sin x = 2\cos x + x$$

$B + 2A = 0$, $2A = 1$, $D - C = 2$, $C + D = 0 \Rightarrow A = 1/2$, $B = -1$, $D = 1$, $C = -1$

$$\therefore y_p = \frac{x^2}{2} - x - \cos x + \sin x$$

$\therefore$ The general solution is $y = c_1 + c_2 e^{-x} + \dfrac{x^2}{2} - x - \cos x + \sin x$.

Working Rule 3:

If the auxiliary equation of (A) has an r multiple root and $Q(x)$ contains a term $x^k f(x)$ (neglecting the constant coefficients) where $f(x)$ is a term in $y_c(x)$ and is obtained from the r multiple roots, then $y_p(x)$ will be a linear combination of $x^{k+r} f(x)$ and all its linearly independent derivatives. In addition if $Q(x)$ contains terms as in the case of working Rules 1 and 2, then the proper terms must also be included accordingly in y_p.

Problem: Solve $(D^2 + 4D + 4)y = 3xe^{-2x}$

Solution: given equation is $(D^2 + 4D + 4)y = 3xe^{-2x}$

The auxiliary equation is $m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2$

$$y_c = c_1 e^{-2x} + c_2 x e^{-2x}.$$

The auxiliary equation has a multiple root $m = -2$.
$Q(x)$ contains the term $x e^{-2x}$ which is x times the term e^{-2x} in y_c and this term in y_c came from a multiple root.
$\therefore r = 2$ and $k = 1$

y_p must be a linear combination of $x^3 e^{-2x}$ and all its linearly independent derivatives. In obtaining this linear combination we can neglect e^{-2x} and $x e^{-2x}$ as they are already in y_c

$$\therefore y_p = A x^3 e^{-2x} + B x^2 e^{-2x}.$$

$$y_p' = 3Ax^2 e^{-2x} - 2Ax^3 e^{-2x} + 2Bx e^{-2x} - 2Bx^2 e^{-2x}.$$

$$y_p'' = (3A - 2B)(2x e^{-2x} - 2x^2 e^{-2x}) - 6Ax^2 e^{-2x} + 4Ax^3 e^{-2x} + 2B e^{-2x} - 4Bx e^{-2x}$$

Substituting the values of y_p, y_p', y_p'' in (1) we get

$$(3A - 2B)(2x e^{-2x} - 2x^2 e^{-2x}) - 6Ax^2 e^{-2x} + 4Ax^3 e^{-2x} + 2B e^{-2x} - 4Bx e^{-2x}$$
$$+ 12Ax^2 e^{-2x} - 8Ax^3 e^{-2x} + 8Bx e^{-2x} - 8Bx^2 e^{-2x} + 4Ax^3 e^{-2x} + 4Bx^2 e^{-2x} = 3x e^{-2x}$$

$$\Rightarrow 6Ax e^{-2x} + 2B e^{-2x} = 3x e^{-2x}.$$

Comparing the like terms, we have

$$6A = 3 \Rightarrow A = \tfrac{1}{2}, \quad B = 0.$$

$$\therefore y_p = \tfrac{1}{2} x^3 e^{-2x}.$$

The general solution is $y = y_c + y_p = c_1 e^{-2x} + c_2 x e^{-2x} + \tfrac{1}{2} x^3 e^{-2x}$.

Linear Differential Equations with non Constant Coefficients

Linear Differential equation of order n:

An equation of the form $a_n(x)\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=Q(x)$ where $a_0, a_1, \ldots a_{n-1}, a_n$ and Q are Continuous real functions in x defined on an interval I is called a linear differential equation of order n over the interval I

Def: An equation of the form $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)=R(x)$ where $P(x)$, $Q(x)$ and $R(x)$ are real valued functions of x defined on an interval I, is called the linear equation of the second order with variable Coefficients.

The linear equation of the second order with variable Coefficients can be solved by the method of Variation of Parameters.

General solution of $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ by the method of variation of Parameters:

given linear differential equation is $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ —(1)

Its homogeneous equation is $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0.$ —(2)

Let $y_c=c_1u+c_2v$ be the general solution of (2), where u and v are functions of x and c_1, c_2 are real constants.

$y=c_1u+c_2v$ satisfies(2) $\Rightarrow (c_1u_2+c_2v_2)+P(c_1u_1+c_2v_1)+Q(c_1u+c_2v)=0$

$\Rightarrow c_1(u_2+Pu_1+Qu)+c_2(v_2+Pv_1+Qv)=0$

$\Rightarrow u_2+Pu_1+Qu=0$ — (3) and $v_2+Pv_1+Qv=0$ —(4)

Let y_p of (1) be $y_p=Au+Bv$ —(5) which is obtained from y_c of (1) by replacing c_1 and c_2 by A and B which are also functions of x

$$\frac{dy_p}{dx}=Au_1+u\frac{dA}{dx}+Bv_1+v\frac{dB}{dx}=(Au_1+Bv_1)+u\frac{dA}{dx}+v\frac{dB}{dx}$$

choose A and B such that $u\frac{dA}{dx}+v\frac{dB}{dx}=0$ — (6)

then $\frac{dy_p}{dx}=Au_1+Bv_1 \Rightarrow \frac{d^2y_p}{dx^2}=(Au_2+Bv_2)+u_1\frac{dA}{dx}+v_1\frac{dB}{dx}$

Substituting these values in (1), we get

$$Au_2 + Bv_2 + u_1\frac{dA}{dx} + v_1\frac{dB}{dx} + P(Au_1 + Bv_1) + Q(Au + Bv) = R \quad -(7)$$

$$\Rightarrow A(u_2 + Pu_1 + Qu) + B(v_2 + Pv_1 + Qv) + \left(u_1\frac{dA}{dx} + v_1\frac{dB}{dx}\right) = R$$

using (3) and (4), $u_1\frac{dA}{dx} + v_1\frac{dB}{dx} = R \quad -(8)$

solving (6) and (8) $\Rightarrow \dfrac{dA/dx}{vR} = \dfrac{dB/dx}{-uR} = \dfrac{1}{vu_1 - uv_1}$

$$\Rightarrow \frac{dA}{dx} = \frac{-vR}{uv_1 - vu_1} \quad \text{and} \quad \frac{dB}{dx} = \frac{uR}{uv_1 - vu_1}$$

$$\Rightarrow A = \int \frac{-vR}{uv_1 - vu_1}\,dx \quad \text{and} \quad B = \int \frac{uR}{uv_1 - vu_1}\,dx \quad -(9)$$

Substituting the values of A and B from (9) and (5), we get y_p

$\therefore$ The general solution of (1) is $y = y_c + y_p = c_1u + c_2v + Au + Bv$

Working Rule to find the general solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad -(i)$ by the method of variation of Parameters:

1. In case the given equation is not in the standard form, reduce it to the standard form.
2. Find the solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

   Let $y_c = c_1u(x) + c_2v(x)$
3. Let $y_p = Au + Bv$, where A and B are functions of x
4. Find $u\frac{dv}{dx} - v\frac{du}{dx} \Rightarrow uv_1 - vu_1$
5. Find A and B by using $A = \int \frac{-vR\,dx}{uv_1 - vu_1}$, $B = \int \frac{uR\,dx}{uv_1 - vu_1}$
6. The general solution of (i) is $y = c_1u + c_2v + Au + Bv$.

Problem: Solve $(D^2 + a^2)y = \tan ax$ by the method of Variation of Parameters

Solution: Given equation is $(D^2 + a^2)y = \tan ax \quad -(1)$

The auxiliary equation is $m^2 + a^2 = 0 \Rightarrow m = \pm ai$

$$\therefore y_c = c_1\cos ax + c_2\sin ax \quad -(2)$$

Let $y_p = A\cos ax + B\sin ax$

Then $u = \cos ax$, $v = \sin ax$

$uv_1 - vu_1 = \cos ax(a\cos ax) - \sin ax(-a\sin ax) = a(\cos^2 ax + \sin^2 ax)$
$= a$

$$A = \int \frac{-vR}{uv_1 - vu_1}\,dx = -\int \frac{\sin ax \tan ax}{a}\,dx = \frac{-1}{a}\int \frac{\sin^2 ax}{\cos ax}\,dx$$

$$= -\frac{1}{a}\int \frac{1 - \cos^2 ax}{\cos ax}\,dx = -\frac{1}{a}\left[\int \sec ax\,dx - \int \cos ax\,dx\right]$$

$$= -\frac{1}{a^2}\log|\sec ax + \tan ax| + \frac{1}{a^2}\sin ax \quad -(3)$$

$$B = \int \frac{uR}{uv_1 - vu_1}\,dx = \int \frac{\cos ax \tan ax}{a}\,dx = \frac{1}{a}\int \sin ax\,dx = -\frac{1}{a^2}\cos ax \quad -(4)$$

$$\therefore\ y_p = Au + Bv = \frac{1}{a^2}\left[\sin ax - \log|\sec ax + \tan ax|\right]\cos ax$$
$$- \frac{1}{a^2}\cos ax \sin ax = -\frac{\cos ax}{a^2}\log|\sec ax + \tan ax|$$

$\therefore$ The general solution is $y = y_c + y_p$

$$= c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\cos ax \log|\sec ax + \tan ax|$$

Problem: Solve $(D^2+1)y = \operatorname{cosec} x$ by the method of variation of parameters

Solution: Given equation is $(D^2+1)y = \operatorname{cosec} x$.

Auxiliary equation is $m^2+1=0 \Rightarrow m = \pm i$

$\therefore y_c = c_1 \cos x + c_2 \sin x$

Let $y_p = A\cos x + B\sin x$

$uv_1 - vu_1 = \cos x(\cos x) - \sin x(-\sin x) = \cos^2 x + \sin^2 x = 1$

$$A = \int \frac{-vR}{uv_1 - vu_1}\,dx = -\int \sin x \operatorname{cosec} x\,dx = -\int \cancel{\cot x}\,dx = \cancel{\log|\sin x|} - x$$

$$B = \int \frac{uR}{uv_1 - vu_1}\,dx = \int \cos x \operatorname{cosec} x\,dx = \int \cot x\,dx = \log|\sin x|$$

$y_p = (-x)\cos x + (\log|\sin x|)\sin x$

$\therefore$ The general solution is $y = y_c + y_p$

$$= c_1\cos x + c_2 \sin x - x\cos x + \sin x \log|\sin x|$$

Problem: Solve $(D^2 - 2D)y = e^x \sin x$ by the method of variation of Parameters

Solution: Given equation is $(D^2-2D)y = e^x \sin x$

Auxiliary equation is $m^2 - 2m = 0 \Rightarrow m(m-2) = 0 \Rightarrow m = 0, 2$

$\therefore y_c = c_1 + c_2 e^{2x}$

Let $y_p = A + Be^{2x}$, where $u = 1$, $v = e^{2x}$

$uv_1 - vu_1 = 1(2e^{2x}) - e^{2x}(0) = 2e^{2x}$

$$A = \int \frac{-vR}{uv_1 - vu_1}\,dx = -\int \frac{e^{2x} e^x \sin x}{2e^{2x}}\,dx = -\frac{1}{2}\int e^x \sin x\,dx$$

$$= -\frac{1}{2}\left[\frac{e^x \sin x - e^x \cos x}{1^2+1^2}\right] = \frac{1}{4}e^x(\cos x - \sin x)$$

$$B = \int \frac{uR}{uv_1 - vu_1}\,dx = \int \frac{e^x \sin x}{2e^{2x}}\,dx = \frac{1}{2}\int e^{-x}\sin x\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-x}(-\sin x) - e^{-x}\cos x}{(-1)^2 + 1^2}\right] = -\frac{1}{4}e^{-x}(\cos x + \sin x)$$

$$y_p = \frac{1}{4}e^x(\cos x - \sin x) - \frac{1}{4}e^{-x}(\cos x + \sin x)e^{2x}$$
$$= \frac{1}{4}e^x(\cos x - \sin x - \cos x - \sin x) = -\frac{1}{2}e^x \sin x$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1 + c_2 e^{2x} - \frac{1}{2} e^{x} \sin x$$

Problem: Solve $[(x-1)D^2 - xD + 1]y = (x-1)^2$ by the method of variation of Parameters.

Solution: given equation is $\frac{d^2y}{dx^2} - \frac{x}{x-1}\frac{dy}{dx} + \frac{1}{x-1}y = x-1$ — (1)

Homogeneous equation of (1) is $\frac{d^2y}{dx^2} - \frac{x}{x-1}\frac{dy}{dx} + \frac{1}{x-1}y = 0$ — (2)

$1+P+Q = 1 - \frac{x}{x-1} + \frac{1}{x-1} = 0 \Rightarrow y = e^x$ is a solution of (2)

$P + Qx = -\frac{x}{x-1} + \frac{x}{x-1} = 0 \Rightarrow y = x$ is a solution of (2)

$$\therefore y_c = c_1 e^x + c_2 x$$

Let $y_p = Ae^x + Bx$, where $u = e^x$, $v = x$

$uv_1 - vu_1 = e^x(1) - xe^x = (1-x)e^x$.

$$A = \int \frac{-vR}{uv_1 - vu_1} dx = \int \frac{-x(x-1)}{(1-x)e^x} dx = \int x e^{-x} dx$$

$$= -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} = -(1+x)e^{-x}$$

$$B = \int \frac{uR}{uv_1 - vu_1} dx = \int \frac{e^x(x-1)}{e^x(1-x)} dx = -\int dx = -x$$

$$y_p = -(1+x)e^{-x}e^x - x(x) = -(1+x+x^2)$$

$\therefore$ the general solution is $y = y_c + y_p = c_1 e^x + c_2 x - (1+x+x^2)$

Problem: Solve $(D^2 - 2D + 2)y = e^x \tan x$ by the method of variation of Parameters

Solution: $(D^2 - 2D + 2)y = e^x \tan x$

Auxiliary equation is $m^2 - 2m + 2 = 0$

$$\Rightarrow m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1-i, 1+i$$

$$\therefore y_c = e^x(c_1 \cos x + c_2 \sin x)$$

Let $y_p = Ae^x \cos x + Be^x \sin x$, where $u = e^x \cos x$, $v = e^x \sin x$

$uv_1 - vu_1 = e^x \cos x(e^x \cos x + e^x \sin x) - e^x \sin x(e^x \cos x - e^x \sin x)$

$= e^{2x}\cos^2 x + e^{2x}\sin x \cos x - e^{2x}\sin x \cos x + e^{2x}\sin^2 x = e^{2x}$

$$A = \int \frac{-vR}{uv_1 - vu_1} dx = -\int \frac{e^x \sin x \, e^x \tan x}{e^{2x}} dx = -\int \frac{\sin^2 x}{\cos x} dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x} dx = \int (\cos x - \sec x) dx = \sin x - \log|\sec x + \tan x|$$

$$B = \int \frac{uR}{uv_1 - vu_1} dx = \int \frac{e^x \cos x \, e^x \tan x}{e^{2x}} dx = \int \sin x \, dx = -\cos x$$

$y_p = e^x \cos x(\sin x - \log|\sec x + \tan x|) - e^x \sin x \cos x$

$= -e^x \cos x \log|\sec x + \tan x|$

The general solution is $y = y_c + y_p$

$= e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x \log|\sec x + \tan x|$

Problem: Solve $(D^2+a^2)y = \sec ax$ by the method of variation of Parameters

Solution: $(D^2+a^2)y = \sec ax$

Auxiliary equation is $m^2+a^2=0 \Rightarrow m = \pm ai$

$\therefore y_c = c_1 \cos ax + c_2 \sin ax$.

Let $y_p = A\cos ax + B\sin ax$, where $u = \cos ax$, $v = \sin ax$.

$uv_1 - vu_1 = \cos ax(a\cos ax) - \sin ax(-a\sin ax) = a\cos^2 ax + a\sin^2 ax$

$= a(\cos^2 ax + \sin^2 ax) = a$

$$A = -\int \frac{vR}{uv_1 - vu_1}dx = -\int \frac{\sin ax \sec ax}{a}dx = -\frac{1}{a}\int \tan ax\, dx$$

$$= -\frac{1}{a^2}\log(\sec ax) = \frac{1}{a^2}\log(\cos ax)$$

$$B = \int \frac{uR}{uv_1 - vu_1}dx = \int \frac{\cos ax \sec ax}{a}dx = \frac{1}{a}\int dx = \frac{x}{a}$$

$$y_p = \frac{1}{a^2}(\cos ax)\log(\cos ax) + \frac{1}{a}x\sin ax.$$

$\therefore$ The general solution is $y = c_1\cos ax + c_2 \sin ax + \frac{1}{a^2}\cos ax \log(\cos ax)$
$+ \frac{1}{a}x \sin ax$.

Problem: Solve $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ by the method of variation of Parameter

Solution: $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$.

Auxiliary equation is $m^2+4=0 \Rightarrow m^2 = -4 \Rightarrow m = \pm 2i$

$\therefore y_c = c_1 \cos 2x + c_2 \sin 2x$.

Let $y_p = A\cos 2x + B\sin 2x$, where $u = \cos 2x$, $v = \sin 2x$.

$uv_1 - vu_1 = \cos 2x(2\cos 2x) - \sin 2x(-2\sin 2x) = 2(\cos^2 2x + \sin^2 2x) = 2$

$$A = -\int \frac{vR\,dx}{uv_1 - vu_1} = -\int \frac{\sin 2x(4\tan 2x)}{2}dx = -2\int \frac{\sin^2 2x}{\cos 2x}dx$$

$$= -2\int \frac{1-\cos^2 2x}{\cos 2x}dx = -2\left[\int \sec 2x\,dx - \int \cos 2x\,dx\right] = -\log(\sec 2x + \tan 2x) + \sin 2x$$

$$B = \int \frac{uR}{uv_1 - vu_1}dx = \int \frac{\cos 2x(4\tan 2x)}{2}dx = 2\int \sin 2x\,dx = -\cos 2x$$

$y_p = \cos 2x(-\log(\sec 2x + \tan 2x) + \sin 2x) + \sin 2x(-\cos 2x)$

$= -\cos 2x \log(\sec 2x + \tan 2x)$

$\therefore$ The general solution is $y = y_c + y_p$

$$= c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$$

Problem: Solve $(D^2 + a^2)y = \operatorname{cosec} ax$ by the method of variation of Parameters

Solution: $(D^2 + a^2)y = \operatorname{cosec} ax$

Auxiliary equation is $m^2 + a^2 = 0 \Rightarrow m = \pm ai$

$\therefore y_c = c_1 \cos ax + c_2 \sin ax$.

Let $y_p = A \cos ax + B \sin ax$, where $u = \cos ax$, $v = \sin ax$.

$$uv_1 - vu_1 = \cos ax\,(a\cos ax) - (-a\sin ax)\sin ax$$
$$= a\cos^2 ax + a\sin^2 ax = a(\cos^2 ax + \sin^2 ax) = a$$

$$A = -\int \frac{vR\,dx}{uv_1 - vu_1} = -\frac{1}{a}\int \sin ax \operatorname{cosec} ax\,dx = -\frac{1}{a}\int dx = -\frac{x}{a}$$

$$B = \int \frac{uR\,dx}{uv_1 - vu_1} = \int \frac{\cos ax \operatorname{cosec} ax}{a}\,dx = \frac{1}{a}\int \cot ax\,dx = \frac{1}{a^2}\log|\sin ax|$$

$$y_p = -\frac{x}{a}\cos ax + \frac{1}{a^2}(\log \sin ax)\sin ax$$

$\therefore$ The general solution is $y = y_c + y_p$

$$= c_1 \cos ax + c_2 \sin ax - \frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax \log(\sin ax)$$

Cauchy-Euler equation:

An equation of the form $x^n \dfrac{d^n y}{dx^n} + P_1 x^{n-1}\dfrac{d^{n-1}y}{dx^{n-1}} + \cdots + P_n y = Q$ where $P_1, P_2, \ldots, P_n$ are constants and Q is a function of x is called Cauchy Euler equation of order n.

We reduce this to linear differential equation with constant coefficients which can be solved.

Put $x = e^z$ so that $z = \log x$, $\dfrac{dz}{dx} = \dfrac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$
$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dx^2} \Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(-\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\right) = \frac{2}{x^3}\frac{dy}{dz} - \frac{1}{x^2}\frac{d^2y}{dz^2}\frac{1}{x} - \frac{2}{x^3}\frac{d^2y}{dx^2} + \frac{1}{x^2}\frac{d^3y}{dz^3}\frac{1}{x}$$
$$= \frac{1}{x^3}\frac{d^3y}{dx^3} - \frac{3}{x^3}\frac{d^2y}{dx^2} + \frac{2}{x^3}\frac{dy}{dz} \Rightarrow x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}$$

Let the differential operator $\dfrac{d}{dz}$ be denoted by θ so that $\dfrac{d}{dz} \equiv \theta$

then $\dfrac{d^2}{dz^2} \equiv \theta^2$, $\dfrac{d^3}{dz^3} \equiv \theta^3$, $\ldots$, $\dfrac{d^n}{dz^n} \equiv \theta^n$

$x\frac{dy}{dx} = xDy = \theta y, \quad x^2\frac{d^2y}{dx^2} = x^2D^2y = \theta^2 y - \theta y = \theta(\theta-1)y$

$x^3\frac{d^3y}{dx^3} = x^3D^3y = \theta^3 y - 3\theta^2 + 2\theta y = \theta(\theta-1)(\theta-2)y$

$\vdots$

$x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} = x^{n-1}D^{n-1}y = (\theta(\theta-1)\cdots(\theta-(n-2)))y$

$x^n\frac{d^ny}{dx^n} = x^nD^ny = [\theta(\theta-1)\cdots(\theta-(n-1))]y$

$xD \equiv \theta,\ x^2D^2 \equiv \theta(\theta-1),\ x^3D^3 \equiv \theta(\theta-1)(\theta-2), \ldots, x^nD^n \equiv \theta(\theta-1)\ldots(\theta-(n-1))$

Substituting the values in Cauchy Euler equation, we get

$\theta(\theta-1)\ldots(\theta-n+1)y + P_1\theta(\theta-1)\ldots(\theta-n+2)y + \cdots + P_ny = Q(e^z)$

$[\theta(\theta-1)\cdots(\theta-n+1) + P_1\theta(\theta-1)\cdots(\theta-n+2) + \cdots + P_n]y = Q(e^z)$

$\Rightarrow f(\theta)y = Z$ where $f(\theta) \equiv \theta(\theta-1)\ldots(\theta-n+1) + P_1\theta(\theta-1)\ldots(\theta-n+2) + \cdots + P_n$ and $Z = Qe^z$

This is a linear differential equation with constant coefficients.

$f(\theta)y = Z$ can be solved by the previous methods

Problem: Solve $3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x$

Solution: given equation is $(3x^2D^2 + xD + 1)y = x$

Put $x = e^z \Rightarrow z = \log x,\ x > 0$

Let $\theta = \frac{d}{dz}$ Then $xD \equiv \theta,\ x^2D^2 = \theta(\theta-1)$

$(3\theta(\theta-1) + \theta + 1)y = e^z \Rightarrow (3\theta^2 - 2\theta + 1)y = e^z$

$f(\theta) = 3\theta^2 - 2\theta + 1$ Then $f(m) = 0 \Rightarrow 3m^2 - 2m + 1 = 0$

$\Rightarrow m = \frac{2 \pm \sqrt{4-12}}{6} = \frac{2 \pm 2\sqrt{2}i}{6} = \frac{1}{3} \pm i\frac{\sqrt{2}}{3}$

$y_c = e^{z/3}\left(c_1\cos\frac{\sqrt{2}}{3}z + c_2\sin\frac{\sqrt{2}}{3}z\right)$

$y_p = \frac{1}{3\theta^2 - 2\theta + 1}e^z = e^z\frac{1}{3(1^2) - 2(1) + 1} = \frac{1}{2}e^z$

The general solution is $y = y_c + y_p = e^{z/3}\left(c_1\cos\frac{\sqrt{2}}{3}z + c_2\sin\frac{\sqrt{2}}{3}z\right) + \frac{e^z}{2}$

$\therefore\ y = x^{1/3}\left(c_1\cos\left(\frac{\sqrt{2}\log x}{3}\right) + c_2\sin\left(\frac{\sqrt{2}\log x}{3}\right)\right) + \frac{x}{2} \quad \left(e^{z/3} = e^{\frac{\log x}{3}} = x^{1/3},\ e^z = e^{\log x} = x\right)$

Problem: Solve $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

Solution: given equation is $(x^3D^3 + 2x^2D^2 + 2)y = 10\left(x + \frac{1}{x}\right)$

Let $x = e^z \Rightarrow z = \log x,\ x > 0$ and $\theta \equiv \frac{d}{dz}$

Then $xD = \theta,\ x^2D^2 = \theta(\theta-1),\ x^3D^3 = \theta(\theta-1)(\theta-2)$

$\Rightarrow [\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 2]y = 10(e^z + e^{-z})$

$\Rightarrow (\theta^3-\theta^2+2)y = 10(e^z+e^{-z})$ where $f(\theta)=\theta^3-\theta^2+2$

Auxiliary equation is $m^3-m^2+2=0 \Rightarrow (m+1)(m^2-2m+2)=0$

$\Rightarrow (m+1)(m^2-2m+2)=0 \Rightarrow m+1=0,\ m^2-2m+2=0$

$\Rightarrow m=-1,\ m=\dfrac{2\pm\sqrt{4-8}}{2}=\dfrac{2\pm 2i}{2}=1\pm i$

$\Rightarrow -1, 1\pm i$ are the roots

$\therefore y_c = c_1e^{-z}+e^z(c_2\cos z+c_3\sin z)$

$y_P = 10\dfrac{1}{\theta^3-\theta^2+2}(e^z+e^{-z}) = 10\left[\dfrac{1}{\theta^3-\theta^2+2}e^z+\dfrac{1}{\theta^3-\theta^2+2}e^{-z}\right]$

$=10\dfrac{e^z}{1-1+2}+10\dfrac{1}{(\theta+1)(\theta^2-2\theta+2)}e^{-z} = 5e^z+\dfrac{10}{1+2+2}\dfrac{1}{\theta+1}e^{-z} = 5e^z+2ze^{-z}$

$\therefore$ the general solution is $y=c_1e^{-z}+e^z(c_2\cos z+c_3\sin z)+5e^z+2ze^{-z}$

$= c_1x^{-1}+x(c_2\cos(\log x)+c_3\sin(\log x))+5x+\frac{2}{x}\log x$

Problem: Solve $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+5y = x^2\sin(\log x)$

Solution: $(x^2D^2-3xD+5)y = x^2\sin(\log x)$

let $x=e^z \Rightarrow z=\log x,\ x>0$ and $\dfrac{d}{dz}\equiv\theta \Rightarrow xD\equiv\theta,\ x^2D^2\equiv\theta(\theta-1)$

$\therefore (\theta(\theta-1)-3\theta+5)y = e^{2z}\sin z \Rightarrow (\theta^2-4\theta+5)y = e^{2z}\sin z$

$f(\theta)=\theta^2-4\theta+5,\ f(m)=0 \Rightarrow m^2-4m+5=0$

$\Rightarrow m=\dfrac{4\pm\sqrt{16-20}}{2}=\dfrac{4\pm 2i}{2}=2\pm i$

$\therefore y_c = e^{2z}(c_1\cos z+c_2\sin z)$

$y_P = \dfrac{1}{\theta^2-4\theta+5}(e^{2z}\sin z) = e^{2z}\dfrac{1}{(\theta+2)^2-4(\theta+2)+5}\sin z$

$= e^{2z}\dfrac{1}{\theta^2+1}\sin z = e^{2z}\left(-\frac{z}{2}\right)\cos z$

$\therefore$ the general solution is $y=e^{2z}(c_1\cos z+c_2\sin z)-\frac{z}{2}e^{2z}\cos z$

i.e., $y = x^2\left[c_1\cos(\log x)+c_2\sin(\log x)\right]-\frac{1}{2}\log x\cdot x^2\cos(\log x)$

Problem:- Solve $x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+y=\dfrac{1}{(1-x)^2}$

Solution: $(x^2D^2+3xD+1)y = \dfrac{1}{(1-x)^2}$

let $x=e^z \Rightarrow z=\log x,\ x>0$ and $\dfrac{d}{dz}\equiv\theta \Rightarrow xD=\theta,\ x^2D^2=\theta(\theta-1)$

$[\theta(\theta-1)+3\theta+1]y = \dfrac{1}{(1-e^z)^2} \Rightarrow (\theta^2+2\theta+1)y = \dfrac{1}{(1-e^z)^2}$

$f(\theta)=\theta^2+2\theta+1,\ f(m)=0 \Rightarrow m^2+2m+1=0 \Rightarrow m=-1,-1$

$\therefore y_c = (c_1+c_2z)e^{-z}$

$y_P = \dfrac{1}{(\theta+1)^2(1-e^z)^2} = \dfrac{1}{\theta+1}\left[\dfrac{1}{\theta+1}\dfrac{1}{(1-e^z)^2}\right] = \dfrac{1}{\theta+1}\left[e^{-z}\int\dfrac{1}{(1-e^z)^2}e^z dz\right]$

Put $e^{-z} = t \Rightarrow e^{-z} dz = dt$

$$\frac{1}{\theta+1} e^{-z} \frac{1}{1-t} = \frac{1}{\theta+1} \frac{e^{-z}}{1-e^{-z}} = e^{-z} \int \frac{e^{-z}}{1-e^{-z}} e^{z} dz$$

$$= e^{-z} \int \frac{dz}{1-e^{-z}} = e^{-z} \int \frac{e^{-z}}{e^{-z}+1} dz = -e^{-z} \log|e^{-z}-1|$$

$$\therefore y = y_c + y_p = (c_1 + c_2 z)e^{-z} - e^{-z} \log(e^{-z}-1)$$

$$= (c_1 + c_2 \log x)\frac{1}{x} - \frac{1}{x} \log\left|\frac{1-x}{x}\right|$$

Problem: Solve $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 2x^2$

Solution: given equation is $(x^2D^2 - 3xD + 4)y = 2x^2$

let $x = e^z$ so that $z = \log x$

$\theta \equiv \frac{d}{dz} \Rightarrow xD \equiv \theta,\ x^2D^2 \equiv \theta(\theta-1)$

$(\theta(\theta-1) - 3\theta + 4)y = 2e^{2z} \Rightarrow (\theta^2 - 4\theta + 4)y = 2e^{2z}$

Auxiliary equation is $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$

$y_c = (c_1 + c_2 z)e^{2z} = (c_1 + c_2 \log x)x^2$

$$y_p = \frac{2e^{2z}}{\theta^2 - 4\theta + 4} = \frac{2z\, e^{2z}}{2\theta - 4} = 2z^2 \frac{e^{2z}}{2} = z^2 e^{2z} = (\log x)^2 x^2$$

The general solution is $y = (c_1 + c_2 \log x)x^2 + x^2(\log x)^2$

Problem: Solve $(x^2D^2 + xD - 1)y = x^3$

Solution: $(x^2D^2 + xD - 1)y = x^3$

let $x = e^z$ so that $z = \log x$

$\theta \equiv \frac{d}{dz}$, $xD \equiv \theta$, $x^2D^2 = \theta(\theta-1)$

$(\theta(\theta-1) + \theta - 1)y = e^{3z} \Rightarrow (\theta^2 - 1)y = e^{3z}$

Auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$y_c = c_1 e^{z} + c_2 e^{-z} = c_1 x + \frac{c_2}{x}$

$$y_p = \frac{e^{3z}}{\theta^2 - 1} = \frac{e^{3z}}{9-1} = \frac{x^3}{8}$$

$\therefore$ the general solution is $y = c_1 x + \frac{c_2}{x} + \frac{x^3}{8}$

Problem: Solve $(x^2D^2 - xD + 2)y = x \log x$

Solution: $(x^2D^2 - xD + 2)y = x \log x$

let $x = e^z \Rightarrow z = \log x$

$\theta \equiv \frac{d}{dz}$ then $xD = \theta$, $x^2D^2 = \theta(\theta-1)$

$(\theta(\theta-1) - \theta + 2)y = z e^{z} \Rightarrow (\theta^2 - 2\theta + 2)y = z e^{z}$

Auxiliary equation is $m^2 - 2m + 2 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$$y_c = e^z(c_1\cos z + c_2 \sin z)$$

$$y_p = \frac{1}{\theta^2 - 2\theta + 2} z e^z = e^z \frac{1}{(\theta+1)^2 - 2(\theta+1)+2} z = e^z \frac{1}{\theta^2+1} z$$

$$= e^z(1+\theta^2)^{-1} z = e^z(1-\theta^2+\cdots)z = z e^z$$

$\therefore$ the general solution is $y = y_c + y_p = e^z(c_1\cos z + c_2 \sin z) + z e^z$

$$= x[c_1\cos(\log x) + c_2\sin(\log x)] + x\log x$$

Problem: Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x\sin(\log x)$

Solution: Put $x = e^z$ so that $z = \log x$

$$\theta = \frac{d}{dz} \Rightarrow xD = \theta, \; x^2D^2 = \theta(\theta-1)$$

$$(\theta(\theta-1) - \theta + 4)y = \cos z + e^z \sin z$$

$$\Rightarrow (\theta^2 - 2\theta + 4)y = \cos z + e^z \sin z$$

Auxiliary equation is $m^2 - 2m + 4 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{3}i$

$$\therefore y_c = e^z(c_1\cos\sqrt{3}z + c_2\sin\sqrt{3}z) = x[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x)]$$

$$y_p = \frac{1}{\theta^2-2\theta+4}(\cos z + e^z\sin z) = \frac{1}{\theta^2-2\theta+4}\cos z + \frac{1}{\theta^2-2\theta+4} e^z \sin z$$

$$\frac{1}{\theta^2-2\theta+4}(\cos z + e^z\sin z) = \frac{1}{\theta^2-2\theta+4}\cos z + \frac{1}{\theta^2-2\theta+4} e^z \sin z$$

$$\frac{1}{\theta^2-2\theta+4}\cos z = \frac{1}{-1-2\theta+4}\cos z = \frac{1}{3-2\theta}\cos z = \frac{3+2\theta}{9-4\theta^2}\cos z$$

$$= \frac{1}{13}(3+2\theta)\cos z = \frac{1}{13}(3\cos z - 2\sin z)$$

$$\frac{1}{\theta^2-2\theta+4} e^z \sin z = e^z \frac{1}{(\theta+1)^2 - 2(\theta+1)+4}\sin z = e^z\frac{1}{\theta^2+3}\sin z = e^z\frac{\sin z}{2}$$

$$y_p = \frac{1}{13}(3\cos z + 2\sin z) + \frac{1}{2}e^z \sin z$$

$$= \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)] + \frac{x}{2}\sin(\log(x))$$

$\therefore$ the general solution is

$$y = x[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x)] + \frac{x}{2}\sin(\log x) + \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)]$$

Problem: Solve $(x^2D^2 + xD - 1)y = x^2e^x$

Solution: given equation is $(x^2D^2 + xD - 1)y = x^2e^x$

$$\text{let } x = e^z \Rightarrow z = \log x$$

$$\frac{d}{dz} \equiv \theta \Rightarrow xD \equiv \theta, \; x^2D^2 = \theta(\theta-1)$$

$$[\theta(\theta-1) + \theta - 1]y = e^{2z}\exp(e^z) \Rightarrow (\theta^2-1)y = e^{2z}\exp(e^z)$$

Auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$y_c = c_1e^z + c_2e^{-z}$$

$$y_p = \frac{1}{\theta^2-1}e^{2z}\exp(e^z) = \frac{1}{2}\left[\frac{1}{\theta-1} - \frac{1}{\theta+1}\right]e^{2z}\exp(e^z)$$

$$= \frac{1}{2}\frac{1}{\theta-1}e^{2z}\exp(e^z) - \frac{1}{2}\frac{1}{\theta+1}e^{2z}\exp(e^z)$$

$\frac{1}{\theta-1} e^{2z}\exp(e^z) = e^{z}\int e^{2z}\exp(e^z)e^{-z}dz = e^{z}\int e^{z}\exp(e^z)\,dz$

Put $e^z = t \Rightarrow e^z dz = dt$

$e^z\int e^t dt = e^z\exp(e^z)$

$\frac{1}{\theta+1} e^{2z}\exp(e^z) = e^{-z}\int e^{z} e^{2z}\exp(e^z)dz$

$= e^{-z}\int e^{3z}\exp(e^z)dz$ Put $e^z = t \Rightarrow e^z dz = dt$

$= e^{-z}\int t^2e^t dt = e^{-z}\left[t^2e^t - \int 2te^t dt\right] = e^{-z}\left[t^2e^t - 2(te^t - \int e^t dt)\right]$

$= e^{-z}\left[e^{2z}\exp(e^z) - 2e^z\exp(e^z) + 2\exp(e^z)\right] = (e^z + 2e^{-z} - 2)\exp(e^z)$

$\therefore y_p = \frac{1}{2}e^z\exp(e^z) - \frac{1}{2}(e^z + 2e^{-z} - 2)\exp(e^z) = (1 - e^{-z})\exp(e^z)$

The general solution is $y = y_c + y_p = c_1e^z + c_2e^{-z} + (1 - e^{-z})\exp(e^z)$

i.e., $y = c_1 x + c_2 x^{-1} + (1 - \frac{1}{x})e^x$

Problem: Solve $(x^4D^3 + 2x^3D^2 - x^2D + x)y = 1$

Solution: $(x^3D^3 + 2x^2D^2 - xD + 1)y = 1/x$

Let $x = e^z$, then $z = \log x$

$\theta \equiv \frac{d}{dz} \Rightarrow xD = \theta,\ x^2D^2 = \theta(\theta-1),\ x^3D^3 = \theta(\theta-1)(\theta-2)$

$(\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) - \theta + 1)y = \frac{1}{e^z}$

$\Rightarrow (\theta^3 - \theta^2 - \theta + 1)y = e^{-z}$

Auxiliary equation is $m^3 - m^2 - m + 1 = 0 \Rightarrow (m-1)(m^2-1) = 0$

$\Rightarrow m = 1, \pm 1$

$\therefore y_c = (c_1 + c_2 z)e^z + c_3e^{-z} = (c_1 + c_2\log x)x + \frac{c_3}{x}$

$y_p = \frac{e^{-z}}{\theta^3 - \theta^2 - \theta + 1} \Rightarrow \frac{ze^{-z}}{3\theta^2 - 2\theta - 1} = \frac{ze^{-z}}{3+2-1} = \frac{ze^{-z}}{4} = \frac{(\log x)\frac{1}{x}}{4}$

$\therefore$ the general solution is $y = (c_1 + c_2\log x)x + \frac{c_3}{x} + \frac{\log x}{4x}$

Legender's Equation:

An equation of the form $(ax+b)^n\frac{d^ny}{dx^n} + P_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_ny = Q$

where $P_1, P_2, \ldots P_n$ are constants and Q is a function of x

is called Legender's linear equation.

Put $ax+b = e^z \Rightarrow z = \log(ax+b) \Rightarrow \frac{dz}{dx} = \frac{a}{ax+b}$

$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{a}{ax+b}\frac{dy}{dz} \Rightarrow (ax+b)\frac{dy}{dx} = a\frac{dy}{dz} = a\theta y$, where $\theta = \frac{d}{dz}$

$\Rightarrow (ax+b)Dy = a\theta y$.

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{a}{ax+b}\frac{dy}{dz}\right] = \frac{dy}{dz}\,a\frac{d}{dx}\left(\frac{1}{ax+b}\right) + \frac{a}{ax+b}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$

$= \frac{-a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a^2}{(ax+b)^2}\frac{d^2y}{dz^2} = \frac{a^2}{(ax+b)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$

$\Rightarrow (ax+b)^2 \frac{d^2y}{dx^2} = a^2(\theta^2 y - \theta y) = a^2\theta(\theta-1)y \Rightarrow (ax+b)^2 D^2 y = a^2\theta(\theta-1)y$

Similarly $(ax+b)^3 \frac{d^3y}{dx^3} = a^3\theta(\theta-1)(\theta-2)y$

$\Rightarrow (ax+b)^3 D^3 y = a^3\theta(\theta-1)(\theta-2)y$ and so on.

Problem: Solve $[(1+x)^2D^2 + (1+x)D + 1]y = 4\cos\log(1+x)$

Solution: $(1+x)^2D^2y + (1+x)Dy + y = 4\cos\log(1+x)$

Let $1+x = e^z \Rightarrow z = \log(1+x)$

$\frac{d}{dz} = \theta \Rightarrow (x+1)D = \theta$, $(x+1)^2D^2 = \theta(\theta-1)$

$\theta(\theta-1)y + \theta y + y = 4\cos z \Rightarrow (\theta^2+1)y = 4\cos z$

$f(\theta) = \theta^2+1$, $f(m)=0 \Rightarrow m^2+1=0 \Rightarrow m = -i, i$

$y_c = c_1\cos z + c_2\sin z = c_1 \cancel{\cos} \log(1+x) + c_2\sin\log(1+x)$

$y_p = 4\frac{1}{\theta^2+1}\cos z = 4\frac{z}{2}\sin z = 2z\sin z = 2\log(1+x)\sin\log(1+x)$

$\therefore y = y_c + y_p \Rightarrow y = c_1\cos z + c_2\sin z + 2z\sin z$

$\Rightarrow y = c_1\cos(\log(1+x)) + c_2\sin(\log(1+x)) + 2[\log(1+x)]\sin(\log(1+x))$

Problem: Solve $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1$

Solution: $[(3x+2)^2D^2 + 3(3x+2)D - 36]y = 3x^2+4x+1$

Let $3x+2 = e^z \Rightarrow z = \log(3x+2)$

$\frac{d}{dz} = \theta \Rightarrow x = \frac{e^z-2}{3}$

$(3x+2)D = 3\theta$ and $(3x+2)^2D^2 = 3^2\theta(\theta-1)$

$(3^2\theta(\theta-1) + 3(3\theta) - 36)y = \frac{1}{3}(e^{2z}+4-4e^z+4e^z-8+3)$

$\Rightarrow (\theta^2-4)y = \frac{1}{27}(e^{2z}-1)$ where $f(\theta) = \theta^2-4$.

$f(m) = 0 \Rightarrow m^2-4=0 \Rightarrow m = -2, 2$

$\therefore y_c = c_1e^{2z} + c_2e^{-2z} = c_1(3x+2)^2 + c_2[1/(3x+2)^2]$

$y_p = \frac{1}{27}\frac{1}{\theta^2-4}(e^{2z}-1) = \frac{1}{27}\left[\frac{1}{\theta^2-4}e^{2z} - \frac{1}{\theta^2-4}e^{0z}\right]$

$= \frac{1}{27}\left[\frac{1}{2+2}\frac{1}{\theta-2}e^{2z} - \frac{1}{0-4}\right] = \frac{1}{27}\left[\frac{1}{4}ze^{2z} + \frac{1}{4}\right]$

$= \frac{1}{108}(ze^{2z}+1) = \frac{1}{108}[(3x+2)^2\log(3x+2)+1]$

$\therefore y = y_c + y_p = c_1e^{2z} + c_2e^{-2z} + \frac{1}{108}(ze^{2z}+1)$

$= c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2\log(3x+2)+1]$

Problem : Solve $[(1+2x)^2D^2 - 6(1+2x)D + 16]y = 8(1+2x)^2$

Solution : Let $1+2x = e^z \Rightarrow z = \log(1+2x)$

$\frac{d}{dz} = \theta \Rightarrow x = \frac{e^z - 1}{2}$

$(1+2x)D = 2\theta$ and $(1+2x)^2D^2 = 2^2\theta(\theta-1)$

$(2^2\theta(\theta-1) - 6(2\theta) + 16)y = 8e^{2z}$

$\Rightarrow (4\theta^2 - 16\theta + 16)y = 8e^{2z} \Rightarrow (\theta^2 - 4\theta + 4)y = 2e^{2z}$

$f(\theta) = \theta^2 - 4\theta + 4$, $f(m) = 0 \Rightarrow m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$

$\therefore y_c = (c_1 + c_2 z)e^{2z} = [c_1 + c_2\log(1+2x)](1+2x)^2$

$y_p = 2\frac{1}{(\theta-2)^2}e^{2z} = 2\frac{z^2}{2!}e^{2z} = [\log(1+2x)]^2(1+2x)^2$

$\therefore$ the general solution is $y = y_c + y_p = (c_1 + c_2 z)e^{2z} + z^2e^{2z}$

$= [c_1 + c_2\log(1+2x)][(1+2x)^2 + [\log(1+2x)]^2(1+2x)^2$