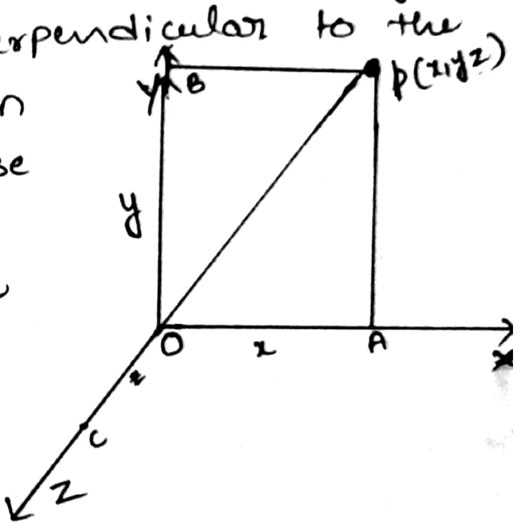
# UNIT - I

## Coordinates [Additional Input]

**Coordinates of a point in space:**

Let O be any point in space S. Let $XOX'$, $ZOZ'$ be two perpendicular straight lines determining the $XOZ$ plane. Through O draw the line $YOY'$ perpendicular to the $XOZ$ plane. So that we have the mutually perpendicular straight lines $XOX'$, $YOY'$, $ZOZ'$ known as rectangular coordinate axes. The three axes taken in pairs determine the planes $XOY$, $YOZ$, $ZOX$ or, breifly $XY$, $YZ$, $ZX$ planes mutually at right angles known as coordinate planes. Through any point P in space draw the three lines perpendicular to the coordinate axes which meet the axes in A, B, C. Let $OA = x$, $OB = y$, $OC = z$. These three numbers $x, y, z$, taken in this order determined by the point P are called the coordinates of the point P.




* Any point on the X-axis is $(a, 0, 0)$
* Any point on the Y-axis is $(0, b, 0)$
* Any point on the Z-axis is $(0, 0, c)$
* Any point on the XY plane is $(a, b, 0)$
  YZ plane is $(0, b, c)$
  ZX plane is $(a, 0, c)$

* The equations of X-axis are $y = 0$, $z = 0$
  Y-axis are $z = 0$, $x = 0$
  Z-axis are $x = 0$, $y = 0$

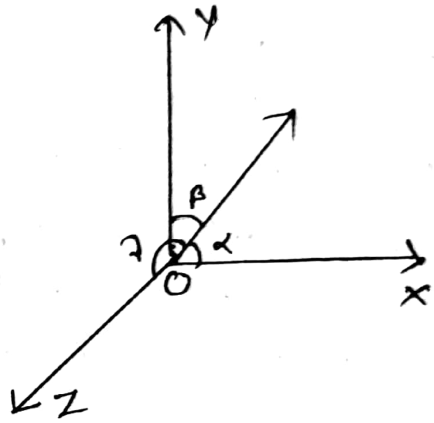
* The equations of XY-plane is $z = 0$
  YZ-plane is $x = 0$
  ZX-plane is $y = 0$

*. **3D-Space:** There is a (one-one, onto) one-to-one ~~coor~~ correspondence between the set of all ordered triads in $\mathbb{R}^3$ and the set of all points in a space then the space is called the 3D-space.

*. **Direction cosines of a line:**

If a ray makes angles $\alpha, \beta, \gamma$ with the coordinate axes in the positive direction then $\cos\alpha, \cos\beta, \cos\gamma$ are called direction cosines of that ray and are generally denoted by $l, m, n$ i.e $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

If $l, m, n$ are d.cs. of a line. Then $-l, -m, -n$ are also d.cs. of that line.




*. **D.cs of Coordinate axes:**

Since x-axis makes $0°, 90°, 90°$ with the coordinate axes.

The D.cs of x-axis are $1, 0, 0$

similarly The D.cs of y-axis are $0, 1, 0$

The D.cs of ~~Z~~ z-axis are $0, 0, 1$

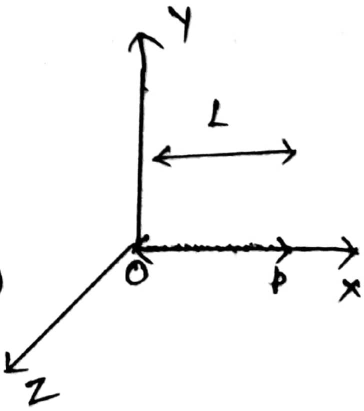
*. **Direction ratios of a Line:**

Suppose $L$ is a given line and $P$ is a point such that $\overleftrightarrow{OP} \parallel L$. Then the coordinates of any point on $\overleftrightarrow{OP}$ are called direction ratios of $L$. A line has infinitely many direction ratios and are proportional to d.cs.

If $a, b, c$ are D.rs of a line.

Then $\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$

are D.cs. of that line.




**Note:** If $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$ are two points. Then d.rs of $\overleftrightarrow{AB}$ are $x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1$.

## *. Distance Between Two points :

The Distance between two points $A=(x_1,y_1,z_1)$ and $B=(x_2,y_2,z_2)$ is $AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

**Problem :** (i). Find the distance between the points $(-1,2,-3)$ and $(5,4,-6)$.

(ii). Find the distance from the origion to the point $(2,-1,3)$.

**Sol :** (i). let $A=(-1,2,-3)$, $B=(5,4,-6)$

The distance between AB is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$$= \sqrt{(5+1)^2+(4-2)^2+(-6+3)^2}$$

$$= \sqrt{(6)^2+(2)^2+(-3)^2}$$

$$= \sqrt{36+4+9} = \sqrt{49} = 7.$$

(ii). let $P=(2,-1,3)$ and $O=(0,0,0)$.

$$OP = \sqrt{(2)^2+(-1)^2+(3)^2} = \sqrt{4+1+9} = \sqrt{14}$$

## *. Section Formula :

The point which divides the line segment joining the points $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$ in the ratio $l:m$.

(i). Internally is $\left(\frac{lx_2+mx_1}{l+m}, \frac{ly_2+my_1}{l+m}, \frac{lz_2+mz_1}{l+m}\right)$ where $l+m\neq 0$

(ii). Externally is $\left(\frac{lx_2-mx_1}{l-m}, \frac{ly_2-my_1}{l-m}, \frac{lz_2-mz_1}{l-m}\right)$ where $l\neq m$

**Problem :** Find the points which divides the line segment joining the points $(3,-2,4)$, $(-1,4,-2)$ in the ratio $3:2$ internally and Externally.

**Sol :** The point which divides $\overline{AB}$ in the ratio $3:2$ internally is

$$\left(\frac{3(-1)+2(3)}{3+2}, \frac{3(4)+2(-2)}{3+2}, \frac{3(-2)+2(4)}{3+2}\right)$$

$$= \left(\frac{-3+6}{5}, \frac{12-4}{5}, \frac{-6+8}{5}\right) = \left(\frac{3}{5}, \frac{8}{5}, \frac{2}{5}\right)$$

The point which divides $\overline{AB}$ in the ratio $3:2$ Externally is

$$\left(\frac{3(-1)-2(3)}{3-2}, \frac{3(4)-2(-2)}{3-2}, \frac{3(-2)-2(4)}{3-2}\right)$$

$$= \left(\frac{-3-6}{1}, \frac{12+4}{1}, \frac{-6-8}{1}\right) = (-9,16,-14)$$

## *. Angle between two lines :

If $l_1, m_1, n_1$ and $l_2, m_2, n_2$ are d.cs of two lines $L_1$ and $L_2$ then an angle $\theta$ between them is given by $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$.

If $(l_1, m_1, n_1)$ are d.rs of $L_1$ and $(l_2, m_2, n_2)$ are d.rs of $L_2$ then $\cos\theta = \dfrac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$

**Problem :** Calculate the cosine of the angle $\theta$ of the triangle with vertices $A(1,-1,2)$, $B(6,11,2)$, $C(1,2,6)$.

**Sol :** Given that $A = (1,-1,2)$, $B = (6,11,12)$ and $C = (1,2,6)$

D.rs of $\vec{AB}$ is $(6-1,\ 11+1,\ 2-2) = (5,12,0)$

D.cs of $\vec{AB}$ is $\left(\dfrac{5}{\sqrt{5^2+12^2+0^2}}, \dfrac{12}{\sqrt{5^2+12^2+0^2}}, \dfrac{0}{\sqrt{5^2+12^2+0^2}}\right) = \left(\dfrac{5}{13}, \dfrac{12}{13}, 0\right)$

D.rs of $\vec{AC}$ is $(1-1,\ 2+1,\ 6-2) = (0,3,4)$.

D.cs of $\vec{AC}$ is $\left(\dfrac{0}{\sqrt{0^2+3^2+4^2}}, \dfrac{3}{\sqrt{0^2+3^2+4^2}}, \dfrac{4}{\sqrt{0^2+3^2+4^2}}\right) = \left(0, \dfrac{3}{5}, \dfrac{4}{5}\right)$

Therefore $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

$$= \frac{5}{13}\cdot 0 + \frac{12}{13}\cdot\frac{3}{5} + \frac{4}{5}\cdot 6 = \frac{36}{65}$$

$$\Rightarrow \cos\theta = \frac{36}{65}$$

**Problem :** Find the angle between the lines whose d.rs are $(1,-1,0)$ and $(1,-2,1)$.

**Sol :** Given that the D.rs are $(1,-1,0)$ and $(1,-2,1)$.

If $\theta$ is an angle between the lines then

$$\cos\theta = \frac{1\cdot 1 + (-1)(-2) + 0(1)}{\sqrt{1^2+(-1)^2+0^2}\sqrt{1^2+(-2)^2+1^2}} = \frac{1+2+0}{\sqrt{1+1}\sqrt{1+4+1}} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \cos^{-1}(\sqrt{3}/2)$$

# THE PLANE

***. Definition :** A plane is a surface such that if any two points are taken on it, the line joining them lies wholly on the surface.

**Theorem :** Every equation of the first degree in $x, y, z$ represents a plane.

**Proof:** Let $ax+by+cz+d=0$, $a^2+b^2+c^2 \neq 0 \rightarrow (1)$ be the first degree equation in $x, y, z$.

To show that (1) represents the equation to the plane.

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points on (1)

Then $ax_1+by_1+cz_1+d=0$, $ax_2+by_2+cz_2+d=0 \rightarrow (2)$

Let $P$ be any point which divides $\overleftrightarrow{AB}$ in the ratio $m:n$

Then $P = \left(\dfrac{mx_2+nx_1}{m+n}, \dfrac{my_2+ny_1}{m+n}, \dfrac{mz_2+nz_1}{m+n}\right)$, $m \neq n$

$$\text{Now } ax+by+cz+d = a\left(\frac{mx_2+nx_1}{m+n}\right)+b\left(\frac{my_2+ny_1}{m+n}\right)+c\left(\frac{mz_2+nz_1}{m+n}\right)+d$$

$$= \frac{(amx_2+bmy_2+cmz_2)+(anx_1+bny_1+cnz_1)+dm+dn}{m+n}$$

$$= \frac{m(ax_2+by_2+cz_2)+n(ax_1+by_1+cz_1)+dm+dn}{m+n}$$

$$= \frac{m(ax_2+by_2+cz_2+d)+n(ax_1+by_1+cz_1+d)}{m+n}$$

$$= 0$$

Therefore $P$ lines on (1). Since $P$ is an arbitrary point every point on $AB$ lies on (1).

Therefore the equation $ax+by+cz+d=0$, $a^2+b^2+c^2 \neq 0$ always represents a plane.

***. General form of a plane :**

The general form of a plane is $ax+by+cz+d=0$ where $a, b$ and $c$ are d.r's of the Normal to the plane

* **Normal form of a plane:**

The Normal form of the plane $ax+by+cz+d=0$ is $lx+my+nz=p$ where $l, m, n$ are d.c's of the normal to the plane.

$$p = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

* **Transformation to the Normal form:**

Transformation to the normal form of a plane in general form

Consider the equation of a plane in general form $ax+by+cz+d=0$ where $a^2+b^2+c^2 \neq 0$. Then the equation of the plane in the normal form is

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x + \frac{b}{\sqrt{a^2+b^2+c^2}}y + \frac{c}{\sqrt{a^2+b^2+c^2}}z = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

i.e $lx+my+nz = p$ where $l = \frac{a}{\sqrt{a^2+b^2+c^2}}$, $m = \frac{b}{\sqrt{a^2+b^2+c^2}}$

$n = \frac{c}{\sqrt{a^2+b^2+c^2}}$, $p = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$

**Problem:** Find the d.r's of the Normal form to the plane $2x-2y+z=5$. Express the equation into Normal form.

**Sol:** Given plane is $2x-2y+z=5 \rightarrow (1)$

(i). clearly the d.r's of Normal to (1) are $2, -2, 1$

(ii). Normal form: Dividing equation (1) with $\sqrt{a^2+b^2+c^2} = \sqrt{4+4+1} = 3$ on both sides.

We get $\left(\frac{2}{3}\right)x + \left(\frac{-2}{3}\right)y + \left(\frac{1}{3}\right)z = \frac{5}{3} \Rightarrow lx+my+nz = p.$

* **Angle Between Two Planes:**

Definition: Angles between two planes are equal to the angles between their normals.

Angles between the planes $a_1x+b_1y+c_1z=d_1$, $a_2x+b_2y+c_2z=d_2$

Let the equation to the planes be

$a_1x+b_1y+c_1z+d_1=0 \rightarrow (1)$, $a_2x+b_2y+c_2z+d_2=0 \rightarrow (2)$

D.c's of the normal to (1) $= m_1 = \left(\frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}\right)$

D.c's of the normal to (2) $= m_2 = \left(\frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}}, \frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}}, \frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}}\right)$

Let $\theta$ be one of the angles between the planes.

$\therefore$ $\theta$ = one of the angles between the normals $m_1, m_2$

$$= \cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

The other angle between the planes is $180° - \theta$

**Problem:** Find the angle between the planes $2x+6y+6z=9$ and $3x+4y-5z=9$.

**Sol:** Given planes are $2x+6y+6z-9=0 \rightarrow (1)$, $3x+4y-5z-9=0 \rightarrow (2)$

Here $a_1=2,\ b_1=6,\ c_1=6,\ a_2=3,\ b_2=4,\ c_2=-5$

$$\cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{|2\cdot 3+6\cdot 4+6\cdot(-5)|}{\sqrt{4+36+36}\sqrt{9+16+25}} = \frac{|30-30|}{\sqrt{76}\sqrt{50}}$$

$\Rightarrow \cos\theta = 0 = \cos 90^\circ$

$\Rightarrow \theta = \pi/2$

**Problem:** Find the angle between the planes $2x-y+z=5$ and $x+y+2z=7$.

**Sol:** Given planes are $2x-y+z-6=0$, $x+y+2z-7=0$.

Here $a_1=2,\ b_1=-1,\ c_1=1$ and $a_2=1,\ b_2=1,\ c_2=2$

$$\cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{|2-1+2|}{\sqrt{4+1+4}\sqrt{1+4+1}} = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2}$$

$\Rightarrow \cos\theta = \frac{1}{2} = \cos 60^\circ \Rightarrow \theta = 60^\circ \Rightarrow \theta = \pi/3$.

## *. Condition of parallelism:

~~Para~~ planes are parallel $\Rightarrow \theta = 0^\circ$ or $180^\circ$

$$\Rightarrow \pm 1 = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$\Rightarrow (a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2) = (a_1a_2+b_1b_2+c_1c_2)^2$

$\Rightarrow a_1^2b_2^2+a_1^2c_2^2+b_1^2a_2^2+b_1^2c_2^2+c_1^2a_2^2+c_1^2b_2^2-2a_1a_2b_1b_2-2b_1b_2c_1c_2-2c_1c_2a_1a_2=0$

$\Rightarrow (a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2=0$

$\Rightarrow a_1b_2-a_2b_1 = b_1c_2-b_2c_1 = c_1a_2-c_2a_1 = 0$

$\Rightarrow a_1:a_2 = b_1:b_2 = c_1:c_2$

OR: planes are parallel $\Rightarrow$ Their normals are parallel

$\Rightarrow$ d.r's of normals are proportional

$\Rightarrow a_1:a_2 = b_1:b_2 = c_1:c_2$

## *. Condition of perpendicularity:

planes are perpendicular $\Rightarrow \theta = 90^\circ \Rightarrow a_1a_2+b_1b_2+c_1c_2=0$

**Example:** The plane $x+2y-3z+4=0$ is perpendicular to the plane $2x+5y+4z+1=0$ since $(1)(2)+(2)(5)+(-3)(4)=0$.

**Note:** 1. The equations $a_1x+b_1y+c_1z+d_1=0$, $a_1x+b_1y+c_1z+d_2=0$ represent a pair of parallel planes.

2. A plane parallel to $ax+by+cz+d=0$ is $ax+by+cz+d=k$, where $k$ is an unknown real number.

Definition: If a plane $\pi$ intersects the cordinate axes at $(a,0,0)$ $(0,b,0)$, $(0,0,c)$ then $a,b,c$ are Respectively the x-intercept, the y-Intercept, the z-Intercept of the plane $\pi$.

If the plane $lx+my+nz=p$ intersects the x-axis at $(a,0,0)$ then its x-Intercept $= a = \frac{p}{l}$.

similarly its y-intercept $= b = \frac{p}{m}$, its z-Intercept $= c = \frac{p}{n}$

Note: Equation to the plane making intercepts $a,b,c$ on the coordinate axes is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Problem: Reduce the plane $6x-2y-3z+7=0$ into Intercept form.

Sol: Given plane is $6x-2y-3z+7=0 \rightarrow (1)$

$$\Rightarrow 6x-2y-3z=-7$$

$$\Rightarrow \frac{6}{-7}x+\frac{2}{7}y+\frac{3}{7}z=\frac{-7}{-7}$$

$$\Rightarrow \frac{-6}{7}x+\frac{2}{7}y+\frac{3}{7}z=1$$

$$\Rightarrow \frac{x}{(-7/6)}+\frac{y}{(7/2)}+\frac{z}{(7/3)}=1 \text{ is in the form } \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Therefore x-Intercept $(a) = -7/6$, y-Intercept $(b) = 7/2$
z-intercept $(c) = 7/3$

Note: 1. The distance of $A(x_1,y_1,z_1)$ from the plane $ax+by+cz+d=0$. i.e. length of the perpendicular from the point $A(x_1,y_1,z_1)$ to the plane $ax+by+cz+d=0$ is

$$\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}.$$

2. Equation to the plane through the point $A(x_1,y_1,z_1)$ and Perpendicular to line with d.rs $(a,b,c)$ is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

3. Equation to the plane through $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$ and $C(x_3,y_3,z_3)$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

4. The four points $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$, $C(x_3,y_3,z_3)$ and $d(x_4,y_4,z_4)$ are coplanar if

$$\frac{1}{6}\begin{vmatrix} x_1-x_4 & y_1-y_4 & z_1-z_4 \\ x_2-x_4 & y_2-y_4 & z_2-z_4 \\ x_3-x_4 & y_3-y_4 & z_3-z_4 \end{vmatrix}=0$$

5. Equation to the plane through $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ and perpendicular to the plane $ax+by+cz+d=0$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix} = 0$$

6. Equation to the plane through $A(x_1,y_1,z_1)$ and perpendicular to the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

7. The distance between parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is $\dfrac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}}$

**Problem:** Find the equation of the plane through $(2,-3,1)$ and whose normal is the line joining the points $(3,4,1)$ and $(2,-1,5)$

**Sol:** Given that $A=(2,-3,1)$.

Let $B=(3,4,1)$ and $C=(2,-1,5)$.

D.rs of the normal $\overline{BC}$ of the required plane is

$(x_2-x_1,\ y_2-y_1,\ z_2-z_1) = (2-3,\ -1-4,\ 5+1) = (-1,-5,6)$

Equation to the required plane is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

$\Rightarrow -1(x-2)-5(y+3)+6(z-1)=0$

$\Rightarrow -x+2-5y-15+6z-6=0$

$\Rightarrow -x-5y+6z-19=0 \Rightarrow x+5y-6z+19=0.$

**Problem:** Find the equation of the plane through the points $(2,2,-1)$ and $(3,4,2)$, $(7,0,6)$.

**Sol:** Let $A(x_1,y_1,z_1)=(2,2,-1)$, $B(x_2,y_2,z_2)=(3,4,2)$ and $C(x_3,y_3,z_3)=(7,0,6)$

Equation to the plane through A, B, C is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z+1 \\ 3-2 & 4-2 & 2+1 \\ 7-2 & 0-2 & 6+1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z+1 \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix} = 0$$

$\Rightarrow (x-2)(14+6)-(y-2)(7-15)+(z+1)(-2-10)=0$

$\Rightarrow (x-2)(20)-(y-2)(-8)+(z+1)(-12)=0$

$\Rightarrow 20x-40+8y-16-12z-12=0$

$\Rightarrow 20x+8y-12z-68=0 \Rightarrow 5x+2y-3z-17=0$

**Problem:** Show that the following four points $(-6,3,2)$, $(3,-2,4)$, $(5,7,3)$ and $(-13,17,-1)$ are coplanar.

**Sol:** Let $A=(-6,3,2)$, $B=(3,-2,4)$, $C=(5,7,3)$, $D=(-13,17,-1)$

Equation to the plane through $A,B,C$ is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$

$$\Rightarrow \begin{vmatrix} x+6 & y-3 & z-2 \\ 3+6 & -2-3 & 4-2 \\ 5+6 & 7-3 & 3-2 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x+6 & y-3 & z-2 \\ 9 & -5 & 2 \\ 11 & 4 & 1 \end{vmatrix}=0$$

$$\Rightarrow (x+6)(-5-8)-(y-3)(9-22)+(z-2)(36+55)=0$$

$$\Rightarrow (x+6)(-13)-(y-3)(-13)+(z-2)(91)=0$$

$$\Rightarrow -13x-78+13y-39+91z-182=0$$

$$\Rightarrow -13x+13y+91z-299=0 \rightarrow ①$$

Put $D=(-13,17,-1)$ in ①. $\Rightarrow -13(-13)+13(17)+91(-1)-299$

$= 169+221-91-299$

$= \cancel{299-299}\ 390-390$

$= 0$

Therefore given four points are coplanar.

**Problem:** Find the equation of the plane through the points $(0,4,3)$, $(-1,-5,-3)$, $(-2,-2,1)$ and Hence show that the four points $(0,4,3)$, $(-1,-5,-3)$, $(-2,-2,1)$, $(1,1,-1)$ are coplanar.

**Sol:** Let $A=(0,4,3)$, $B=(-1,-5,-3)$, $C=(-2,-2,1)$, $D=(1,1,-1)$.

Equation to the plane through $A,B,C$ is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$

$$\Rightarrow \begin{vmatrix} x-0 & y-4 & z-3 \\ -1-0 & -5-4 & -3-3 \\ -2-0 & -2-4 & 1-3 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x & y-4 & z-3 \\ -1 & -9 & -6 \\ -2 & -6 & -2 \end{vmatrix}=0$$

$$\Rightarrow x(18-36)-(y-4)(2-12)+(z-3)(6-18)=0$$

$$\Rightarrow -18x+10y-40-12z+36=0$$

$$\Rightarrow 9x-5y+6z+2=0 \rightarrow ①$$

Put $D=(1,1,-1)$ in ① $\Rightarrow 9(1)-5(1)+6(-1)+2 = 9-5-6+2 = 11-11=0$

Therefore given four points are coplanar.

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**Problem:** Find the equation of the plane through the points $(2,2,1)$, $(9,3,6)$ and perpendicular to the plane $2x+6y+6z=9$

**Sol:** Let $A=(2,2,1)$, $B=(9,3,6)$

Given plane is $2x+6y+6z=9$.

The D.rs of the normal to the plane are $2,6,6$

Equation to the required plane is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix}=0$

$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 9-2 & 3-2 & 6-1 \\ 2 & 6 & 6 \end{vmatrix}=0$

$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix}=0$

$\Rightarrow (x-2)(6-30)-(y-2)(42-10)+(z-1)(42-2)=0$

$\Rightarrow (x-2)(-24)-(y-2)(32)+(z-1)(40)=0$

$\Rightarrow -24x+48-32y+64+40z-40=0$

$\Rightarrow -24x-32y+40z+72=0 \Rightarrow 3x+4y-5z-9=0$

**Problem:** Show that the equation of the plane passing through the points $(1,-2,4)$, $(3,-4,5)$ and perpendicular to xy-plane is $x+y+1=0$.

**Sol:** Let $A=(1,-2,4)$, $B=(3,-4,5)$.

Given plane is xy-plane $\Rightarrow z=0$

The D.rs of normal to the xy-plane are $0,0,1$

The equation to the required plane is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix}=0$

$\Rightarrow \begin{vmatrix} x-1 & y+2 & z-4 \\ 3-1 & -4+2 & 5-4 \\ 0 & 0 & 1 \end{vmatrix}=0$

$\Rightarrow \begin{vmatrix} x-1 & y+2 & z-4 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{vmatrix}=0$

$\Rightarrow (x-1)(-2-0)-(y+2)(2-0)+(z-4)(0+0)=0$

$\Rightarrow -2x+2-2y-4=0$

$\Rightarrow -2x-2y-2=0$

$\Rightarrow 2x+2y+2=0$

$\Rightarrow x+y+1=0.$

**Problem:** Find the equation of the plane through $(4,4,0)$ and perpendicular to the planes $x+2y+2z=5$ and $3x+3y+2z=8$

**Sol** let $A = (4,4,0)$

Given planes are $x+2y+2z=5$ and $3x+3y+2z=8$

The D.rs of normal to the planes are $a_1, b_1, c_1$ is $1,2,2$ and $a_2, b_2, c_2$ is $3,3,2$.

The equation to the required plane is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x-4 & y-4 & z-0 \\ 1 & 2 & 2 \\ 3 & 3 & 2 \end{vmatrix} = 0$$

$\Rightarrow (x-4)(4-6) - (y-4)(2-6) + (z-0)(3-6) = 0$

$\Rightarrow (x-4)(-2) - (y-4)(-4) + (z-0)(-3) = 0$

$\Rightarrow -2x+8+4y-16-3z = 0$

$\Rightarrow -2x+4y-3z-8 = 0 \Rightarrow 2x-4y+3z+8=0.$

**Problem:** Find the equation of the plane through the point $(-1,3,2)$ and perpendicular to the planes $x+2y+2z=5$ and $3x+3y+2z=8$.

**Sol:** let $A = (-1,3,2)$

Given planes are $x+2y+2z=5$ and $3x+3y+2z=8$

The D.rs of normal to the planes are $a_1, b_1, c_1$ is $1,2,2$ and $a_2, b_2, c_2$ is $3,3,2$.

The equation to the required plane is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x+1 & y-3 & z-2 \\ 1 & 2 & 2 \\ 3 & 3 & 2 \end{vmatrix} = 0$$

$\Rightarrow (x+1)(4-6) - (y-3)(2-6) + (z-2)(3-6) = 0$

$\Rightarrow (x+1)(-2) - (y-3)(-4) + (z-2)(-3) = 0$

$\Rightarrow -2x-2+4y-12-3z+6 = 0$

$\Rightarrow -2x+4y-3z+8 = 0$

$\Rightarrow 2x-4y+3z+8 = 0.$

**Problem:** Find the distance between the parallel planes $2x-2y+z+3=0$ and $4x-4y+2z+5=0$.

**Sol:** Given planes are $2x-2y+z+3=0 \rightarrow (1)$, $4x-4y+2z+5=0 \rightarrow (2)$

Now $(2) \times (1) \rightarrow 4x-4y+2z+6=0 \rightarrow (3)$

The distance between (2) and (3) is $\dfrac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}} = \dfrac{|6-5|}{\sqrt{16+16+4}} = \dfrac{1}{\sqrt{36}} = \dfrac{1}{6}$

**Problem:** Two systems of rectangular axes have the same origin. If a plane intersects them at distances, $a, b, c$ and $a_1, b_1, c_1$ respectively from the origin then prove that $a^{-2}+b^{-2}+c^{-2} = a_1^{-2}+b_1^{-2}+c_1^{-2}$

**Sol:** Let $OXYZ$ and $OX'Y'Z'$ be two systems of rectangular axes have the same origin.

Let $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1 \rightarrow (1)$ be the plane intersects the axes $OX, OY, OZ$ at $a, b, c$. Respetively from the origin.

Let $\frac{x}{a_1}+\frac{y}{b_1}+\frac{z}{c_1} = 1 \rightarrow (2)$ be the plane intersects the axes $OX', OY', OZ'$ at $a_1, b_1, c_1$ respectively from the origin.

Since the distance from $(0,0,0)$ to (1) is equal to the distance from $(0,0,0)$ to (2).

We know that the distance from origin to a plane is

$$\frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

So,

$$\frac{|-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = \frac{|-1|}{\sqrt{\frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2}}}$$

$$\Rightarrow \sqrt{\frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2}} = \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$$

$$\Rightarrow \frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

$$\Rightarrow a^{-2}+b^{-2}+c^{-2} = a_1^{-2}+b_1^{-2}+c_1^{-2}$$

**Problem:** A plane meets the coordinate axes in $A, B, C$. If the centroid of $\Delta ABC$ is $(a, b, c)$, Then show that the equation to the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 3$.

**Sol:** If a plane meet the coordinate axes at $A = (a_1, 0, 0)$, $B = (0, b_1, 0)$, $C = (0, 0, c_1)$ respectively.

Then the equation of the plane is $\frac{x}{a_1}+\frac{y}{b_1}+\frac{z}{c_1}=1 \rightarrow (1)$

The Centroid of the $\Delta ABC$ is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

$$\rightarrow \left(\frac{a_1+0+0}{3}, \frac{0+b_1+0}{3}, \frac{0+0+c_1}{3}\right)=(a,b,c)$$

$$\rightarrow \left(\frac{a_1}{3}, \frac{b_1}{3}, \frac{c_1}{3}\right)=(a,b,c) \rightarrow \frac{a_1}{3}=a,\ \frac{b_1}{3}=b,\ \frac{c_1}{3}=c$$

$$\rightarrow a_1=3a,\ b_1=3b,\ c_1=3c$$

From (1), $\frac{x}{3a}+\frac{y}{3b}+\frac{z}{3c}=1 \rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3.$

Problem: A variable plane moves so that the sum of the reciprocals of its intercepts on the coordinate axes is a constant. Then show that it passes through a fixed point.

Sol: Suppose a variable plane meets the coordinate axes at $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ respectively.

Then the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \rightarrow (1)$

Also given, The sum of the reciprocals of intercepts of a plane on the coordinate axes is constant.

$$\rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=k \rightarrow \frac{1}{ka}+\frac{1}{kb}+\frac{1}{kc}=1 \rightarrow \frac{\frac{1}{k}}{a}+\frac{\frac{1}{k}}{b}+\frac{\frac{1}{k}}{c}=1 \rightarrow (2)$$

From (1) and (2), the plane (1) passes through a fixed point $\left(\frac{1}{k},\frac{1}{k},\frac{1}{k}\right)$.

Problem: A variable plane is at a constant distance $3p$ from the origin and meets the axes in $A, B, C$. Then show that the locus of the centroid of the $\Delta ABC$ is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$

Sol: If a variable plane meets the coordinate axes at $A(\alpha,0,0)$, $B(0,\beta,0)$, $C(0,0,\gamma)$ respectively.

Then the equation of the plane is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1 \rightarrow (1)$

$$\rightarrow \left(\frac{1}{\alpha}\right)x+\left(\frac{1}{\beta}\right)y+\left(\frac{1}{\gamma}\right)z+(-1)=0 \rightarrow (2)$$ is in the form $ax+by+cz+d=0$. Here $a=\frac{1}{\alpha}$, $b=\frac{1}{\beta}$, $c=\frac{1}{\gamma}$, $d=-1$.

Since $3p$ = The distance from $(0,0,0)$ to the plane (2)

$$\rightarrow 3p=\frac{|d|}{\sqrt{a^2+b^2+c^2}} \rightarrow 3p=\frac{|-1|}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}}$$

$$\rightarrow 3p\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}=1 \rightarrow \sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}=\frac{1}{3p}$$

$\rightarrow \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{1}{9p^2} \rightarrow (3)$

Let $Q(x_1, y_1, z_1)$ be the centroid of the $\Delta ABC$.

Then $(x_1, y_1, z_1) = \left(\frac{\alpha+0+0}{3}, \frac{0+\beta+0}{3}, \frac{0+0+\gamma}{3}\right)$

$\Rightarrow (x_1, y_1, z_1) = \left(\frac{\alpha}{3}, \frac{\beta}{3}, \frac{\gamma}{3}\right)$

$\rightarrow x_1 = \frac{\alpha}{3}, \quad y_1 = \frac{\beta}{3}, \quad z_1 = \frac{\gamma}{3} \Rightarrow \alpha = 3x_1, \beta = 3y_1, \gamma = 3z_1$

From (3), $\frac{1}{9x_1^2}+\frac{1}{9y_1^2}+\frac{1}{9z_1^2} = \frac{1}{9p^2}$

$\rightarrow \frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2} = \frac{1}{p^2} \Rightarrow x_1^{-2}+y_1^{-2}+z_1^{-2} = p^{-2}$

Therefore the locus of Q is $x^{-2}+y^{-2}+z^{-2} = p^{-2}$.

Problem: A variable plane is at a constant distance $p$ from the origin and meets the axes in A, B, C. Then show that the locus of the centroid of the tetrahedron OABC is $x^{-2}+y^{-2}+z^{-2} = 16p^{-2}$.

Sol: If a variable plane meets the coordinate axes at $A(\alpha,0,0)$ $B(0,\beta,0)$, $C(0,0,\gamma)$ respectively.

Then the equation of the plane is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma} = 1 \rightarrow (1)$

$\rightarrow \left(\frac{1}{\alpha}\right)x+\left(\frac{1}{\beta}\right)y+\left(\frac{1}{\gamma}\right)z+(-1) = 0$. is of the form $ax+by+cz+d=0$

Here $a = \frac{1}{\alpha}, \quad b = \frac{1}{\beta}, \quad c = \frac{1}{\gamma}, d = -1$.

Since $p$ = The distance from $(0,0,0)$ to the plane (2).

$\Rightarrow p = \frac{|d|}{\sqrt{a^2+b^2+c^2}} \rightarrow p = \frac{|-1|}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}} \Rightarrow \frac{1}{p} = \sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}} \Rightarrow \frac{1}{p^2} = \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$ $\rightarrow (3)$

Let $Q(x_1, y_1, z_1)$ be the centroid of the tetrahedron.

Then $(x_1, y_1, z_1) = \left(\frac{\alpha}{4}, \frac{\beta}{4}, \frac{\gamma}{4}\right) \Rightarrow x_1 = \frac{\alpha}{4}, y_1 = \frac{\beta}{4}, z_1 = \frac{\gamma}{4}$

$\rightarrow \alpha = 4x_1, \beta = 4y_1, \gamma = 4z_1$

From (3), $\frac{1}{16x_1^2}+\frac{1}{16y_1^2}+\frac{1}{16z_1^2} = \frac{1}{p^2}$

$\rightarrow \frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2} = \frac{16}{p^2}$

$\rightarrow x_1^{-2}+y_1^{-2}+z_1^{-2} = 16p^{-2}$

Therefore the locus of Q is $x^{-2}+y^{-2}+z^{-2} = 16p^{-2}$.

**Problem:** $p$ is a point such that the sum of the squares of its distances from the planes $x+y+z=0$, $x+y-2z=0$, $x-y=0$ is 5. then Show that the locus of $p$ is $x^2+y^2+z^2=5$.

**Sol:** Given planes are $x+y+z=0 \rightarrow (1)$, $x+y-2z=0 \rightarrow (2)$, $x-y=0 \rightarrow (3)$

Also given, The sum of the squares of its distances of a variable Point $P(x_1, y_1, z_1)$ from the planes (1), (2), (3) is 5.

$$\Rightarrow \left(\frac{|1\cdot x_1+1\cdot y_1+1\cdot z_1|}{\sqrt{1^2+1^2+1^2}}\right)^2 + \left(\frac{|1\cdot x_1+1\cdot y_1+(-2)(z_1)|}{\sqrt{1^2+1^2+(-2)^2}}\right)^2 + \left(\frac{|1\cdot x_1+(-1)y_1+(0)(z_1)|}{\sqrt{1^2+(-1)^2+(0)^2}}\right)^2 = 5$$

$$\Rightarrow \frac{(x_1+y_1+z_1)^2}{3} + \frac{(x_1+y_1-2z_1)^2}{6} + \frac{(x_1-y_1)^2}{2} = 5$$

$$\Rightarrow 2(x_1+y_1+z_1)^2 + (x_1+y_1-2z_1)^2 + 3(x_1-y_1)^2 = 30$$

$$\Rightarrow 2x_1^2+2y_1^2+2z_1^2+4x_1y_1+4y_1z_1+4z_1x_1+x_1^2+y_1^2+4z_1^2+2x_1y_1-4y_1z_1-4z_1x_1+3x_1^2+3y_1^2-6x_1y_1=30$$

$$\Rightarrow 6x_1^2+6y_1^2+6z_1^2=30 \Rightarrow x_1^2+y_1^2+z_1^2=5$$

Therefore the locus of $P(x_1, y_1, z_1)$ is $x^2+y^2+z^2=5$

## *. Systems of Planes:

Any plane passing through the line of intersection of the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2$ is $(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$ where $\lambda$ is a parameter.

**Problem:** Find the equation of the plane through $(1,-2,-3)$ and is parallel to $3x-y+z=10$.

**Sol:** Given equation of the plane is $3x-y+z=10 \rightarrow (1)$

The equation of any plane parallel to (1) is ~~3y~~ $3x-y+z+k=0 \rightarrow (2)$

The plane (2) passes through the point $(1,-2,-3)$

$$\rightarrow 3(1)-(-2)+(-3)+k=0$$

$$\rightarrow 2+k=0 \Rightarrow k=-2$$

From (2), $3x-y+z=2$

Therefore the equation of the required plane is $3x-y+z=2$.

**Problem:** Find the equation of the plane through the line of intersection of the planes $x+y+z-6=0$ and $2x+3y+4z+5=0$ and through the point $(1,1,1)$.

**Sol:** Given planes are $x+y+z-6=0 \rightarrow (1)$ and $2x+3y+4z+5=0 \rightarrow (2)$

Equation to the required plane is $\Pi_1 + \lambda \Pi_2 = 0$

$\Rightarrow (x+y+z-6) + \lambda(2x+3y+4z+5) = 0 \rightarrow (3)$

The plane (3) passes through $(1,1,1)$.

$\Rightarrow (1+1+1-6) + \lambda(2+3+4+5) = 0 \Rightarrow -3+14\lambda = 0 \Rightarrow \lambda = \frac{3}{14}$

From (3), $(x+y+z-6) + \frac{3}{14}(2x+3y+4z+5) = 0$

$\rightarrow 14x + 14y + 14z - 84 + 6x + 9y + 12z + 15 = 0$

$\rightarrow 20x + 23y + 26z - 69 = 0$

**Problem:** Find the equation of the plane through the line of intersection of the planes $x+2y+3z+4=0$ and $4x+3y+3z+1=0$ and perpendicular to the plane $x+y+z+9=0$.

**Sol:** Given planes are $x+2y+3z+4=0 \rightarrow (1)$, $4x+3y+3z+1=0 \rightarrow (2)$

The perpendicular plane is $x+y+z+9=0 \rightarrow (3)$

The equation to the plane through the line of intersection of the planes (1) and (2) is

$(x+2y+3z+4) + \lambda(4x+3y+3z+1) = 0$

$\rightarrow (1+4\lambda)x + (2+3\lambda)y + (3+3\lambda)z + (4+\lambda) = 0 \rightarrow (4)$

The plane (4) is perpendicular to (3).

$\Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow (1+4\lambda)\cdot 1 + (2+3\lambda)\cdot 1 + (3+3\lambda)\cdot 1 = 0$

$\Rightarrow 6 + 10\lambda = 0 \Rightarrow \lambda = -3/5$

Therefore the equation of the required plane is

$(x+2y+3z+4) - \frac{3}{5}(4x+3y+3z+1) = 0$

$\rightarrow 5x + 10y + 15z + 20 - 12x - 9y - 9z - 3 = 0$

$\Rightarrow -7x + y + 6z + 17 = 0$

$\Rightarrow 7x - y - 6z - 17 = 0.$

**Problem:** Find the equation of the plane passing through the line of intersection of the planes $2x-y=0$ and $y-3z=0$ and perpendicular to the plane $3z-4x-5y+8=0$

**Sol:** Given planes are $2x-y=0 \rightarrow (1)$, $y-3z=0 \rightarrow (2)$

The perpendicular plane is $3z-4x-5y+8=0 \rightarrow (3)$

The equation to the plane through the line of intersection of the planes (1) and (2) is

$$(2x-y)+\lambda(y-3z)=0 \Rightarrow 2x+(\lambda-1)y-3\lambda z=0 \rightarrow (4)$$

The plane (4) is perpendicular to (3).

$$\rightarrow a_1a_2+b_1b_2+c_1c_2=0$$

$$\Rightarrow (-4)(2)+(-5)(\lambda-1)+(3)(-3\lambda)=0$$

$$\Rightarrow -8-5\lambda+5-9\lambda=0 \Rightarrow -14\lambda-3=0 \Rightarrow \lambda=-3/14$$

Therefore the equation of the required plane is

$$(2x+y)-3/14(y-3z)=0$$

$$\Rightarrow 28x-14y-3y+9z=0 \Rightarrow 28x-17y+9z=0$$

**Problem:** Find the equation to the plane through the line of intersection of $x-2y-z+3=0$, $-3x-5y+2z+1=0$ and perpendicular to $yz$-plane.

**Sol:** Given planes are $x-2y-z+3=0 \rightarrow (1)$, $-3x-5y+2z+1=0 \rightarrow (2)$

The perpendicular plane is $yz$-plane i.e. $x=0 \rightarrow (3)$

The equation to the plane through the line of intersection of the planes (1) and (2) is

$$(x-2y-z+3)+\lambda(-3x-5y+2z+1)=0$$

$$\rightarrow (1-3\lambda)x-(2+5\lambda)y-(1-2\lambda)z+(3+\lambda)=0 \rightarrow (4)$$

The plane (4) is perpendicular to (3)

$$\rightarrow a_1a_2+b_1b_2+c_1c_2=0$$

$$\Rightarrow 1-3\lambda=0 \Rightarrow 1/3=\lambda$$

Therefore the equation of the required plane is

$$(x-2y-z+3)+1/3(-3x-5y+2z+1)=0$$

$$\rightarrow 3x-6y-3z+9-3x-5y+2z+1=0$$

$$\Rightarrow -11y-z+10=0 \Rightarrow 11y+z-10=0$$

**Problem:** Determine the planes through the intersection of the planes $2x+3y-z+4=0$, $x+y+z-1=0$ and which are parallel to the coordinate axes.

**Sol:** Given planes are $2x+3y-z+4=0 \rightarrow (1)$, $x+y+z-1=0 \rightarrow (2)$

The equation to the plane through the intersection of the planes (1) and (2) is $(2x+3y-z+4)+\lambda(x+y+z-1)=0 \rightarrow (3)$

$\Rightarrow \quad (2+\lambda)x+(3+\lambda)y+(\lambda-1)z+(4-\lambda)=0 \rightarrow (4)$

Since (4) is parallel to $x$-axis $\Rightarrow 2+\lambda=0 \Rightarrow \lambda=-2$

From (3), $(2x+3y-z+4)-2(x+y+z-1)=0$

$\Rightarrow 2x+3y-z+4-2x-2y-2z+2=0 \Rightarrow y-3z+6=0$

Since (4) is parallel to $y$-axis $\Rightarrow 3+\lambda=0 \Rightarrow \lambda=-3$

From (3), $(2x+3y-z+4)-3(x+y+z-1)=0$

$\Rightarrow 2x+3y-z+4-3x-3y-3z+3=0 \Rightarrow -x-4z+7=0$

Since (4) is parallel to $z$-axis $\Rightarrow \lambda-1=0 \Rightarrow \lambda=1$

From (3), $(2x+3y-z+4)+1(x+y+z-1)=0$

$\Rightarrow 2x+3y-z+4+x+y+z-1=0 \Rightarrow 3x+4y+3=0$

**Problem:** Find the equations to the planes through the intersection of $2x+y+3z=2$, $x-y+z+4=0$ such that each plane is at a distance of 2 units from the origin.

**Sol:** Given plane equations are $2x+y+3z=2 \rightarrow (1)$, $x-y+z+4=0 \rightarrow (2)$

The equations to the planes through the intesection of (1) and (2) are $(2x+y+3z-2)+\lambda(x-y+z+4)=0 \rightarrow (3)$

$\Rightarrow (2+\lambda)x+(1-\lambda)y+(3+\lambda)z+(-2+4\lambda)=0 \rightarrow (4)$

Since $2=$ The distance from $(0,0,0)$ to (4)

$$\Rightarrow 2=\frac{|d|}{\sqrt{a^2+b^2+c^2}} \Rightarrow 2=\frac{|4\lambda-2|}{\sqrt{(2+\lambda)^2+(1-\lambda)^2+(3+\lambda)^2}}$$

$$\Rightarrow 2\sqrt{(2+\lambda)^2+(1-\lambda)^2+(3+\lambda)^2}=2(2\lambda+1)$$

$$\Rightarrow (2+\lambda)^2+(1-\lambda)^2+(3+\lambda)^2=(2\lambda-1)^2$$

$$\Rightarrow 4+\lambda^2+4\lambda+1+\lambda^2-2\lambda+9+\lambda^2+6\lambda=4\lambda^2+1-4\lambda$$

$$\Rightarrow 3\lambda^2+8\lambda+14=4\lambda^2-4\lambda+1$$

$$\Rightarrow \lambda^2-12\lambda-13=0 \Rightarrow \lambda^2-13\lambda+\lambda-13=0 \Rightarrow \lambda(\lambda-13)+1(\lambda-13)=0$$

$$\Rightarrow (\lambda+1)(\lambda-13)=0 \Rightarrow \lambda=-1, 13$$

From (4), If $\lambda=-1$ then $(2-1)x+(1+1)y+z(3-1)+(-2-4)=0 \Rightarrow x+2y+2z-6=0$

If $\lambda=13$ then $(2+13)x+(1-13)y+(3+13)z+(-2+52)=0 \Rightarrow 15x-12y+16z+50=0$

**Problem:** Find the equations of the planes through the intersection of the planes $x+3y+6=0$ and $3x-y-4z=0$ such that the perpendicular distance of each from the origin is unity.

**Sol:** Given plane equations are $x+3y+6=0 \to (1)$, $3x-y-4z=0 \to (2)$

The equations of the planes through the intersection of (1) and (2) are $(x+3y+6)+\lambda(3x-y-4z)=0 \Rightarrow (1+3\lambda)x+(3-\lambda)y+(-4\lambda)z+6=0 \to (3)$

Since $1$ = The distance from $(0,0,0)$ to (3)

$$\Rightarrow 1 = \frac{|d|}{\sqrt{a^2+b^2+c^2}} \Rightarrow 1 = \frac{6}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}$$

$$\Rightarrow \sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2} = 6$$

$$\Rightarrow (1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2 = 36$$

$$\Rightarrow 1+9\lambda^2+6\lambda+9+\lambda^2-6\lambda+16\lambda^2 = 36 \Rightarrow 26\lambda^2 = 26$$

$$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

From (3), If $\lambda = 1$ then $(1+3)x+(3-1)y+(-4)z+6=0$

$$\Rightarrow 4x+2y-4z+6=0 \Rightarrow 2x+y-2z+3=0$$

If $\lambda=-1$ then $(1-3)x+(3+1)y+(+4)z+6=0$

$$\Rightarrow -2x+4y+4z+6=0 \Rightarrow x-2y-2z-3=0.$$

**Problem:** A variable plane passes through a fixed point $(a,b,c)$. It meets the axes at A, B and C. Then show that the locus of the point of intersection of the planes through A, B, C and Parallel to the coordinate planes is $ax^{-1}+by^{-1}+cz^{-1}=1$.

**Sol:** Let the variable plane meets the coordinate axes in $A(\alpha,0,0)$, $B(0,\beta,0)$, $C(0,0,\gamma)$ be $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1 \to (1)$

Also equation (1) passes through the fixed point $(a,b,c)$

$$\rightarrow \frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1 \to (2)$$

Any plane through $A(\alpha,0,0)$ and parallel to $yz$-plane is $x=\alpha$

Any plane through $B(0,\beta,0)$ and parallel to $zx$-plane is $y=\beta$

Any plane through $C(0,0,\gamma)$ and parallel to $xy$-plane is $z=\gamma$

Clearly the planes intersects at $P(\alpha,\beta,\gamma)$

Therefore the locus of $P$ from equation (2) is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1 \Rightarrow ax^{-1}+by^{-1}+cz^{-1}=1.$$

**Problem:** A variable plane is at a constant distance $p$ from the origin. It meets the coordinate axes at A, B, C. Then show that the locus of the point of intersection of the planes through A, B, C and parallel to the coordinate planes is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$

Sol: A variable plane ~~is~~ meets the coordinate axes at $A(\alpha,0,0)$ $B(0,\beta,0)$, $C(0,0,\gamma)$ be $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1 \rightarrow (1) \Rightarrow (\frac{1}{\alpha})x+(\frac{1}{\beta})y+(\frac{1}{\gamma})z+(-1)=0$

Since $p$ = The distance from $(0,0,0)$ to the plane (1)

$$\Rightarrow p = \frac{|d|}{\sqrt{a^2+b^2+c^2}} \Rightarrow p = \frac{1}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}} \Rightarrow \sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}} = \frac{1}{p}$$

$$\Rightarrow \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{1}{p^2} \rightarrow (2)$$

Any plane through $A(\alpha,0,0)$ and parallel to $yz$-plane is $x=\alpha$

Any plane through $B(0,\beta,0)$ and parallel to $xz$-plane is $y=\beta$

Any plane through $C(0,0,\gamma)$ and parallel to ~~xy~~ $xy$-plane is $z=\gamma$

Clearly the planes intersects at $P(\alpha,\beta,\gamma)$.

from (2), $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$

Therefore the locus of $P$ is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

## *. Planes Bisecting The Angles Between Two Planes :

$\pi_1 = a_1x+b_1y+c_1z+d_1=0$, $\pi_2 = a_2x+b_2y+c_2z+d_2=0$ and $d_1d_2>0$.

Equation to the plane bisecting the angle containing the origin between the planes $\pi_1, \pi_2$ is $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = +\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \rightarrow (1)$

and to the plane bisecting the other angle between the planes $\pi_1, \pi_2$ is $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = -\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \rightarrow (2)$

Note: In the bisecting planes (1) and (2), one bisects the acute and the other bisects the obtuse angle between the given planes.

The bisecting plane of the acute angle makes with either of the planes $\pi_1, \pi_2$ an angle less than $45°$ and the bisecting plane of the obtuse angle makes with either of of the planes $\pi_1, \pi_2$ an angle greater than $45°$. This gives a test for determining which angle each bisecting plane bisects.

Even if $d_1<0, d_2<0$, the theorem holds.

But if $d_1>0, d_2<0$ or, $d_1<0, d_2>0$, the equation to the plane bisecting the angle containing the origin is

$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = -\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$ and the other equation gives the other bisecting plane.

**Problem**: Find the equations of the planes bisecting the angles between the planes $3x-2y+6z+2=0$ and $-2x+y-2z-2=0$ Point out which bisects the acute angle. Also point out the plane bisecting the angle containing the origin.

**Sol**: Given plane equations are $3x-2y+6z+2=0 \rightarrow (1)$

$-2x+y-2z-2=0 \Rightarrow 2x-y+2z+2=0 \rightarrow (2)$

Equations to the bisecting planes are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\Rightarrow \frac{3x-2y+6z+2}{\sqrt{9+4+36}} = \pm \frac{2x-y+2z+2}{\sqrt{4+1+4}}$$

$$\Rightarrow \frac{3x-2y+6z+2}{\sqrt{49}} = \pm \frac{2x-y+2z+2}{\sqrt{9}}$$

$$\Rightarrow 3(3x-2y+6z+2) = \pm 7(2x-y+2z+2)$$

$$\Rightarrow 9x-6y+18z+6 = \pm 14x-7y+14z+14$$

$\Rightarrow 9x-6y+18z+6 = 14x-17y+14z+14$ ; $9x-6y+18z+6 = -14x+7y-14z-14$

$\Rightarrow 5x-y-4z+8=0 \rightarrow (3)$ ; $23x-13y+32z+20=0 \rightarrow (4)$

Let $\theta$ be the angle between (1) and (3), Then $\cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

$$\Rightarrow \frac{|15+2-24|}{\sqrt{9+4+36}\sqrt{25+1+16}} = \frac{7}{7\sqrt{42}} = \frac{1}{\sqrt{42}}$$

$\cos\theta = \frac{1}{\sqrt{42}} \Rightarrow \tan\theta = \frac{\sqrt{41}}{1} = \sqrt{41} > 1$

$\Rightarrow \tan\theta > 1 = \tan 45° \Rightarrow \theta > 45°$

The plane (3) bisects obtuse angle between the planes (1) and (2)

The plane (4) bisects ~~ob~~ acute angle between the planes (1) and (2)

From (1) and (2), $d_1=2$ and $d_2=2 \Rightarrow d_1d_2>0$

Therefore $5x-y-4z+8=0$ is the bisecting plane angle containing the origin.

**Problem**: Find the equations of the planes bisecting the angles between the planes $3x-6y+2z+5=0$, $4x-12y+3z-3=0$. Point out which plane bisects the acute angle. Also point out the plane bisecting the angle containing the origin.

Sol: Given plane equations are $3x-6y+2z+5=0 \rightarrow (1)$, $4x-12y+3z-3=0 \rightarrow (2)$

Equations to the bisecting planes are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\Rightarrow \frac{3x-6y+2z+5}{\sqrt{9+36+4}} = \pm \frac{4x-12y+3z-3}{\sqrt{16+144+9}}$$

$$\Rightarrow \frac{3x-6y+2z+5}{7} = \pm \frac{4x-12y+3z-3}{13}$$

$$\Rightarrow 13(3x-6y+2z+5) = \pm 7(4x-12y+3z-3)$$

$$\Rightarrow 39x-78y+26z+65 = \pm 28x-84y+21z-21$$

$$\Rightarrow 39x-78y+26z+65 = 28x-84y+21z-21 \; ; \; 39x-78y+26z+65 = -28x+84y-21z+21$$

$$\Rightarrow 11x+6y+5z+86=0 \rightarrow (3) \; ; \; 67x-162y+47z+44=0 \rightarrow (4)$$

Let $\theta$ be the angle between (1) and (3) Then

$$\cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{|33-36+10|}{\sqrt{49}\cdot\sqrt{182}} = \frac{7}{7\sqrt{182}} = \frac{1}{\sqrt{182}}$$

$$\cos\theta = \frac{1}{\sqrt{182}} \Rightarrow \tan\theta = \frac{\sqrt{181}}{1} = \sqrt{181} > 1 \Rightarrow \tan\theta > 1 = \tan 45^\circ$$

$$\Rightarrow \theta > 45^\circ$$

The plane (3) bisects obtuse angle between the planes (1) and (2)

The plane (4) bisects Acute angle between the planes (1) and (2)

From (1) and (2), $d_1 = 5$, $d_2 = -3 \Rightarrow d_1d_2 < 0$

Therefore $67x-162y+47z+44=0$ is the bisecting plane angle containing the origin.

Problem: Find the equations of the planes bisecting the angles between the planes $x+2y+2z=19$, $4x-3y+12z+3=0$. point out which plane bisects the acute angle. Also point out the plane bisecting the angle containing the origin.

Sol: Given plane equations are $x+2y+2z-19=0 \rightarrow (1)$, $4x-3y+12z+3=0 \rightarrow (2)$

Equations to the bisecting planes are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\Rightarrow \frac{x+2y+2z-19}{\sqrt{1+4+4}} = \pm \frac{4x-3y+12z+3}{\sqrt{16+9+144}}$$

$$\Rightarrow \frac{x+2y+2z-19}{3} = \pm \frac{4x-3y+12z+3}{13} \Rightarrow 13(x+2y+2z-19) = \pm 3(4x-3y+12z+3)$$

$$\Rightarrow 13x+26y+26z-247 = \pm 12x-9y+36z+9$$

$\Rightarrow 13x+26y+26z-247 = 12x-9y+36z+9$ ; $13x+26y+26z-247 = -12x+9y-36z-9$

$\Rightarrow x+35y-10z-256 = 0 \rightarrow$ (3) ; $25x+17y+62z-238 = 0 \rightarrow$ (4)

Let $\theta$ be the angle between (1) and (3) then

$$\cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{|1+70-20|}{\sqrt{9}\cdot\sqrt{1326}} = \frac{51}{3\sqrt{1326}} = \frac{17}{\sqrt{1326}}$$

$\cos\theta = \frac{17}{\sqrt{1326}} \rightarrow \tan\theta = \frac{\sqrt{1037}}{17} > 1 \Rightarrow \tan\theta > 1 = \tan 45^\circ$

$\Rightarrow \theta > 45^\circ$

The plane (3) bisects obtuse angle between the planes (1) and (2)

The plane (4) bisects Acute angle between the planes (1) and (2)

From (1) and (2), $d_1 = -19$, $d_2 = 3 \Rightarrow d_1d_2 < 0$

Therefore $25x+17y+62z-238 = 0$ is the bisecting plane angle containing the origin.

## * Joint Equation of a pair of planes :

Definition: The equation $(a_1x+b_1y+c_1z+d_1)(a_2x+b_2y+c_2z+d_2)=0$ is called the joint equation of the planes $\Pi_1$ and $\Pi_2$.

If $H = ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$ and $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Then D is called the Determinate of H.

Note: The equation $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ represents a pair of planes if $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$, $h^2 \geq ab$, $g^2 > ac$, $f^2 \geq bc$

If $\theta\ (\leq \pi/2)$ is the acute angle between the pair of planes

Then $$\cos\theta = \left| \frac{a+b+c}{\sqrt{\{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)\}}} \right|$$

Problem: Prove that the equation $2x^2-6y^2-12z^2+18yz+2zx+xy=0$ represents a pair of planes and find the angle between them.

Sol: Given equation is $2x^2-6y^2-12z^2+18yz+2zx+xy = 0 \rightarrow$ (1)

Comparing (1) with $ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$

Then $a = 2$ $\quad 2f = 18 \quad 2g = 2 \quad 2h = 1$

$b = -6 \quad f = 9 \quad g = 1 \quad h = 1/2$

$c = -12$

Now $h^2 = (1/2)^2 = \frac{1}{4}$ ; $ab = -12 \Rightarrow h^2 > ab$

$g^2 = 1$ ; $ac = -24 \Rightarrow g^2 > ac$

$f^2 = 81$ ; $bc = 72 \Rightarrow f^2 > bc$

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 2 & 1/2 & 1 \\ 1/2 & -6 & 9 \\ 1 & 9 & -12 \end{vmatrix} = 2(72-81) - 1/2(-6-9) + 1(9/2+6)$$

$$= -18 + \frac{15}{2} + \frac{21}{2} = \frac{-36+15+21}{2} = 0$$

Therefore equation ① represents a pair of planes.

Let $\theta$ be the angle between the planes then

$$\cos\theta = \left| \frac{a+b+c}{\sqrt{[(a+b+c)^2 + 4(f^2+g^2+h^2-ab-bc-ca)]}} \right|$$

$$= \left| \frac{2-6-12}{\sqrt{[(2-6-12)^2 + 4(81+1+\frac{1}{4}+12-72+24)]}} \right|$$

$$= \left| \frac{-16}{\sqrt{(-16)^2 + 4\left(\frac{185}{4}\right)}} \right| = \left| \frac{-16}{\sqrt{256+185}} \right| = \left| \frac{-16}{\sqrt{441}} \right| = \frac{16}{21}$$

$$\Rightarrow \cos\theta = \frac{16}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{16}{21}\right).$$

Problem: Show that the following equations represents a pair of planes. Also find the angles between them.

②. $2x^2 - 3y^2 + 4z^2 + xy + 6zx - yz = 0$

Sol: Given equation is $2x^2 - 3y^2 + 4z^2 + xy + 6zx - yz = 0 \rightarrow$ ①

Here $a = 2$, $b = -3$, $c = 4$; $2f = -1$, $f = -1/2$; $2g = 6$, $g = 3$; $2h = 1$, $h = 1/2$

Now
$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 2 & 1/2 & 3 \\ 1/2 & -3 & -1/2 \\ 3 & -1/2 & 4 \end{vmatrix} = 2\left(-12-\frac{1}{4}\right) - 1/2\left(2+\frac{3}{2}\right) + 3\left(-\frac{1}{4}+9\right)$$

$$= 2\left(-\frac{49}{4}\right) - \frac{1}{2}\left(\frac{7}{4}\right) + 3\left(\frac{35}{4}\right)$$

$$= -\frac{98}{4} - \frac{7}{4} + \frac{105}{4} = 0$$

$f^2 = \frac{1}{4}$ ; $bc = -12 \Rightarrow f^2 \geq bc$

$g^2 = 9$ ; $ac = 8 \Rightarrow g^2 \geq ac$

$h^2 = \frac{1}{4}$ ; $ab = -6 \Rightarrow h^2 \geq ab$

Therefore equation ① represents a pair of planes.

Let $\theta$ be the angle between the planes then

$$\cos\theta = \left| \frac{a+b+c}{\sqrt{[(a+b+c)^2 + 4(f^2+g^2+h^2-ab-bc-ca)]}} \right|$$

$$= \left| \frac{2-3+4}{\sqrt{[(2-3+4)^2 + 4(\frac{1}{4}+9+\frac{1}{4}+6-8+12)]}} \right| = \left| \frac{3}{\sqrt{9+4(39/2)}} \right| = \left| \frac{3}{\sqrt{9+78}} \right| = \frac{3}{\sqrt{87}}$$

$$\Rightarrow \cos\theta = \frac{3}{\sqrt{87}} \Rightarrow \theta = \cos^{-1}\left(\frac{3}{\sqrt{87}}\right).$$

(ii). $6x^2+4y^2-10z^2-11xy+3yz+4zx=0$

Sol: Given equation is $6x^2+4y^2-10z^2-11xy+3yz+4zx=0 \rightarrow (1)$

Here $a=6$, $b=4$, $c=-10$; $2f=3$, $f=3/2$; $2g=4$, $g=2$; $2h=-11$, $h=-11/2$

$$D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 6 & -11/2 & 2 \\ -11/2 & 4 & 3/2 \\ 2 & 3/2 & -10 \end{vmatrix} = 6\left(-40-\frac{9}{4}\right)-\left(-11/2\right)\left(\frac{110}{2}-\frac{6}{2}\right)+2\left(\frac{-33}{4}-8\right)$$

$$= 6\left(-\frac{169}{4}\right)+11/2\left(\frac{104}{2}\right)+2\left(-\frac{65}{4}\right)$$

$$= -\frac{1144}{4}+\frac{1144}{4}=0$$

$f^2 = 9/4$ ; $bc = -40$ $\Rightarrow$ $f^2 \geq bc$

$g^2 = 4$ ; $ac = -60$ $\Rightarrow$ $g^2 \geq ac$

$h^2 = \frac{121}{4}$ ; $ab = 24$ $\Rightarrow$ $h^2 \geq ab$

Therefore equation (1) represents a pair of planes.

Let $\theta$ be the angle between the planes then

$$\cos\theta = \left|\frac{a+b+c}{\sqrt{[(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)]}}\right|$$

$$= \left|\frac{6+4-10}{\sqrt{[(6+4-10)^2+4(9/4+4+121/4-24+60+40)]}}\right| = 0$$

$\cos\theta = 0 \Rightarrow \theta = \cos^{-1}(0) = \pi/2.$

(iii). $x^2+4y^2-z^2+4xy=0$

Sol: Given equation is $x^2+4y^2-z^2+4xy=0 \rightarrow (1)$

Here $a=1, b=4, c=-1$, $2g=0$, $g=0$; $2f=0$, $f=0$; $2h=4$, $h=2$

$$D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 1(-4-0)-2(-2-0)+0 = 0$$

$f^2 = 0$ ; $bc=-4$ $\rightarrow$ $f^2 \geq bc$

$g^2 = 0$ ; $ac=-1$ $\rightarrow$ $g^2 \geq ac$

$h^2 = 4$ ; $ab = 4$ $\rightarrow$ $h^2 \geq ab$

Therefore the equation (1) represents a pair of planes.

Let $\theta$ be the angle between the planes then

$$\cos\theta = \left|\frac{a+b+c}{\sqrt{[(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)]}}\right| = \left|\frac{1+4-1}{\sqrt{[(1+4-1)^2+4(0+0+4+4+1-4)]}}\right|$$

$$= \left|\frac{4}{\sqrt{16+20}}\right| = \left|\frac{4}{\sqrt{36}}\right| = \frac{4}{6} = \frac{2}{3}$$

$\rightarrow \cos\theta = \frac{2}{3} \Rightarrow \theta = \cos^{-1}(2/3).$

**Problem:** Find the planes represented by the equation $x^2-2y^2-z^2-xy+3yz-6x+3y+9=0$ and hence find the angle between them.

**Sol:** Given equation is $x^2-2y^2-z^2-xy+3yz-6x+3y+9=0$

$$\Rightarrow x^2+x(-6-y)+(-2y^2-z^2+3yz+3y+9)=0$$

$$\Rightarrow x = \frac{-(-6-y)\pm\sqrt{(-6-y)^2-4(-2y^2-z^2+3yz+3y+9)}}{2}$$

$$\Rightarrow 2x = (6+y)\pm\sqrt{36+y^2+12y+8y^2+4z^2+12yz-12y-36}$$

$$\Rightarrow 2x = (6+y)\pm\sqrt{9y^2+4z^2-12yz}$$

$$\Rightarrow 2x = (6+y)\pm\sqrt{(3y-2z)^2}$$

$$\Rightarrow 2x = (6+y)\pm(3y-2z)$$

$$\Rightarrow 2x=(6+y)+(3y-2z) \; ; \; 2x=(6+y)-(3y-2z)$$

$$\Rightarrow 2x-4y+2z-6=0 \; ; \; 2x+2y-2z-6=0$$

$$\Rightarrow x-2y+z-3=0 \rightarrow (1) \; ; \; x+y-z-3=0 \rightarrow (2)$$

Let $\theta$ be the angle between (1) and (2) then

$$\cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{|1.1+(-2).1+1.(-1)|}{\sqrt{1+4+1}\;\sqrt{1+1+1}}$$

$$= \frac{|1-2-1|}{\sqrt{6}.\sqrt{3}} = \frac{|-2|}{\sqrt{18}} = \frac{2}{\sqrt{9}.\sqrt{2}} = \frac{\sqrt{2}}{3}$$

$$\Rightarrow \cos\theta = \frac{\sqrt{2}}{3} \Rightarrow \theta = \cos^{-1}(\sqrt{2}/3).$$

**Problem:** Show that the equation $x^2+4y^2+9z^2-12yz-6zx+4xy+5x+10y-15z+6=0$ represents a pair of parallel planes and find the distance between them.

**Sol:** Given equation is $x^2+4y^2+9z^2-12yz-6zx+4xy+5x+10y-15z+6=0 \rightarrow (1)$

$$x^2+4y^2+9z^2-12yz-6zx+4xy = (x+2y-3z)^2$$

$$x^2+4y^2+9z^2-12yz-6zx+4xy+5x+10y-15z+6 = (x+2y-3z+l)(x+2y-3z+m).$$

$$\rightarrow (x+2y-3z+l)(x+2y-3z+m)=0$$

$$\rightarrow x^2+2xy-3xz+xm+2xy+4y^2-6yz+2ym-3xz-6yz+9z^2-3zm+lx+2yl-3zl+lm=0$$

$$\Rightarrow x^2+4y^2+9z^2+4xy-12yz-6zx+x(l+m)+2(l+m)y-3(l+m)z+lm=0$$

Then we get $lm=6 \rightarrow (2)$, $x(l+m)=5x$

$$\Rightarrow l+m=5 \rightarrow (3)$$

From ③, $l = 5 - m$

From ②, $(5-m)m = 6$

$\Rightarrow 5m - m^2 - 6 = 0$

$\Rightarrow m^2 - 5m + 6 = 0$

$\Rightarrow m^2 - 3m - 2m + 6 = 0 \Rightarrow m(m-3) - 2(m-3) = 0 \Rightarrow (m-3)(m-2) = 0$

$\Rightarrow m = 3, 2$

If $m = 2$ then $l = 5 - m = 5 - 2 = 3$

If $m = 3$ then $l = 5 - m = 5 - 3 = 2$

choose $l = 3$, $m = 2$

The required parallel planes are $x + 2y - 3z + 3 = 0 \rightarrow$ ④, $x + 2y - 3z + 2 = 0 \rightarrow$ ⑤

Therefore the given equation represents a pair of parallel planes

The Distance between ④ and ⑤ is $\dfrac{|d_2 - d_1|}{\sqrt{a^2+b^2+c^2}}$

$$\rightarrow \frac{|2-3|}{\sqrt{1+4+9}} = \frac{|-1|}{\sqrt{14}} = \frac{1}{\sqrt{14}}$$

# UNIT – II

# THE LINE

**DEFINITION:** A straight line is the locus of a moving point which moves in space in a specific direction.

(OR)

The intersection of two planes together represents a straight line.

**SYMMETRIC FORM:** Equation to the line through $A(x_1, y_1, z_1)$ and having dr's $l, m, n$ is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

**UNSYMMETRIC FORM:** $a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$

**NOTE:** Equation to the line through the points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

**PARAMETRIC EQUATION OF A STRAIGHT LINE:**

Consider the straight line $L: \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ (say) where $r$ is a parameter

$\Rightarrow \frac{x-x_1}{l} = r \;;\; \frac{y-y_1}{m} = r \;;\; \frac{z-z_1}{n} = r$

$\Rightarrow x-x_1 = lr \;;\; y-y_1 = mr \;;\; z-z_1 = nr$

$\Rightarrow x = lr+x_1 \;;\; y = mr+y_1 \;;\; z = nr+z_1$

**TRANSFORMATION FROM THE UNSYMMETRICAL FORM TO THE SYMMETRICAL FORM:**

Consider the equations of a line in unsymmetrical form

$$\left.\begin{array}{l} a_1x+b_1y+c_1z+d_1=0 \\ a_2x+b_2y+c_2z+d_2=0 \end{array}\right\} \rightarrow (1)$$

To transform the equation of a line (1) into symmetrical form we have to find the following.

→ The Dr's of the line (1)

→ The coordinates of a point on the line (1)

TO FIND THE DR'S OF THE LINE (1):

Let $l, m_1, n_1$ be the dc's of the line (1)

Then $a_1 l + b_1 m + c_1 n = 0 \rightarrow (2)$, $a_2 l + b_2 m + c_2 n = 0 \rightarrow (3)$

From equations (2) & (3)

$$\begin{array}{cccc} b_1 \;{}^{l} & c_1 \;{}^{m} & a_1 \;{}^{n} & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{array}$$

$$\Rightarrow \frac{l}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - a_1c_2} = \frac{n}{a_1b_2 - b_1a_2}$$

$\therefore$ The Dr's of the line (1) are $(b_1c_2 - c_1b_2,\ c_1a_2 - a_1c_2,\ a_1b_2 - b_1a_2)$

TO FIND THE EQUATIONS TO L, WE REQUIRE A POINT ON L:

Suppose the line (1) intersects the xy-plane Then $z = 0$

from equation (1) $a_1x + b_1y + c_1z + d_1 = 0 \rightarrow$ i,
$a_2x + b_2y + c_2z + d_2 = 0 \rightarrow$ ii,

From i, and ii,

$$\begin{array}{cccc} b_1 \;{}^{x} & d_1 \;{}^{y} & a_1 \;{}^{1} & b_1 \\ b_2 & d_2 & a_2 & b_2 \end{array}$$

$$\Rightarrow \frac{x}{b_1d_2 - d_1b_2} = \frac{y}{d_1a_2 - a_1d_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\frac{x}{b_1d_2 - d_1b_2} = \frac{1}{a_1b_2 - a_2b_1} \Rightarrow x = \frac{b_1d_2 - d_1b_2}{a_1b_2 - a_2b_1}$$

$$\frac{y}{d_1a_2 - a_1d_2} = \frac{1}{a_1b_2 - a_2b_1} \Rightarrow y = \frac{d_1a_2 - a_1d_2}{a_1b_2 - a_2b_1}$$

The coordinates of one point on the line (1) are

$$\left(\frac{b_1d_2 - d_1b_2}{a_1b_2 - a_2b_1}, \frac{d_1a_2 - a_1d_2}{a_1b_2 - a_2b_1}, 0\right)$$

Therefore the equations of the line (1) in Symmetrical form are

$$\frac{x - \frac{b_1d_2 - d_1b_2}{a_1b_2 - a_2b_1}}{b_1c_2 - b_2c_1} = \frac{y - \frac{a_2d_1 - a_1d_2}{a_1b_2 - b_1a_2}}{c_1a_2 - a_1c_2} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

PROBLEMS :

1\. 1. Find the equation of the line through the points $(3,4,-7),(1,-1,6)$

Sol: Sol: Given points are $(3,4,-7),(1,-1,6)$

Equation to the line through the points $(x_1,y_1,z_1),(x_2,y_2,z_2)$

is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

$\Rightarrow \frac{x-3}{1-3}=\frac{y-4}{-1-4}=\frac{z+7}{6+7}$

$\Rightarrow \frac{x-3}{-2}=\frac{y-4}{-5}=\frac{z+7}{13}$

2\. Find the value of K if the lines $\frac{x-1}{1}=\frac{y-2}{K}=\frac{z+1}{-1}$ and $\frac{x+1}{-K}=\frac{y+1}{2}=\frac{z-2}{1}$ are perpendicular.

Sol: Given lines $\frac{x-1}{1}=\frac{y-2}{K}=\frac{z+1}{-1}\rightarrow(1)$

$\frac{x+1}{-K}=\frac{y+1}{2}=\frac{z-2}{1}\rightarrow(2)$

Since (1) and (2) are perpendicular then $l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow 1.(-K)+K(2)+(-1)(1)=0$

$\Rightarrow -K+2K-1=0$

$\Rightarrow K-1=0$

$\Rightarrow K=1$

3\. Write the equations of the line $x=ay+b$ and $z=cy+d$ in the symmetrical form.

Sol: Given line is $x=ay+b\rightarrow(1)$

$z=cy+d\rightarrow(2)$

From (1) $ay=b-x \Rightarrow y=\frac{x-b}{a}\rightarrow(3)$

From (2) $cy=z-d \Rightarrow y=\frac{z-d}{c}\rightarrow(4)$

From (3) and (4) we get $\frac{x-b}{a}=\frac{y-0}{1}=\frac{z-d}{c}$

4. Find in symmetrical form the equation of the line

$x+y+z+1=0=4x+y-2z+2$

Sol: Given line is $x+y+z+1=0=4x+y-2z+2 \rightarrow L$

let $l, m, n$ be the dr's of $L$

Then $l+m+n=0 \rightarrow (1)$

$4l+m-2n=0 \rightarrow (2)$

Solving (1) & (2)

$$\begin{array}{cccc} l & m & n & \\ 1 & 1 & 1 & 1 \\ 1 & -2 & 4 & 1 \end{array}$$

$\Rightarrow \frac{l}{-2-1} = \frac{m}{4+2} = \frac{n}{1-4}$

$\Rightarrow \frac{l}{-3} = \frac{m}{6} = \frac{n}{-3}$

If $z=0$ meets (L) Then $x+y+1=0 \rightarrow (3)$

$4x+y+2=0 \rightarrow (4)$

$-3x-1=0$

$\Rightarrow x = -1/3$

From (3) $y=-1-x \Rightarrow y=-1+\frac{1}{3} = -2/3$

Therefore the equation of the line (1) in symmetrical form is

$$\frac{x+\frac{1}{3}}{-3} = \frac{y+\frac{2}{3}}{6} = \frac{z-0}{-3}$$

5. Find the distance of the point $(1,-2,3)$ from the plane $x-y+z=5$ measured parallel to the line whose dc's are proportional to $2, 3, -6$

Sol: let $P=(1,-2,3)$

Given plane is $x-y+z=5$

Any line through $P(1,-2,3)$ and having dr's $(2,3,-6)$ is

$\frac{x-x_1}{l_1} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ (say)

$\Rightarrow \frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = r$

$\Rightarrow \frac{x-1}{2} = r, \frac{y+2}{3} = r, \frac{z-3}{-6} = r$

$\Rightarrow x-1 = 2r, y+2 = 3r, z-3 = -6r$

$\Rightarrow x = 2r+1, y = 3r-2, z = -6r+3$

let $Q = (2r+1, 3r-2, -6r+3)$

If $Q$ lies on $x-y+z=5$

Then $(2r+1)-(3r-2)+(-6r+3)=5$

$\Rightarrow 2r+1-3r+2-6r+3=5$

$\Rightarrow -7r+6=5 \Rightarrow r=1/7$

$\Rightarrow Q=(2(1/7)+1,\ 3(1/7)-2,\ -6(1/7)+3)=(9/7, -11/7, 15/7)$

$\therefore |PQ|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$=\sqrt{(9/7-1)^2+(-11/7+2)^2+(15/7-3)^2}=\sqrt{\frac{4+9+36}{49}}=\sqrt{\frac{49}{49}}=\frac{7}{7}=1$

$\therefore$ Required distance $=1$

6. Find the image of the point $(2,-1,3)$ in the plane $3x-2y+z=9$

Sol: Let $P=(2,-1,3)$

Given plane is $3x-2y+z=9$

Let $Q=(x_2, y_2, z_2)$ be the image of $P$ in the given plane.

Equation to the line through $P(1,3,4)$ and having d.r's $(3,-2,1)$ is

$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r$ (say)

$\Rightarrow \frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-3}{1}=r$

$\Rightarrow \frac{x-2}{3}=r,\ \frac{y+1}{-2}=r,\ \frac{z-3}{1}=r$

$\Rightarrow x-2=3r,\ y+1=-2r,\ z-3=r$

$\Rightarrow x=3r+2,\ y=-2r-1,\ z=r+3$

If M lies on the plane $3x-2y+z=9$

$\Rightarrow 3(3r+2)-2(-2r-1)+(r+3)=9$

$\Rightarrow 9r+6+4r+2+r+3=9$

$\Rightarrow 14r+7=9 \Rightarrow 14r=-2$

$\Rightarrow r=-1/7$

$M=\left(3\left(\frac{-1}{7}\right)+2,\ -2\left(\frac{-1}{7}\right)-1,\ -1/7+3\right)=\left(\frac{-3+14}{7}, \frac{2-7}{7}, \frac{-1+21}{7}\right)=\left(\frac{11}{7}, \frac{-5}{7}, \frac{20}{7}\right)$

Since $M=$ mid point of $PQ$

$\Rightarrow \left(\frac{11}{7}, \frac{-5}{7}, \frac{20}{7}\right)=\left(\frac{x_2+2}{2}, \frac{y_2-1}{2}, \frac{z_2+3}{2}\right)$

$\Rightarrow \frac{x_2+2}{2}=\frac{11}{7},\ \frac{y_2-1}{2}=\frac{-5}{7},\ \frac{z_2+3}{2}=\frac{20}{7}$

$\Rightarrow 7(x_2+2)=22,\ 7(y_2-1)=-10,\ 7(z_2+3)=40$

$\Rightarrow 7x_2+14=22,\ 7y_2-7=-10,\ 7z_2=19$

$\Rightarrow 7x_2=8,\ 7y_2=-3,\ z_2=19/7$

$\Rightarrow x_2 = 8/7, \quad y_2 = -3/7, \quad z_2 = 19/7$

$\therefore Q(x_2, y_2, z_2) = (8/7, -3/7, 19/7)$

7: Find the image of the point $(1,3,4)$ in the plane $2x-y+z+3=0$.

Sol: Let $P=(1,3,4)$

Given plane is $2x-y+z+3=0$

Let $Q=(x_2,y_2,z_2)$ be the image of $P$ in the given plane

Equation to the line through $P(1,3,4)$ and having d.r's $(2,-1,1)$ is

$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ (say)

$\Rightarrow \frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = r$

$\Rightarrow \frac{x-1}{2} = r, \quad \frac{y-3}{-1} = r, \quad \frac{z-4}{1} = r$

$\Rightarrow x-1 = 2r, \quad y-3 = -r, \quad z-4 = r$

$\Rightarrow x = 2r+1, \quad y = -r+3, \quad z = r+4$

Let $M = (2r+1, -r+3, r+4)$. If $M$ lies on the plane $2x-y+z+3=0$

Then $2(2r+1) - (-r+3) + (r+4) + 3 = 0$

$\Rightarrow 4r+2+r-3+r+4+3=0$

$\Rightarrow 6r+6=0 \Rightarrow r=-1$

$M = (-2+1, 1+3, -1+4) = (-1,4,3)$

Since $M$ = Mid point of $PQ$

$\rightarrow (-1,4,3) = \left(\frac{x_2+1}{2}, \frac{y_2+3}{2}, \frac{z_2+4}{2}\right)$

$\rightarrow \frac{x_2+1}{2} = -1, \quad \frac{y_2+3}{2} = 4, \quad \frac{z_2+4}{2} = 3$

$\rightarrow x_2 = -3, \quad y_2 = 5, \quad z_2 = 2$

Therefore $Q(x_2,y_2,z_2) = (-3,5,2)$

8: Find the image of the line $\frac{x-1}{9} = \frac{y-2}{-1} = \frac{z+3}{-3}$ in the plane $3x-3y+10z-26=0$

Sol: Given equation of the line is $\frac{x-1}{9} = \frac{y-2}{-1} = \frac{z+3}{-3} \rightarrow (1)$

Given equation of the plane is $3x-3y+10z-26=0 \rightarrow (2)$

(4)

To Find the image of the point $A(1,2,-3)$ in the plane:

Point on the line (1) is $A(1,2,-3)$

The equation of the line through $A(1,2,-3)$ and perpendicular to the plane (2)

is $\frac{x-1}{3}=\frac{y-2}{-3}=\frac{z+3}{10}=r \rightarrow (3)$

Any point on the line (3) is $(3r+1, -3r+2, 10r-3)$

If the point $(3r+1, -3r+2, 10r-3)$ lies on the plane (2)

Then $3(3r+1)-3(-3r+2)+10(10r-3)-26=0$

$\Rightarrow 9r+3+9r-6+100r-30-26=0$

$\Rightarrow 118r=59$

$\Rightarrow r=1/2$

The point of intersection of plane (2) and line (3) is

$\Rightarrow (3(1/2)+1, -3(1/2)+2, 10(1/2)-3)=(5/2, 1/2, 2)$

let $A'(x_1, y_1, z_1)$ be the image of $A(1,2,-3)$ in the plane (2) Then

$(5/2, 1/2, 2)$ is the midpoint of $AA'$

$$\left(5/2, 1/2, 2\right)=\left(\frac{1+x_1}{2}, \frac{2+y_1}{2}, \frac{-3+z_1}{2}\right)$$

$$\Rightarrow \frac{5}{2}=\frac{1+x_1}{2} ; \quad \frac{1}{2}=\frac{2+y_1}{2} ; \quad 2=\frac{-3+z_1}{2}$$

$$\Rightarrow x_1=4 ; \quad y_1=-1 ; \quad z_1=7$$

The image of the point $A(1,2,-3)$ in the plane (2) is $A'(4,-1,7)$

To Find the equation of image line:

The image line passes through $A'(4,-1,7)$ and is parallel to the given line whose dr's are $(9,-1,-3)$

$\therefore$ The equation of the image line is $\frac{x-4}{9}=\frac{y+1}{-1}=\frac{z-7}{-3}$

9: Find the angle between the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-4}{6}$ and

$\frac{x+1}{1}=\frac{y+2}{2}=\frac{z-4}{2}$

Sol: Given lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-4}{6} \rightarrow (1)$

$$\frac{x+1}{1}=\frac{y+2}{2}=\frac{z-4}{2} \rightarrow (2)$$

The dr's of the line (1) is $(2,3,6)$

The dr's of the line (2) is $(1,2,2)$

Then the angle between (1) & (2) is

$$\cos\theta = \frac{|l_1l_2+m_1m_2+n_1n_2|}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$$

$$= \frac{|2\cdot1+3\cdot2+6\cdot2|}{\sqrt{4+9+36}\sqrt{1+4+4}} = \frac{|20|}{\sqrt{49}\sqrt{9}} = \frac{20}{7\cdot3} = \frac{20}{21}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{20}{21}\right)$$

10: Find the angle between the lines $x+2y-2z=0=x-2y+z-7$ and $\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z}{2}$

Sol: Given lines $x+2y-2z=0=x-2y+z-7 \rightarrow (L_1)$

$$\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z}{2} \rightarrow L_2$$

let $(l_1, m_1, n_1)$ be the dr's of $L_1$

Then $l_1+2m_1-2n_1=0$

$l_1-2m_1+n_1=0$

$$\begin{array}{cccc} & l_1 & m_1 & n_1 \\ 2 & -2 & 1 & 2 \\ -2 & 1 & 1 & -2 \end{array}$$

$$\Rightarrow \frac{l_1}{2-4} = \frac{m_1}{-2-1} = \frac{n_1}{-2-2}$$

$$\Rightarrow \frac{l_1}{-2} = \frac{m_1}{-3} = \frac{n_1}{-4}$$

$\therefore l_1=2, m_1=-3, n_1=-4$

If $\theta=(L_1, L_2)$

Then $$\cos\theta = \frac{|l_1l_2+m_1m_2+n_1n_2|}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$$

$$= \frac{|-2+6-8|}{\sqrt{1+4+4}\sqrt{4+9+16}} = \frac{4}{3\sqrt{29}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{4}{3\sqrt{29}}\right)$$

11: Find the angle between the lines $x-2y+z=0=x+y-z-3$, $x+2y+z-5=0=8x+12y+5z$

Sol: Given lines are $x-2y+z=0=x+y-z-3 \rightarrow L_1$

$x+2y+z-5=0=8x+12y+5z \rightarrow L_2$

Let $(l_1, m_1, n_1)$ be the dr's of $L_1$ and
Let $(l_2, m_2, n_2)$ be the dr's of $L_2$

Then $l_1 - 2m_1 + n_1 = 0$ $\qquad$ $l_2 + 2m_2 + n_2 = 0$
$l_1 + m_1 - n_1 = 0$ $\qquad$ $8l_2 + 12m_2 + 5n_2 = 0$

$$\begin{array}{cccc} & l_1 & m_1 & n_1 \\ -2 & 1 & 1 & -2 \\ 1 & -1 & 1 & 1 \end{array} \qquad \begin{array}{cccc} & l_2 & m_2 & n_2 \\ 2 & 1 & 1 & 2 \\ 12 & 5 & 8 & 12 \end{array}$$

$\Rightarrow \frac{l_1}{2-1} = \frac{m_1}{1+1} = \frac{n_1}{1+2}$ $\qquad$ $\frac{l_2}{10-12} = \frac{m_2}{8-5} = \frac{n_2}{12-16}$

$\Rightarrow \frac{l_1}{1} = \frac{m_1}{2} = \frac{n_1}{3}$ $\qquad$ $\frac{l_2}{-2} = \frac{m_2}{3} = \frac{n_2}{-4}$

$l_1 = 1, m_1 = 2, n_1 = 3$ $\qquad$ $l_2 = -2, m_2 = 3, n_2 = -4$

If $\theta = (L_1, L_2)$

Then $\cos\theta = \frac{|l_1l_2 + m_1m_2 + n_1n_2|}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$

$= \frac{|-2+6-12|}{\sqrt{14}\sqrt{29}} = \frac{8}{\sqrt{406}}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{8}{\sqrt{406}}\right)$

**Q:** Show that the lines $2x+3y-4z=0=3x-4y+z-7$; and $5x-y-3z+12=0=x-7y+5z+06$ are parallel

**Sol:** Given lines are $2x+3y-4z=0=3x-4y+z-7 \rightarrow L_1$
$5x-y+3z+12=0=x-7y+5z+06 \rightarrow L_2$

Let $l_1, m_1, n_1$ be the dr's of $L_1$ and
Let $l_2, m_2, n_2$ be the dr's of $L_2$

$2l_1 + 3m_1 - 4n_1 = 0$ $\qquad$ $5l_2 - m_2 - 3n_2 = 0$
$3l_1 - 4m_1 + n_1 = 0$ $\qquad$ $l_2 - 7m_2 + 5n_2 = 0$

$$\begin{array}{cccc} & l_1 & m_1 & n_1 \\ 3 & -4 & 2 & 3 \\ -4 & 1 & 3 & -4 \end{array} \qquad \begin{array}{cccc} & l_2 & m_2 & n_2 \\ -1 & -3 & 5 & -1 \\ -7 & 5 & 1 & -7 \end{array}$$

$\frac{l_1}{-13} = \frac{m_1}{-14} = \frac{n_1}{-17}$ $\qquad$ $\frac{l_2}{-26} = \frac{m_2}{-28} = \frac{n_2}{-34}$

$l_1 = -13, m_1 = -14, n_1 = -17$ $\qquad$ $l_2 = -26, m_2 = -28, n_2 = -34$

$\therefore \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{1}{2}$

Therefore the given lines are parallel.

13: If the lines $x = az + b$, $y = cz + d$ and $x = a_1 z + b_1$, $y = c_1 z + d_1$ are perpendicular Then show that $aa_1 + cc_1 + 1 = 0$

Sol: Given lines $x = az + b,\ y = cz + d \rightarrow (1)$

$x = a_1 z + b_1,\ y = c_1 z + d_1 \rightarrow (2)$

from (1), $az = x - b \Rightarrow \frac{z}{1} = \frac{x-b}{a}$

$cz = y - d \Rightarrow \frac{z}{1} = \frac{y-d}{c}$

$\Rightarrow \frac{x-b}{a} = \frac{y-d}{c} = \frac{z}{1} \rightarrow (L_1)$

from (2), $a_1 z = x - b_1 \Rightarrow \frac{z}{1} = \frac{x-b_1}{a_1}$

$c_1 z = y - d_1 \Rightarrow \frac{z}{1} = \frac{y-d_1}{c_1}$

$\Rightarrow \frac{x-b_1}{a_1} = \frac{y-d_1}{c_1} = \frac{z}{1} \rightarrow (L_2)$

Given that $L_1$ and $L_2$ are perpendicular Then $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\Rightarrow aa_1 + cc_1 + 1 = 0.$

ANGLE BETWEEN A LINE AND A PLANE:

If $\theta$ is an angle between the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane $ax + by + cz + d = 0$ Then $\sin\theta = \pm \frac{al + bm + cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$

CONDITIONS FOR A LINE TO LIE IN A PLANE:

The conditions for a line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ to lie in a plane $ax + by + cz + d = 0$ are

i, $ax_1 + by_1 + cz_1 + d = 0$

ii, $al + bm + cn = 0$

14: Show that the line $\frac{x+1}{-1} = \frac{y+2}{3} = \frac{z+5}{5}$ lies in the plane $x + 2y - z = 0$

Sol: Given line is $\frac{x+1}{-1} = \frac{y+2}{3} = \frac{z+5}{5}$

Given plane is $x + 2y - z = 0$

Now $ax_1 + by_1 + cz_1 + d_1 = 0$

$\Rightarrow 1(-1) + 2(-2) + (-1)(-5) = -1 - 4 + 5 = 0$

Next, $al+bm+cn=0$

$\Rightarrow 1(-1)+2(3)+(-1)(5) = -1+6-5=0$

15: Find the distance of the point $(3,-4,5)$ from the plane $2x+5y-6z=16$ measured along a line with dc's proportional to $(2,1,-2)$

Sol: Any line through $P(3,-4,5)$ and having dr's $(2,1,-2)$ is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$$

$$\Rightarrow \frac{x-3}{2} = \frac{y+4}{1} = \frac{z-5}{-2} = r \rightarrow (1)$$

$\Rightarrow x = 2r+3,\ y = r-4,\ z = -2r+5$

Any point on (1) is $Q(2r+3, r-4, -2r+5)$

If $Q$ lies on $2x+5y-6z-16=0$

Then $2(2r+3)+5(r-4)-6(-2r+5)-16=0$

$\Rightarrow 4r+6+5r-20+12r-30-16=0$

$\Rightarrow 21r-60=0$

$\Rightarrow r = 20/7$

$Q = \left(2(20/7)+3, (20/7)-4, -2(20/7)+5\right)$

$= (61/7, -8/7, -5/7)$

$$|PQ| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \sqrt{\left(\frac{61}{7}-3\right)^2+\left(\frac{-8}{7}+4\right)^2+\left(\frac{-5}{7}-5\right)^2}$$

$$= \sqrt{\frac{1600+400+1600}{49}} = \sqrt{\frac{3600}{49}} = \frac{60}{7}$$

16: Find the equation of the plane containing the line $2x-5y+2z-6=0=2x+3y-z-5$ and parallel to the line $x = \frac{-y}{6} = \frac{z}{7}$

Sol: The equation of the line is $2x-5y+2z-6=0=2x+3y-z-5 \rightarrow (1)$

The equation of the plane containing the line (1) is $\pi_1+\lambda\pi_2=0$

$(2x-5y+2z-6)+\lambda(2x+3y-z-5)=0$

$\Rightarrow (2+2\lambda)x+(-5+3\lambda)y+(2-\lambda)z+(-6-5\lambda)=0 \rightarrow (2)$

If the plane (2) is parallel to the line $\frac{x}{1} = \frac{y}{-6} = \frac{z}{7}$

Then $(2+2\lambda)(1)+(-5+3\lambda)(-6)+(2-\lambda)(7)=0 \Rightarrow \lambda = 2$

$\therefore$ The equation of the required plane is $(2x-5y+2z-6)+2(2x+3y-z-5)=0$

$\Rightarrow 2x-5y+2z-6+4x+6y-2z-10=0$

$\Rightarrow 6x+y-16=0$

## COPLANARITY OF LINES:

**DEFINITION:** Two lines $L_1$ and $L_2$ are said to be "Coplanar" if they lie in the same plane.

**NOTE:** 1. $L_1, L_2$ are lines whose equations are

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1};\quad \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

$L_1, L_2$ are coplanar $\Rightarrow \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$

2. Equation to the plane containing the line $L_1$, with equations $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and parallel to the line $L_2$ with equations $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$

3. $L_1, L_2$ are lines whose equations are $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$; $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$. Then the equation to the plane containing $L_1, L_2$ is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$

**17:** Prove that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$; $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ are coplanar. Also find their point of intersection and the plane containing the lines

**Sol:** Given lines are $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=r$ (say) $\rightarrow$ (1)

and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}=t \rightarrow$ (2)

- Any point on the line (1) is $P(2r+1, 3r+2, 4r+3)$
- Any point on the line (2) is $Q(3t+2, 4t+3, 5t+4)$

Suppose that (1) & (2) are intersect

$\Rightarrow P=Q.$

$\Rightarrow (2r+1, 3r+2, 4r+3)=(3t+2, 4t+3, 5t+4)$

$\rightarrow 2r+1=3t+2,\ 3r+2=4t+3,\ 4r+3=5t+4$

$\Rightarrow 2r-3t-1=0 \rightarrow (3) \quad 3r-4t-1=0 \rightarrow (4) \quad 4r-5t-1=0 \rightarrow (5)$

Solving the equations (3) & (4)

$(3)\times 3 \Rightarrow \quad 6r-9t-3=0$

$(4)\times 2 \rightarrow \quad 6r-8t-2=0$

$\qquad\qquad -t-1=0$

$\qquad\qquad \Rightarrow t=-1$

from (3), $2r-3(-1)-1=0$

$\Rightarrow 2r+3-1=0$

$\Rightarrow r=-1$

clearly the values of $r$ and $t$ satisfies the equation (5)

The given lines (1) & (2) are intersect and the point of intersection is $(2(-1)+1,\ 3(-1)+2,\ 4(-1)+3)=(-1,-1,-1)$

The equation of the plane containing the lines (1) and (2) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}=0$$

$\Rightarrow (x-1)(15-16)-(y-2)(10-12)+(z-3)(8-9)=0$

$\Rightarrow -x+1+2y-4-z+3=0$

$\Rightarrow x-2y+z=0$

Hence the given lines are coplanar. The point of intersection of given lines is $(-1,-1,-1)$ and the plane containing the given lines is $x-2y+z=0$

18: prove that the lines $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}$ and $x+2y+3z-8=0=2x+3y+4z-11$ are coplanar also find their point of intersection and the plane containing the lines

Sol: Given lines are $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}=r\text{ (say)}\rightarrow L_1$

and $x+2y+3z-8=0=2x+3y+4z-11 \rightarrow L_2$

Any point on $L_1$ is $P(r-1,\ 2r-1,\ 3r-1)$

If $P$ lies on $x+2y+3z-8=0$

$\rightarrow (r-1)+2(2r-1)+3(3r-1)-8=0$

$\Rightarrow 14r-14=0$

$\Rightarrow r=1$

Therefore $P=(r-1, 2r-1, 3r-1)=(1-1, 2(1)-1, 3(1)-1)=(0,1,2)$

Now $2x+3y+4z-11=2(0)+3(1)+4(2)-11=0$

Therefore P lies on $L_2$

Therefore $L_1$ & $L_2$ are intersect at $P(0,1,2)$.

Hence $L_1$ and $L_2$ are coplanar.

Plane Equation: let $l_2, m_2, n_2$ be the dr's of $L_2$

Then $l_2+2m_2+3n_2=0 \rightarrow (1)$ and $2l_2+3m_2+4n_2=0 \rightarrow (2)$

Solving (1) and (2)

$$\begin{array}{cccc} l_2 & m_2 & n_2 & \\ 2 & 3 & 1 & 2 \\ 3 & 4 & 2 & 3 \end{array}$$

$$\Rightarrow \frac{l_2}{8-9}=\frac{m_2}{6-4}=\frac{n_2}{3-4} \Rightarrow \frac{l_2}{-1}=\frac{m_2}{2}=\frac{n_2}{-1}$$

From $L_1$, $(l_1, m_1, n_1)=(1,2,3)$

Equation to the plane containing $L_1$ and $L_2$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x+1 & y+1 & z+1 \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix}=0$$

$\Rightarrow (x+1)(-2-6)-(y+1)(-1+3)+(z+1)(2+2)=0$

$\Rightarrow 4x+y-2z+3=0$

Hence the given lines are coplanar. The point of intersection of given lines is $(0,1,2)$ and the plane containing the given lines is $4x+y-2z+3=0$

19: Show that the lines $x+2y+3z-4=0=2x+3y+4z-5$ and $2x-3y+3z-5=0=3x-2y+4z-6$ are coplanar and find the equation of the plane containing them.

Therefore clearly $Q$ lies on $(L_2)$

Therefore the lines $L_1$ and $L_2$ are intersect at $Q\left(\frac{-4}{11}, \frac{-3}{11}, \frac{18}{11}\right)$

Hence $L_1$ and $L_2$ are coplanar

Plane Equation: let $l_2, m_2, n_2$ be the d.r's of $(L_2)$

Then $2l_2-3m_2+3n_2=0 \to (5)$, $3l_2-2m_2+4n_2=0 \to (6)$

Solving (5) & (6)

$$\begin{array}{cccc} & l_2 & m_2 & n_2 \\ -3 & 3 & 2 & -3 \\ -2 & 4 & 3 & -2 \end{array}$$

$$\Rightarrow \frac{l_2}{-12+6} = \frac{m_2}{9-8} = \frac{n_2}{-4+9} \Rightarrow \frac{l_2}{-6} = \frac{m_2}{1} = \frac{n_2}{5}$$

Equation to the plane containing (L1) and (L2) is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$

$$\rightarrow \begin{vmatrix} x+2 & y-3 & z-0 \\ -1 & 2 & -1 \\ -6 & 1 & 5 \end{vmatrix} = 0$$

$\rightarrow (x+2)(11)-(y-3)(-11)+(z-0)(11)=0$

$\Rightarrow 11x+11y+11z-11=0 \Rightarrow x+y+z-1=0$

20: Find the equation of the line through the point $(1,1,1)$ and intersecting the lines $2x-y-z-2=0=x+y+z-1$ and $x-y-z-3=0=2x+4y-z-4$

Sol: Given equations of the lines are $2x-y-z-2=0=x+y+z-1 \to (L_1)$

$x-y-z-3=0=2x+4y-z-4 \to (L_2)$

Equation to the plane containing $(L_1)$ is $\pi_1+\lambda_1\pi_2=0$

$\Rightarrow (2x-y-z-2)+\lambda_1(x+y+z-1)=0$

It passes through $(1,1,1)$ Then $(2-1-1-2)+\lambda_1(1+1+1-1)=0$

$\Rightarrow -2+2\lambda_1=0 \Rightarrow \lambda_1=1$

From (1), we get $(2x-y-z-2)+1(x+y+z-1)=0$

$\rightarrow 2x-y-z-2+x+y+z-1=0$

$\rightarrow 3x-3=0 \Rightarrow x-1=0$

Equation to the plane containing $(L_2)$ is $\pi_3+\lambda\pi_4=0$

$\rightarrow (x-y-z-3)+\lambda_2(2x+4y-z-4)=0$

It passes through $(1,1,1) \Rightarrow (1-1-1-3)+\lambda_2(2+4-1-4)=0$

$\rightarrow -4+\lambda_2=0 \Rightarrow \lambda_2=4$

**Sol:** Given lines are $x+2y+3z-4=0=2x+3y+4z-5 \rightarrow L_1$

and $2x-3y+3z-5=0=3x-2y+4z-6 \rightarrow L_2$

Reduce $L_1$ into symmetric form

Let $l_1, m_1, n_1$ are the dr's of $L_1$

Then $l_1+2m_1+3n_1=0 \rightarrow (1)$ and $2l_1+3m_1+4n_1=0 \rightarrow (2)$

Solving (1) and (2)

$l_1 \quad m_1 \quad n_1$

$2 \quad 3 \quad 1 \quad 2$

$3 \quad 4 \quad 2 \quad 3$

$$\Rightarrow \frac{l_1}{8-9} = \frac{m_1}{6-4} = \frac{n_1}{3-4} \Rightarrow \frac{l_1}{-1} = \frac{m_1}{2} = \frac{n_1}{-1}$$

If $z=0$ meets $L_1$ Then $x+2y-4=0 \rightarrow (3)$, $2x+3y-5=0 \rightarrow (4)$

Solving (3) and (4)

$(3)\times 2 \Rightarrow 2x+4y-8=0$

$(4) \Rightarrow 2x+3y-5=0$

$y-3=0 \Rightarrow y=3$

From (3), $x+2(3)-4=0 \Rightarrow x+2=0 \Rightarrow x=-2$

Therefore $P(-2,3,0)$ and $(l_1,m_1,n_1)=(-1,2,-1)$

$$L_1: \frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$$

$$\frac{x+2}{-1} = \frac{y-3}{2} = \frac{z-0}{-1} = r \text{ (say)}$$

$L_2: 2x-3y+3z-5=0=3x-2y+4z-6$

Any point on $L_1$ is $Q(-r-2,\ 2r+3,\ -r)$

If $Q$ lies on $2x-3y+3z-5=0$

Then $2(-r-2)-3(2r+3)+3(-r)=5$

$\Rightarrow -2r-4-6r-9-3r=5$

$\Rightarrow r=-18/11$

Now $Q = \left(\frac{18}{11}-2,\ \frac{-36}{11}+3,\ \frac{18}{11}\right)$

$= \left(\frac{18-22}{11},\ \frac{-36+33}{11},\ \frac{18}{11}\right)$

$= \left(\frac{-4}{11},\ \frac{-3}{11},\ \frac{18}{11}\right)$

Now $3x-2y+4z-6 = 3\left(\frac{-4}{11}\right)-2\left(\frac{-3}{11}\right)+4\left(\frac{8}{11}\right)-6$

$= \frac{-12}{11}+\frac{6}{11}+\frac{72}{11}-6=0$

From (2) $(x-y-z-3)+4(2x+4y-z-4)=0$

$\Rightarrow x-y-z-3+8x+16y-4z-16=0$

$\Rightarrow 9x+15y-5z-19=0$

Equation to the required plane is $x-1=0=9x+15y-5z-19$

21: Find the equation of line with dc's proportional to $(7,4,-1)$ which intersecting the lines $x-1=-9+3y=3z+6$ and $\frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-5}{4}$

Sol: Given lines are $x-1=-9+3y=3z+6$

$\Rightarrow x-1=3(y-3)=3(z+2)$

$\Rightarrow \frac{x-1}{3}=\frac{y-3}{1}=\frac{z+2}{1}=r \text{ (say)} \rightarrow L_1$

and $\frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-5}{4}=t \text{ (say)} \rightarrow L_2$

Any point on $L_1$ is $P(3r+1, r+3, r-2)$

Any point on $L_2$ is $Q(-3t-3, 2t+3, 4t+5)$

Dr's of $\overleftrightarrow{PQ}$ are $(x_2-x_1, y_2-y_1, z_2-z_1)=(-3t-3r-4, 2t-r, 4t+7-r)$

If $\overleftrightarrow{PQ}$ is the required line then $\frac{-3t-3r-4}{7}=\frac{2t-r}{4}=\frac{4t-r+7}{-1}$

$\Rightarrow \frac{-3t-3r-4}{7}=\frac{2t-r}{4}, \quad \frac{2t-r}{4}=\frac{4t-r+7}{-1}$

$\Rightarrow -12t-12r-16=14t-7r \; ; \quad -2t+r=16t-4r+28$

$\Rightarrow 26t+5r+16=0 \rightarrow (1) \; ; \quad 18t-5r+28=0 \rightarrow (2)$

$(1)+(2) \Rightarrow 44t+44=0 \Rightarrow t=-1$

From (1), $-26+16=-5r \Rightarrow -10=-5r \Rightarrow r=2$

Therefore $P=(3r+1, r+3, r-2)=(7,5,0)$

$Q=(-3t-3, 2t+3, 4t+5)=(0,1,1)$

Equation to the required line $\overleftrightarrow{PQ}$ is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

$\Rightarrow \frac{x-7}{-7}=\frac{y-5}{4}=\frac{z-0}{1}$

22: Find the equation of the line intersecting the lines $2x+y-1=0=x-2y+3z$ and $3x-y+z+2=0=4x+5y-2z-3$ and is parallel to the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$

Sol: Given lines $2x+y-1=0=x-2y+3z \rightarrow L_1$

$3x-y+z+2=0=4x+5y-2z-3 \rightarrow L_2$

Any plane containing (1) is $\pi_1+\lambda_1\pi_2=0$

$\Rightarrow (2x+y-1)+\lambda_1(x-2y+3z)=0 \rightarrow (1) \Rightarrow (2+\lambda_1)x+(1-2\lambda_1)y+(3\lambda_1)z=0 \rightarrow (2)$

Since (2) is parallel to $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$

$\Rightarrow l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow (2+\lambda_1)\cdot 1+(1-2\lambda_1)2+3\lambda_1\cdot 3=0$

$\rightarrow 6\lambda_1+4=0 \Rightarrow \lambda_1=-2/3$

From (1) $(2x+y-1)-\frac{2}{3}(x-2y+3z)=0$

$\Rightarrow 6x+3y-3-2x+4y-6z=0$

$\rightarrow 4x+7y-6z-3=0 \rightarrow (3)$

Any plane containing (L2) is $\Pi_3+\lambda_2\Pi_4=0$

$\Rightarrow (3x-y+z+2)+\lambda_2(4x+5y-2z-3)=0 \rightarrow (4)$

$\Rightarrow (3+4\lambda_2)x+(-1+5\lambda_2)y+(1-2\lambda_2)z+(2-3\lambda_2)=0 \rightarrow (5)$

Since (5) is parallel to $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$

$\Rightarrow l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow (3+4\lambda_2)-2+10\lambda_2+3-6\lambda_2=0$

$\Rightarrow 8\lambda_2+4=0 \Rightarrow \lambda_2=-1/2$

From (4), $2(3x-y+z+2)+1(4x+5y-2z-3)=0$

$\Rightarrow 6x-2y+2z+4-4x-5y+2z+3=0$

$\Rightarrow 2x-7y+4z+7=0 \rightarrow (6)$

Equation to the required line is $4x+7y-6z-3=0=2x-7y+4z+7$

SHORTEST DISTANCE BETWEEN TWO SKEW LINES:

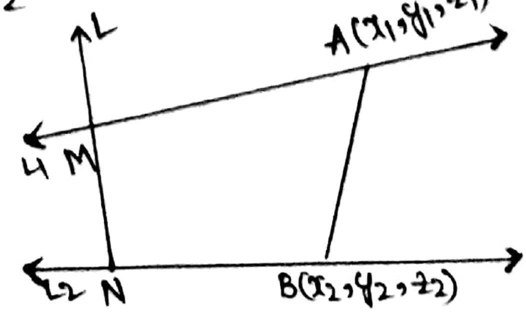
SKEW LINES: Any two non-parallel and non-intersecting lines are called skew lines. Skew lines are non-coplanar

DEFINITION: Let $L_1$ and $L_2$ be two skew lines then there is one and only line which is common perpendicular to both of them. This common perpendicular is called "shortest distance between $L_1$ and $L_2$".

NOTE: The shortest distance between two lines

$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ ; $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ is

$$\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\sum (m_1n_2-m_2n_1)^2}}$$



23: Find the magnitude and equation of the line of shortest distance between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$

Sol: Given lines are $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}=r \rightarrow (L_1)$

and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}=s\ (\text{say}) \rightarrow (L_2)$

Any point on $L_1$ is $P(3r+3, -r+8, r+3)$

Any point on $L_2$ is $Q(-3s-3, 2s-7, 4s+6)$

Let $\overleftrightarrow{PQ}$ be the line of shortest distance

Now Dr's of $\overleftrightarrow{PQ} = (x_2-x_1, y_2-y_1, z_2-z_1) = (-3s-3r-6, 2s+r-15, 4s-r+3)$

Since $\overleftrightarrow{PQ}$ is perpendicular to $(L_1)$ Then $l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow 3(-3s-3r-6)+(-1)(2s+r-15)+1(4s-r+3)=0$

$\Rightarrow -9s-9r-18-2s-r+15+4s-r+3=0$

$\Rightarrow -7s-11r=0 \Rightarrow 7s+11r=0 \rightarrow (1)$

Since $\overleftrightarrow{PQ}$ is perpendicular to $L_2$ Then $l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow (-3)(-3s-3r-6)+2(2s+r-15)+4(4s-r+3)=0$

$\Rightarrow 9s+9r+18+4s+2r-30+16s-4r+12=0$

$\Rightarrow 29s+7r=0 \rightarrow (2)$

Solving (1) and (2) $\Rightarrow$ $(1)\times 7 \Rightarrow 49s+77r=0$

$(2)\times 11 \Rightarrow 319s+77r=0$

$-270s=0 \Rightarrow s=0$

From equation (1), $7(0)+11r=0 \rightarrow r=0$

Therefore $P=(3,8,3)$, $Q=(-3,-7,6)$

Dr's of $\overleftrightarrow{PQ}$ are $(-6,-15,3)$

Therefore shortest distance between the lines $(L_1)$ and $(L_2)$

$PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}=\sqrt{(-3-3)^2+(-7-8)^2+(6-3)^2}$

$=\sqrt{36+225+9}=\sqrt{270}=3\sqrt{30}$ units

Therefore equation of the line of shortest distance between $L_1$ and $L_2$ is $\frac{x-3}{-6}=\frac{y-8}{-15}=\frac{z-3}{3}$

24: Find the magnitude and the equation of the line of shortest distance between the lines $\frac{x-3}{-1}=\frac{y-4}{2}=\frac{z+2}{1}$; $\frac{x-1}{1}=\frac{y+7}{3}=\frac{z+2}{1}$

Sol: Given lines are $\frac{x-3}{-1}=\frac{y-4}{2}=\frac{z+2}{1}=r \rightarrow (L_1)$

and $\frac{x-1}{1}=\frac{y+7}{3}=\frac{z+2}{1}=s\ (\text{say}) \rightarrow (L_2)$

Any point on $L_1$ is $P(-r+3, 2r+4, r-2)$

Any point on $L_2$ is $Q(s+1, 3s-7, s-2)$

Let $\overleftrightarrow{PQ}$ be the line of shortest distance

Now Dr's of $\overleftrightarrow{PQ} = (x_2-x_1, y_2-y_1, z_2-z_1) = (s+r-2, 3s-2r-11, s-r)$

Since $\overleftrightarrow{PQ}$ is perpendicular to (L1) Then $l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow (-1)(s+r-2)+(2)(3s-2r-11)+(1)(s-r)=0$

$\Rightarrow -s-r+2+6s-4r-22+s-r=0 \Rightarrow 6s-4r-20=0 \Rightarrow 3s-2r-10=0 \rightarrow (1)$

Since $\overleftrightarrow{PQ}$ is perpendicular to (L2) Then $l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow (s+r-2)(1)+(3s-2r-11)(3)+(s-r)(1)=0$

$\Rightarrow s+r-2+9s-6r-33+s-r=0$

$\Rightarrow 11s-6r-35=0 \rightarrow (2)$

Solving (1) and (2) $\Rightarrow$ $(1)\times 3 \Rightarrow 9s-6r-30=0$

$(2) \Rightarrow 11s-6r-35=0$

$-2s+5=0$

$\Rightarrow s=5/2$

From (1), $3(5/2)-2r-10=0 \Rightarrow 15-4r-20=0$

$\Rightarrow r=-5/4$

Therefore $P=(5/4+3, \frac{-10}{4}+4, -\frac{5}{4}-2) = (\frac{17}{4}, \frac{6}{4}, \frac{-13}{4})$

$Q=(\frac{5}{2}+1, 3(\frac{5}{2})-7, \frac{5}{2}-2) = (\frac{7}{2}, \frac{1}{2}, 1/2)$

Dr's of $\overleftrightarrow{PQ}$ are $(\frac{7}{2}-\frac{17}{4}, \frac{1}{2}-\frac{6}{4}, \frac{1}{2}+\frac{13}{4}) = (\frac{-3}{4}, -1, 15/4)$

Therefore shortest distance between the lines $L_1$ and $L_2$ is

$$PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$$= \sqrt{(\frac{-3}{4}-\frac{7}{2})^2+(-1-1/2)^2+(15/4-1/2)^2} = \sqrt{\frac{289}{16}+\frac{9}{4}+\frac{169}{16}} = \sqrt{\frac{494}{16}} = \frac{\sqrt{494}}{4}$$

Therefore equation of the line of shortest distance between $L_1$ and $L_2$ is $\frac{x-3}{(289/16)} = \frac{y-4}{(9/4)} = \frac{z+2}{(169/16)}$

25: Find the shortest distance between the lines $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$; $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$. Hence show that the lines are coplanar.

Sol: Given lines are $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}=r \rightarrow L_1$

$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=s \rightarrow L_2$

Any point on $L_1$ is $P(3r+2, 4r+3, 5r+4)$

Any point on $L_2$ is $Q(2s+1, 3s+2, 4s+3)$

Let $\overleftrightarrow{PQ}$ be the line of shortest distance

Now Dr's of $\overleftrightarrow{PQ}$ is $(x_2-x_1, y_2-y_1, z_2-z_1) = (2s-3r-1, 3s-4r-1, 4s-5r-1)$

Since $\overleftrightarrow{PQ}$ is perpendicular to $L_1$ Then $l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow 41s-50r-12=0 \rightarrow (1)$

Since $\overleftrightarrow{PQ}$ is perpendicular to $L_2$ Then $l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow 31s-38r-9=0 \rightarrow (2)$

Solving (1) and (2)

$$\begin{array}{cccc} & s & r & 1 \\ -50 & -12 & 41 & -50 \\ -38 & -9 & 31 & -38 \end{array}$$

$$\Rightarrow \frac{s}{450-456} = \frac{r}{-372+369} = \frac{1}{-1558+1550}$$

$$\Rightarrow \frac{s}{-6} = \frac{r}{-3} = \frac{1}{-8}$$

$$\Rightarrow \frac{s}{-6} = \frac{1}{-8} \; ; \; \frac{r}{-3} = \frac{-1}{8}$$

$$\Rightarrow s = 3/4 \; ; \; r = 3/8$$

Therefore $P = \left(9/8+2, \frac{12+3}{88}, \frac{15}{8}+4\right) = \left(\frac{25}{8}, \frac{9}{2}, \frac{51}{8}\right)$

$Q = \left(2\left(\frac{3}{4}\right)+1, 3(3/4)+2, 4(3/4)+3\right) = (5/2, 17/4, 8)$

Dr's of $\overleftrightarrow{PQ} = (x_2-x_1, y_2-y_1, z_2-z_1)$

$$= \left(\frac{5}{2}-\frac{25}{8}, \frac{17}{4}-\frac{9}{2}, 8-\frac{51}{8}\right) = \left(\frac{-5}{8}, \frac{-1}{4}, \frac{3}{8}\right)$$

Therefore the shortest distance between the lines $L_1$ and $L_2$ is

$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \sqrt{\left(\frac{5}{2}-\frac{25}{8}\right)^2+\left(\frac{17}{4}-\frac{9}{2}\right)^2+\left(8-\frac{51}{8}\right)^2} = \sqrt{\frac{25}{64}+\frac{1}{16}+\frac{9}{64}}$$

$$= \sqrt{\frac{25+4+9}{64}} = \frac{\sqrt{38}}{8}$$

If $L_1$ and $L_2$ are coplanar Then $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

Now $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$

$\Rightarrow 1(16-15)-1(12-10)+1(9-8)$

$\Rightarrow 1-2+1=0$

26: Find the length and equations of the line of shortest distance between the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$ and $x+y+2z-3=0=2x+3y+3z-4$

Sol: Given lines are $\frac{x}{1}=\frac{y}{2}=\frac{z}{1} \rightarrow (L_1)$

and $x+y+2z-3=0=2x+3y+3z-4 \rightarrow (L_2)$

Any plane containing $(L_2)$ is $\pi_1+\lambda\pi_2=0$

$\Rightarrow (x+y+2z-3)+\lambda(2x+3y+3z-4)=0 \rightarrow (*)$

$\Rightarrow (1+2\lambda)x+(1+3\lambda)y+(2+3\lambda)z+(-3-4\lambda)=0 \rightarrow (1)$

If (1) is parallel to $(L_1)$ Then $(1+2\lambda)(1)+(1+3\lambda)(2)+(2+3\lambda)(1)=0$

$\Rightarrow 1+2\lambda+2+6\lambda+2+3\lambda=0$

$\Rightarrow \lambda=-5/11$

From $(*)$ $11x+11y+22z-33-10x-15y-15z+20=0$

$\Rightarrow x-4y+7z-13=0 \rightarrow (2)$

Shortest distance = Distance from (0,0,0) to the plane (2)

$=\frac{|d|}{\sqrt{a^2+b^2+c^2}}=\frac{|-13|}{\sqrt{1+16+49}}=\frac{13}{\sqrt{66}}$ units

Line equation of shortest distance:

Equation to the plane containing $(L_1)$ and perpendicular to (2) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

$\Rightarrow \begin{vmatrix} x & y & z \\ 1 & 2 & 1 \\ 1 & -4 & 7 \end{vmatrix}=0 \Rightarrow x(14+4)-y(7-1)+z(-4-2)=0$

$\Rightarrow 3x-y-z=0 \rightarrow (3)$

Equation to the plane containing $(L_2)$ and perpendicular to (2)

Then $l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow (1+2\lambda).1+(1+3\lambda).(-4)+(2+3\lambda).7=0$

$\Rightarrow 11\lambda=-11 \Rightarrow \lambda=-1$

From $(*)$ $(x+y+2z-3)+(-1)(2x+3y+3z-4)=0$

$\Rightarrow x+y+2z-3-2x-3y-3z+4=0$

$\Rightarrow x+2y+z-1=0 \rightarrow (4)$

From (3) and (4) we get the equation to the required line shortest distance is $3x-y-z=0=x+2y+z-1$

27: Find the shortest distance and the equation of the line of shortest distance between the lines $3x-9y+5z=0=x+y-z$ and $6x+8y+3z-10=0=x+2y+z-3$

Sol: Given lines are $3x-9y+5z=0=x+y-z \rightarrow (L_1)$

and $6x+8y+3z-10=0=x+2y+z-3 \rightarrow (L_2)$

Now Reduce $(L_1)$ into symmetric form

Let $l_1, m_1, n_1$ be the dr's of $L_1$ Then $3l_1-9m_1+5n_1=0 \rightarrow (1)$

$l_1+m_1-n_1=0 \rightarrow (2)$

Solving (1) and (2)

$$\begin{array}{cccc} l_1 & m_1 & n_1 & \\ -9 & 5 & 3 & -9 \\ 1 & -1 & 1 & 1 \end{array}$$

$$\Rightarrow \frac{l_1}{9-5} = \frac{m_1}{5+3} = \frac{n_1}{3+9}$$

$$\Rightarrow \frac{l_1}{4} = \frac{m_1}{8} = \frac{n_1}{12} \Rightarrow \frac{l_1}{1} = \frac{m_1}{2} = \frac{n_1}{3}$$

clearly $p\ (0,0,0)$

$$L_1: \frac{x-0}{1} = \frac{y-0}{2} = \frac{z-0}{3}$$

$L_2: 6x+8y+3z-10=0=x+2y+z-3$

Any plane containing $(L_2)$ is $\pi_1+\lambda\pi_2=0$

$(6x+8y+3z-10)+\lambda(x+2y+z-3)=0 \rightarrow (*)$

$\Rightarrow (6+\lambda)x+(8+2\lambda)y+(3+\lambda)z+(-10-3\lambda)=0 \rightarrow (1)$

If (1) is parallel to $(L_1)$ Then $(6+\lambda).1+(8+2\lambda).2+(3+\lambda).3=0$

$\Rightarrow 6+\lambda+16+4\lambda+9+3\lambda=0$

$\Rightarrow 8\lambda+31=0 \Rightarrow \lambda=-31/8$

From $(*)$ $(6x+8y+3z-10)-\frac{31}{8}(x+2y+z-3)=0$

$48x+64y+24z-80-31x-62y-31z+93=0$

$\Rightarrow 17x+2y-7z+13=0 \rightarrow (2)$

Shortest distance = Distance from $p\ (0,0,0)$ to the plane $17x+2y-7z+13=0$ is $\frac{|d|}{\sqrt{a^2+b^2+c^2}} = \frac{|13|}{\sqrt{289+4+49}} = \frac{13}{\sqrt{342}}$ units

Equation to the plane containing $(L_1)$ and perpendicular to equation (2) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 17 & 2 & -7 \end{vmatrix} = 0$$

$\Rightarrow x(-14-6) - y(-7-51) + z(2-34) = 0$

$\Rightarrow -20x + 58y - 32z = 0$

$\Rightarrow 10x - 29y + 16z = 0 \rightarrow (3)$

Equation to the plane containing (L2) and perpendicular to eq (2)

is $l_1l_2 + m_1m_2 + n_1n_2 = 0$

$17(6+\lambda) + 2(8+2\lambda) + (-7)(3+\lambda) = 0$

$\rightarrow 102 + 17\lambda + 16 + 4\lambda - 21 - 7\lambda = 0$

$\Rightarrow 14\lambda = -97 \Rightarrow \lambda = -97/14$

From (*) $(6x+8y+3z-10) + \left(\frac{-97}{14}\right)(x+2y+z-3) = 0$

$\Rightarrow 84x + 112y + 42z - 140 - 97x - 194y - 97z + 291 = 0$

$\Rightarrow -13x - 82y - 55z + 151 = 0$

$\Rightarrow 13x + 82y + 55z - 151 = 0 \rightarrow (4)$

From (3) and (4) we get the equation to the required line of shortest distance is

$10x - 29y + 16z = 0 = 13x + 82y + 55z - 151$

LENGTH OF THE PERPENDICULAR FROM A POINT TO A LINE:

* If L is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and $P = (x_1, y_1, z_1)$

Then ~~length~~ length of the perpendicular from p to L is

$$\frac{1}{\sqrt{l^2+m^2+n^2}} \left[ \sum \{ m(z_1-\gamma) - n(y_1-\beta) \}^2 \right]^{1/2}$$

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28: Find the length of the perpendicular from the point (1,2,3) to the line through the point (6,7,7) whose dr's are 3,2,-2

Sol: Equation of the line through the point (6,7,7) and having the dr's (3,2,-2) is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

$$\Rightarrow \frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2} \rightarrow (1)$$

The dc's of the line (1) are $\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$

$$\Rightarrow \frac{3}{\sqrt{9+4+4}}, \frac{2}{\sqrt{9+4+4}}, \frac{-2}{\sqrt{9+4+4}}$$

i.e., $\frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}$

The length of the perpendicular from the point (1,2,3) to the line (1) is

$$\sqrt{[(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2]-[l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)]^2}$$

$$= \sqrt{[(1-6)^2+(2-7)^2+(3-7)^2]-\left[\frac{3}{\sqrt{17}}(1-6)+\frac{2}{\sqrt{17}}(2-7)+\left(\frac{-2}{\sqrt{17}}\right)(3-7)\right]^2}$$

$$= \sqrt{25+25+16-\left[\frac{-15-10+8}{\sqrt{17}}\right]^2}$$

$$= \sqrt{66-\left(\frac{-17}{\sqrt{17}}\right)^2}$$

$$= \sqrt{66-(\sqrt{17})^2}$$

$$= \sqrt{49} = 7$$

29: Find the length of the perpendicular from the point (-2,1,5) to the line $\frac{x-2}{-2} = \frac{y-3}{3} = \frac{z-5}{-6}$

Sol: Given equation of the line is $\frac{x-2}{-2} = \frac{y-3}{3} = \frac{z-5}{-6} \rightarrow (1)$

The dc's of the line (1) are

$$\Rightarrow \frac{-2}{\sqrt{4+9+36}}, \frac{3}{\sqrt{4+9+36}}, \frac{-6}{\sqrt{4+9+36}}$$

$$\Rightarrow \frac{-2}{7}, \frac{3}{7}, \frac{-6}{7}$$

$\therefore$ The length of the perpendicular from the point $(-2,1,5)$ to the line (1) is

$$\sqrt{[(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2]-[l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)]^2}$$

$$= \sqrt{[(-2-2)^2+(1-3)^2+(5-5)^2]-[\frac{-2}{7}(-2-2)+\frac{3}{7}(1-3)-\frac{6}{7}(5-5)]^2}$$

$$= \sqrt{[(-4)^2+(-2)^2+0]-[\frac{-2}{7}(-4)+\frac{3}{7}(-2)+\frac{6}{3}(0)]^2}$$

$$= \sqrt{(16+4)-(\frac{8}{7}-\frac{6}{7})^2}$$

$$= \sqrt{20-(2/7)^2}$$

$$= \sqrt{20-\frac{4}{49}}$$

$$= \sqrt{\frac{980-4}{49}}$$

$$= \sqrt{\frac{976}{49}} = \sqrt{\frac{61\times16}{49}} = \frac{4\sqrt{61}}{7}$$

# UNIT-3

## THE SPHERE

**DEFINITION:** The set of points in space which are at a constant distance $a\ (\geq 0)$ from a fixed point $C$ is called a sphere.

In other words a sphere is the locus of the points in space which are at a constant distance $a\ (\geq 0)$ from a fixed point $C$.

$C$ is called the centre and $a$ is called the radius of the sphere.

If $a=0$ the sphere is called a point sphere.

**EQUATION OF A SPHERE:**

* Equation to the sphere with centre $(x_1, y_1, z_1)$ and radius $a$ is
$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=a^2$
* The equation of the sphere whose centre is the origin and radius 'a' is
$x^2+y^2+z^2=a^2$
* The equation of the sphere in general form is $x^2+y^2+z^2+2ux+2vy+2wz+d=0$
* The equation of the sphere is centre of the sphere is $(-u,-v,-w)$ and radius of the sphere is $\sqrt{u^2+v^2+w^2-d}$

**Concentric spheres:** spheres with the same centre are known as concentric spheres. If $S=0$ is a sphere, Then it's concentric sphere is always $x^2+y^2+z^2+2ux+2vy+2wz+k=0$ Where $k$ is an unknown constant.

**PROBLEMS:**

1: Find the equation of the sphere of radius 3 concentric with the sphere $x^2+y^2+z^2-2x-2y-2z=1$

Sol: Given equation of the sphere is $x^2+y^2+z^2-2x-2y-2z-1=0 \rightarrow (1)$

$2u=-2$ , $2v=-2$ , $2w=-2$ , $d=-1$
$u=-1$ , $v=-1$ , $w=-1$ ,

centre of the sphere (1) is $(1,1,1)$

centre of concentric sphere is same as the centre of the sphere (1)

$\therefore$ Equation of the required concentric sphere is

$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=a^2$

$\Rightarrow (x-1)^2+(y-1)^2+(z-1)^2=3^2 \Rightarrow x^2+1-2x+y^2+1-2y+z^2+1-2z=9$

$\Rightarrow x^2+y^2+z^2-2x-2y-2z-6=0$

**NOTE:** The equation $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ represents a sphere if $a=b=c\neq 0$, $f=g=h=0$ and $u^2+v^2+w^2>ad$

**2:** Find the centre and radius of the sphere $x^2+y^2+z^2-6x+8y-10z+1=0$

**Sol:** Given equation of the sphere is $x^2+y^2+z^2-6x+8y-10z+1=0 \rightarrow (1)$

Here $2u=-6$ ; $2v=+8$ ; $2w=-10$ ; $d=1$
$u=-3$ ; $v=4$ ; $w=-5$ ;

centre of the sphere is $(3,-4,5)$

Radius of the sphere ① is $\sqrt{u^2+v^2+w^2-d}=\sqrt{9+16+25-1}=\sqrt{49}=7$

**3:** Find t, if the radius of the sphere $x^2+y^2+z^2+6x-8y-t=0$ is 6.

**Sol:** Given sphere is $x^2+y^2+z^2+6x-8y-t=0 \rightarrow (1)$

$2u=6$ , $2v=-8$ , $2w=0$ , $d=-t$
$u=3$ , $v=-4$ , $w=0$ ,

Radius of the sphere (1) is $\sqrt{u^2+v^2+w^2-d}=\sqrt{(3)^2+(-4)^2+(0)^2+t}=\sqrt{25+t}$

Given that radius of the sphere ① is 6.

$\Rightarrow \sqrt{25+t}=6$

squaring on Both sides

$\Rightarrow 25+t=36$

$\rightarrow t=11$

**4:** Find the equation to the sphere through $O=(0,0,0)$ and making intercepts $a,b,c$ on the axes

**Sol:** let the sphere through $O(0,0,0)$ intersects the axes $a,b,c$ at $A,B,C$

$\therefore A=(a,0,0)$ , $B=(0,b,0)$ , $C=(0,0,c)$

let the equation to the sphere through $O,A,B,C$ be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \rightarrow (1)$ $(\because d=0)$

Since (1) passes through $A(a,0,0)$

$\Rightarrow a^2+0+0+2ua+2v(0)+2w(0)=0 \Rightarrow a^2+2ua=0$

$\Rightarrow a(a+2u)=0$

$\Rightarrow a+2u=0 \Rightarrow u=-a/2$

Since (1) passes through $B(0,b,0)$

$\Rightarrow 0+b^2+0+2u(0)+2vb+2w(0)=0$

$\Rightarrow b^2+2vb=0 \rightarrow b(b+2v)=0$

$\rightarrow b+2v=0 \Rightarrow v=-b/2$

Since the sphere (1) passes through $C(0,0,c)$

$\Rightarrow 0+0+c^2+2u(0)+2v(0)+2wc=0$

$\Rightarrow c^2+2wc=0 \Rightarrow c(c+2w)=0 \Rightarrow c+2w=0 \Rightarrow w=-c/2$

$\therefore$ Equation to the sphere through the origin and making intercepts $a,b,c$ on the axes is $x^2+y^2+z^2+2(-a/2)x+2(-b/2)y+2(-c/2)z=0$

$\Rightarrow x^2+y^2+z^2-ax-by-cz=0.$

5: A plane passes through a fixed point $(a,b,c)$ and intersects the axes in $A,B,C$. Show that the centre of the sphere $OABC$ lies on $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

Sol: Let the sphere through $O,A,B,C$ be $x^2+y^2+z^2+2ux+2vy+2wz=0 \rightarrow (1)$

Since $A\in$ x-axis Then $x^2+2ux=0$ $(\because y=z=0)$

$\Rightarrow x(x+2u)=0$

$\Rightarrow x+2u=0$

$\Rightarrow x=-2u$

Since $B\in$ y-axis Then $y^2+2vy=0$ $(\because x=z=0)$

$\Rightarrow y(y+2v)=0$

$\Rightarrow y+2v=0$

$\Rightarrow y=-2v$

Since $C\in$ z-axis Then $z^2+2wz=0$ $(\because x=y=0)$

$\Rightarrow z(z+2w)=0$

$\Rightarrow z+2w=0$

$\Rightarrow z=-2w$

Equation of the plane $\overleftrightarrow{ABC}$ is $\frac{x}{-2u}+\frac{y}{-2v}+\frac{z}{-2w}=1 \rightarrow (2)$

Since the plane (2) passes through the point $(a,b,c)$ we have

$\frac{a}{-2u}+\frac{b}{-2v}+\frac{c}{-2w}=1 \Rightarrow \frac{a}{-u}+\frac{b}{-v}+\frac{c}{-w}=2$

Therefore the centre $(-u,-v,-w)$ of the sphere $OABC$ lies on $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

6: A sphere of constant radius $k$ passes through the origin and intersects the axes in $A,B,C$. Prove that the centroid of the $\Delta ABC$ lies on the sphere $9(x^2+y^2+z^2)=4k^2$

Sol: Let $OA=a$, $OB=b$, $OC=c$

$\Rightarrow$ co-ordinates of $A,B,C$ are $(a,0,0),(0,b,0),(0,0,c)$

The equation of the sphere $OABC$ is $x^2+y^2+z^2-ax-by-cz=0 \rightarrow (1)$

$2u=-a$ , $2v=-b$ , $2w=-c$

$u=-a/2$ , $v=-b/2$ , $w=-c/2$

The radius of the sphere (1) is $\sqrt{(-a/2)^2+(-b/2)^2+(-c/2)^2} = k$

$\Rightarrow \sqrt{a^2/4+b^2/4+c^2/4} = k$

$\Rightarrow \frac{a^2+b^2+c^2}{4} = k^2$

$\Rightarrow a^2+b^2+c^2 = 4k^2 \longrightarrow (*)$

Let $(x_1, y_1, z_1)$ be the centroid of $\Delta ABC$ Then

$\left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right) = (x_1, y_1, z_1)$

$\Rightarrow \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right) = (x_1, y_1, z_1)$

$\Rightarrow \frac{a}{3} = x_1, \ \frac{b}{3} = y_1, \ \frac{c}{3} = z_1$

$\Rightarrow a = 3x_1, \ b = 3y_1, \ c = 3z_1$

From (*) we get $\Rightarrow (3x_1)^2+(3y_1)^2+(3z_1)^2 = 4k^2$

$\Rightarrow 9x_1^2+9y_1^2+9z_1^2 = 4k^2$

$\Rightarrow 9(x_1^2+y_1^2+z_1^2) = 4k^2$

The locus of $(x_1, y_1, z_1)$ is $9(x^2+y^2+z^2) = 4k^2$

$\therefore$ The centroid of the $\Delta ABC$ lies on the sphere $9(x^2+y^2+z^2) = 4k^2$

7: Find the equation of the sphere through $(4,-1,2), (0,-2,3), (1,5,-1)$ $(2,0,1)$

Sol: Let the given points $A(4,-1,2), B(0,-2,3), C(1,5,-1), D(2,0,1)$

Let the equation of the sphere through $A, B, C, D$ is

$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \longrightarrow (1)$

Since the sphere (1) passes through $A(4,-1,2)$ is

$(4)^2+(-1)^2+(2)^2+2u(4)+2v(-1)+2w(2)+d=0$

$\Rightarrow 16+1+4+8u-2v+4w+d=0$

$\Rightarrow 8u-2v+4w+d+21=0 \longrightarrow (2)$

Since the sphere (1) passes through $B(0,-2,3)$ is

$0+4+9+0+2v(-2)+2w(3)+d=0$

$\Rightarrow -4v+6w+d+13=0 \longrightarrow (3)$

Since the sphere (1) passes through $C(1,5,-1)$ is

$\Rightarrow 1+25+1+2u(1)+2v(5)+2w(-1)+d=0$

$\Rightarrow 2u+10v-2w+d+27=0 \longrightarrow (4)$

Since the sphere (1) passes through $D(2,0,1)$ is

$\Rightarrow 4+0+1+2u(4)+2v(0)+2w(1)+d=0$

$\Rightarrow 4u+2w+d+5=0 \rightarrow (5)$

Solving the equations (2), (3), (4) and (5)

$(2) \Rightarrow 8u-2v+4w+d=-21$

$(3) \Rightarrow -4v+6w+d=-13$

$8u+2v-2w=-8 \rightarrow (6)$

$(4) \Rightarrow 2u+10v-2w+d=-27$

$(5) \Rightarrow 4u+0v+2w+d=-5$

$-2u+10v-4w=-22 \rightarrow (7)$

$(6)\times 2 \Rightarrow 16u+4v-4w=-16$

$(7) \Rightarrow -2u+10v-4w=-22$

$18u-6v=6$

$\Rightarrow 3u-v=1 \rightarrow (8)$

$(3) \Rightarrow -4v+6w+d=-13$

$(5) \Rightarrow -4u+2w+d=-5$

$-4u-4v+4w=-8$

$\Rightarrow -u-v+w=-2 \rightarrow (9)$

$(2) \Rightarrow 8u-2v+4w+d=-21$

$(4) \Rightarrow 2u+10v-2w+d=-27$

$6u-12v+6w=6$

$\Rightarrow u-2v+w=1 \rightarrow (10)$

$(9) \Rightarrow -u-v+w=-2$

$(10) \Rightarrow u-2v+w=1$

$-2u+v=-3 \rightarrow (11)$

$(8) \Rightarrow 3u-v=1$

$(11) \Rightarrow -2u+v=-3$

$\boxed{u=-2}$

From (11), $-2(-2)+v=-3$

$\Rightarrow 4+v=-3$

$\Rightarrow \boxed{v=-7}$

From (9),

$-(-2)-(-7)+w=-2$

$\Rightarrow 2+7+w=-2$

$\Rightarrow w=-2-9$

$\Rightarrow \boxed{w=-11}$

From (5) $4(-2)+2(-11)+d=-5$

$\Rightarrow -8-22+d=-5$

$\Rightarrow d=25$

$\therefore$ The equation of the sphere through A, B, C and D be

$$x^2+y^2+z^2+2(-2)x+2(-7)y+2(-11)z+25=0$$

$$\Rightarrow x^2+y^2+z^2-4x-14y-22z+25=0$$

<u>8:</u> Find the equation of the sphere through the points $(1,-4,+3)$, $(1,-5,2)$, $(1,-3,0)$ and whose centre lies on the plane $x+y+z=0$

<u>Sol:</u> Let the given points $A(1,-4,3)$, $B(1,-5,2)$, $C(1,-3,0)$

Let the equation of the sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0 \rightarrow (1)$

centre of the sphere is $(-u,-v,-w)$

Since the centre of the sphere (1) lies on the plane $x+y+z=0$

$$\Rightarrow -u-v-w=0 \Rightarrow u+v+w=0 \rightarrow (2)$$

Since the sphere (1) passes through $A(1,-4,3)$

$$\Rightarrow 1+16+9+2u(1)+2v(-4)+2w(3)+d=0$$

$$\Rightarrow 2u-8v+6w+d+26=0 \rightarrow (3)$$

Since the sphere (1) passes through $B(1,-5,2)$

$$\Rightarrow 1+25+4+2u(1)+2v(-5)+2w(2)+d=0$$

$$\Rightarrow 2u-10v+4w+d+30=0 \rightarrow (4)$$

Since the sphere (1) passes through $C(1,-3,0)$

$$\Rightarrow 1+9+0+2u(1)+2v(-3)+2w(0)+d=0$$

$$\Rightarrow 2u-6v+d+10=0 \rightarrow (5)$$

solving the equation (3), (4) and (5)

$$(3) \Rightarrow \quad 2u-8v+6w+d+26=0$$

$$(4) \Rightarrow \quad 2u-10v+4w+d+30=0$$

$$2v+2w-4=0$$

$$\Rightarrow v+w-2=0 \rightarrow (6)$$

$$(2) \Rightarrow \quad u+v+w=0$$

$$(6) \Rightarrow \quad 0u+v+w-2=0$$

$$u+2=0 \Rightarrow u=-2$$

$$(4) \Rightarrow \quad 2u-10v+4w+d+30=0$$

$$(5) \Rightarrow \quad 2u-6v+0w+d+10=0$$

$$-4v+4w+20=0$$

$$\Rightarrow -v+w+5=0 \rightarrow (7)$$

$(6) \Rightarrow \quad v+w-2=0$

$(7) \Rightarrow \quad -v+w+5=0$

$$2w+3=0$$

$$w=-3/2$$

From (6), $\quad v-3/2-2=0$

$\Rightarrow 2v-3-4=0$

$\Rightarrow 2v=7 \quad \Rightarrow v=7/2$

From eqn (5), $2(-2)-6(7/2)+d+10=0$

$\Rightarrow -4-21+d+10=0$

$\Rightarrow d=15$

$\therefore$ The equation of the required sphere be

$x^2+y^2+z^2+2(-2)x+2(7/2)y+2(-3/2)z+15=0$

$\Rightarrow x^2+y^2+z^2-4x+7y-3z+15=0$

9: Find the equation of the sphere through the points $(1,0,0)$, $(0,1,0)$ $(0,0,1)$ and having the least radius

Sol: Let the equation of the sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0 \rightarrow (1)$

Since the sphere (1) passes through $A(1,0,0)$ is

$1+0+0+2u(1)+2v(0)+2w(0)+d=0$

$\Rightarrow 2u+d+1=0 \Rightarrow 2u=-(1+d) \Rightarrow u=\frac{-(1+d)}{2}$

Since the sphere (1) passes through $B(0,1,0)$ is

$0+1+0+2u(0)+2v(1)+2w(0)+d=0$

$\Rightarrow 2v+d+1=0 \Rightarrow 2v=-(1+d) \rightarrow v=\frac{-(1+d)}{2}$

Since the sphere (1) passes through $C(0,0,1)$ is

$0+0+1+2u(0)+2v(0)+2w(1)+d=0$

$\Rightarrow 2w+d+1=0 \Rightarrow 2w=-(1+d) \Rightarrow w=\frac{-(1+d)}{2}$

Let $r$ be the radius of the sphere (1)

$r^2=u^2+v^2+w^2-d$

$r^2=\left(\frac{1+d}{2}\right)^2+\left(\frac{1+d}{2}\right)^2+\left(\frac{1+d}{2}\right)^2$

$r^2=\frac{1}{4}(1+d^2+2d+1+d^2+2d+1+d^2+2d-4d)$

$r^2=\frac{1}{4}(3d^2+2d+3)$

$r^2=\frac{3}{4}\left(d^2+\frac{2}{3}d+1\right)$

$r^2=\frac{3}{4}\left(d^2+\frac{2}{3}d+1/9-1/9+1\right)$

$r^2 = \frac{3}{4}(d+1/3)^2 + 8/9$

The radius of the sphere (1) is least ~~maximum~~ if $(d+1/3)^2 = 0$

$\Rightarrow d + 1/3 = 0 \Rightarrow d = -1/3$

Therefore $u = -\left(\frac{1+d}{2}\right) = -\left(\frac{1-1/3}{2}\right) = \frac{-2/3}{2} = -1/3$

$v = -\left(\frac{1+d}{2}\right) = -\left(\frac{1-1/3}{2}\right) = \frac{-2/3}{2} = -1/3$

$w = -\left(\frac{1+d}{2}\right) = -\left(\frac{1-1/3}{2}\right) = \frac{-2/3}{2} = -1/3$

Therefore the equation of the required sphere is

$x^2+y^2+z^2+2(-1/3)x+2(-1/3)y+2(-1/3)z-1/3=0$

$\Rightarrow x^2+y^2+z^2-2/3x-\frac{2}{3}y-\frac{2}{3}z-\frac{1}{3}=0$

$\Rightarrow 3(x^2+y^2+z^2)-2x-2y-2z-1=0.$

10: A plane passes through a fixed point $(a,b,c)$. Show that the locus of the foot of the perpendicular drawn to it from the origin is the sphere $x^2+y^2+z^2-ax-by-cz=0$.

Sol: Equation of a plane passes through $(a,b,c)$ and having the dr's as $a_1, b_1, c_1$ is $a_1(x-a)+b_1(y-b)+c_1(z-c)=0 \rightarrow (1)$

Let $P(x_1,y_1,z_1)$ be the foot of the perpendicular from origin to the plane

Since $\overleftrightarrow{OP}$ is perpendicular to the plane (1)

The equation of $\overleftrightarrow{OP}$ is $\frac{x}{a_1}=\frac{y}{b_1}=\frac{z}{c_1}=r \rightarrow (2)$

Any point on the line (2) is $(a_1r, b_1r, c_1r)$.

Since $P(x_1,y_1,z_1)$ lies on the plane (1).

$a_1(x_1-a)+b_1(y_1-b)+c_1(z_1-c)=0 \rightarrow (3)$

If $(a_1r, b_1r, c_1r) = (x_1,y_1,z_1)$ Then

$a_1r = x_1 \qquad b_1r = y_1 \qquad c_1r = z_1$

$a_1 = \frac{x_1}{r} \qquad b_1 = \frac{y_1}{r} \qquad c_1 = \frac{z_1}{r}$

From equation (3) $\frac{x_1}{r}(x_1-a)+\frac{y_1}{r}(y_1-b)+\frac{z_1}{r}(z_1-c)=0$

$\Rightarrow x_1^2-ax_1+y_1^2-by_1+z_1^2-cz_1=0$

$\Rightarrow x_1^2+y_1^2+z_1^2-x_1a-y_1b-z_1c=0$

Therefore the locus of $P(x_1,y_1,z_1)$ is

$x^2+y^2+z^2-ax-by-cz=0.$

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## PLANE SECTIONS OF A SPHERE:

**DEFINITION:** Consider a sphere and a plane. The non-empty set of points common to the sphere and the plane is called a plane section of the sphere. Then we say that the plane intersects the sphere.

**DEFINITION: (GREAT CIRCLE)**

If a plane passes through the centre of a sphere then the plane section of the sphere is called a "Great Circle".

The centre and radius of the great circle are respectively the centre and radius of the sphere

**DEFINITION: (SMALL CIRCLE)**

If a plane does not pass through the centre of a sphere and intersects the sphere then the plane section is called a "small circle".

The centre of the small circle is the foot of the perpendicular from the centre of the sphere to the plane and radius of the small circle is less than the radius of the sphere.

**CONDITION FOR A PLANE TO INTERSECT A SPHERE:**

The sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ will intersect the plane $lx+my+nz=p$ if and only if $\frac{|l(-u)+m(-v)+n(-w)-p|}{\sqrt{l^2+m^2+n^2}} \le \sqrt{u^2+v^2+w^2-d}$

i.e, if and only if $(lu+mv+nw+p)^2 \le (l^2+m^2+n^2)(u^2+v^2+w^2-d)$

**INTERSECTION OF TWO SPHERES:**

Suppose two spheres intersect then the locus of the points of the intersection of two spheres is a circle.

**NOTE:** * $S = x^2+y^2+z^2+2ux+2vy+2wz+d=0$,

$S' = x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$ are two spheres intersecting in a circle. Then equations to circle 'c' are $S=0, S-S'=0$ (or) $S'=0, S-S'=0$

* $U=0$ is a plane and $S=0$ is a sphere such that the plane section of the sphere is the circle 'c'. Then the equations $S=0, U=0$ together represents the circle 'c'.

11: Find the centre and radius of the circle $x^2+y^2+z^2-2y-4z=11$, $x+2y+2z=15$

Sol: Given equation of the circle is $x^2+y^2+z^2-2y-4z=11 \rightarrow (1)$

and $x+2y+2z=15 \rightarrow (2)$

centre of the sphere (1) is $(-u,-v,-w)=(0,1,2)$

Radius of the sphere (1) is $\sqrt{u^2+v^2+w^2-d}=\sqrt{0+(-1)^2+(-2)^2+11}=\sqrt{1+4+11}=\sqrt{16}=4$

The equation of a line through the centre of the sphere (1) and perpendicular to the plane (2) is $\frac{x-0}{1}=\frac{y-1}{2}=\frac{z-2}{2}=r \rightarrow (3)$

Any point on the line (3) is $(r, 2r+1, 2r+2)$

If the point $(r,2r+1,2r+2)$ lies on the plane (2) then $r+2(2r+1)+2(2r+2)=15$

$\Rightarrow r+4r+2+4r+4=15 \Rightarrow 9r=9 \Rightarrow r=1$

$\therefore$ The centre of the circle is $(1,3,4)$

Let M be the perpendicular distance from the centre of the sphere (1) to the plane (2) is $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

$$\Rightarrow \frac{|1(0)+2(1)+2(2)-15|}{\sqrt{1^2+2^2+2^2}}=\frac{|2+4-15|}{\sqrt{9}}=\frac{|-9|}{3}=3$$

$\therefore$ Radius of the circle is $\sqrt{r^2-m^2}=\sqrt{4^2-3^2}=\sqrt{16-9}=\sqrt{7}$

12: Prove that the circle $x^2+y^2+z^2-4x-2y+5z+6=0$, $x+y+2z+2=0$ is a great circle.

Sol: Given equation of the circle is $x^2+y^2+z^2-4x-2y+5z+6=0 \rightarrow (1)$

and $x+y+2z+2=0 \rightarrow (2)$

centre of the sphere (1) is $(-u,-v,-w)=(2,1,-5/2)$

from (2), $2+1+2(-5/2)+2=2+1-5+2=0$

$\therefore$ centre of the sphere (1) lies on the plane (2)

Therefore, it is a great circle

DEFINITION:

Equation to the sphere having $A=(x_1,y_1,z_1)$ and $B=(x_2,y_2,z_2)$ as the ends of the diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$.

13: Find the equation of the sphere with (1,2,3) and (2,3,4) as the ends of a diameter.

Sol: The equation of a sphere with (1,2,3),(2,3,4) as the ends of a diameter is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

$\Rightarrow (x-1)(x-2)+(y-2)(y-3)+(z-3)(z-4)=0$

$\Rightarrow x^2-2x-x+2+y^2-3y-2y+6+z^2-4z-3z+12=0$

$\Rightarrow x^2+y^2+z^2-3x-5y-7z+20=0$

SPHERE THROUGH A GIVEN CIRCLE:

let $S=0, U=0$ be the equations of circle then any sphere through the circle is $S+\lambda U=0$

If $\lambda$ is a constant then $S+\lambda U=0$ represents a sphere through the given circle. If $\lambda$ is a parameter. Then $S+\lambda U=0$ represents a system of spheres which pass through the given circle.

14: Show that the four points $(-8,5,2), (-5,2,2), (-7,6,6), (-4,3,6)$ are concyclic.

Sol: let $A(-8,5,2), B(-5,2,2), C(-7,6,6), D(-4,3,6)$

To show that A,B,C,D are concyclic

If any circle passes through any three of the points then the fourth point lies on the same circle.

For this We can find

i, The equation of the plane $\overleftrightarrow{ABC}$ ii, The equation of the sphere OABC

let the equation of the plane ABC be $a(x-x_1)+b(y-y_1)+c(z-z_1)=0 \rightarrow (1)$

Since the plane (1) passes through the point $(-8,5,2)$

$\Rightarrow a(x+8)+b(y-5)+c(z-2)=0 \rightarrow (2)$

Since the plane (2) passes through the point $B(-5,2,2) \Rightarrow a(-5+8)+b(2-5)+c(2-2)=0$

$\Rightarrow 3a-3b=0 \Rightarrow a-b=0 \rightarrow (3)$

Since the plane (2) passes through the point $C(-7,6,6) \Rightarrow a(-7+8)+b(6-5)+c(6-2)=0$

$\Rightarrow a+b+4c=0 \rightarrow (4)$

Solving (3) and (4)

$$\begin{array}{cccc} a & b & c & \\ -1 & 0 & 1 & -1 \\ 1 & 4 & 1 & 1 \end{array} \Rightarrow \frac{a}{-4+0}=\frac{b}{0-4}=\frac{c}{1+1}$$

$\Rightarrow \frac{a}{-4}=\frac{b}{-4}=\frac{c}{2}$

The equation of the plane ABC is $-4(x+8)-4(y-5)+2(z-2)=0$

$\Rightarrow -4x-32-4y+20+2z-4=0$

$\Rightarrow -4x-4y+2z-16=0 \Rightarrow 2x+2y-z+8=0 \rightarrow (5)$

(ii) Let the equation of the sphere OABC be $x^2+y^2+z^2+2ux+2vy+2wz=0 \rightarrow (6)$

Since the sphere (6) passes through $A(-8,5,2)$

$\Rightarrow 64+25+4-16u+10v+4w=0 \Rightarrow -16u+10v+4w+93=0 \rightarrow (7)$

Since the sphere (6) passes through $B(-5,2,2)$

$\rightarrow 25+4+4-10u+4v+4w=0 \Rightarrow -10u+4v+4w+33=0 \rightarrow (8)$

Since the sphere (6) passes through $C(-7,6,6)$

$\rightarrow 49+36+36-14u+12v+12w=0 \Rightarrow -14u+12v+12w+121=0 \rightarrow (9)$

Solving (7), (8), (9)

$(7) \Rightarrow -16u+10v+4w+93=0$

$(8) \Rightarrow -10u+4v+4w+33=0$

$-6u+6v+60=0$

$\Rightarrow -u+v+10=0 \rightarrow (10)$

$(8)\times 3 \Rightarrow -30u+12v+12w+99=0$

$(9) \Rightarrow -14u+12v+12w+121=0$

$-16u-22=0$

$\Rightarrow 8u+11=0$

$u=-11/8$

From equation (10), $-(-11/8)+v+10=0 \rightarrow 11+8v+80=0$

$\rightarrow 8v+91=0$

$\Rightarrow v=-91/8$

From equation (8), $-10\left(\frac{-11}{8}\right)+4(-91/8)+4w+33=0$

$\Rightarrow 55-182+16w+132=0$

$\Rightarrow 16w=-5$

$\Rightarrow w=-5/16$

The equation of the sphere OABC is

$x^2+y^2+z^2+2(-11/8)x+2(-91/8)y+2(-5/16)z=0$

$\Rightarrow 8x^2+8y^2+8z^2-22x-182y-5z=0 \rightarrow (11)$

Clearly the point $D(-4,3,6)$ lies on the plane (5) and sphere (11)

$\therefore$ The given points A, B, C, D are concyclic.

15: Find the equations of the spheres passing through the circle $x^2+y^2=4, z=0$ and is intersected by the plane $x+2y+2z=0$ in a circle of radius 3.

Sol: Any sphere through the circle $x^2+y^2=4, z=0$ is $(x^2+y^2-4)+\lambda(z)=0 \rightarrow (1)$

Centre of the sphere (1) is $(0,0,-\lambda/2)$

Radius of the sphere (1) is $\sqrt{0+0+(-\lambda/2)^2+4}=\sqrt{\lambda^2/4+4}$

The perpendicular distance from the centre of sphere (1) to the plane $x+2y+2z=0$ is $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}=0$

$$= \frac{|1.(0)+2(0)+2(-\lambda/2)|}{\sqrt{1+4+4}} = \frac{\lambda}{3}.$$

$\therefore$ Radius of the circle formed by (1) and the plane

$x+2y+2z=0$ is $\sqrt{\left(\sqrt{\lambda^2/4+4}\right)^2 - \lambda^2/9} = 3$

$$\Rightarrow \sqrt{\frac{\lambda^2}{4}+4-\lambda^2/9} = 3$$

$$\Rightarrow \sqrt{\frac{9\lambda^2+144-4\lambda^2}{36}} = 3$$

$$\Rightarrow \sqrt{\frac{5\lambda^2+144}{36}} = 3$$

$$\Rightarrow \frac{5\lambda^2+144}{36} = 9 \Rightarrow 5\lambda^2+144 = 324$$

$$\Rightarrow 5\lambda^2 = 180$$

$$\Rightarrow \lambda^2 = 36$$

$$\Rightarrow \lambda = \pm 6$$

$\therefore$ The equations of the required spheres are $x^2+y^2+z^2 \pm 6z - 4 = 0$

16: prove that the plane $x+2y-z=4$ intersects the sphere $x^2+y^2+z^2-x+z-2=0$ in a circle of radius unity. Also find the equation of the sphere which has this circle for one of the great circles.

Sol: Given equation of the plane is $x+2y-z-4=0 \rightarrow (1)$

Given equation of the sphere is $x^2+y^2+z^2-x+z-2=0 \rightarrow (2)$

$2u=-1$, $2v=0$, $2w=1$, $d=-2$

$u=-1/2$, $v=0$, $w=1/2$,

Centre of the sphere (2) is $C = (1/2, 0, -1/2)$

Radius of the sphere (2) is $a = \sqrt{\frac{1}{4}+0+\frac{1}{4}+2} = \sqrt{\frac{1}{2}+2} = \sqrt{5/2}$

Let M be the centre of the circle given by (1) and (2)

$$CM = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} = \frac{|1(1/2)+2(0)+(-1)(-1/2)-4|}{\sqrt{1+4+1}} = \frac{|-3|}{\sqrt{6}} = \frac{3}{\sqrt{6}}$$

Let $r$ be the radius of the circle. $(CM)^2 + r^2 = a^2$

$$\Rightarrow r^2 = a^2 - (CM)^2 = (\sqrt{5/2})^2 - (3/\sqrt{6})^2 = 5/2 - 9/6 = \frac{15-9}{6} = 1$$

$$\Rightarrow r^2 = 1 \Rightarrow r = 1$$

Any sphere through the circle given by (1) and (2) is

$(x^2+y^2+z^2-x+z-2)+\lambda(x+2y-z-4)=0$

$\rightarrow x^2+y^2+z^2+(-1+\lambda)x+2\lambda y+(1-\lambda)z+(-2-4\lambda)=0 \rightarrow (3)$

$2u=-1+\lambda \quad 2v=2\lambda \quad 2w=1-\lambda \quad d=-2-4\lambda$

$u=\frac{-1+\lambda}{2} \quad v=\lambda \quad w=\frac{1-\lambda}{2}$

centre of the sphere (3) is $\left(\frac{-(-1+\lambda)}{2}, -\lambda, \frac{-(1-\lambda)}{2}\right)$

If the circle is a great circle then the centre of the sphere (3) must lies on the plane (1) $\frac{-(-1+\lambda)}{2}+2(-\lambda)+\left(\frac{1-\lambda}{2}\right)-4=0$

$\Rightarrow 1-\lambda-4\lambda+1-\lambda-8=0 \Rightarrow -6\lambda-6=0 \Rightarrow \lambda=-1$

$\therefore$ Equation to the required sphere is $(x^2+y^2+z^2-x+z-2)-1(x+2y-z-4)=0$

$\rightarrow x^2+y^2+z^2-2x-2y+2z+2=0.$

17: Show that the two circles $x^2+y^2+z^2-y+2z=0$, $x-y+z=2$ and $x^2+y^2+z^2+x-3y+z-5=0$, $2x-y+4z-1=0$ lies on the same sphere and find it's equation

Sol: The equation of the given circles are $x^2+y^2+z^2-y+2z=0, x-y+z-2=0 \rightarrow (1)$

and $x^2+y^2+z^2+x-3y+z-5=0, \; 2x-y+4z-1=0 \rightarrow (2)$

Any sphere through the circle (1) is $(x^2+y^2+z^2-y+2z)+\lambda(x-y+z-2)=0$

$\Rightarrow x^2+y^2+z^2+x\lambda+(-1-\lambda)y+(2+\lambda)z-2\lambda=0 \rightarrow (3)$

Any sphere through the circle (2) is $(x^2+y^2+z^2+x-3y+z-5)+\mu(2x-y+4z-1)=0$

$\Rightarrow x^2+y^2+z^2+(1+2\mu)x+(-3-\mu)y+(1+4\mu)z+(-5-\mu)=0 \rightarrow (4)$

If (3) and (4) represents the same sphere then

$\lambda=1+2\mu \quad -1-\lambda=-3-\mu \quad 2+\lambda=1+4\mu \quad -2\lambda=-5-\mu$

$\rightarrow \lambda-2\mu=1 \rightarrow i, \quad -\lambda+\mu=-2 \rightarrow ii, \quad \lambda-4\mu=-1 \rightarrow iii, \quad -2\lambda+\mu=-5 \rightarrow iv,$

solving the equation i, and ii,

i, $\Rightarrow \lambda-2\mu=1$

ii, $\Rightarrow -\lambda+\mu=-2$

$-\mu=-1 \Rightarrow \mu=1$

From equation i, $\lambda-2(1)=1 \Rightarrow \lambda=3$

clearly the values of $\lambda,\mu$ satisfy the equations iii, and iv,

$\therefore$ The given circles (1) and (2) lies on the same sphere and it's equation is $(x^2+y^2+z^2-y+2z)+3(x-y+z-2)=0$

$\rightarrow x^2+y^2+z^2+3x-4y+5z-6=0.$

18: Find the equation of the sphere through the circle $x^2+y^2+z^2=9$, $2x+3y+2z=5$ and the point $(-1,-2,3)$

Sol: The equation of a given circle is $x^2+y^2+z^2-9=0,\ 2x+3y+2z-5=0 \rightarrow 1,$

Any sphere through the circle (1) is $S+\lambda U=0$

$\rightarrow (x^2+y^2+z^2-9)+\lambda(2x+3y+2z-5)=0 \rightarrow (2)$

If the sphere (2) passes through the point $(-1,-2,3)$

$(1+4+9-9)+\lambda(-2-6+6-5)=0$

$5+\lambda(-7)=0 \Rightarrow \lambda=5/7$

The equation of the required sphere is

$(x^2+y^2+z^2-9)+\frac{5}{7}(2x+3y+2z-5)=0$

$\Rightarrow 7(x^2+y^2+z^2-9)+5(2x+3y+2z-5)=0$

$\rightarrow 7x^2+7y^2+7z^2-63+10x+15y+10z-25=0$

$\Rightarrow 7x^2+7y^2+7z^2+10x+15y+10z-88=0$

19: prove that the circles $x^2+y^2+z^2-2x+3y+4z-5=0$, $5y+6z+1=0$, $x^2+y^2+z^2-3x-4y+5z-6=0, x+2y-7z=0$ lies on the same sphere and find it's equation. Also find the value of 'a' for which $x+y+z=\sqrt{3}a$ touches the sphere.

Sol: The equation of the given circles are

$x^2+y^2+z^2-2x+3y+4z-5=0,\ 5y+6z+1=0 \rightarrow (1)$

$x^2+y^2+z^2-3x-4y+5z-6=0,\ x+2y-7z=0 \rightarrow (2)$

Any sphere through the circle (1) is $S+\lambda U=0$

$(x^2+y^2+z^2-2x+3y+4z-5)+\lambda(5y+6z+1)=0$

$\Rightarrow x^2+y^2+z^2+(-2)x+(3+5\lambda)y+(4+6\lambda)z+(-5+\lambda)=0 \rightarrow (3)$

Any sphere through the circle (2) is $S+\mu U=0$ is

$(x^2+y^2+z^2-3x-4y+5z-6)+\mu(x+2y-7z)=0$

$\Rightarrow x^2+y^2+z^2+(-3+\mu)x+(-4+2\mu)y+(5-7\mu)z-6=0 \rightarrow (4)$

If (3) and (4) represents the same sphere then

$-2=-3+\mu$, $3+5\lambda=-3+\mu$, $4+6\lambda=5-7\mu$

$\mu-1=0$, $3+5\lambda=-3+1$, $6\lambda+7\mu=1$

$5\lambda=-5$,

$\mu=1$, $\lambda=-1$

clearly the values of $\lambda, \mu$ satisfy the equations

$$5\lambda - 2\mu = -7 \quad , \quad 6\lambda + 7\mu = 1$$

$\therefore$ The given circles (1) and (2) lies on the same sphere and it's equation is $(x^2+y^2+z^2-2x+3y+4z-5) - 1(5y+6z+1) = 0$

$$\Rightarrow x^2+y^2+z^2-2x+3y+4z-5-5y-6z-1=0$$

$$\Rightarrow x^2+y^2+z^2-2x-2y-2z-6=0 \longrightarrow (5)$$

$$2u=-2 \quad , \quad 2v=-2 \quad , \quad 2w=-2 \quad , \quad d=-6$$

$$u=-1 \quad , \quad v=-1 \quad , \quad w=-1 \quad ,$$

Centre of the sphere (5) is $(1,1,1)$

Radius of the sphere (5) is $\sqrt{1^2+1^2+1^2+6} = \sqrt{1+1+1+6} = \sqrt{9} = 3$

If the plane $x+y+z-a\sqrt{3}=0$ touches the sphere (5) then the perpendicular distance from the centre of the sphere (5) to the plane $x+y+z-a\sqrt{3}=0$ is equal to the radius of the sphere (5)

i.e., $$\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} = \text{radius of the sphere}$$

$$\Rightarrow \frac{|1(1)+1(1)+1(1)-a\sqrt{3}|}{\sqrt{1^2+1^2+1^2}} = 3$$

$$\Rightarrow \frac{|3-a\sqrt{3}|}{\sqrt{3}} = 3 \Rightarrow 3-a\sqrt{3} = \pm 3\sqrt{3}$$

$$\Rightarrow 3-a\sqrt{3} = 3\sqrt{3} \quad , \quad 3-a\sqrt{3} = -3\sqrt{3}$$

$$\Rightarrow \sqrt{3}-a = 3 \quad , \quad \sqrt{3}-a = -3$$

$$\Rightarrow a = \sqrt{3}-3 \quad , \quad a = 3+\sqrt{3}$$

## INTERSECTION OF A SPHERE AND A PLANE :

**NOTATION:** $S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d = F(x,y,z)$

$$S_1 \equiv xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d$$

$$S_2 \equiv xx_2+yy_2+zz_2+u(x+x_2)+v(y+y_2)+w(z+z_2)+d$$

$$S_{11} \equiv x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d = F(x_1,y_1,z_1)$$

**DEFINTION:** If line L through a given point B has only one common point T with a given sphere, Then $L(=\overleftrightarrow{BT})$ is Called a tangent line to the sphere from B. T is called the point of contact of the tangent line $\overleftrightarrow{BT}$ with the sphere. $\overleftrightarrow{BT}$ is said to touch the sphere at T and it is called a tangent line to the sphere at T.

**DEFINITION:** If $\overleftrightarrow{BT}$ is an tangent line from an External point B to a sphere touches the sphere at T Then BT is called length of the tangent line to the sphere from B.

**NOTE:** length of the tangent line $B(x_1, y_1, z_1)$ to the sphere $S=0$ is $\sqrt{S_{11}}$

**NOTE:** The locus of the tangent line at a point on a sphere of non-zero radius is a plane.

**DEFINITION:** The locus of the tangent lines at a point B on a sphere of non-zero radius is a plane called the "tangent plane" to the sphere at B". B is called the point of contact of the plane with the sphere

**NOTE:** Equation to the tangent plane to the sphere $S=0$ at $(x_1, y_1, z_1)$ on the sphere is $xx_1+yy_1+zz_1+u(x+x_1)+(y+y_1)v+w(z+z_1)+d=0$ (or) $S_1=0$

**DEFINTION:** Two spheres $S=0, S'=0$ are said to touch each other if $S=0$, $S'=0$ have only one common point B. The common point B is called the point of contact of $S=0, S'=0$

**DEFINTION:** If two spheres $S=0, S'=0$ touch each other at B Then the tangent plane at B to S is same as the tangent plane at B to $S'=0$. This plane is called common tangent plane at B to $S=0, S'=0$.

**POWER OF A POINT:**

**DEFINTION:** B is a point on a line L intersecting a sphere with centre 'C' and radius 'a' in P, Q. Then the power of the point B with respect to the sphere is

i, BP.BQ if B is an External point to the sphere

ii, −BP.BQ if B is an Internal point to the sphere

iii. 0 if B is on the sphere.

20: Find the length of the tangent line from $(3,1,-1)$ to the sphere $x^2+y^2+z^2-3x+5y+7=0$.

Sol: The length of the tangent line from $(3,1,-1)$ to the sphere $x^2+y^2+z^2-3x+5y+7=0$ is $\sqrt{S_{11}}$

$2u=-3$, $2v=5$, $2w=0$, $d=7$

$u=-3/2$, $v=5/2$, $w=0$,

$\Rightarrow \sqrt{(3)^2+(1)^2+(-1)^2+2(-3/2)(3)+2(5/2)(1)+2(0)(-1)+7}$

$\Rightarrow \sqrt{9+1+1-9+5+7}=\sqrt{14}$.

21: Find the points of intersection the line $\frac{x+3}{4}=\frac{y+4}{3}=\frac{z-8}{-5}$ with the sphere $x^2+y^2+z^2+2x-10y=23$

Sol: Given equation of the line $\frac{x+3}{4}=\frac{y+4}{3}=\frac{z-8}{-5}=t \rightarrow (1)$

Given equation of the sphere is $x^2+y^2+z^2+2x-10y=23 \rightarrow (2)$

Any point on line (1) is $(4t-3, 3t-4, -5t+8)$

If it belongs to the sphere (2) Then

$(4t-3)^2+(3t-4)^2+(-5t+8)^2+(4t-3)2+(-10)(3t-4)-23=0$

$\Rightarrow 16t^2+9-24t+9t^2+16-24t+25t^2+64-80t+8t-6-30t+40-23=0$

$\Rightarrow 50t^2-150t+100=0$

$\Rightarrow t^2-3t+2=0 \Rightarrow t^2-2t-t+2=0 \Rightarrow t(t-2)-1(t-2)=0$

$\Rightarrow (t-2)(t-1)=0$

$\Rightarrow t=1,2$

$\therefore$ The point of intersection of the line (1) with the plane (2) are

$(4(2)-3, 3(2)-4, -5(2)+8) = (5,2,-2)$

$(4(1)-3, 3(1)-4, -5(1)+8) = (1,-1,3)$

22: Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$ and find the point of contact

Sol: Equation of the given sphere is $x^2+y^2+z^2-2x-4y+2z-3=0 \rightarrow (1)$

$2u=-2$, $2v=-4$, $2w=2$, $d=-3$

$u=-1$, $v=-2$, $w=1$,

Equation of the plane is $2x-2y+z+12=0 \rightarrow (2)$

centre of the sphere (1) is $(1,2,-1)$

Radius of the sphere (1) is $\sqrt{(-1)^2+(-2)^2+(1)^2+3}=\sqrt{9}=3$

The perpendicular distance from (1,2,-1) to the plane (2) is

$$\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

$$\Rightarrow \frac{|2(1)+(-2)2+(1)(-1)+12|}{\sqrt{2^2+(-2)^2+(1)^2}} = \frac{|2-4-1+12|}{\sqrt{9}} = \frac{9}{3} = 3$$

The plane (2) touches the sphere (1)

Let the line through the centre of the sphere (1) and perpendicular to the plane (2) be $\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1} = r \rightarrow (3)$

Any point on (3) is $(2r+1, -2r+2, r-1)$

If it is point of contact of plane (2) with the sphere (1) then this point must lies on the plane (2)

$$2(2r+1)-2(-2r+2)+1(r-1)+12=0$$

$$\Rightarrow 4r+2+4r-4+r-1+12=0$$

$$\Rightarrow 9r+9=0 \Rightarrow r=-1$$

$\therefore$ point of contact of sphere (1) and plane (2) is

$$(2(-1)+1, -2(-1)+2, -1-1) = (-1,4,-2)$$

**23:** Find the equation of the tangent plane to the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$ at $(-1,4,-2)$

**Sol:** Given sphere is $x^2+y^2+z^2-2x-4y+2z-3=0 \rightarrow (1)$

$$2u=-2, \quad 2v=-4, \quad 2w=2, \quad d=-3$$

$$u=-1, \quad v=-2, \quad w=1$$

The equation of the tangent plane to the sphere at $(-1,4,-2)$ is $S_1=0$

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$$

$$\Rightarrow x(-1)+y(4)+z(-2)+(-1)(x-1)+(-2)(y+4)+1(z-2)-3=0$$

$$\Rightarrow -x+4y-2z-x+1-2y-8+z-2-3=0$$

$$\Rightarrow -2x+2y-z-12=0$$

$$\Rightarrow 2x-2y+z+12=0.$$

**24:** Find the value of K for which the plane $x+y+z=K\sqrt{3}$ touches the sphere $x^2+y^2+z^2-2x-2y-2z=6$.

**Sol:** The equation of the given sphere is $x^2+y^2+z^2-2x-2y-2z=6 \rightarrow (1)$

$$2u=-2, \quad 2v=-2, \quad 2w=-2, \quad d=-6$$

$$u=-1, \quad v=-1, \quad w=-1,$$

Centre of the sphere (1) is $(1,1,1)$

Radius of the sphere (1) is $\sqrt{(-1)^2+1^2+1^2+6} = \sqrt{9} = 3.$

If the plane $x+y+z-k\sqrt{3}=0$ touches the sphere (1) Then the perpendicular distance from the Centre of the Sphere (1) to the plane $x+y+z-k\sqrt{3}=0$ is equal to the radius of the sphere (1) is $\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

$$\Rightarrow \frac{|(1)(1)+(1)(1)+(1)(1)-k\sqrt{3}|}{\sqrt{1^2+1^2+1^2}} = 3$$

$$\Rightarrow \frac{|3-k\sqrt{3}|}{\sqrt{3}} = 3 \Rightarrow 3-k\sqrt{3} = \pm 3\sqrt{3}$$

$$3-k\sqrt{3} = 3\sqrt{3} \quad , \quad 3-k\sqrt{3} = -3\sqrt{3}$$

$$\Rightarrow \sqrt{3}-k = 3 \quad , \Rightarrow \sqrt{3}-k = -3$$

$$\Rightarrow k = \sqrt{3}-3 \quad , \quad \Rightarrow k = \sqrt{3}+3.$$

25: Find the equation of the sphere touches the sphere $x^2+y^2+z^2-x+3y+2z=3$ at $(1,1,-1)$ and passing through the origin

Sol: The equation of the given sphere is $x^2+y^2+z^2-x+3y+2z-3=0 \rightarrow (1)$

$$u = -1/2 \, , \, v = 3/2 \, , \, w = 1 \, , \, d = -3$$

The equation of the tangent plane to the sphere (1) at $(1,1,-1)$

$$x(1)+y(1)+z(-1)+(-1/2)(x+1)+(3/2)(y+1)+(1)(z-1)-3=0$$

$$\Rightarrow 2x+2y-2z-x-1+3y+3+2z-2-6=0$$

$$\Rightarrow x+5y-6=0 \longrightarrow (2)$$

The equation of the sphere through (1) and (2) is

$$(x^2+y^2+z^2-x+3y+2z-3)+\lambda(x+5y-6)=0 \rightarrow (3)$$

If the sphere (3) passing through the origin Then

$$-3-6\lambda=0 \Rightarrow -6\lambda=3 \Rightarrow \lambda=-1/2$$

The equation of the required sphere is

$$(x^2+y^2+z^2-x+3y+2z-3)-\frac{1}{2}(x+5y-6)=0$$

$$\rightarrow 2x^2+2y^2+2z^2-2x+6y+4z-6-x-5y+6=0$$

$$\Rightarrow 2(x^2+y^2+z^2)-3x+y+4z=0.$$

**26:** Find the equation of the sphere which passes through the circle $x^2+y^2+z^2-1=0$, $2x+4y+5z=6$ and touch the plane $z=0$

**Sol:** The given equation of circle are $x^2+y^2+z^2-1=0$, $2x+4y+5z-6=0$ $\rightarrow(1)$

Any sphere through the circle (1) is $(x^2+y^2+z^2-1)+\lambda(2x+4y+5z-6)=0$

$x^2+y^2+z^2+2\lambda x+4\lambda y+5\lambda z+(-1-6\lambda)=0 \rightarrow (2)$

$2u=2\lambda$, $2v=4\lambda$, $2w=5\lambda$, $d=-1-6\lambda$

$u=\lambda$, $v=2\lambda$, $w=5\lambda/2$,

centre of the sphere (2) is $(-\lambda, -2\lambda, -5\lambda/2)$

Radius of the sphere (2) is $\sqrt{(\lambda)^2+(2\lambda)^2+(\frac{5}{2}\lambda)^2-(-1-6\lambda)}$

$$= \sqrt{\lambda^2+4\lambda^2+\frac{25\lambda^2}{4}+1+6\lambda}$$

$$= \sqrt{5\lambda^2+\frac{25\lambda^2}{4}+1+6\lambda}$$

$$= \sqrt{\frac{20\lambda^2+25\lambda^2+4+24\lambda}{4}}$$

$$= \sqrt{\frac{45\lambda^2+24\lambda+4}{4}}$$

If the sphere (2) touches the plane $z=0$

The perpendicular distance from the centre of the sphere (2) through the plane $z=0$ is equation of the radius of the sphere (2)

$$\frac{|0(-\lambda)+0(-2\lambda)+1(\frac{-5\lambda}{2})+0|}{\sqrt{0+0+1}} = \sqrt{\frac{45\lambda^2+24\lambda+4}{4}}$$

$$\Rightarrow \frac{|\frac{-5\lambda}{2}|}{1} = \sqrt{\frac{45\lambda^2+24\lambda+4}{4}}$$

$$\Rightarrow \frac{25\lambda^2}{4} = \frac{45\lambda^2+24\lambda+4}{4}$$

$\Rightarrow 45\lambda^2+24\lambda+4=25\lambda^2$

$\Rightarrow 20\lambda^2+24\lambda+4=0$

$\Rightarrow 5\lambda^2+6\lambda+1=0$

$\Rightarrow 5\lambda^2+5\lambda+\lambda+1=0 \Rightarrow 5\lambda(\lambda+1)+(\lambda+1)1=0$

$\Rightarrow (\lambda+1)(5\lambda+1)=0$

$\Rightarrow \lambda=-1, -1/5$

Therefore the equation of the required spheres are

$$x^2+y^2+z^2-1-1(2x+4y+5z-6)=0 \Rightarrow x^2+y^2+z^2-2x-4y-5z+5=0$$

$$x^2+y^2+z^2-1-\frac{1}{5}(2x+4y+5z-6)=0 \rightarrow 5(x^2+y^2+z^2)-2x-4y-5z+11=0.$$

<u>27:</u> Find the equation of the sphere which passes through the circle $x^2+y^2+z^2=5$, $x+2y+3z-3=0$ and touch the plane $4x+3y+15=0$

<u>Sol:</u> The given equation of a circle are $x^2+y^2+z^2-5=0$, $x+2y+3z-3=0 \rightarrow (1)$

Any sphere through the circle (1) is $(x^2+y^2+z^2-5)+\lambda(x+2y+3z-3)=0$

$$\Rightarrow x^2+y^2+z^2+x\lambda+2\lambda y+3\lambda z+(-5-3\lambda)=0 \rightarrow (2)$$

$2u=\lambda$, $2v=2\lambda$, $2w=3\lambda$, $d=-5-3\lambda$

$u=\lambda/2$, $v=\lambda$, $w=3\lambda/2$,

Centre of the sphere (1) is $C=(-\lambda/2, -\lambda, -3\lambda/2)$

Radius of the sphere (1) is $\sqrt{(\lambda/2)^2+(\lambda)^2+(3\lambda/2)^2-(-5-3\lambda)}$

$$=\sqrt{\frac{\lambda^2}{4}+\lambda^2+\frac{9\lambda^2}{4}+5+3\lambda}$$

$$=\sqrt{\frac{\lambda^2+4\lambda^2+9\lambda^2+20+12\lambda}{4}}$$

$$=\sqrt{\frac{14\lambda^2+12\lambda+20}{4}}$$

If the sphere (2) touches the plane $4x+3y-15=0$. The perpendicular distance from the centre of the sphere (2) through the plane $4x+3y-15=0$ is equation of the radius of the sphere (2)

$$\frac{|4(-\lambda/2)+3(-\lambda)+0(-3\lambda/2)-15|}{\sqrt{4^2+3^2+0}}=\sqrt{\frac{14\lambda^2+12\lambda+20}{4}}$$

$$\Rightarrow \frac{|-2\lambda-3\lambda-15|}{\sqrt{25}}=\sqrt{\frac{14\lambda^2+12\lambda+20}{4}}$$

$$\Rightarrow \frac{|-5\lambda-15|}{\sqrt{25}}=\sqrt{\frac{14\lambda^2+12\lambda+20}{4}}$$

$$\rightarrow \frac{|+5(-\lambda-3)|}{5}=\sqrt{\frac{14\lambda^2+12\lambda+20}{4}}$$

$$\rightarrow |-\lambda-3|=\sqrt{\frac{14\lambda^2+12\lambda+20}{4}}$$

$\Rightarrow (\lambda+3)^2 = \frac{14\lambda^2+12\lambda+20}{4}$

$\Rightarrow 4(\lambda^2+9+6\lambda) = 14\lambda^2+12\lambda+20$

$\Rightarrow 4\lambda^2+36+24\lambda = 14\lambda^2+12\lambda+20$

$\Rightarrow 14\lambda^2-4\lambda^2+12\lambda-24\lambda+20-36=0$

$\Rightarrow 10\lambda^2-12\lambda-16=0$

$\Rightarrow 5\lambda^2-6\lambda-8=0$

$\Rightarrow 5\lambda^2-10\lambda+4\lambda-8=0$

$\Rightarrow 5\lambda(\lambda-2)+4(\lambda-2)=0$

$\Rightarrow (\lambda-2)(5\lambda+4)=0$

$\Rightarrow \lambda=2, -4/5$

The equation of the required sphere

If $\lambda=2$; $(x^2+y^2+z^2-5)+2(x+2y+3z-3)=0 \Rightarrow x^2+y^2+z^2+2x+4y+6z-11=0$

If $\lambda=-\frac{4}{5}$; $(x^2+y^2+z^2-5)-\frac{4}{5}(x+2y+3z-3)=0 \Rightarrow 5(x^2+y^2+z^2)-4x-8y-12z-13=0.$

<u>28:</u> Find the tangent plane and to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ Which are parallel to $2x+2y-z=0$

<u>Sol:</u> Any plane parallel to $2x+2y-z=0$ is $2x+2y-z+k=0 \rightarrow (1)$

Given equation of the sphere is $x^2+y^2+z^2-4x+2y-6z+5=0 \rightarrow (2)$

centre of the sphere (2) is $(2,1,-3)$

Radius of the sphere (2) is $\sqrt{4+1+9-5} = \sqrt{9} = 3$

If the plane (1) is a tangent plane to the sphere (2). Then the perpendicular distance from the centre of the sphere (2) to the plane (1) is equal to the radius of the sphere

$$\frac{|2(2)+2(-1)+(-1)(3)+k|}{\sqrt{4+4+1}} = 3$$

$\Rightarrow \frac{|4-2-3+k|}{\sqrt{9}} = 3 \Rightarrow \frac{|k-1|}{3} = 3 \Rightarrow k-1 = \pm 9$

$k=-8, k=10$

The equation of the tangent plane to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ which are parallel to $2x+2y-z=0$ are $2x+2y-z+10=0$, $2x+2y-z-8=0$.

29: Show that the sphere $x^2+y^2+z^2-25=0$ ; $x^2+y^2+z^2-24x-40y-18z+225=0$ touch externally at the point $(\frac{12}{5}, 4, 9/5)$

Sol: Given equation of the spheres are $x^2+y^2+z^2-25=0 \rightarrow (1)$

$x^2+y^2+z^2-24x-40y-18z+225=0 \rightarrow (2)$

Centre of the sphere (1) is $A=(0,0,0)$

Radius of the sphere (1) is $r_1 = \sqrt{25} = 5$

Centre of the sphere (2) is $B=(12,20,9)$

Radius of the sphere (2) is $r_2 = \sqrt{144+400+81-225}$

$= \sqrt{400} = 20$

$AB = \sqrt{(12-0)^2+(20-0)^2+(9-0)^2}$

$= \sqrt{144+400+81}$

$= \sqrt{625}$

$= 25$

$\therefore C_1C_2 = r_1+r_2 \Rightarrow AB = r_1+r_2$

The two spheres touch externally say ~~to~~ at B

let $(P, A, B) = r_1 : r_2$

$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$

$P = \left(\frac{1(12)+4(0)}{1+4}, \frac{1(20)+4(0)}{1+4}, \frac{1(9)+4(0)}{1+4}\right)$

$= (12/5, 20/5, 9/5)$

$= (12/5, 4, 9/5)$

$\therefore$ The point of contact of the sphere (1) and (2) is $(\frac{12}{5}, 4, \frac{9}{5})$

## PLANE OF CONTACT:

DEFINITION: Through a external point B planes are drawn touch in sphere. The locus of contact of the tangent plane of the sphere is a plane is called the "plane of contact" of the point with respect to the sphere. If B is a point on the sphere. Then the tangent plane at 'p' to the sphere is called the "plane of contact" of B with respect to the sphere.

NOTE: Equation of the plane of contact of the point $(x_1, y_1, z_1)$ w.r.to the sphere $S=0$ of non-zero radius is $S_1=0$

## POLAR PLANE, POLE OF A PLANE:

DEFINITION: Let $S=0$ be a sphere the locus of the points, so that the plane of contact of each point with respect to the sphere $S=0$ passes through a point B is a plane called the polar plane of B with respect to the sphere $S=0$. B is called the pole of polar plane

NOTE: The equation to the polar plane of the point $(x_1, y_1, z_1)$ with respect to the sphere $S=0$ is $S_1=0$

30: Find the plane of contact of the point $(2,-1,1)$ with the respect to the sphere $2(x^2+y^2+z^2)+10x+6y+4z+5=0$

Sol: Given sphere is $2(x^2+y^2+z^2)+10x+6y+4z+5=0 \rightarrow (1)$

$2u=10$, $2v=6$, $2w=4$, $d=5$
$u=5$, $v=3$, $w=2$,

The equation of the plane of contact with to the sphere (1) at $(2,-1,1)$ is

$S_1=0$ $xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$

$\Rightarrow x(2)+y(-1)+z(1)+5(x+2)+(3)(y-1)+2(z+1)+5=0$

$\Rightarrow 2x-y+z+5x+10+3y-3+2z+2+5=0$

$\Rightarrow 7x+2y+3z+12=0$

31: If $x^2+y^2+z^2-a^2=0$ is a sphere, Then the pole of the plane $lx+my+nz=p$ $(p\neq 0)$ is $\left(\frac{a^2l}{p}, \frac{a^2m}{p}, \frac{a^2n}{p}\right)$

Sol: The equation of a given sphere is $x^2+y^2+z^2-a^2=0 \rightarrow (1)$

The equation of the plane is $lx+my+nz-p=0 \rightarrow (2)$

Let $P(x_1, y_1, z_1)$ be the pole of the plane (2) with respect to the sphere (1)

Then the polar plane of $P$ with respect to the sphere (1) is $xx_1+yy_1+zz_1-a^2=0$ $\rightarrow (3)$

Since equation (2) and (3) represents the same polar plane

$$\frac{x_1}{l}=\frac{y_1}{m}=\frac{z_1}{n}=\frac{a^2}{p} \Rightarrow \frac{x_1}{l}=\frac{a^2}{p}, \frac{y_1}{m}=\frac{a^2}{p}, \frac{z_1}{n}=\frac{a^2}{p}$$

$$x_1=\frac{a^2l}{p}, y_1=\frac{a^2m}{p}, z_1=\frac{a^2n}{p}$$

$\therefore$ The required pole is $P(x_1,y_1,z_1)=\left(\frac{a^2l}{p}, \frac{a^2m}{p}, \frac{a^2n}{p}\right)$

**32:** Find the pole of the plane $x+2y+3z=7$ with respect to the sphere $x^2+y^2+z^2-2x-4y-6z+11=0$

**Sol:** The given equation of the plane is $x+2y+3z-7=0 \rightarrow (1)$

The given equation of the sphere is $x^2+y^2+z^2-2x-4y-6z+11=0 \rightarrow (2)$

polar plane of $P(x_1,y_1,z_1)$ w.r.to the sphere (2) is

$$xx_1+yy_1+zz_1-1(x+x_1)-2(y+y_1)-3(z+z_1)+11=0$$

$$\Rightarrow (x_1-1)x+(y_1-2)y+(z_1-3)z-(x_1+2y_1+3z_1-11)=0 \rightarrow (3)$$

Since equation (1) and (3) represents the same polar plane.

$$\frac{x_1-1}{1}=\frac{y_1-2}{2}=\frac{z_1-3}{3}=\frac{-(x_1+2y_1+3z_1-11)}{-7}=t \text{ (say)}$$

$\Rightarrow (x_1,y_1,z_1)=(1+t, 2+2t, 3+3t)$ and $x_1+2y_1+3z_1-11=7t$

$\Rightarrow (1+t)+2(2+2t)+3(3+3t)-11=7t \Rightarrow 1+4+9-11=7t-14t$

$\Rightarrow 3=7t \Rightarrow t=-3/7$

pole of the plane (1) w.r.to the sphere (2) is $\left(\frac{-3}{7}+1, 2\left(\frac{-3}{7}\right)+2, 2\left(\frac{-3}{7}\right)+3\right)$

$=(4/7, 8/7, 12/7)$

**33:** Find the pole of the plane $x-y+5z-3=0$ w.r.to the sphere $x^2+y^2+z^2=9$

**Sol:** Given equation of the plane is $x-y+5z-3=0 \rightarrow (1)$

Given equation of the sphere is $x^2+y^2+z^2-9=0 \rightarrow (2)$

$$\left(\frac{a^2l}{p}, \frac{a^2m}{p}, \frac{a^2n}{p}\right)=\left(\frac{9(1)}{3}, \frac{9(-1)}{3}, \frac{9(5)}{3}\right)=(3,-3,15)$$

**CONJUGATE POINTS; CONJUGATES PLANES:**

**DEFINITION:** Let $S=0$ be a sphere if $A, B$ are two points such that the polar plane of $B$ with respect to the sphere $S=0$ passes through $A$. Then $A, B$ are called "conjugate points" with respect to the sphere $S=0$.

The polar planes of $A$ and $B$ are called "conjugate plane"

# UNIT - IV

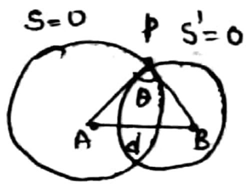
## The Sphere and cones

### *. Angles of Intersection of Spheres, Orthogonal Spheres :

**Definition:** P is common point to two spheres. Any angle $\theta$ between the tangent planes at P to two spheres is called an angle of intersection of the spheres at P. The other angle between the spheres is $\pi - \theta$.

If $\theta = \pi/2$ the spheres are said to intersect orthogonally at P and the spheres are called orthogonal spheres.

**Theorem:** If d is the distance between the centre of two intersecting spheres $S=0$, $S'=0$ of radius $r_1, r_2$ and $\theta$ is the angle of intersection of the spheres at a common point P. Then $\cos\theta = \pm\left(\dfrac{r_1^2+r_2^2-d^2}{2r_1r_2}\right)$.




**Proof:** Let A, B be centres of the two spheres. The tangent planes at P to the two spheres are perpendicular to AP, BP respectively.

Since the angle between the planes is the angle between their normals.

$$\angle APB = \theta \text{ or, } \pi - \theta$$

$$AP = r_1 \,,\quad BP = r_2 \text{ and } AB = d$$

From $\triangle APB$, $(AB)^2 = (AP)^2 + (BP)^2 - 2AP \cdot BP \cos(\angle APB)$

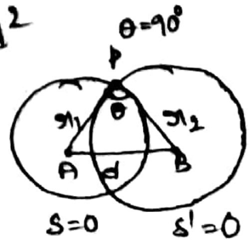
$$d^2 = r_1^2 + r_2^2 \pm 2r_1r_2\cos\theta \qquad \because \cos(\pi-\theta) = -\cos\theta$$

$$\cos\theta = \pm\left(\frac{r_1^2+r_2^2-d^2}{2r_1r_2}\right)$$

**Note:** Let d be the distance between the centres of two intersecting spheres $S=0$ and $S'=0$ with radii $r_1, r_2$.

Then two spheres cut orthogonally $\Leftrightarrow r_1^2 + r_2^2 = d^2$

i.e. $\theta = 90^\circ \Leftrightarrow r_1^2 + r_2^2 = d^2$



**Theorem:** $S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$S' \equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$ are two orthogonal spheres $\Leftrightarrow$ $2uu'+2vv'+2ww' = d+d'$.

**Proof:** The equations of the given spheres are

$$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0 \rightarrow (1)$$

$$S' \equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0 \rightarrow (2)$$

Centre of the sphere (1) is $A=(-u,-v,-w)$

Radius of the sphere (1) is $r_1 = \sqrt{u^2+v^2+w^2-d}$

Centre of the sphere (2) is $B=(-u',-v',-w')$

Radius of the sphere (2) is $r_2 = \sqrt{u'^2+v'^2+w'^2-d'}$

The spheres (1), (2) cut orthogonally.

$$\Leftrightarrow \quad AB^2 = r_1^2+r_2^2$$

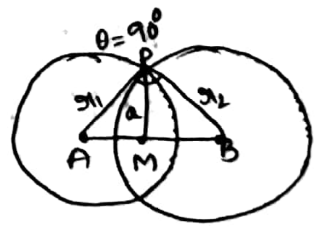
$$\Leftrightarrow \left[\sqrt{(-u'+u)^2+(-v'+v)^2+(-w'+w)^2}\right]^2 = \left[\sqrt{u^2+v^2+w^2-d}\right]^2+\left[\sqrt{u'^2+v'^2+w'^2-d'}\right]^2$$

$$\Leftrightarrow u'^2+u^2-2uu'+v'^2+v^2-2vv'+w'^2+w^2-2ww' = u^2+v^2+w^2-d+u'^2+v'^2+w'^2-d'$$

$$\Leftrightarrow -2uu'-2vv'-2ww' = -d-d'$$

$$\Leftrightarrow 2uu'+2vv'+2ww' = d+d'$$

**Theorem:** If $r_1, r_2$ are ~~two~~ the radii of two orthogonal spheres then the radius of the circle of their intersection is $\dfrac{r_1 r_2}{\sqrt{r_1^2+r_2^2}}$




**Proof:** Let A, B are the centres of the two orthogonal spheres.

Let M is the centre of the circle and 'a' is the radius of the circle common to the spheres.

Therefore A, M, B are collinear and $MP \perp AB$ and P is a common point of intersection of the spheres.

$$AP = r_1, \quad BP = r_2, \quad \angle APB = 90^\circ$$

$$(AB)^2 = r_1^2+r_2^2$$

$$(AM+MB)^2 = r_1^2+r_2^2$$

$$AM^2+MB^2+2AM\cdot MB = r_1^2+r_2^2$$

$\Delta APM$ $\qquad\qquad\qquad$ $\Delta BPM$

$AP^2 = AM^2 + MP^2$ $\qquad$ $BP^2 = MP^2 + BM^2$

$r_1^2 = AM^2 + a^2$ $\qquad$ $r_2^2 = a^2 + BM^2$

$AM^2 = r_1^2 - a^2$ $\qquad$ $MB^2 = r_2^2 - a^2$

$$\Rightarrow (r_1^2 - a^2) + (r_2^2 - a^2) + 2(\sqrt{r_1^2 - a^2})(\sqrt{r_2^2 - a^2}) = r_1^2 + r_2^2$$

$$\Rightarrow -2a^2 + 2\sqrt{r_1^2 - a^2}\cdot\sqrt{r_2^2 - a^2} = 0$$

$$\Rightarrow \sqrt{r_1^2 - a^2}\cdot\sqrt{r_2^2 - a^2} = a^2$$

$$\Rightarrow (r_1^2 - a^2)(r_2^2 - a^2) = a^4$$

$$\Rightarrow r_1^2 r_2^2 - r_1^2 a^2 - r_2^2 a^2 + a^4 = a^4$$

$$\Rightarrow r_1^2 r_2^2 - a^2(r_1^2 + r_2^2) = 0$$

$$\Rightarrow a^2 = \frac{r_1^2 r_2^2}{r_1^2 + r_2^2} \Rightarrow a = \frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$$

## *. Radical Plane :

**Definition:** The locus of points each of whose powers w.r.t. two non-concentric spheres are equal is a plane called the radical plane of the two spheres.

**Note:** Equation to the radical plane of spheres $S=0$, $S'=0$ is $S - S' = 0$.

**Problem:** Find the equation of the spheres through the circle $x^2+y^2+z^2-2x+3y-4z+6=0$, $3x-4y+5z-15=0$ and cutting the sphere $x^2+y^2+z^2+2x+4y-6z+11=0$ orthogonally.

**Sol:** Given equation of the circle is

$$x^2+y^2+z^2-2x+3y-4z+6=0,\ 3x-4y+5z-15=0 \rightarrow (1)$$

Given equation of the sphere is $x^2+y^2+z^2+2x+4y-6z+11=0 \rightarrow (2)$

$2u = 2$ $\qquad$ $2v = 4$ $\qquad$ $2w = -6$ $\qquad$ $d = +11$

$u = 1$ $\qquad$ $v = 2$ $\qquad$ $w = -3$

Any sphere through the circle (1) is

$$(x^2+y^2+z^2-2x+3y-4z+6) + \lambda(3x-4y+5z-15) = 0$$

$$\Rightarrow x^2+y^2+z^2+(-2+3\lambda)x + (3-4\lambda)y + (-4+5\lambda)z + (6-15\lambda) = 0 \rightarrow (3)$$

$2u' = -2+3\lambda$ $\qquad$ $2v' = 3-4\lambda$ $\qquad$ $2w' = -4+5\lambda$ $\qquad$ $d' = 6-15\lambda$

$u' = \dfrac{-2+3\lambda}{2}$ $\qquad$ $v' = \dfrac{3-4\lambda}{2}$ $\qquad$ $w' = \dfrac{-4+5\lambda}{2}$

The sphere (3) cuts the sphere (2) orthogonally.

$\Rightarrow$ $2uu' + 2vv' + 2ww' = d + d'$

$\Rightarrow$ $2(1)\left(\frac{-2+3\lambda}{2}\right) + 2(2)\left(\frac{3-4\lambda}{2}\right) + 2(-3)\left(\frac{4+5\lambda}{2}\right) = 11 + (6-15\lambda).$

$\Rightarrow$ $-2 + 3\lambda + 6 - 8\lambda + 12 - 15\lambda = 17 - 15\lambda$

$\Rightarrow$ $-5\lambda + 16 = 17$ $\Rightarrow$ $-5\lambda = 1$ $\Rightarrow$ $\lambda = -1/5$

Therefore the equation of the required sphere is

$(x^2+y^2+z^2-2x+3y-4z+6) - \frac{1}{5}(3x-4y+5z-15) = 0$

$\Rightarrow$ $5(x^2+y^2+z^2) - 10x + 15y - 20z + 30 - 3x + 4y - 5z + 15 = 0$

$\Rightarrow$ $5(x^2+y^2+z^2) - 13x + 19y - 25z + 45 = 0.$

Problem: Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at $(1,-2,1)$ and cuts orthogonally the sphere $x^2+y^2+z^2-4x+6y+4=0$.

Sol: Given equation of the sphere is $x^2+y^2+z^2-4x+6y+4=0 \rightarrow (1)$

$2u = -4$ $\quad 2v = 6$ $\quad 2w = 0$ $\quad d = 4$

$u = -2$ $\quad v = 3$ $\quad w = 0$

Centre of the sphere (1) is $(2, -3, 0)$

Radius of the sphere (1) is $\sqrt{(-2)^2+(3)^2+(0)^2-4} = 3$

Since the plane $3x+2y-z+2=0$ at $(1,-2,1)$ is the tangent plane to the required sphere.

Equations of the normal to the given plane $3x+2y-z+2=0$ through $(1,-2,1)$ are $\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-1}{-1} = t$

Centre of the required sphere can be taken as $(3t+1, 2t-2, -t+1)$ and radius is $\sqrt{(3t+1-1)^2+(2t-2+2)^2+(-t+1-1)^2}$

$= \sqrt{9t^2+4t^2+t^2} = \sqrt{14t^2} = \sqrt{14}\,|t|$

Since the required sphere cuts orthogonally the sphere (1)

$r_1^2 + r_2^2 = d^2$

$\Rightarrow$ $9 + 14t^2 = (3t+1-2)^2 + (2t-2+3)^2 + (-t+1-0)^2$

$\Rightarrow$ $9 + 14t^2 = (3t-1)^2 + (2t+1)^2 + (-t+1)^2$

$\Rightarrow$ $9 + 14t^2 = 9t^2 + 1 - 6t + 4t^2 + 1 + 4t + t^2 + 1 - 2t$

$\Rightarrow$ $9 + 14t^2 = 14t^2 - 4t + 3$ $\Rightarrow$ $-4t = 6$ $\Rightarrow$ $t = -3/2$

The centre of the required sphere is

$\Rightarrow \left(3(-3/2)+1,\ 2(-3/2)-2,\ 3/2+1\right) = \left(-7/2, -5, 5/2\right)$

Radius of the required sphere is $\sqrt{14}\,\left|-\frac{3}{2}\right| = \frac{3}{2}\sqrt{14}$

Therefore the equation of the required sphere is

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2 = a^2$$

$$\Rightarrow (x+7/2)^2+(y+5)^2+(z-5/2)^2 = \left(\frac{3}{2}\sqrt{14}\right)^2$$

$$\Rightarrow x^2+y^2+z^2+7x+10y-5z+\frac{49}{4}+25+\frac{25}{4} = \frac{9}{4}\cdot 14$$

$$\Rightarrow 4(x^2+y^2+z^2)+28x+40y-20z+49+100+25 = 126$$

$$\Rightarrow 4(x^2+y^2+z^2)+28x+40y-20z+48 = 0$$

$$\Rightarrow x^2+y^2+z^2+7x+10y-5z+12 = 0.$$

**Problem:** Show that the spheres $x^2+y^2+z^2+6y+2z+8=0$, $x^2+y^2+z^2+6x+8y+4z+20=0$ are orthogonal.

**Sol:** Given equations of the spheres are

$x^2+y^2+z^2+6y+2z+8=0 \rightarrow (1)$, $x^2+y^2+z^2+6x+8y+4z+20=0 \rightarrow (2)$

$\rightarrow u=0,\ v=3,\ w=1,\ d=8$ ; $u'=3,\ v'=4,\ w'=2,\ d'=20$

The spheres (1) and (2) are orthogonal if $2uu'+2vv'+2ww'=d+d'$

$$\Rightarrow 2(0)(3)+2(3)(4)+2(1)(2) = 8+20$$

$$\Rightarrow 0+24+4 = 8+20 \Rightarrow 28=28$$

Therefore given spheres are orthogonal.

## Coaxial System of Spheres:

**Definition:** A system of spheres such that any two spheres of the system have the same radical plane is called a coaxial system of spheres.

$S=0$, $S'=0$ are two spheres of a coaxial system of spheres $\Rightarrow S-S'=0$ is the radical plane of the coaxial system of spheres.

**Note:** If $S=0$ is a sphere and $U=0$ is a plane, then the equation $S+\lambda U=0$ represents a coaxial system of spheres with radical plane $U=0$.

**Limiting points:**

**Definition:** Point spheres of a coaxial system of spheres are called limiting points of the system.

**Problem:** Find the limiting points of the coaxial system of spheres $x^2+y^2+z^2+4x-2y+2z+6=0$, $x^2+y^2+z^2+2x-4y-2z+6=0$

**Sol:** The radical plane of the spheres of coaxial system is

$$(x^2+y^2+z^2+4x-2y+2z+6)-(x^2+y^2+z^2+2x-4y-2z+6)=0$$

$$\Rightarrow 2x+2y+4z=0 \Rightarrow x+y+2z=0=U$$

Equation of any sphere belonging to the system defined by the given spheres is $S+\lambda U=0$

$$\Rightarrow (x^2+y^2+z^2+4x-2y+2z+6)+\lambda(x+y+2z)=0$$

$$\Rightarrow x^2+y^2+z^2+(4+\lambda)x+(-2+\lambda)y+(2+2\lambda)z+6=0 \rightarrow (1)$$

$$2u=4+\lambda \qquad 2v=-2+\lambda \qquad 2w=2+2\lambda \qquad d=6$$

$$u=\frac{4+\lambda}{2} \qquad v=\frac{-2+\lambda}{2} \qquad w=\frac{2+2\lambda}{2}$$

Centre of the sphere (1) is $\left[-\left(\frac{4+\lambda}{2}\right), -\left(\frac{-2+\lambda}{2}\right), -\left(\frac{2+2\lambda}{2}\right)\right]$

Radius of the sphere (1) is $\sqrt{\left(\frac{4+\lambda}{2}\right)^2+\left(\frac{-2+\lambda}{2}\right)^2+\left(\frac{2+2\lambda}{2}\right)^2-6}$

For limiting points of the system, radius $=0$

$$\Rightarrow \sqrt{\left(\frac{4+\lambda}{2}\right)^2+\left(\frac{-2+\lambda}{2}\right)^2+\left(\frac{2+2\lambda}{2}\right)^2-6}=0$$

$$\Rightarrow \sqrt{\frac{16+\lambda^2+8\lambda+4+\lambda^2-4\lambda+4+4\lambda^2+8\lambda-24}{4}}=0$$

$$\Rightarrow \sqrt{\frac{6\lambda^2+12\lambda}{4}}=0 \Rightarrow \frac{6\lambda^2+12\lambda}{4}=0$$

$$\Rightarrow 6\lambda^2+12\lambda=0 \Rightarrow \lambda^2+2\lambda=0 \Rightarrow \lambda=0,-2$$

Therefore limiting points are $(-2,1,-1)$, $(-1,2,1)$.

**Problem:** Find the equation of the spheres of the coaxial system $(x^2+y^2+z^2-5)+\lambda(2x+y+3z-3)=0$ which touch the plane $3x+4y=15$

**Sol:** Any sphere of the given system is

$$(x^2+y^2+z^2-5)+\lambda(2x+y+3z-3)=0$$

$$\Rightarrow x^2+y^2+z^2+2\lambda x+\lambda y+3\lambda z-5-3\lambda=0 \rightarrow (1)$$

Centre of the sphere (1) is $\left(-\lambda, -\frac{\lambda}{2}, -\frac{3\lambda}{2}\right)$

Radius of the sphere ① is $\sqrt{\lambda^2+\frac{\lambda^2}{4}+\frac{9\lambda^2}{4}+5+3\lambda}$

$$= \sqrt{\frac{4\lambda^2+\lambda^2+9\lambda^2+12\lambda+20}{4}} = \sqrt{\frac{14\lambda^2+12\lambda+20}{4}}$$

If the sphere ① touchs the plane $3x+4y=15$ then the length of the perpendicular from the centre of the sphere ① to the plane $3x+4y-15=0$ must be equal to the radius of the sphere ①.

$$\Rightarrow \frac{|3(-\lambda)+4(-\lambda/2)-0(-3\lambda/2)-15|}{\sqrt{(3)^2+(4)^2+(0)^2}} = \sqrt{\frac{14\lambda^2+12\lambda+20}{4}}$$

$$\Rightarrow \frac{|-3\lambda-2\lambda-15|}{5} = \sqrt{\frac{14\lambda^2+12\lambda+20}{4}}$$

$$\Rightarrow |-(\lambda+3)| = \sqrt{\frac{14\lambda^2+12\lambda+20}{4}}$$

$$\Rightarrow 4(\lambda+3)^2 = 14\lambda^2+12\lambda+20$$

$$\Rightarrow 4\lambda^2+24\lambda+36 = 14\lambda^2+12\lambda+20$$

$$\Rightarrow -10\lambda^2+12\lambda+16=0 \Rightarrow 5\lambda^2-6\lambda-8=0$$

$$\Rightarrow 5\lambda^2-10\lambda+4\lambda-8=0$$

$$\Rightarrow 5\lambda(\lambda-2)+4(\lambda-2)=0$$

$$\Rightarrow (\lambda-2)(5\lambda+4)=0$$

$$\Rightarrow \lambda = 2,\ -\frac{4}{5}$$

Put these values of $\lambda$ in equation ①.

The required equation of the spheres are

$$(x^2+y^2+z^2-5)+2(2x+y+3z-3)=0;\ (x^2+y^2+z^2-5)-\frac{4}{5}(2x+y+3z-3)=0$$

$$\Rightarrow x^2+y^2+z^2+4x+2y+6z-6=0;\ 5(x^2+y^2+z^2)-8x-4y-12z-13=0$$

**Problem:** Find the limiting points of the coaxial system of the spheres $(x^2+y^2+z^2-20x+30y-40z+29)+\lambda(2x+4z-3y)=0$

**Sol:** Given coaxial system of spheres is

$$(x^2+y^2+z^2-20x+30y-40z+29)+\lambda(2x+3y+4z)=0 \quad \to ①$$

$$\Rightarrow x^2+y^2+z^2+(-20+2\lambda)x+(30-3\lambda)y+(-40+4\lambda)z+29=0$$

centre of the sphere ① is $\left(-\frac{(-20+2\lambda)}{2}, -\frac{(30-3\lambda)}{2}, -\frac{(-40+4\lambda)}{2}\right)$

Radius of the sphere ① is $\sqrt{\left(\frac{-20+2\lambda}{2}\right)^2+\left(\frac{30-3\lambda}{2}\right)^2+\left(\frac{-40+4\lambda}{2}\right)^2-29}$

For limiting points of the system, radius $=0$

$$\Rightarrow \sqrt{\left(\frac{-20+2\lambda}{2}\right)^2+\left(\frac{30-3\lambda}{2}\right)^2+\left(\frac{-40+4\lambda}{2}\right)^2-29} = 0$$

$$\Rightarrow (-20+2\lambda)^2+(30-3\lambda)^2+(-40+4\lambda)^2-116=0$$

$$\Rightarrow 400+4\lambda^2-8\lambda+900+9\lambda^2-180\lambda+1600+16\lambda^2-320\lambda-116=0$$

$$\Rightarrow 29\lambda^2-580\lambda+2784=0$$

$$\Rightarrow \lambda^2-20\lambda+96=0 \Rightarrow \lambda^2-12\lambda-8\lambda+96=0$$

$$\Rightarrow \lambda(\lambda-12)-8(\lambda-12)=0$$

$$\Rightarrow (\lambda-12)(\lambda-8)=0 \Rightarrow \lambda=12, 8$$

Therefore limiting points are $(-2,3,-4)$, $(2,-3,4)$.

**Problem:** Find the limiting points of the coaxial system of spheres of which two members are $x^2+y^2+z^2+3x-3y+6=0$, $x^2+y^2+z^2-6y-6z+6=0$

**Sol:** The radical plane of the coaxial system is

$$\Rightarrow (x^2+y^2+z^2+3x-3y+6)-(x^2+y^2+z^2-6y-6z+6)=0$$

$$\Rightarrow 3x+3y+6z=0 \Rightarrow x+y+2z=0 =U$$

Equation to the spheres of the coaxial system with radical plane is $(x^2+y^2+z^2+3x-3y+6)+\lambda(x+y+2z)=0 \rightarrow ①$

$$\Rightarrow x^2+y^2+z^2+(3+\lambda)x+(-3+\lambda)y+(2\lambda)z+6=0$$

$2u = 3+\lambda \qquad 2v=-3+\lambda \qquad 2w=2\lambda \qquad d=6$

$u=\frac{3+\lambda}{2} \qquad v=\frac{-3+\lambda}{2} \qquad w=\lambda$

Centre of the sphere ① is $\left(-\left(\frac{3+\lambda}{2}\right), -\left(\frac{-3+\lambda}{2}\right), -(\lambda)\right)$

Radius of the sphere ① is $\sqrt{\left(\frac{3+\lambda}{2}\right)^2+\left(\frac{-3+\lambda}{2}\right)^2+(\lambda)^2-6}$

For limiting points of the system, radius $=0$

$$\Rightarrow \sqrt{\frac{(3+\lambda)^2}{4}+\frac{(-3+\lambda)^2}{4}+\lambda^2-6}=0 \Rightarrow \sqrt{(3+\lambda)^2+(-3+\lambda)^2+4\lambda^2-24}=0$$

$$\Rightarrow 9+\lambda^2+6\lambda+9+\lambda^2-6\lambda+4\lambda^2-24=0$$

$$\Rightarrow 6\lambda^2-6=0 \Rightarrow \lambda^2-1=0 \Rightarrow \lambda=\pm1$$

Therefore limiting point are $(-2,1,-1)$, $(-1,2,1)$.

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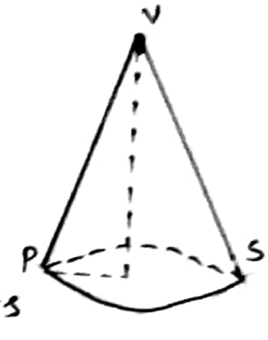
## The Cone :

**Definition :** The Surface generated by a straight line that which Passes through a fixed point and intersecting a given curve or touching a given Surface is called a cone.




The fixed point is called the vertex and the given curve the guiding Curve of the cone.

An individual Straight line on the surface of a cone is called a generator.

Thus a cone is the set of lines called generators through a given point.

**Note :** 1. If $V$ is the vertex of the cone $S$ and $P$ is a point on '$S$' then $\overleftrightarrow{VP}$ is a generator.

2. If $L$ is a generator of the cone $S$ then every point of $L$ lies on $S$.

**Example :** The equation $2x^2+3y^2-z^2=0$ represents a cone with vertex as origin.

Intersecting pairs of planes form a cone with every point on the common line as vertex.

**Note :** If $f(x,y,z)$ is a homogeneous polynomial of $n^{th}$ degree then the Surface $S$ represented by $f(x,y,z)=0$ is a cone with vertex at the origin.

If $f(x,y,z)$ is a homogeneous polynomial of degree $n$ then the cone $f(x,y,z)=0$ is called a cone of $n^{th}$ degree.

**Example :** 1. $2x^3-y^3+3x^2z+2z^3=0$ is a cone of third degree.

2. $x^2+y^2+z^2=0$ is a cone of $2^{nd}$ degree.

Cones of Second degree are also called Quadric cones.

**Theorem :** The equation of a cone whose vertex is the origin is Homogeneous in $x,y,z$.

**Proof :** Let $f(x,y,z)=0 \rightarrow (1)$ be the equation of the cone with vertex is origin.

Let $P(x_1,y_1,z_1)$ be any point on the cone (1). $\rightarrow f(x_1,y_1,z_1)=0 \rightarrow (2)$

Also the equation of the generator $\overleftrightarrow{OP}$ is $\frac{x-0}{x_1-0}=\frac{y-0}{y_1-0}=\frac{z-0}{z_1-0}=t$

$$\rightarrow \frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}=t \rightarrow (3)$$

Any point on the line (3) is $(tx_1, ty_1, tz_1)$.

The generator completely lies on the cone.

The point $(tx_1, ty_1, tz_1)$ must lie on the cone $f(tx_1, ty_1, tz_1) = 0$ for all values of $t$ $\rightarrow$ (4)

From equations (2) and (3) the equation $f(x,y,z) = 0$ homogeneous in $x, y, z$.

Thus the equation of a cone whose vertex is the origin is homogeneous in $x, y, z$.

<u>Note:</u> Every homogeneous equation of the second degree represents a cone with its vertex at the origin.

<u>Problem:</u> Find the equation of the cone whose vertex is the point $(\alpha, \beta, \gamma)$ and whose generators intersect the conic $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0,\ z = 0$.

<u>Sol:</u> The equation of any line through $(\alpha, \beta, \gamma)$ and having D.r's as $l, m, n$ are $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ $\rightarrow$ (1)

Equation (1) is a generator of the cone $\Leftrightarrow$ equation (1) intersects the conic.

Clearly equation (1) intersects $z=0$ at the point $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$

This point will lie on the given conic

$$a\left(\alpha - \frac{l\gamma}{n}\right)^2 + b\left(\beta - \frac{m\gamma}{n}\right)^2 + 2h\left(\alpha - \frac{l\gamma}{n}\right)\left(\beta - \frac{m\gamma}{n}\right) + 2g\left(\alpha - \frac{l\gamma}{n}\right) + 2f\left(\beta - \frac{m\gamma}{n}\right) + c = 0 \rightarrow (2)$$

This is the condition for the line (1) to intersects the given conic

Eliminating $l, m, n$ from equation (1) and (2) we get

$$a\left(\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right)^2 + b\left(\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right)^2 + 2h\left(\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right)\left(\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right) + 2g\left(\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right) + 2f\left(\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right) + c = 0$$

$$\Rightarrow a\left(\frac{\alpha(z-\gamma) - (x-\alpha)\gamma}{z-\gamma}\right)^2 + b\left(\frac{\beta(z-\gamma) - (y-\beta)\gamma}{z-\gamma}\right)^2 + 2h\left(\frac{\alpha(z-\gamma) - (x-\alpha)\gamma}{z-\gamma}\right)\left(\frac{\beta(z-\gamma) - (y-\beta)\gamma}{z-\gamma}\right) + 2g\left(\frac{\alpha(z-\gamma) - (x-\alpha)\gamma}{z-\gamma}\right) + 2f\left(\frac{\beta(z-\gamma) - (y-\beta)\gamma}{z-\gamma}\right) + c = 0$$

$$\rightarrow a\left(\frac{\alpha z - \alpha\gamma - x\alpha + \alpha\gamma}{z-\gamma}\right)^2 + b\left(\frac{\beta z - \beta\gamma - y\gamma + \beta\gamma}{z-\gamma}\right)^2 + 2h\left(\frac{\alpha\gamma + \alpha z + \alpha\gamma - x\gamma}{z-\gamma}\right)\left(\frac{\beta z - \beta\gamma - y\gamma + \beta\gamma}{z-\gamma}\right) + 2g\left(\frac{\alpha z - \alpha\gamma - x\gamma + \alpha\gamma}{z-\gamma}\right) + 2f\left(\frac{\beta z - \beta\gamma - \gamma y - \beta\gamma}{z-\gamma}\right) + c = 0$$

$$\rightarrow a(\alpha z - x\gamma)^2 + b(\beta z - y\gamma)^2 + 2h(\alpha z - x\gamma)(\beta z - y\gamma) + 2g(\alpha z - x\gamma)(z-\gamma) + 2f(\beta z - y\gamma)(z-\gamma) + c(z-\gamma)^2 = 0$$

which is the required equation of the cone.

**Problem:** Find the equation of the cone whose generators pass through the point $(\alpha, \beta, \gamma)$ and have their direction cosines satisfying the relation $al^2+bm^2+cn^2=0$.

**Sol:** The equation to the generator passing through $(\alpha, \beta, \gamma)$ and having d.c.s $l, m, n$ is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = k$

$\rightarrow l = \frac{x-\alpha}{k}, \ m = \frac{y-\beta}{k}, \ n = \frac{z-\gamma}{k}$

But $l, m, n$ satisfy $al^2+bm^2+cn^2=0$

$\Rightarrow a\left(\frac{x-\alpha}{k}\right)^2 + b\left(\frac{y-\beta}{k}\right)^2 + c\left(\frac{z-\gamma}{k}\right)^2 = 0$

$\Rightarrow a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0$

Thus the equation of the required cone is

$$a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0$$

**Problem:** Show that $x=-y=-z$ is a generator of the cone $5yz+8zx-3xy=0$.

**Sol:** Given equation of the cone is $5yz+8zx-3xy=0 \rightarrow (1)$

$\rightarrow f(x,y,z)=0$

Given generator is $x=-y=-z \Rightarrow \frac{x}{1} = \frac{y}{-1} = \frac{z}{-1} \rightarrow (2)$

Now $f(l,m,n) = 5mn+8nl-3lm$

$= 8-8 = 0$

$\rightarrow f(l,m,n)=0$

Therefore (2) is a generator of the given cone (1).

**Problem:** Find the equation of the cone whose vertex is the point $(1,1,0)$ and guiding curve is $y=0, x^2+z^2=4$.

**Sol:** The equation of the line through $(1,1,0)$ and having d.r.s as $l, m, n$ is $\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-0}{n} \rightarrow (1)$

clearly equation (1) intersect $y=0$ at $\left(1-\frac{l}{m}, 0, -\frac{n}{m}\right)$

this point lie on the surface $x^2+y^2=4$

$\Rightarrow \left(1-\frac{l}{m}\right)^2 + \left(-\frac{n}{m}\right)^2 = 4 \Rightarrow \left(1-\frac{l}{m}\right)^2 + \left(\frac{n}{m}\right)^2 = 4 \rightarrow (2)$

Eliminating $l, m, n$ from equations (1) and (2)

we get $\left(1-\left(\frac{x-1}{y-1}\right)\right)^2 + \left(\frac{z}{y-1}\right)^2 = 4$

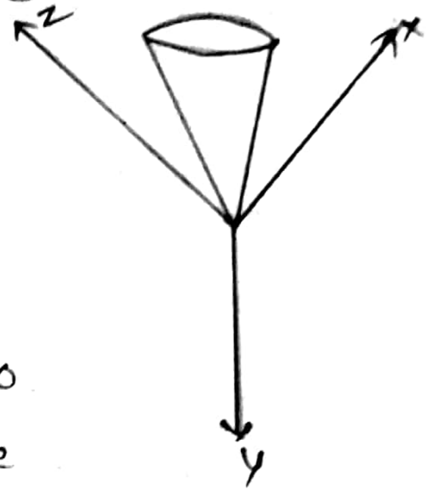
$\rightarrow (y-1-x+1)^2 + z^2 = 4(y-1)^2 \Rightarrow (y-x)^2+z^2 = 4(y-1)^2$

which is the required equation of the cone.

**Theorem:** Show that the general equation of the cone of the second degree which pass through the coordinate axes is $fyz + gzx + hxy = 0$.

**Proof:** The equation of the cone of the second degree is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \rightarrow (1)$$




x-axis is a generator of the cone.

$\Rightarrow$ The direction cosines $(1,0,0)$ of the x-axis satisfies (1).

$\Rightarrow \quad a = 0$

Similarly, y-axis is a generator $\Rightarrow b = 0$

Similarly, z-axis is a generator $\Rightarrow c = 0$

Hence the general equation of the cone containing the three axes is $2fyz + 2gzx + 2hxy = 0$

$\rightarrow fyz + gzx + hxy = 0$

**Problem:** Find the equation to the cone which passes through the coordinate axes as well as the three lines $\frac{1}{2}x = y = -z$, $x = \frac{1}{3}y = \frac{1}{5}z$ and $\frac{1}{8}x = -\frac{1}{11}y = \frac{1}{5}z$.

**Sol:** Given lines are $\frac{x}{2} = \frac{y}{1} = \frac{z}{-1}$ and $\frac{x}{1} = \frac{y}{3} = \frac{z}{5}$, $\frac{x}{8} = \frac{y}{-11} = \frac{z}{5}$

Equation to the required cone is $fyz + gzx + hxy = 0$

Since $\frac{x}{2} = \frac{y}{1} = \frac{z}{-1}$ is a generator of (1)

$\Rightarrow -f - 2g + 2h = 0 \rightarrow (2)$

Since $\frac{x}{1} = \frac{y}{3} = \frac{z}{5}$ is a generator of (1)

$\rightarrow 15f + 5g + 3h = 0 \rightarrow (3)$

Solving (2) and (3)

$$\begin{array}{cccc} & f & g & h \\ -2 & 2 & -1 & -2 \\ 5 & 3 & 15 & 5 \end{array}$$

$\Rightarrow f = -6 - 10 = -16$

$g = 30 + 3 = 33$

$h = -5 + 30 = 25$

From (1), we get the equation of the required cone is

$$-16yz + 33zx + 25xy = 0$$

Verification: $x = 8$, $y = -11$, $z = 5$

$\Rightarrow -16(-55) + 33(40) + 25(-88)$

$= 2200 - 2200 \quad \therefore \frac{x}{8} = \frac{y}{-11} = \frac{z}{5}$ is a generator of the required cone

$= 0$

**Problem:** Find the equation to the cone which passes through the three coordinate axes and the lines $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and $\frac{x}{2} = \frac{y}{1} = \frac{z}{1}$

**Sol** Given lines are $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and $\frac{x}{2} = \frac{y}{1} = \frac{z}{1}$

The equation of the cone which passes through the three coordinate axes is $fyz + gzx + hxy = 0 \rightarrow (1)$

If the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ lies on the cone (1) then

$$-6f + 3g - 2h = 0 \rightarrow (2)$$

If the line $\frac{x}{2} = \frac{y}{1} = \frac{z}{1}$ lies on the cone (1) then

$$f + 2g + 2h = 0 \rightarrow (3)$$

Solving the equations (2) and (3)

$$\begin{array}{cccccc} f & & g & & h & \\ +3 & -2 & & -6 & & 3 \\ 2 & 2 & & 1 & & 2 \end{array}$$

$$\Rightarrow \frac{f}{6+4} = \frac{g}{-2+12} = \frac{h}{-12-3}$$

$$\Rightarrow \frac{f}{10} = \frac{g}{10} = \frac{h}{-15} \Rightarrow \frac{f}{2} = \frac{g}{2} = \frac{h}{-3}$$

Therefore the equation of the required cone is ~~2yz+2zx+~~

$$2yz + 2zx - 3xy = 0.$$

**Problem:** Find the equation of the quadric cone through the coordinate axes and the three lines $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$, $\frac{x}{-1} = \frac{y}{1} = \frac{z}{1}$ and $\frac{x}{5} = \frac{y}{4} = \frac{z}{1}$.

**Sol** Given lines are $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$, $\frac{x}{-1} = \frac{y}{1} = \frac{z}{1}$ and $\frac{x}{5} = \frac{y}{4} = \frac{z}{1}$

The equation to the cone passing through the three coordinate axis is $fyz + gzx + hxy = 0 \rightarrow (1)$

If the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ lies on the Cone (1) then

$$-6f + 3g - 2h = 0 \rightarrow (2)$$

If the line $\frac{x}{-1} = \frac{y}{1} = \frac{z}{1}$ lies on the cone (1) then

$$f - g - h = 0 \rightarrow (3)$$

Solving the equations (2) and (3)

$$\begin{array}{cccccc} f & & g & & h & \\ 3 & -2 & & -6 & & 3 \\ -1 & -1 & & 1 & & -1 \end{array}$$

$$\rightarrow \frac{f}{-5} = \frac{g}{-8} = \frac{h}{3}$$

Thus the equation of a cone which passes through the coordinate axes and the lines $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$, $\frac{x}{-1} = \frac{y}{1} = \frac{z}{1}$ is

$$-5yz-8zx+3xy=0 \Rightarrow 5yz+8zx-3xy=0$$

Clearly the D.rs of the line $\frac{x}{5}=\frac{y}{4}=\frac{z}{1}$ satisfy the above equation.

Thus the equation of the required cone is

$$5yz+8zx-3xy=0.$$

Problem: Find the equation of the lines of intersection of the plane $2x+y-z=0$ and the cone $4x^2-y^2+3z^2=0$

Sol: Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \rightarrow ①$ be the equation of one of the two lines in which the plane $2x+y-z=0 \rightarrow ②$ intersects the cone $4x^2-y^2+3z^2=0 \rightarrow ③$

Since the line ① lies on the cone ③, $4l^2-m^2+3n^2=0$

Since the line ① lies on the plane ②, the line ① is perpendicular to the normal of the plane ②

$$2l+m-n=0 \rightarrow (i)$$

$$\Rightarrow n=2l+m$$

$$4l^2-m^2+3(2l+m)^2=0$$

$$\Rightarrow 4l^2+3(4l^2+m^2+4lm)-m^2=0$$

$$\Rightarrow 4l^2-m^2+12l^2+3m^2+12lm=0$$

$$\Rightarrow 16l^2+12lm+2m^2=0$$

$$\Rightarrow 8l^2+6lm+m^2=0$$

$$\Rightarrow 8l^2+4lm+2lm+m^2=0$$

$$\Rightarrow 4l(2l+m)+m(2l+m)=0$$

$$\Rightarrow (2l+m)(4l+m)=0$$

$$\Rightarrow 2l+m=0 \rightarrow (ii), \quad 4l+m=0 \rightarrow (iii)$$

Solving (i), (ii) $\Rightarrow$

$$\begin{array}{cccccc} & l & & m & & n \\ 1 & \times & -1 & \times & 2 & \times & 1 \\ 1 & & 0 & & 2 & & 1 \end{array}$$

$$\Rightarrow \frac{l}{1}=\frac{m}{-2}=\frac{n}{0}$$

Solving (i), (iii) $\Rightarrow$

$$\begin{array}{cccccc} & l & & m & & n \\ 1 & \times & -1 & \times & 2 & \times & 1 \\ 1 & & 0 & & 4 & & 1 \end{array} \Rightarrow \frac{l}{1}=\frac{m}{-4}=\frac{n}{-2}$$

Therefore the equations of the required lines are

$$\frac{x}{1}=\frac{y}{-2}=\frac{z}{0} \quad \text{and} \quad \frac{x}{1}=\frac{y}{-4}=\frac{z}{2}$$

**Problem:** Find the equations of the lines of intersection of the plane $3x+4y+z=0$ and the cone $15x^2-32y^2-7z^2=0$

**Sol:** Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \rightarrow (1)$ be the equation of one of the two lines in which the plane $3x+4y+z=0 \rightarrow (2)$ intersects the cone $15x^2-32y^2-7z^2=0 \rightarrow (3)$

Since the line (1) lies on the cone (3), $15l^2-32m^2-7n^2=0$

Since the line (1) lies on the plane (2), the line (1) is perpendicular to the normal of the plane (2)

$$3l+4m+n=0 \rightarrow (i) \rightarrow n=-(3l+4m)$$

$$15l^2-32m^2-7\left(-(3l+4m)\right)^2=0$$

$$\Rightarrow 15l^2-32m^2-7(9l^2+16m^2+24lm)=0$$

$$\Rightarrow 15l^2-32m^2-63l^2-112m^2-168lm=0$$

$$\Rightarrow -48l^2-144m^2-168lm=0$$

$$\Rightarrow 4l^2+14lm+12m^2=0$$

$$\Rightarrow 2l^2+7lm+6m^2=0$$

$$\Rightarrow 2l^2+4lm+3lm+6m^2=0$$

$$\Rightarrow 2l(l+2m)+3m(l+2m)=0$$

$$\rightarrow (l+2m)(2l+3m)=0$$

$$\Rightarrow l+2m=0 \rightarrow (ii), \quad 2l+3m=0 \rightarrow (iii)$$

Solving (i), (ii) $\Rightarrow$

$$\begin{array}{cccc} l & & m & n \\ 4 & 1 & 3 & 4 \\ 2 & 0 & 1 & 2 \end{array} \Rightarrow \frac{l}{-2}=\frac{m}{1}=\frac{n}{2}$$

Solving (i), (iii) $\Rightarrow$

$$\begin{array}{cccc} l & & m & n \\ 4 & 1 & 3 & 4 \\ 3 & 0 & 2 & 3 \end{array} \Rightarrow \frac{l}{-3}=\frac{m}{2}=\frac{n}{1}$$

Therefore the equations of the required lines are

$$\frac{x}{-2}=\frac{y}{1}=\frac{n}{2} \quad \text{and} \quad \frac{x}{-3}=\frac{y}{2}=\frac{z}{1}$$

**Problem:** Find the angle between the lines of intersection of the plane $x-3y+z=0$ and the cone $x^2-5y^2+z^2=0$.

**Sol:** Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \rightarrow (1)$ be the equation of one of the two lines in which the plane $x-3y+z=0 \rightarrow (2)$ intersects the cone $x^2-5y^2+z^2=0 \rightarrow (3)$.

Since the line (1) lies on the cone (3), $l^2-5m^2+n^2=0$
Since the line (1) lies on the plane (2), the line (1) is perpendicular to the normal of the plane (2)

$$l-3m+n=0 \rightarrow (i) \Rightarrow n=3m-l$$

$$l^2-5m^2+(3m-l)^2=0$$

$$\Rightarrow l^2-5m^2+9m^2+l^2-6lm=0$$

$$\Rightarrow 2l^2+4m^2-6lm=0$$

$$\Rightarrow 2l^2-2lm-4lm+4m^2=0$$

$$\Rightarrow 2l(l-m)-4m(l-m)=0$$

$$\Rightarrow (2l-4m)(l-m)=0$$

$$\Rightarrow 2l-4m=0 \rightarrow (ii), \quad l-m=0 \rightarrow (iii)$$

$$l-2m=0$$

Solving (i), (ii) $\rightarrow$

$$\begin{array}{cccc} l & m & n & \\ -3 & 1 & 1 & -3 \\ -2 & 0 & 1 & -2 \end{array} \Rightarrow \frac{l}{2}=\frac{m}{1}=\frac{n}{1}$$

Solving (i), (iii) $\Rightarrow$

$$\begin{array}{cccc} l & m & n & \\ -3 & 1 & 1 & -3 \\ -1 & 0 & 1 & -1 \end{array} \Rightarrow \frac{l}{1}=\frac{m}{1}=\frac{n}{2}$$

Therefore the equations of the two lines in which the plane intersect the cone are $\frac{x}{2}=\frac{y}{1}=\frac{z}{1}$ and $\frac{x}{2}=\frac{y}{1}=\frac{z}{1}$

Let $\theta$ be the angle between the lines

$$\cos\theta = \frac{|(1)(2)+(1)(1)+(2)(1)|}{\sqrt{1+1+4}\,\sqrt{4+1+1}} = \frac{|2+1+2|}{\sqrt{6}\sqrt{6}} = \frac{5}{6}$$

$$\Rightarrow \cos\theta = \frac{5}{6} \Rightarrow \theta = \cos^{-1}(5/6).$$

**Problem:** Show that the equation of the quadratic cone which contains the three coordinate axes the lines in which the plane $x-5y-3z=0$ cuts the cone $7x^2+5y^2-3z^2=0$ is $yz+10zx+18xy=0$.

**Sol:** Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \rightarrow (1)$ be one of the two lines in which the plane $x-5y-3z=0 \rightarrow (2)$ cuts the cone $7x^2+5y^2-3z^2=0 \rightarrow (3)$

Since the line (1) lies one the cone (3), $7l^2+5m^2-3n^2=0$

Since the line (1) lies on the plane (2) the line (1) is perpendicular to the normal of the plane (2).

$l - 5m - 3n = 0 \Rightarrow l = 5m + 3n \rightarrow (i)$

$7(5m+3n)^2 + 5m^2 - 3n^2 = 0$

$\Rightarrow 7(25m^2 + 9n^2 + 30mn) + 5m^2 - 3n^2 = 0$

$\Rightarrow 180m^2 + 60n^2 + 210mn = 0$

$\Rightarrow 6m^2 + 7mn + 2n^2 = 0$

$\Rightarrow 6m^2 + 3mn + 4mn + 2n^2 = 0$

$\Rightarrow 3m(2m+n) + 2n(2m+n) = 0$

$\Rightarrow (2m+n)(3m+2n) = 0$

$\Rightarrow 2m+n = 0 \rightarrow (ii), \quad 3m+2n = 0 \rightarrow (iii)$

Solving (i), (iii) $\Rightarrow$

$$\begin{array}{cccc} l & m & n & \\ -5 & -3 & 1 & -5 \\ 3 & 2 & 0 & 3 \end{array} \Rightarrow \frac{l}{-1} = \frac{m}{-2} = \frac{n}{3}$$

Solving (i), (ii) $\Rightarrow$

$$\begin{array}{cccc} l & m & n & \\ -5 & -3 & 0 & -5 \\ 2 & 1 & 0 & 2 \end{array} \Rightarrow \frac{l}{1} = \frac{m}{-1} = \frac{n}{2}$$

Therefore the equations of the lines in which plane (2) cuts the cone (3) are $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$ and $\frac{x}{-1} = \frac{y}{-2} = \frac{z}{+3}$.

The equation of the cone which contains three mud coordinate axes are is $fyz + gzx + hxy = 0 \rightarrow (4)$

If the line $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$ lies on the cone (4) $\Rightarrow -2f + 2g - h = 0 \rightarrow (iv)$

If the line $\frac{x}{-1} = \frac{y}{-2} = \frac{z}{+3}$ lies on the cone (4) $\Rightarrow$ ~~$-6yz - g2x + 8$~~

$-6f - 3g + 2h = 0 \Rightarrow 6f + 3g - 2h = 0 \rightarrow (v)$

Solving (iv) and (v) $\Rightarrow \frac{f}{4-3} = \frac{g}{6+4} = \frac{h}{6+12}$

$\Rightarrow \frac{f}{1} = \frac{g}{10} = \frac{h}{18}$

Hence the required cone is $yz + 10zx + 18xy = 0$.

**Problem:** prove that if the angle between the ~~two~~ lines of intersection of the plane $x+y+z = 0$ and the cone $ayz + bzx + cxy = 0$ is $\frac{\pi}{2}$ then $a+b+c = 0$.

Sol: Let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \rightarrow (1)$ be the equations of one of the two lines in which the plane $x+y+z = 0 \rightarrow (2)$ intersects the cone $ayz + bzx + cxy = 0 \rightarrow (3)$

Since the line (1) lies on the cone (3) $\rightarrow amn + bnl + clm = 0$

Since the line (1) lies on the plane (2), the line (1) is perpendicular to the normal of the plane (2). $l+m+n = 0 \Rightarrow l = -(m+n)$

$amn + bn(-(m+n)) + c(-(m+n))m = 0$

$\rightarrow amn - bmn - bn^2 - cm^2 - cmn = 0$

$\rightarrow -bn^2 + mn(a-b-c) - cm^2 = 0$

$\rightarrow bn^2 - mn(a-b-c) + cm^2 = 0$

$\rightarrow b\left(\frac{n}{m}\right)^2 - \left(\frac{n}{m}\right)(a-b-c) + c = 0$

This is a quadratic equation in $n/m$.

Let the roots be $\frac{n_1}{m_1}, \frac{n_2}{m_2}$

$\rightarrow \left(\frac{n_1}{m_1}\right)\left(\frac{n_2}{m_2}\right) = \frac{c}{b} \rightarrow \frac{n_1 n_2}{c} = \frac{m_1 m_2}{b} \Rightarrow \frac{l_1 l_2}{a} = \frac{m_1 m_2}{b} = \frac{n_1 n_2}{c} = k$

The angle between the lines with d.cs $(l_1, m_1, n_1)$ and $(l_2, m_2, n_2)$ is $\pi/2$

Then $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \rightarrow k(a+b+c) = 0 \rightarrow a+b+c = 0$

Problem: Show that the condition that the plane $ux+vy+wz=0$ may cut the cone $ax^2+by^2+cz^2=0$ in particular perpendicular generators is $(b+c)u^2 + (c+a)v^2 + (a+b)w^2 = 0$.

Sol: Let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \rightarrow ①$ be the equation of one of the two lines in which the plane $ux+vy+wz=0 \rightarrow ②$ intersects the cone $ax^2+by^2+cz^2=0 \rightarrow ③$

Since the line ① lies on the cone ③, $al^2+bm^2+cn^2=0$

Since the line ① lies on the plane ②, $ul+vm+wn=0$

$\rightarrow ul = -(vm+wn) \rightarrow l = \frac{-(vm+wn)}{u}$

$a\left(\frac{-(vm+wn)}{u}\right)^2 + bm^2 + cn^2 = 0$

$\rightarrow a(vm+wn)^2 + bu^2m^2 + cu^2n^2 = 0$

$\rightarrow av^2m^2 + aw^2n^2 + 2amnvw + bu^2w^2 + cu^2n^2 = 0$

$\rightarrow m^2(v^2a + bu^2) + n^2(w^2a + cu^2) + 2amnvw = 0$

$\rightarrow v^2a + bu^2 + \frac{n^2}{m^2}(w^2a + cu^2) + \frac{2amnvw}{m^2} = 0$

$\rightarrow \left(\frac{n}{m}\right)^2(w^2a + cu^2) + \left(\frac{n}{m}\right)(2avw) + v^2a + bu^2 = 0$

This is a quadratic equation in $\frac{n}{m}$

Let the roots be $\frac{n_1}{m_1}, \frac{n_2}{m_2}$

$\rightarrow \frac{n_1 n_2}{m_1 m_2} = \frac{v^2a + bu^2}{w^2a + cu^2} \Rightarrow \frac{n_1 n_2}{v^2a + bu^2} = \frac{m_1 m_2}{w^2a + cu^2}$

$$\Rightarrow \frac{l_1l_2}{bw^2+cv^2} = \frac{m_1m_2}{aw^2+cu^2} = \frac{n_1n_2}{av^2+bu^2} = k$$

If the lines of intersection is right angle then $l_1l_2+m_1m_2+n_1n_2=0$

$$\rightarrow k(bw^2+cv^2+aw^2+cu^2+av^2+bu^2)=0$$
$$\rightarrow k[u^2(b+c)+v^2(c+a)+w^2(a+b)]=0$$
$$\Rightarrow u^2(b+c)+v^2(c+a)+w^2(a+b)=0$$

**Problem:** If the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines then prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

**Sol:** Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \rightarrow (1)$ be the equations of one of the two lines in which the plane $ax+by+cz=0 \rightarrow (2)$ intersects the cone $yz+zx+xy=0 \rightarrow (3)$

Since the line (1) lies on the cone (3), $mn+nl+lm=0$

Since the line (1) lies on the plane (2), the line (1) is perpendicular to the normal of the plane (2).

$$al+bm+cn=0 \Rightarrow al=-(bm+cn) \Rightarrow l=-\frac{(bm+cn)}{a}$$

$$mn+n\left(-\frac{(bm+cn)}{a}\right)+\left(-\frac{(bm+cn)}{a}\right)m=0$$

$$\Rightarrow amn-bmn-cn^2-bm^2-cmn=0$$
$$\Rightarrow -bm^2+mn(a-b-c)-cn^2=0$$
$$\Rightarrow bm^2-mn(a-b-c)+cn^2=0$$
$$\Rightarrow b\left(\frac{m}{n}\right)^2-\left(\frac{m}{n}\right)(a-b-c)+c=0$$

This is a quadratic equation in $\frac{m}{n}$

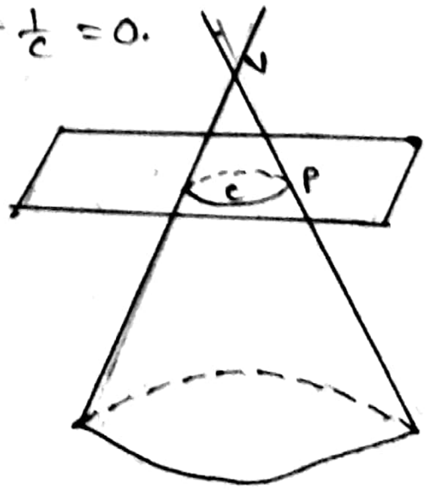
Let the roots be $\frac{m_1}{n_1}, \frac{m_2}{n_2}$

$$\Rightarrow \left(\frac{m_1}{n_1}\right)\left(\frac{m_2}{n_2}\right)=\frac{c}{b} \Rightarrow \frac{m_1m_2}{n_1n_2}=\frac{1/b}{1/c} \Rightarrow \frac{l_1l_2}{1/a}=\frac{m_1m_2}{1/b}=\frac{n_1n_2}{1/c}=k$$

If the lines are perpendicular then $l_1l_2+m_1m_2+n_1n_2=0$

$$k(1/a+1/b+1/c)=0 \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0.$$

**Cone with a base curve:**




**Definition:** Let S be the set of lines concurrent at V and C be a curve not containing V.

If $P \in C \rightarrow \overleftrightarrow{VP} \subset S$ then S is called the cone with vertex at V.

C is called the base curve or, guiding curve. $\overleftrightarrow{VP}$ is called a generator of the cone.

<u>Theorem</u>: The equation of a cone with vertex at $(\alpha,\beta,\gamma)$ & xy plane and the guiding curve $f(x,y)=0,\ z=0$ is

$$(z-\gamma)^2 \cdot f\left(\alpha-\gamma\frac{x-\alpha}{z-\gamma},\ \beta-\gamma\frac{y-\beta}{z-\gamma}\right)=0$$

<u>Proof</u>: Let the equation to a line through $(\alpha,\beta,\gamma)$ with direction ratios $(l,m,n)$ be $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \rightarrow ①$

Any point on the line ① is $(lr+\alpha,\ mr+\beta,\ nr+\gamma)$

Let the equation to the curve is

$$f(x,y) = ax^2+2hxy+by^2+2gx+2fy+c=0,\quad z=0$$

The line passes through the conic.

$\Leftrightarrow$ The point $P(lr+\alpha,\ mr+\beta,\ nr+\gamma)$ lies on $f(x,y)=0$ and on the plane $z=0$.

$$z=0 \Rightarrow nr+\gamma=0 \Rightarrow r=-\gamma/n$$

Hence the point $P=\left(\alpha-\frac{l\gamma}{n},\ \beta-\frac{m\gamma}{n},\ 0\right)$ which will lie on the given conic.

$$\Rightarrow a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\cdot\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta-\frac{m\gamma}{n}\right)^2+2g\left(\alpha-\frac{l\gamma}{n}\right)+2f\left(\beta-\frac{m\gamma}{n}\right)+c=0 \rightarrow ②$$

This is the condition for the line ① to intersect the conic.

Eliminating $l,m,n$ from ① and ② we get

$$\Rightarrow a\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)^2+2h\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)\left(\beta-\frac{y-\alpha}{z-\gamma}\cdot\gamma\right)+b\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)^2$$
$$+2g\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)+2f\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)+c=0$$

$$\Rightarrow a(\alpha z-x\gamma)^2+2h(\alpha z-x\gamma)(\beta z-\gamma y)+b(\beta z-\gamma y)^2+2g(\alpha z-\gamma x)(z-\gamma)+2f(\beta z-\gamma y)(z-\gamma)+c(z-\gamma)^2=0$$

$$\Rightarrow (z-\gamma)^2\cdot f\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma,\ \beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)=0$$

which is the required equation of the conic.

<u>Note</u>: The guiding curve of the cone may be $f(y,z)=0,\ x=0$ or, $f(z,x)=0,\ y=0$

**Problem:** Find the equation of the cone whose vertex is $(1,1,0)$ and whose guiding curve is $y=0,\ x^2+z^2=4$

**Sol** Let the equation to the line through $(1,1,0)$ and having d.r's $l,m,n$ be $\frac{x-1}{l}=\frac{y-1}{m}=\frac{z-0}{n}=r \rightarrow (1)$

Any point on the line (1) is $(lr+1, mr+1, nr)$

If this point lies on the curve $y=0,\ x^2+z^2=4$

Then $y=0 \Rightarrow mr+1=0 \Rightarrow r=-1/m$

$$\left(l(-1/m)+1,\ m(-1/m)+1,\ n(-1/m)\right)=\left(1-l/m,\ 0,\ -n/m\right)$$

$$x^2+z^2=4 \rightarrow \left(1-\frac{l}{m}\right)^2+\left(\frac{-n}{m}\right)^2=4 \Rightarrow (m-l)^2+n^2=4m^2 \rightarrow (2)$$

Eliminating $l,m,n$ from (1) and (2).

$$[(y-1)-(x-1)]^2+(z)^2=4(y-1)^2$$

$$\Rightarrow (y-x)^2+z^2=4(y-1)^2$$

$$\Rightarrow y^2+x^2-2xy+z^2=4(y^2+1-2y)$$

$$\Rightarrow y^2+x^2-2xy+z^2=4y^2+4-8y \Rightarrow x^2-3y^2+z^2+8y-2xy-4=0$$

**Problem:** Find the equation of the cone with vertex $(5,4,3)$ and $3x^2+2y^2=6,\ y+z=0$ as base

**Sol:** Let the equation to the line through $(5,4,3)$ and having d.r's $(l,m,n)$ be $\frac{x-5}{l}=\frac{y-4}{m}=\frac{z-3}{n}=r \rightarrow (1)$

Any point on the line (1) is $(lr+5, mr+4, nr+3)$

If this point lies on the base $3x^2+2y^2=6,\ y+z=0$.

$$y+z=0 \Rightarrow mr+4+nr+3=0 \Rightarrow r=\frac{-7}{m+n}$$

$$3x^2+2y^2-6=0 \Rightarrow 3(lr+5)^2+2(mr+4)^2-6=0$$

$$\Rightarrow 3(l^2r^2+25+10lr)+2(m^2r^2+16+8mr)-6=0$$

$$\Rightarrow 3l^2r^2+75+30lr+2m^2r^2+32+16mr-6=0$$

$$\Rightarrow 3l^2\left(\frac{-7}{m+n}\right)^2+75+30l\left(\frac{-7}{m+n}\right)+2m^2\left(\frac{-7}{m+n}\right)^2+32+16m\left(\frac{-7}{m+n}\right)-6=0$$

$$\Rightarrow 147l^2-210l(m+n)+98m^2-112m(m+n)+101(m+n)^2=0 \rightarrow (2)$$

Eliminating $l,m,n$ from (1) and (2)

$$\Rightarrow 147(x-5)^2-210(x-5)(y-4+z-3)+98(y-4)^2-112(y-4)(y-4+z-3)+101(y-4+z-3)^2=0$$

Problem: obtain the Locus of the lines which passes through a point $(\alpha, \beta, \gamma)$ and through the points of the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0$

Sol let the equation to the line through $(\alpha, \beta, \gamma)$ and having d.r.s $l, m, n$ be $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \rightarrow ①$

Any point on the line ① is $(lr+\alpha,\ mr+\beta,\ nr+\gamma)$

If this point lies on the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0$

$z = 0 \Rightarrow nr+\gamma = 0 \rightarrow r = \frac{-\gamma}{n}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow \frac{(lr+\alpha)^2}{a^2} + \frac{(mr+\beta)^2}{b^2} = 1$

$\Rightarrow \frac{\left(-l\cdot(\frac{\gamma}{n})+\alpha\right)^2}{a^2} + \frac{\left(m(-\gamma/n)+\beta\right)^2}{b^2} = 1$

$\rightarrow \frac{(-l\gamma+n\alpha)^2}{n^2a^2} + \frac{(-n\gamma+n\beta)^2}{n^2b^2} = 1$

$\rightarrow \frac{b^2(n\alpha-l\gamma)^2 + a^2(n\beta-n\gamma)^2}{n^2a^2b^2} = 1$

$\rightarrow b^2(n\alpha-l\gamma)^2 + a^2(n\beta-n\gamma)^2 = n^2a^2b^2 \rightarrow ②$

Eliminating $l, m, n$ from ① and ②

$b^2[(z-\gamma)\alpha - (x-\alpha)\gamma]^2 + a^2[(z-\gamma)\beta - (y-\beta)\gamma]^2 = a^2b^2(z-\gamma)^2$

$\rightarrow b^2[z\alpha - \gamma\alpha - x\gamma + \alpha\gamma]^2 + a^2[z\beta - \gamma\beta - y\gamma + \beta\gamma]^2 = a^2b^2(z-\gamma)^2$

$\rightarrow b^2(z\alpha - x\gamma)^2 + a^2(z\beta - y\gamma)^2 = a^2b^2(z-\gamma)^2.$

Problem: Find the equation of the cone whose vertex is $(1,2,3)$ and base $y^2 = 4ax$ and $z = 0$

Sol: let the equation to the line through $(1,2,3)$ and having d.r.s $l, m, n$ be $\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} = r \rightarrow ①$

Any point on the line ① is $(lr+1,\ mr+2,\ nr+3)$

If this point lies on the base $y^2 = 4ax,\ z = 0$

$z = 0 \Rightarrow nr+3 = 0 \rightarrow r = -3/n$

$y^2 = 4ax \Rightarrow (mr+2)^2 = 4a(lr+1).$

$\Rightarrow (m(-3/n)+2)^2 = 4a(l(-3/n)+1).$

$\Rightarrow (-3m+2n)^2 = 4an(-3l+n) \rightarrow (2)$

Eliminating $l, m, n$ from (1) and (2)

$\rightarrow [-3(y-2)+2(z-3)]^2 = 4a(z-3)[-3(x-1)+(z-3)]$

$\rightarrow (-3y+6+2z-6)^2 = 4a(z-3)(-3x+3+z-3)$

$\Rightarrow (2z-3y)^2 = 4a(z-3)(z-3x).$

Problem : Find the equation of the cone with vertex at $(-1,1,2)$ and guiding curve $3x^2-y^2=1,\ z=0.$

Sol: Let the equation to the line through $(-1,1,2)$ and having d.rs $l, m, n$ be $\frac{x+1}{l} = \frac{y-1}{m} = \frac{z-2}{n} = r \rightarrow (1)$

Any point on the line (1) is $(lr-1, mr+1, nr+2)$

If this point lies on the base $3x^2-y^2=1,\ z=0$

$z=0 \Rightarrow nr+2=0 \Rightarrow r = \frac{-2}{n}$

$3x^2-y^2=1 \rightarrow 3(lr-1)^2-(mr+1)^2=1$

$\rightarrow 3\left(l\left(\frac{-2}{n}\right)-1\right)^2 - \left(m\left(\frac{-2}{n}\right)+1\right)^2 = 1$

$\Rightarrow 3(-2l-n)^2 - (-2m+n)^2 = n^2 \rightarrow (2)$

Eliminating $l, m, n$ from (1) and (2)

$\rightarrow 3[-2(x+1)-(z-2)]^2 - [-2(y-1)+(z-2)]^2 = (z-2)^2$

$\rightarrow 3(-2x+2-z+2)^2 - (-2y+2+z-2)^2 = (z-2)^2$

$\Rightarrow 3(-2x-z)^2 - (-2y+z)^2 = (z-2)^2$

$\rightarrow 3(4x^2+z^2+4xz) - (4y^2+z^2-4yz) = z^2+4-4z$

$\Rightarrow 12x^2+3z^2+12x\cancel{y}z - 4y^2 - z^2 + 4yz = z^2-4z+4$

$\Rightarrow 12x^2+3z^2+12xz-4y^2-z^2+4yz-z^2+4z-4=0$

$\Rightarrow 12x^2-4y^2+z^2+12zx+4yz+4z-4=0$

which is the equation to the required cone.

Problem: Show that the equation of the cone whose vertex is the origin and whose base is the circle through three points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ is
$\Sigma a(b^2+c^2)yz = 0$

Sol The equation of the circle through $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ is $x^2+y^2+z^2-ax-by-cz=0 \rightarrow ①$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \rightarrow ②$

Making Homogeneous equation ① by using equation ②.

$$x^2+y^2+z^2-(ax+by+cz)(1)=0$$

$$\rightarrow x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

$$\Rightarrow x^2+y^2+z^2-\left(x^2+\frac{a}{b}xy+\frac{a}{c}zx+\frac{b}{a}xy+y^2+\frac{b}{c}yz+\frac{c}{a}zx+\frac{c}{b}yz+z^2\right)=0$$

$$\Rightarrow x^2+y^2+z^2-x^2-y^2-z^2-\frac{a}{b}xy-\frac{b}{a}xy-\frac{c}{b}yz-\frac{b}{c}yz-\frac{c}{a}zx-\frac{a}{c}zx=0$$

$$\Rightarrow -xy\left(\frac{a}{b}+\frac{b}{a}\right)-yz\left(\frac{b}{c}+\frac{c}{b}\right)-zx\left(\frac{c}{a}+\frac{a}{c}\right)=0$$

$$\Rightarrow cxy(a^2+b^2)+ayz(b^2+c^2)+bzx(c^2+a^2)=0$$

$$\Rightarrow \Sigma a(b^2+c^2)yz=0.$$

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Problem: Show that the equation of the cone whose vertex is the origin and whose base is the circle through three points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ is $\sum a(b^2+c^2)yz = 0$

Sol The equation of the circle through $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ is $x^2+y^2+z^2-ax-by-cz=0$ $\rightarrow$ ①, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ $\rightarrow$ ②

Making Homogeneous equation ① by using equation ②

$$x^2+y^2+z^2-(ax+by+cz)(1)=0$$

$$\Rightarrow x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

$$\Rightarrow x^2+y^2+z^2-\left(x^2+\frac{a}{b}xy+\frac{a}{c}zx+\frac{b}{a}xy+y^2+\frac{b}{c}yz+\frac{c}{a}zx+\frac{c}{b}yz+z^2\right)=0$$

$$\Rightarrow x^2+y^2+z^2-x^2-y^2-z^2-\frac{a}{b}xy-\frac{b}{a}xy-\frac{c}{b}yz-\frac{b}{c}yz-\frac{c}{a}zx-\frac{a}{c}zx=0$$

$$\Rightarrow -xy\left(\frac{a}{b}+\frac{b}{a}\right)-yz\left(\frac{b}{c}+\frac{c}{b}\right)-zx\left(\frac{c}{a}+\frac{a}{c}\right)=0$$

$$\Rightarrow cxy(a^2+b^2)+ayz(b^2+c^2)+bzx(c^2+a^2)=0$$

$$\Rightarrow \sum a(b^2+c^2)yz=0.$$

# UNIT-5

# CONES



## ENVELOPING CONE OF A SPHERE :

**DEFINITION:** Let S be a surface and P be a point not ~~point~~ ~~oth~~ on the surface the set of tangent line to the surface S and passing through P from a cone with vertex at P.

This is called the enveloping cone (or) Tangent cone of the given surface.

**NOTATION:** If $S = x^2+y^2+z^2-a^2$

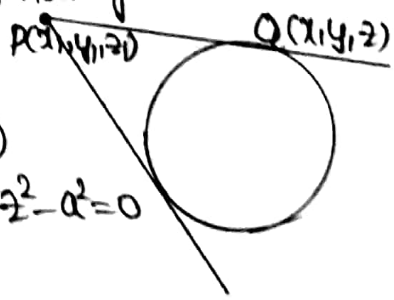
$$S_1 = xx_1+yy_1+zz_1-a^2$$
$$S_{12} = x_1x_2+y_1y_2+z_1z_2-a^2$$
$$S_{11} = x_1^2+y_1^2+z_1^2-a^2$$

If $S = x^2+y^2+z^2+2ux+2vy+2wz+d$

$$S_1 = xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d$$
$$S_{12} = x_1x_2+y_1y_2+z_1z_2+u(x_1+x_2)+v(y_1+y_2)+w(z_1+z_2)+d$$
$$S_{11} = x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d.$$

**THEOREM:** The enveloping cone of the sphere $x^2+y^2+z^2=a^2$ with vertex at $(x_1,y_1,z_1)$ is $(xx_1+yy_1+zz_1-a^2)^2 = (x^2+y^2+z^2-a^2)(x_1^2+y_1^2+z_1^2-a^2)$

**proof:** The equation of a line through $(x_1,y_1,z_1)$ and having direction ratios $l,m,n$ is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r \rightarrow (1)$




Any point on the line (1) is $(lr+x_1, mr+y_1, nr+z_1)$

If this point ~~on the line~~ lies on the surface $x^2+y^2+z^2-a^2=0$

Then $(lr+x_1)^2+(mr+y_1)^2+(nr+z_1)^2-a^2=0$

$\Rightarrow l^2r^2+x_1^2+2lrx_1+m^2r^2+y_1^2+2mry_1+n^2r^2+z_1^2+2nrz_1-a^2=0$

$\Rightarrow r^2(l^2+m^2+n^2)+2r(lx_1+my_1+nz_1)+(x_1^2+y_1^2+z_1^2-a^2)=0 \rightarrow (2)$

Which is quadratic equation in 'r'.

If the line (1) touches the sphere $x^2+y^2+z^2-a^2=0$

Then the two values of r in equation (2) are same.

The discriminant of the equation (2) is zero. $[b^2-4ac=0]$

$[2(lx_1+my_1+nz_1)]^2-4(l^2+m^2+n^2)(x_1^2+y_1^2+z_1^2-a^2)=0$

$4(lx_1+my_1+nz_1)^2=4(l^2+m^2+n^2)(x_1^2+y_1^2+z_1^2-a^2) \rightarrow (3)$

Which is the condition for the line (1) to touches the sphere $x^2+y^2+z^2=a^2$

Eliminating $l, m, n$ from equation (1) and equation (3)

$$[(x-x_1)x_1+(y-y_1)y_1+(z-z_1)z_1]^2=[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2][x_1^2+y_1^2+z_1^2-a^2]$$

$$[xx_1-x_1^2+yy_1-y_1^2+zz_1-z_1^2]^2=[x^2+x_1^2-2xx_1+y^2+y_1^2-2yy_1+z^2+z_1^2-2zz_1][x_1^2+y_1^2+z_1^2-a^2]$$

$$[xx_1+yy_1+zz_1-(x_1^2+y_1^2+z_1^2)]^2=[(x^2+y^2+z^2)-2(xx_1+yy_1+zz_1)+x_1^2+y_1^2+z_1^2][x_1^2+y_1^2+z_1^2-a^2]$$

$$[xx_1+yy_1+zz_1-a^2-(x_1^2+y_1^2+z_1^2-a^2)]^2=[x^2+y^2+z^2-a^2-2(xx_1+yy_1+zz_1-a^2)+x_1^2+y_1^2+z_1^2-a^2][x_1^2+y_1^2+z_1^2-a^2]$$

$$[S_1-S_{11}]^2=(S-2S_1+S_{11})(S_{11})$$

$$S_1^2+S_{11}^2-2S_1S_{11}=SS_{11}-2S_1S_{11}+S_{11}^2$$

$$S_1^2=SS_{11}$$

Thus the equation to the enveloping cone is

$$[xx_1+yy_1+zz_1-a^2]^2=(x^2+y^2+z^2-a^2)(x_1^2+y_1^2+z_1^2-a^2).$$

1: Find the enveloping cone of the sphere $x^2+y^2+z^2+2x-2y=2$ with its vertex at $(1,1,1)$

Sol: Given equation of the sphere is $x^2+y^2+z^2+2x-2y-2=0 \rightarrow (1)$

$$2u=2,\quad 2v=-2,\quad 2w=0,\quad d=-2$$
$$u=1,\quad v=-1,\quad w=0,$$

Given vertex is $(1,1,1)$

$$S_1=xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d$$
$$=x(1)+y(1)+z(1)+1(x+1)+(-1)(y+1)+0(z+1)-2$$
$$=x+y+z+x-y-2$$
$$=2x+z-2$$

$$S_{11}=x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d$$
$$=(1)^2+(1)^2+(1)^2+2(1)(1)+2(-1)(1)+2(0)(1)-2$$
$$=1+1+1+2-2-2$$
$$=1$$

Thus the equation to the enveloping cone of the sphere (1) with vertex at $(1,1,1)$ is $S_1^2=SS_{11}$

$$\Rightarrow (2x+z-2)^2=(x^2+y^2+z^2+2x-2y-2)(1)$$
$$\Rightarrow 3x^2-y^2+4xz-10x+2y-4z+6=0.$$

2: Find the enveloping cone with vertex at the Origin and generators touching the sphere $x^2+y^2+z^2-2x+4z-1=0$.

Sol: Given equation of the sphere $x^2+y^2+z^2-2x+4z-1=0 \rightarrow (1)$

$2u=-2, \quad 2v=0, \quad 2w=4, \quad d=-1$

$u=-1, \quad v=0, \quad w=2,$

Given vertex is $(0,0,0)$

$S_1 = xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d$

$= x(0)+y(0)+z(0)+(-1)(x+0)+0(y+0)+2(z+0)+(-1)$

$= -x+2z-1$

$= x-2z+1$

$S_{11} = x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d$

$= (0)^2+(0)^2+(0)^2+2(-1)(0)+2(0)(0)+2(2)(0)-1$

$= -1$

The equation to the enveloping cone of the sphere (1) with vertex at $(0,0,0)$

$S_1^2 = S.S_{11}$

$(x-2z+1)^2 = (x^2+y^2+z^2-2x+4z-1)(-1)$

$\Rightarrow x^2+4z^2+1-4xz-4z+2x = -x^2-y^2-z^2+2x-4z+1$

$\rightarrow 2x^2+y^2+5z^2-4xz=0.$

3: Find the enveloping cone of the sphere $x^2+y^2+z^2+2x-4y=0$ with it's vertex at $(1,1,1)$

Sol: Given equation of the sphere $x^2+y^2+z^2+2x-4y=0 \rightarrow (1)$

$2u=2, \quad 2v=(-4), \quad 2w=0, \quad d=0$

$u=1, \quad v=(-2), \quad w=0,$

Given vertex is $(1,1,1)$

$S_1 = xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d$

$= x(1)+y(1)+z(1)+1(x+1)-2(y+1)+0(z+1)+0$

$= 2x-y+z-1$

$S_{11} = x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d$

$= 1+1+1+2(1)(1)+(-2)(2)(1)+2(0)+0$

$= 1+1+1+2-4=1$

The equation to the enveloping cone of the sphere (1) with vertex at $(1,1,1)$

$S_1^2 = SS_{11}$

$(2x-y+z-1)^2 = (x^2+y^2+z^2+2x-4y)(1)$

$\Rightarrow 4x^2+y^2+(z-1)^2-4xy-2y(z-1)+4x(z-1)=x^2+y^2+z^2+2x-4y$

$\Rightarrow 4x^2+y^2+z^2+1-2z-4xy-2yz+2y+4xz-4x=x^2+y^2+z^2+2x-4y$

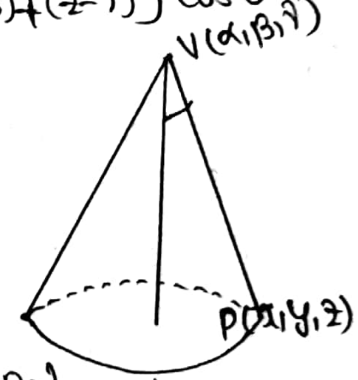
$\Rightarrow 3x^2-4xy-2yz+4xz-6x+6y-2z+1=0.$

## RIGHT CIRCULAR CONE:

**DEFINITION:** A Right Circular cone is a surface generated by a line which passes through a fixed point and makes constant angle with fixed line through the fixed point. The fixed point is called the vertex. The fixed line the axis and the fixed angle the semi-vertical angle of the cone.

**THEOREM:** The equation of a Right circular cone with vertex at $(\alpha,\beta,\gamma)$ semi-vertical angle '$\theta$' and axis having direction ratios $l,m,n$ is

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2=(l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]\cos^2\theta$$




**proof:** Let $V(\alpha,\beta,\gamma)$ be the vertex and $VL$ be the axis of the cone.

The direction ratios of the axis are $l,m,n$

Let $P(x,y,z)$ be a point on the cone

Dr's of VP are $=$ ~~(x-α, y-β~~ $(x-\alpha,y-\beta,z-\gamma)$

Semi-vertical angle '$\theta$' $\cos\theta=\dfrac{|l_1l_2+m_1m_2+n_1n_2|}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$

$$\Rightarrow \cos\theta=\frac{|l(x-\alpha)+m(y-\beta)+n(z-\gamma)|}{\sqrt{l^2+m^2+n^2}\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

$$\Rightarrow \left[\sqrt{l^2+m^2+n^2}\cdot\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}\right]\cos\theta=l[x-\alpha]+m(y-\beta)+n(z-\gamma)$$

$$\Rightarrow [l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2=(l^2+m^2+n^2)((x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2)\cos^2\theta$$

Then the equation of the right circular cone is

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2=(l^2+m^2+n^2)((x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2)\cos^2\theta.$$

**NOTE:** If the vertex is the origin then the equation of the right circular cone becomes

$$(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2)\cos^2\theta.$$

**NOTE:** If the vertex is the origin and the axis of the cone is $z$-axis and semi-vertical angle '$\theta$'. Then the right circular cone becomes

$$[0(x-0)+0(y-0)+1(z-0)]^2 = (0+0+1)[(x-0)^2+(y-0)^2+(z-0)^2]\cos^2\theta$$

$$\Rightarrow z^2 = (x^2+y^2+z^2)\cos^2\theta$$

$$\Rightarrow z^2 = (x^2+y^2)\cos^2\theta + z^2\cos^2\theta$$

$$\Rightarrow z^2 - z^2\cos^2\theta = (x^2+y^2)\cos^2\theta$$

$$\Rightarrow (1-\cos^2\theta)z^2 = (x^2+y^2)\cos^2\theta$$

$$\Rightarrow z^2\sin^2\theta = (x^2+y^2)\cos^2\theta$$

$$\Rightarrow x^2+y^2 = z^2\tan^2\theta$$

$y$-axis $\Rightarrow x^2+z^2 = y^2\tan^2\theta$

$x$-axis $\Rightarrow y^2+z^2 = x^2\tan^2\theta$

**4:** Find the equation to the right circular cone whose vertex is $P(2,-3,5)$ axis $PQ$ which makes equal angle with the co-ordinate axis and which passes through $A(1,-2,3)$

**Sol:** The axis of the cone makes equal angle $\theta$ with the coordinate axis

Direction ratios of the axis are $(\cos\theta, \cos\theta, \cos\theta)$ i.e., $(1,1,1)$

Let $A(1,-2,3)$ be a point on the cone

The dr's of the $AP = (1-2, -2+3, 3-5)$
$= (-1,1,-2)$

Let '$\theta$' be the semi-vertical angle then

$$\cos\theta = \frac{|l_1l_2+m_1m_2+n_1n_2|}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$$

$$= \frac{|1(-1)+1(1)+1(-2)|}{\sqrt{1+1+1}\sqrt{1+1+4}}$$

$$= \frac{|-1+1-2|}{\sqrt{3}\sqrt{6}} = \frac{|-2|}{\sqrt{18}} = \frac{2}{\sqrt{18}}$$

$$\Rightarrow \cos^2\theta = \frac{4}{18} = 2/9$$

The equation of the right circular cone is

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2 = (l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]\cos^2\theta$$

$$[1(x-2)+1(y+3)+1(z-5)]^2 = (1+1+1)[(x-2)^2+(y+3)^2+(z-5)^2]\frac{2}{9}$$

$\Rightarrow 3(x+y+z-4)^2 = \frac{2}{3}(x^2+4-4x+y^2+9+6y+z^2+25-10z)$

$\Rightarrow 3[x^2+y^2+(z-4)^2+2xy+2y(z-4)+2x(z-4)] = 2(x^2+y^2+z^2-4x+6y-10z+38)$

$\Rightarrow 3x^2+3y^2+3z^2+48-24z+6xy+6y(z-4)+6x(z-4) =$
$2x^2+2y^2+2z^2-8x+12y-20z+76$

$\Rightarrow 3x^2+3y^2+3z^2+48-24z+6xy+6yz-24y+6xz-24x =$
$2x^2+2y^2+2z^2-8x+12y-20z+76$

$\Rightarrow x^2+y^2+z^2+6xy+6yz+6xz-16x-36y-4z-28=0.$

5: Find the equation of the Right circular cone whose vertex is $(3,2,1)$, axis line $\frac{x-3}{4} = \frac{y-2}{1} = \frac{z-1}{3}$ and semi-vertical angle $30°$

Sol: Given vertex of the cone $(3,2,1)$

The equation of the axis are $\frac{x-3}{4} = \frac{y-2}{1} = \frac{z-1}{3}$

Direction ratios of the axis are $(4,1,3)$

Semi-vertical angle is $30°$

Thus, The equation of right circular cone is

$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2 = (l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)+(z-\gamma)^2]\cos^2\theta$

$\Rightarrow [4(x-3)+1(y-2)+3(z-1)]^2 = (16+1+9)[(x-3)^2+(y-2)^2+(z-1)^2]\cos^2 30°$

$\Rightarrow [4x-12+y-2+3z-3]^2 = (26)[x^2+9-6x+y^2+4-4y+z^2+1-2z](3/4)$

$\Rightarrow 4[4x+y+(3z-17)]^2 = 78[x^2+y^2+z^2-6x-4y-2z+14]$

$\Rightarrow 2[16x^2+y^2+(3z-17)^2+8xy+8x(3z-17)+2y(3z-17)]$
$= 39[x^2+y^2+z^2-6x-4y-2z+14]$

$\Rightarrow 2(16x^2+y^2+9z^2+289-102z+8xy+24xz-136x+6yz-34y)$
$= 39x^2+39y^2+39z^2-234x-156y-78z+546$

$\Rightarrow 32x^2+2y^2+18z^2 \cancel{-58}\, 578-204z+16xy+48xz-272x+12yz-68y$
$= 39x^2+39y^2+39z^2-234x-156y-78z+546$

$\Rightarrow 7x^2+37y^2+21z^2-16xy-12yz-48zx+38x-88y+126z-32=0.$

6: Find the equation of the Right Circular cone with vertex at $(2,1,-3)$ and whose axis is parallel to oy and whose semi-vertical angle is $45°$. (4)

Sol: Axis of the cone is parallel to oy

Direction ratios of the axis are $(0,1,0)$

Given vertex of the cone $(2,1,-3)$

semi-vertical angle $\theta = 45°$

Thus the equation of the right circular cone is

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2 = (l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]\cos^2\theta$$

$$\Rightarrow [0(x-2)+1(y-1)+0(z+3)]^2 = (0+1+0)[(x-2)^2+(y-1)^2+(z+3)^2]\cos^2 45°$$

$$\Rightarrow (y-1)^2 = 1[x^2+4-4x+y^2+1-2y+z^2+9+6z][1/2]$$

$$\Rightarrow 2(y-1)^2 = x^2+y^2+z^2-4x-2y+6z+14$$

$$\Rightarrow 2y^2+2-4y = x^2+y^2+z^2-4x-2y+6z+14$$

$$\Rightarrow x^2-y^2+z^2-4x+2y+6z+12=0.$$

7: Find the equation of the right circular cone whose vertex is the origin, axis as the line $x=t, y=2t, z=3t$ and whose semi-vertical angle is $60°$.

Sol: Vertex $(\alpha,\beta,\gamma)=(0,0,0)$

Equation to the axis $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}=t$

$\Rightarrow$ Direction ratios of the axis $(l,m,n)=(1,2,3)$

semi-vertical angle $=60°$

$\therefore$ Equation to the required cone is $[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2 = \cos^2\theta\,(l^2+m^2+n^2)\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\}$

$$\Rightarrow [1(x-0)+2.(y-0)+3(z-0)]^2 = \cos^2 60°[1+4+9][(x-0)^2+(y-0)^2+(z-0)^2]$$

$$\Rightarrow [x+2y+3z]^2 = \frac{1}{4}(14)(x^2+y^2+z^2)$$

$$\Rightarrow 2(x+2y+3z)^2 = 7(x^2+y^2+z^2)$$

$$\Rightarrow 2[x^2+4y^2+9z^2+4xy+12yz+6xz] = 7(x^2+y^2+z^2)$$

$$\Rightarrow 2x^2+8y^2+18z^2+8xy+24yz+12xz = 7x^2+7y^2+7z^2$$

$$\Rightarrow 2x^2+8y^2+18z^2+8xy+24yz+12xz-7x^2-7y^2-7z^2$$

$$\Rightarrow -5x^2+y^2+11z^2+8xy+24yz+12xz=0$$

$$\Rightarrow 5x^2-y^2-11z^2-8xy-24yz-12xz=0.$$

8: Find the equation of the right circular cone whose vertex is $P(2,-3,5)$ axis PQ which makes equal angle with the co-ordinate axes and semi-vertical angle $30^\circ$.

Sol: The axis of the cone makes equal angle $\theta$ with the coordinate axes direction ratios of the axis are $(\cos\theta, \cos\theta, \cos\theta)$ i.e., $(1,1,1)$

The equation of the right circular cone is

$$\Rightarrow [l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2 = (l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]\cos^2\theta$$

$$\Rightarrow [1(x-2)+1(y+3)+1(z-5)]^2 = (1+1+1)[(x-2)^2+(y+3)^2+(z-5)^2]\cos^2 30^\circ$$

$$\Rightarrow [(x-2)+y+3+z-5]^2 = 3(x^2+4-4x+y^2+9+6y+z^2+25-10z)(3/4)$$

$$\Rightarrow 4(x+y+z-4)^2 = 9(x^2+y^2+z^2-4x+6y+(-10z)+38)$$

$$\Rightarrow 4(x^2+y^2+(z-4)^2+2xy+2y(z-4)+2x(z-4)) = 9(x^2+y^2+z^2-4x+6y-10z+38)$$

$$\Rightarrow 4(x^2+y^2+z^2+16-8z+2xy+2yz-8y+2xz-8x] = 9(x^2+y^2+z^2-4x+6y-10z+38)$$

$$\rightarrow 4x^2+4y^2+4z^2-32z-32y+8xz-32z+8yz+8xy+64 = 9x^2+9y^2+9z^2-36x+54y-90z+342$$

$$\Rightarrow 5x^2+5y^2+5z^2-8xz-8yz-8xy-4x+86y-58z+278=0.$$

9: If $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represents a cone prove that $u^2/a+v^2/b+w^2/c=d$

Sol: Given equation is $f(x,y,z) = ax^2+by^2+cz^2+2ux+2vy+2wz+d=0 \rightarrow (1)$

Making the equation (1) homogeneous by using a variable $t$ we get $f(x,y,z,t) = ax^2+by^2+cz^2+2uxt+2vyt+2wzt+dt^2=0 \rightarrow (2)$

Differentiating equation (2) partially with respect to $x,y,z,t$ and equating to zero and put $t=1$

$2ax+2ut=0$, $2by+2vt=0$, $2cz+2wt=0$, $2ux+2vy+2wz+2dt=0$

$2ax+2u=0$, $2by+2v=0$, $2cz+2w=0$, $2ux+2vy+2wz+2d=0$

$ax=-u$, $by+v=0$, $cz+w=0$, $ux+vy+wz+d=0$

$x=-u/a$, $y=-v/b$, $z=-w/c$,

Since the given equation represents a cone then the values of $f(x,y,z)$ satisfied

$$u\left(\frac{-u}{a}\right)+v\left(\frac{-v}{b}\right)+w\left(\frac{-w}{c}\right)+d=0$$

$$-\frac{u^2}{a}-\frac{v^2}{b}-\frac{w^2}{c}+d=0$$

$$\Rightarrow u^2/a+v^2/b+w^2/c=d$$

10: Find the vertex of the cone $7x^2+2y^2+2z^2-10zx+10xy+26x-2y+2z-17=0$

Sol: Given equation is $S(x,y,z)=7x^2+2y^2+2z^2-10zx+10xy+26x-2y+2z-17=0 \rightarrow (1)$

making the equation (1) homogeneous by using a variable $t$ we get

$S(x,y,z,t)=7x^2+2y^2+2z^2-10zx+10xy+26xt-2yt+2zt-17t^2=0 \rightarrow (2)$

Differentiating the equation (2) partially with respect to $x,y,z,t$ and equating to zero and putting $t=1$ we get

$14x-10z+10y+26t=0$

$\Rightarrow 14x-10z+10y+26=0 \rightarrow (a)$

$4y+10x-2t=0$

$\Rightarrow 4y+10x-2=0 \rightarrow (b)$

$4z-10x+2t=0$

$\Rightarrow 4z-10x+2=0 \rightarrow (c)$

$26x-2y+2z-34t=0$

$\Rightarrow 26x-2y+2z-34=0 \rightarrow (d)$

Solving the equations (a) $\Rightarrow 14x-10z+10y+26=0$

(d)×5 $\Rightarrow 130x+10z-10y-170=0$

$144x-144=0$

$\Rightarrow x=1$

Substitute in equation (b) $\Rightarrow 4y+10(1)-2=0$

$\Rightarrow 4y=-8 \Rightarrow y=-2$

(c) $\Rightarrow 4z-10(1)+2=0 \Rightarrow 4z-8=0 \Rightarrow z=2$

Thus the vertex of the cone is $(1,-2,2)$

11: show that the two lines of intersection of the plane $ax+by+cz=0$ with the cone $yz+zx+xy=0$ will be perpendicular if $1/a+1/b+1/c=0$

Sol: Given cone is $yz+zx+xy=0$

In this equation coeff. of $x^2$ + coeff. of $y^2$ + coeff. of $z^2=0$

$\therefore$ The cone contains sets of three mutually perpendicular generators

The plane $ax+by+cz=0$ cuts the cone in perpendicular generators if it's normal line through the vertex $(0,0,0)$

i.e., $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ is a generator of the cone

$\Rightarrow bc+ca+ab=0$

$\Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

12: If the line $x=\frac{1}{2}y=z$ represents one of the three mutually perpendicular generators of the cone $11yz+6zx-14xy=0$. find the equation of the other two

Sol: Given equation of the cone $11yz+6zx-14xy=0 \rightarrow (1)$

The equation to the plane through the origin and perpendicular to the generator $\frac{x}{1}=\frac{y}{2}=\frac{z}{1} \rightarrow (2)$ is $x+2y+z=0$

Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be the one of the two lines in which plane intersects the cone (1)

The line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ lies on the given cone and plane Then

$11mn+6nl-14lm=0$

$l+2m+n=0 \rightarrow (3)$

$\Rightarrow n=-(l+2m)$

$11m(-(l+2m))+6l(-(l+2m))-14lm=0$

$\Rightarrow -11lm-22m^2-6l^2-12lm-14lm=0$

$\Rightarrow -6l^2-37lm-22m^2=0$

$\Rightarrow 6l^2+37lm+22m^2=0$

$\Rightarrow 6l^2+33lm+4lm+22m^2=0$

$\Rightarrow 3l(2l+11m)+2m(2l+11m)=0$

$\Rightarrow (3l+2m)(2l+11m)=0$

$2l+11m=0 \rightarrow (4)$, $3l+2m=0 \rightarrow (5)$

Solving (3) and (4)

$$\begin{matrix} l & m & n & \\ 2 & 1 & 1 & 2 \\ 11 & 0 & 2 & 11 \end{matrix}$$

$$\Rightarrow \frac{l}{0-11} = \frac{m}{2-0} = \frac{n}{11-4} \Rightarrow \frac{l}{-11} = \frac{m}{2} = \frac{n}{7}$$

Solving the equation (3) and (5)

$$\begin{matrix} l & m & n & \\ 2 & 1 & 1 & 2 \\ 2 & 0 & 3 & 2 \end{matrix}$$

$$\Rightarrow \frac{l}{0-2} = \frac{m}{3-0} = \frac{n}{2-6} \Rightarrow \frac{l}{-2} = \frac{m}{3} = \frac{n}{-4}$$

The other two perpendicular generators are

$$\frac{x}{-11} = \frac{y}{2} = \frac{z}{7} \text{ and } \frac{x}{-2} = \frac{y}{3} = \frac{z}{-4}$$

**TANGENT PLANE:**

**DEFINITION:** Let $S=0$ be the cone and $L$ be a tangent line to the cone at $P$ on it. The locus of the line $L$ is called the tangent plane to the cone at $P$

**NOTE:** If $P(x_1,y_1,z_1)$ is a point on the surface $S=0$, the equation of the tangent plane to the cone at $P$ is $S_1=0$

**RECIPROCAL CONE:**

**DEFINITION:** The locus of the lines through the vertex of a given cone perpendicular to the tangent plane is called reciprocal cone of the given cone.

**NOTE:** The reciprocal cone of $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ is $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$ where $A,B,C,F,G,H$ are the ~~coefficients~~ cofactors of $a,b,c,f,g,h$ respectively in the determinate $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

The reciprocal cone of $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$ is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

**13:** Prove that the semivertical angle of a right circular cone having 3 mutually perpendicular

a, generators is $\tan^{-1}(\sqrt{2})$ b, Tangent planes is $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

**Sol:** (a) Let the equation of the right circular cone be $x^2+y^2=z^2\tan^2\alpha$

$$\Rightarrow x^2+y^2-z^2\tan^2\alpha=0 \rightarrow (1)$$

If the cone (1) has three mutually perpendicular generators

Then coeff. of $x^2$ + coeff. of $y^2$ + coeff. of $z^2=0$.

$\Rightarrow 1+1-\tan^2\alpha=0$

$\Rightarrow 2-\tan^2\alpha=0$

$\Rightarrow \tan^2\alpha=2$

$\Rightarrow \tan\alpha=\sqrt{2}$

$\Rightarrow \alpha=\tan^{-1}(\sqrt{2})$

Thus the semi-vertical angle of a right circular cone having three mutually perpendicular generators is $\tan^{-1}(\sqrt{2})$

(b) Let the equation of the right circular cone be $x^2+y^2=z^2\tan^2\alpha$

$\Rightarrow x^2+y^2-z^2\tan^2\alpha=0 \rightarrow (1)$

cone (1) contains three mutually perpendicular tangent planes if and only if it's reciprocal cone contains three mutually perpendicular generators

Thus the reciprocal cone of equation (1) is

$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0 \rightarrow (2)$

Where $A=bc-f^2$ , $B=ca-g^2$ , $C=ab-h^2$

$A=-\tan^2\alpha$, $B=\tan^2\alpha$ , $C=1$

$f=gh-af$ , $G=fh-bg$ , $H=fg-ch$

$=0$ , $=0$ , $=0$

$-x^2\tan^2\alpha-y^2\tan^2\alpha+z^2=0$

Thus the equation to the reciprocal cone of eq$^n$ (1) is

$-x^2\tan^2\alpha-y^2\tan^2\alpha+z^2=0$

This cone have three mutually perpendicular generators if

co-eff. of $x^2$ + coeff. of $y^2$ + coeff. of $z^2$

$-\tan^2\alpha-\tan^2\alpha+1=0$

$\Rightarrow -2\tan^2\alpha+1=0$

$\Rightarrow 2\tan^2\alpha=1$

$\Rightarrow \tan^2\alpha=1/2$

$\Rightarrow \alpha=\tan^{-1}(1/\sqrt{2})$

Thus the semi-vertical angle of a right circular cone having three mutually perpendicular tangent planes is $\tan^{-1}(1/\sqrt{2})$

14: Show that the general equation of the cone which touches the three coordinates plane is $\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$ f,g,h being parameters. (7)

Sol: We know that a cone has three mutually tangent planes iff reciprocal cone has three mutually perpendicular generators which normals to the above tangent planes

The equation of the cone which passes through the three co-ordinate axes is $fyz + gzx + hxy = 0$

$\Rightarrow 2(fyz + gzx + hxy) = 0 \longrightarrow (1)$

The reciprocal cone of equation (1) is $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \longrightarrow (2)$

Where $A = bc - f^2$, $B = ca - g^2$, $C = ab - h^2$

$= -f^2$, $B = -g^2$, $= -h^2$

$F = gh - af$, $G = fh - bg$, $H = fg - ch$

$= gh$, $= fh$, $= fg$

$-f^2x^2 - g^2y^2 - h^2z^2 + \cancel{2hfzx} + 2hfzx + 2ghyz + 2fgxy = 0$

$\Rightarrow f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx + 2fgxy = 4fgxy$

$\Rightarrow (fx + gy - hz)^2 = 4fgxy$

$\Rightarrow fx + gy - hz = \pm 2\sqrt{fgxy}$

$\Rightarrow fx + gy \pm 2\sqrt{fgxy} = hz$

$\Rightarrow (\sqrt{fx} \pm \sqrt{gy})^2 = hz$

$\Rightarrow \sqrt{fx} \pm \sqrt{gy} = \pm\sqrt{hz}$

$\Rightarrow \sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0.$

15: Show that if a right circular cone has set of three mutually perpendicular generators, it's semi-vertical angle must be $\tan^{-1}(\sqrt{2})$

Sol: Let the origin be the vertex, l,m,n be dc's of the axis of the cone and $\alpha$ be it's semi-vertical angle

Then the equation to cone is $(lx+my+nz)^2 - (l^2+m^2+n^2)(x^2+y^2+z^2)\cos^2\alpha$

Since the cone contains three mutually perpendicular generators Then coeff of $x^2$ + coeff of $y^2$ + Coeff. of $z^2 = 0 \rightarrow (1)$

coeff of $x^2 = l^2-(l^2+m^2+n^2)\cos^2\alpha$
coeff of $y^2 = m^2-(l^2+m^2+n^2)\cos^2\alpha$
coeff of $z^2 = n^2-(l^2+m^2+n^2)\cos^2\alpha$
Adding (1), we have $(l^2+m^2+n^2)-3(l^2+m^2+n^2)\cos^2\alpha=0$

$$\Rightarrow 1-3\cos^2\alpha=0$$
$$\Rightarrow 3\cos^2\alpha=1$$
$$\Rightarrow \sec^2\alpha=3$$
$$\Rightarrow 1+\tan^2\alpha=3 \Rightarrow \tan^2\alpha=2 \Rightarrow \alpha=\tan^{-1}(\sqrt{2})$$

16: Show that the reciprocal cone of $ax^2+by^2+cz^2=0$ is $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$

Sol: Given cone is $ax^2+by^2+cz^2=0 \rightarrow (1)$
comparing (1) with $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$
$f=0, g=0, h=0$
The reciprocal cone of eqn (1) is $Ax^2+By^2+Cz^2=0 \rightarrow (2)$
Here $A=bc-f^2=bc$
$B=ac-g^2=ac$
$C=ab-h^2=ab$
from (2) $bcx^2+acy^2+abz^2=0$
Divided by "abc"
$$\frac{x^2}{a}+y^2/b+\frac{z^2}{c}=0.$$

17: Find the equations of the tangent planes to the cone $9x^2-4y^2+16z^2=0$ which containing the line $\frac{x}{32}=\frac{y}{72}=\frac{z}{27}$

Sol: Given cone is $9x^2-4y^2+16z^2=0 \rightarrow (1)$
it's reciprocal cone is $\frac{x^2}{9}+\frac{y^2}{-4}+\frac{z^2}{16}=0 \rightarrow (2)$
Now, consider $\frac{x}{32}=\frac{y}{72}=\frac{z}{27}$

$72x-32y=0$ $\qquad$ $27y=72z$
$9x-4y=0 \rightarrow (\pi_1)$ $\qquad$ $3y-8z=0 \rightarrow (\pi_2)$

Equation to the required tangent plane $\pi_1+\lambda\pi_2=0$
$$(9x-4y)+\lambda(3y-8z)=0$$
$$9x-4y+3\lambda y-8\lambda z=0$$
$$\Rightarrow 9x+(-4+3\lambda)y+(-8\lambda)z=0 \rightarrow (3)$$

Since $\frac{x}{9} = \frac{y}{3\lambda-4} = \frac{z}{8\lambda}$ lies on equation (2)

$$\frac{x^2}{9} + \frac{y^2}{-4} + \frac{z^2}{16} = 0$$

$$\Rightarrow \frac{9^2}{9} + \frac{(3\lambda-4)^2}{-4} + \frac{(-8\lambda)^2}{16} = 0$$

$$\Rightarrow 9 - \frac{(9\lambda^2+16-24\lambda)}{4} + \frac{64\lambda^2}{16} = 0$$

$$\Rightarrow \cancel{36}\ 36 - 9\lambda^2 - 16 + 24\lambda + 16\lambda^2 = 0$$

$$\Rightarrow 7\lambda^2 + 24\lambda + 20 = 0$$

$$\Rightarrow 7\lambda^2 + 14\lambda + 10\lambda + 20 = 0$$

$$\Rightarrow 7\lambda(\lambda+2) + 10(\lambda+2) = 0$$

$$\Rightarrow (7\lambda+10)(\lambda+2) = 0$$

$$7\lambda+10 = 0, \quad \lambda+2 = 0$$

$$\lambda = -10/7, \quad \lambda = -2$$

If $\lambda = -2$, $(9x-4y) + (-2)(3y-8z) = 0$

$$\Rightarrow 9x - 4y - 6y + 16z = 0$$

$$\Rightarrow 9x - 10y + 16z = 0$$

If $\lambda = \frac{-10}{7} \Rightarrow (9x-4y) - \frac{10}{7}(3y-8z) = 0$

$$\Rightarrow 7(9x-4y) - 10(3y-8z) = 0$$

$$\Rightarrow 63x - 28y - 30y + 80z = 0$$

$$\Rightarrow 63x - 58y + 80z = 0.$$

The required tangent planes are $9x - 10y + 16z = 0$ and $63x - 58y + 80z = 0$