UNIT : I

THE REAL NUMBERS

Well ordering principle: Every non-empty subset of N has a least element.

Finite and infinite subset of R: A non-empty subset S of
R is said to be finite. If there exists a bijective
function $f: S \rightarrow \{1, 2, 3, \ldots n\}$ for some $n \in N$.
A subset S of R which is not finite is said to be
infinite.

Aggregate: A non-empty subset of R is called an Aggregate

Boundedness of subsets of R:

Upper bound: An aggregate S is said to be bounded above,
if there exists $k_1 \in R$ such that $x \in S \Rightarrow x \leq k_1$. The number
k_1 is called an upper bound of S

Least upper bound (l.u.b) or Supremum: If u is an upper
bound of an aggregate S and any real number less than
u is not an upper bound of S, then u is called least
upper bound or supremum of S

Lower bound: An aggregate S is said to be bounded
below, if there exists $k_2 \in R$ such that $x \in S \Rightarrow x \geq k_2$. The
number k_2 is called a lower bound of S. If k_2 is a lower
bound of S, then any real number less than k_2 is also
a lower bound.

Greatest lower bound (g.l.b) or Infimum: If v is a lower
bound of an aggregate S and any real number
greater than v is not a lower bound of S, then v is
called greatest lower bound or Infimum of S.

Boundedness: An aggregate S is said to be bounded if
it is both bounded below and bounded above.

The Completeness axiom: Every non-empty set of real numbers which is bounded above has supremum in R.

Archimedian property: If $x, y \in R$ and $x>0$, then Exists $n \in Z^+$. Such that $nx > y$.

Integral Part of a real number: If x is real number, then the integral part of x denoted by $[x]$ is the integer n so that $n \leq x < n+1$.

Neighbourhood of a point: If $a \in R$ and $\epsilon > 0$; then the set $\{x \in R : |x-a| < \epsilon\}$ is called ϵ- neighbourhood of 'a' in R

ϵ-nbd of $a = \{x \in R : |x-a| < \epsilon\}$

$= \{x \in R : a-\epsilon < x < a+\epsilon\}$

$= (a-\epsilon, a+\epsilon)$; an upper interval.

ϵ-nbd of 'a' is denoted as $N_\epsilon(a)$ or $N(\epsilon, a)$.

Limit Point of a Subset of R: A point $P \in R$ is said to be a limit point of a subset S of R, if every nbd of P has a Point of S other than P itself.

Bolzano - Weierstrass theorem: Every infinite bounded Set of real numbers has a limit Point.

Real Sequences

Sequence: A function $S : Z^+ \to R$ is Called a Sequence of real numbers or a sequence. Thus we shall denote a sequence $S : Z^+ \to R$ or S by $S_1, S_2, S_3, \ldots, S_n \ldots$

Method of defining Sequences: 1. Defining a sequence by one (or) more formulae so that n^{th} term for each $n \in N$ can be found.

Example 1: The sequence $\{S_n\}$ is defined by the formula $S_n = \frac{1}{\sqrt{n}}$
Here $\{S_n\} = 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \dots, \frac{1}{\sqrt{n}}, \dots$ So that $S_1 = 1$, $S_2 = \frac{1}{\sqrt{2}}$, $S_3 = \frac{1}{\sqrt{3}}$

Example 2: $S_n = \frac{1}{n}$, if n is even and $-\frac{1}{n}$, if n is odd
Here sequence $\{S_n\} = -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots$ so that $S_1 = -1$, $S_2 = \frac{1}{2}$, $S_3 = -\frac{1}{3}$, $S_4 = \frac{1}{4}$

2. Defining a sequence by a Recursion formula (or) Inductive formula i.e. by a formula which express n^{th} term in terms of $(n-1)^{th}$ term.

Example 1: The sequence $\{S_n\} = \{2n\}$ of even natural numbers can be defined as $S_1 = 2$ and $S_{n+1} = S_n + 2$

Example 2: $S_1 = \sqrt{2}$, $S_{n+1} = \sqrt{(2+S_n)}$ for all $n \in Z^+$.
For the above formula, $S_2 = \sqrt{(2+S_1)} = \sqrt{(2+\sqrt{2})}$, $S_3 = \sqrt{2+S_2}$
$= \sqrt{2+\sqrt{2+\sqrt{2}}}$, and so on.

$\therefore \{S_n\} = \sqrt{2}, \sqrt{(2+\sqrt{2})}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \dots$

Constant sequence: The sequence $\{S_n\}$ defined by $S_n = c \in R$ is called a constant sequence.

Subsequence: If $\{S_n\}$ is a sequence and $\{n_n\}$ is a sequence of positive integers such that $n_1 < n_2 < n_3 < n_4 < \dots$, then the sequence $\{S_{n_n}\}$ is called subsequence of $\{S_n\}$.

Range of sequence: The set of all terms of a sequence is called range set or range of the sequence.

Boundedness of a sequence:

Lower bound of sequence: A sequence $\{S_n\}$ is said to be bounded below if the range of the sequence is bounded below i.e. if there exists $K_1 \in R$ such that $K_1 \le S_n$ for every $n \in Z^+$. The number K_1 is called a lower bound of sequence $\{S_n\}$.

Greatest lower bound of sequence: If l is lower bound of a sequence $\{S_n\}$ and any real number greater than l is not a lower bound of $\{S_n\}$, then l is called greatest lower bound of $\{S_n\}$. It is also called infimum of $\{S_n\}$. we write $l = g.l.b\{S_n\}$ or $Inf\{S_n\}$.

upper bound of sequence: A sequence $\{S_n\}$ is said to
be bounded above if the range of the sequence $\{S_n\}$ is
bounded above. i.e; if there exists $K_2 \in R$ such that $S_n \le K_2$
$\forall\, n \in Z^+$. The number K_2 is called an upper bound of
sequence $\{S_n\}$.

Least upper bound: If u is an upper bound of
sequence $\{S_n\}$ and any real number less than u is
not an upper bound of $\{S_n\}$ then u is called least
upper bound. It is also called supremum of $\{S_n\}$. we
write $u = lub\{S_n\}$ or $Sup\{S_n\}$.

Bounded Sequence: A sequence $\{S_n\}$ is said to be bounded
if its range is both bounded below and bounded
above, i.e; if there exists $K_1, K_2 \in R$ such that $K_1 \le S_n \le K_2$
for every $n \in Z^+$.

1. Write the n^{th} term of the sequence $1, -4, 9, -16, 25, \ldots$

Sol:- $1, -4, 9, -16, 25, \ldots$ are respectively $1^2, -2^2, 3^2, -4^2, 5^2, \ldots$

$\therefore S_1 = 1^2,\ S_2 = -2^2,\ S_3 = 3^2,\ S_4 = -4^2,\ S_5 = 5^2 \ldots$

$\Rightarrow S_n = (-1)^{n-1} \cdot n^2$

2. write the sequence given its n^{th} term $= \dfrac{12+5n}{11n+12}$

Sol:- Let $S_n = \dfrac{12+5n}{11n+12}$

we have $S_1 = \dfrac{17}{23},\ S_2 = \dfrac{22}{34},\ S_3 = \dfrac{27}{45}, \ldots$

$\therefore \{S_n\} = \left\{\dfrac{17}{23}, \dfrac{22}{34}, \dfrac{27}{45}, \ldots\right\}$

3. write the sequence given $S_1=1$, $S_2=1$ and $S_{n+2}=S_{n+1}+S_n$ for all $n\in N$

Sol: Putting $n=1,2,3,\ldots$ in the recursion formula

$S_{n+2}=S_{n+1}+S_n$

We have $S_3=S_2+S_1=2$, $S_4=S_3+S_2=3$, $S_5=S_4+S_3=5,\ldots$

$\therefore \{S_n\}=1,1,2,3,5,\ldots$

4. show that $\{\frac{1}{2^n}\}$ is a subsequence of $\{\frac{1}{n}\}$

Sol:- Let $\{S_n\}=\{\frac{1}{n}\}=1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7},\frac{1}{8},\ldots$

$n_1=2$, $n_2=2^2$, $n_3=2^3,\ldots$ are positive integers such that $2<2^2<2^3<\ldots$

i.e; $n_1<n_2<n_3<\ldots$

Thus $\{S_{n_k}\}=\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3},\ldots$ is a subsequence of $\{S_n\}$

5. show that $\{\frac{1}{n!}\}$ is a subsequence of $\{\frac{1}{n}\}$

Sol: Let $n_1=n!$ $\forall$ $n\in N$

Then $n_2=(n+1)!=(n+1)n!$ and $n_3=(n+2)!=(n+2)(n+1)n!,\ldots$

so that $n_1<n_2<n_3<\ldots$ is increasing sequence of positive integers.

$\therefore \{\frac{1}{n!}\}$ is a subsequence of $\{\frac{1}{n}\}$.

__Limit of a sequence and convergent sequence:__

Let $\{S_n\}$ be a sequence and $l\in R$. l is said to be the limit of the sequence. $\{S_n\}$, if to each $\epsilon>0$ there exists $m\in Z^+$ such that $|S_n-l|<\epsilon$ for all $n\geq m$. we also say that the sequence $\{S_n\}$ converges to $l\in R$. If the sequence $\{S_n\}$ has the limit 'l' then we write $S_n\to l$ as $n\to\infty$

(or) $\lim_{n\to\infty} S_n=l$

__convergent sequence:__ A sequence $\{S_n\}$ is said to be convergent if there exists $l\in R$ such that for each $\epsilon>0$ there exists a positive integer m such that $|S_n-l|<\epsilon$ for all $n\geq m$.

Theorem: uniqueness of limit:

statement: A sequence can have atmost one limit (or)
A convergence sequence has unique limit.

proof: Suppose that l and l' are both limits of sequence $\{S_n\}$

Let $l \neq l'$, so that $|l-l'|>0$.

Let $\epsilon = \frac{1}{2}|l-l'|$

$\{S_n\}$ converges to l $\Rightarrow$ there exists $m_1 \in Z^+$ such that
$|S_n - l| < \epsilon \quad \forall \quad n \geq m_1$

$\{S_n\}$ converges to l' $\Rightarrow$ there exists $m_2 \in Z^+$ such that
$|S_n - l'| < \epsilon \quad \forall \quad n \geq m_2$

Let $m = \max\{m_1, m_2\}$

$\therefore |S_n - l| < \epsilon \; , \; |S_n - l'| < \epsilon \quad \forall \quad n \geq m.$

$$\therefore |l-l'| = |(S_n-l)-(S_n-l')| \leq |S_n-l|+|S_n-l'|$$
$$< \epsilon + \epsilon = 2\epsilon$$
$$< |l-l'|$$

which is contraduction to our assumption $l \neq l'$

Therefore $l = l'$

Theorem: Every convergent sequence is bounded

proof: Let the sequence $\{S_n\}$ be convergent to l

$\lim S_n = l$

for $\epsilon = 1$ there exists $m \in Z^+$ such that $|S_n - l| < 1$
for all $n \geq m$.

$\Rightarrow l-1 < S_n < l+1$ for all $n \geq m$

Let $K_1 = \min\{S_1, S_2, \ldots, S_{m-1}, l-1\}$

$K_2 = \max\{S_1, S_2, \ldots, S_{m-1}, l+1\}$

$\therefore K_1 < S_n < K_2$ for all $n \in \mathbb{Z}^+$

Therefore the sequence $\{S_n\}$ is bounded.

$\therefore$ Every convergent sequence is bounded.

Problem 1: Prove that the sequence $\left\{\frac{(-1)^{n-1}}{n}\right\}$ converges to 0

Solution: Let $S_n = \frac{(-1)^{n-1}}{n}$ so that $\{S_n\} = 1, -\frac{1}{2}, \frac{1}{3}, \cdots$

$S_n = \frac{1}{n}$ if n is odd and $S_n = \frac{1}{n}$ if n is even

The subsequence $\{S_{2n-1}\}$ is $1, \frac{1}{3}, \frac{1}{5}, \cdots$ and the subsequence $\{S_{2n}\}$ is $-\frac{1}{2}, \frac{-1}{4}, \frac{-1}{6}, \frac{-1}{8}, \cdots$

But $1, \frac{1}{3}, \frac{1}{5}, \cdots$ and $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$ are subsequences of $\{\frac{1}{n}\}$ which converges to 0

$\therefore \{S_{2n-1}\}$ and $\{S_{2n}\}$ converges to the same limit 0

By known theorem,

$\{S_n\}$ converges to 0

Problem 2: Find $m \in \mathbb{Z}^+$ such that $\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5}$ for all $n \geq m$

Solution: $\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5}$

$$\Rightarrow \left|\frac{2n - 2(n+3)}{n+3}\right| < \frac{1}{5}$$

$$\Rightarrow \left|\frac{2n - 2n - 6}{n+3}\right| < \frac{1}{5}$$

$$\Rightarrow \frac{6}{n+3} < \frac{1}{5}$$

$$\Rightarrow \frac{n+3}{6} > \frac{5}{1}$$

$$\Rightarrow n+3 > 30$$

$$\Rightarrow n > 27$$

If we choose, $m > 27$. i.e, $m = 28$, we have $\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5}$, for all $n \geq m$.

$\therefore$ for $\epsilon = \frac{1}{5}$ the value of $m = 28$

Infact, for each $\epsilon>0$ we can find $m\in\mathbb{Z}^+$ such that $\left|\frac{2n}{n+3}-2\right|<\epsilon$ for all $n\geq m$ and hence the sequence $\left\{\frac{2n}{n+3}\right\}$ converges to 2

Problem 3: Show $\sqrt[n]{n}=1$

Solution: Let $S_n=\sqrt[n]{n}$. clearly $S_n>0$

clearly $S_n\geq 0$

$S_n=n^{1/n}-1 \Rightarrow n^{1/n}=1+S_n$

$\therefore n=(1+S_n)^n$

$=1+nS_n+\frac{n(n-1)}{2}S_n^2+\cdots+S_n^n$

$\geq \frac{n(n-1)}{2}$ for all $n\geq 2$

$\Rightarrow S_n^2\leq\frac{2}{n-1}$ for all $n\geq 2$

$\Rightarrow S_n\leq\sqrt{\frac{2}{n-1}}$ for all $n\geq 2$

for a given $\epsilon>0$, $|S_n-0|=S_n\leq\sqrt{\frac{2}{n-1}}<\epsilon$

i.e., $\frac{2}{n-1}<\epsilon^2$

$\frac{n-1}{2}>\frac{1}{\epsilon^2}$

$n-1>\frac{2}{\epsilon^2}$

$n>\frac{2}{\epsilon^2}+1$

If we choose $m\in\mathbb{Z}^+$ such that $m>\frac{2}{\epsilon^2}+1$ then $|S_n-0|<\epsilon$ for all $n\geq m$.

$\Rightarrow |\sqrt[n]{n}-1|<\epsilon$ for all $n\geq m$.

$\therefore \lim\sqrt[n]{n}=1$

Theorem: Sandwich theorem (or) Squeeze theorem:

Statement: If $\{s_n\}$, $\{t_n\}$, $\{u_n\}$ are three sequences such that (a) $s_n \le u_n \le t_n$ for $n \ge k$ where k is some positive integer and (b) $\lim t_n = l$, then $\lim u_n = l$

proof: Let $\epsilon > 0$

$|s_n - l| < \epsilon$ for all $n \ge m_1$

$\Rightarrow l - \epsilon < s_n < l + \epsilon$ for all $n \ge m_1$

$\lim t_n = l \Rightarrow$ there exists $m_2 \in Z^+$ such that

$|t_n - l| < \epsilon$ for all $n \ge m_2$

$\Rightarrow l - \epsilon < t_n < l + \epsilon$ for all $n \ge m_2$

By hypothesis, $s_n \le u_n \le t_n$ for all $n \ge k$.

Let $m = \max\{m_1, m_2, k\}$

$\therefore l - \epsilon < s_n \le u_n \le t_n < l + \epsilon$ for all $n \ge m$.

$\Rightarrow l - \epsilon < u_n < l + \epsilon$ for all $n \ge m$.

$\therefore |u_n - l| < \epsilon$ for all $n \ge m$.

Hence $\lim u_n = l$.

problem 1: If $s_n = \dfrac{(3n-1)(n^4-n)}{(n^2+2)(n^3+1)}$, prove that $\lim s_n = 3$

solution: Let $s_n = \dfrac{(3n-1)(n^4-n)}{(n^2+2)(n^3+1)}$

$$s_n = \frac{n(3-\frac{1}{n})(1-\frac{n}{n^4})n^4}{n^2(1+\frac{2}{n})(1+\frac{1}{n^3})n^3}$$

$$s_n = \frac{(3-\frac{1}{n})(1-\frac{1}{n^3})}{(1+\frac{2}{n^2})(1+\frac{1}{n^3})}$$

$$\lim s_n = \lim\left(\frac{(3-\frac{1}{n})(1-\frac{1}{n^3})}{(1+\frac{2}{n^2})(1+\frac{1}{n^3})}\right)$$

$$= \frac{(3-0)(1-0)}{(1+0)(1+0)} = 3$$

Problem 2: Show that $\lim \sqrt{\left(\frac{n+1}{n}\right)} = 1$

Solution : We have $\sqrt{\frac{n+1}{n}} = \sqrt{1+\frac{1}{n}} = \left(1+\frac{1}{n}\right)^{1/2}$

$$< 1+\frac{1}{2n} - \frac{1}{4n^2} + \cdots$$

$$< 1+\frac{1}{2n}$$

Since $1+\frac{1}{n} > 1$; $\sqrt{1+\frac{1}{n}} > 1$

and hence $1 < \sqrt{\frac{n+1}{n}} < 1+\frac{1}{2n}$ for all $n \in Z^+$

$\lim 1 = 1$ and $\lim \left(1+\frac{1}{2n}\right) = 1$

$\therefore$ By sandwich theorem, $\lim \sqrt{\frac{n+1}{n}} = 1$

Problem 3: using sandwich theorem. prove that $\lim\limits_{n\to\infty} (\sqrt{n^2+n} - n) =$

Solution: Let $S_n = \sqrt{n^2+n} - n = \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n}$

$$S_n = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n}$$

$$S_n = \frac{n}{\sqrt{(1+\frac{1}{n}}+1)}$$

$$1 < \sqrt{1+\frac{1}{n}} < 1+\frac{1}{2n}$$

$$2 < \sqrt{1+\frac{1}{n}} + 1 < 2+\frac{1}{2n}$$

$$\frac{1}{2} > \frac{1}{\sqrt{1+\frac{1}{n}}+1} > \frac{2n}{4n+1}$$

$\lim\limits_{n\to\infty} \frac{1}{2} = \frac{1}{2}$ and $\lim\limits_{n\to\infty} \frac{2n}{4n+1} = \lim\limits_{n\to\infty} \frac{2}{4+\frac{1}{n}} = \frac{2}{4} = \frac{1}{2}$

$$\therefore \lim_{n\to\infty} S_n = \frac{1}{2}$$

Problem 4: prove that $\lim \frac{\sin(n\pi/3)}{\sqrt{(n)}} = 0$

Solution : $S_n = \sin\frac{n\pi}{3}$ and $t_n = \frac{1}{\sqrt{n}}$ for all $n \in Z^+$

$$-1 \leq \sin\frac{n\pi}{3} \leq 1$$

$\Rightarrow \{s_n\}$ is bounded

we know that $\lim t_n = \lim \frac{1}{\sqrt{n}} = 0$

By known theorem $\lim (s_n t_n) = 0$

$\therefore \lim \dfrac{\sin \frac{n\pi}{3}}{\sqrt{n}} = 0$

<u>problem 5</u>: prove that $\lim \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}\right] = 0$

<u>solution</u>: Let $S_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$

for $1 \le m \le n$, $(n+1)^2 \le (n+m)^2 \le (n+n)^2$

for $1 \le m \le n$, $\frac{1}{(n+1)^2} \ge \frac{1}{(n+m)^2} \ge \frac{1}{(n+n)^2}$

Putting $m = 1, 2, \ldots n$ and adding the n inequalities,

we have $\frac{n}{(n+1)^2} \ge S_n \ge \frac{n}{(n+n)^2}$

$\Rightarrow \frac{n}{4n^2} \le S_n \le \frac{n}{(n+1)^2} < \frac{n}{n^2}$

$\Rightarrow \frac{1}{4n} \le S_n \le \frac{1}{n}$ for all $n \in \mathbb{Z}^+$

$\lim \frac{1}{4n} = 0$ and $\lim \frac{1}{n} = 0$

By sandwich theorem

$\lim S_n = 0$

$\therefore \lim \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}\right] = 0$

<u>problem 6</u>: Discuss the nature of the sequence $\{r^n\}$ for all $-1 < r < 1$.

<u>solution</u>: <u>case 1</u>: Let $r = 1$. Then $r^n = 1$ for all n

$\therefore \{r^n\}$ is a constant sequence and converges to 1.

<u>case 2</u>: Let $0 < r < 1$. Put $r = \frac{1}{1+h}$ where $h > 0$

For $n \in Z^+$; $r^n = \frac{1}{1+nh+\frac{n(n-1)}{2!}h^2+\cdots+h^n} \leq \frac{1}{1+nh} < \frac{1}{nh}$

$\therefore \quad 0 < r^n < \frac{1}{nh}$ for all $n \in Z^+$

$\therefore$ By sandwich theorem, $\lim r^n = 0$ and hence $\{r^n\}$ converges to 0

Case 3: Let $r=0$. Then $r^n = 0$ for all n

$\therefore \{r^n\}$ converges to 0

Case 4: Let $-1 < r < 0$. Put $r = -s$

$\therefore -1 < r < 0 \Rightarrow -1 < -s < 0 \Rightarrow 0 < s < 1$

$\therefore \lim r^n = \lim (-s)^n = \lim (-1)^n s^n = 0$ {by case 2}

$\therefore \{r^n\}$ converges to 0.

Hence $\{r^n\}$ converges when $-1 < r \leq 1$.

Problem 7: If $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$, prove that $\{S_n\}$ is convergent.

Solution: Let $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$

$$S_n = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$S_n = 1 - \frac{1}{n+1}$$

$$S_n = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n(1+1/n)} = \frac{1}{1+0} = \frac{1}{1} = 1$$

$$\therefore \lim_{n\to\infty} S_n = 1$$

$\therefore \{S_n\}$ is converges to 1

Problem 8: prove that $\lim\left\{\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right\}=1$

Solution: Given that $\lim \frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}$

let $S_n=\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}$

for $1\le m\le n$, $1+n^2\le m+n^2\le n+n^2$

$$\sqrt{1+n^2}\le\sqrt{m+n^2}\le\sqrt{n+n^2}$$

$$\frac{1}{\sqrt{1+n^2}}\le\frac{1}{\sqrt{m+n^2}}\le\frac{1}{\sqrt{n^2+1}}\longrightarrow (1)$$

Putting $m=1,2,\cdots n$ in Eqn (1) and adding n

$$\frac{n}{\sqrt{n+n^2}}\le\frac{1}{\sqrt{1+n^2}}+\frac{1}{\sqrt{2+n^2}}+\cdots+\frac{1}{\sqrt{n+n^2}}\le\frac{n}{\sqrt{n^2+1}}$$

$$\frac{n}{\sqrt{n^2(\frac{1}{n}+1)}}\le S_n\le\frac{n}{\sqrt{n^2(1+\frac{1}{n^2})}}$$

$$\frac{n}{n\sqrt{\frac{1}{n}+1}}\le S_n\le\frac{n}{n\sqrt{1+\frac{1}{n^2}}}$$

$$\frac{1}{\sqrt{\frac{1}{n}+1}}\le S_n\le\frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{\frac{1}{n}+1}}=\frac{1}{\sqrt{0+1}}=1$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}}=\frac{1}{\sqrt{1+0}}=1$$

By sandwich theorem $\lim_{n\to\infty}S_n=1$

$$\lim_{n\to\infty}\frac{1}{\sqrt{1+n^2}}+\frac{1}{\sqrt{2+n^2}}+\cdots+\frac{1}{\sqrt{n+n^2}}=1$$

Properly divergent sequences: A sequence which is not convergent is called a divergent sequence.

Definition: A sequence $\{S_n\}$ is said to be diverges to infinity, if for each $G>0$ there exists $m\in\mathbb{Z}^+$ such that $S_n>G$ for all $n\ge m$. If the sequence $\{S_n\}$ diverges to infinity, we write $S_n\to\infty$ as $n\to\infty$

Definition: A sequence $\{S_n\}$ is said to be diverge to minus infinity, if for each $G>0$ there exists $m\in Z^+$ such that $S_n<-G$ for all $n\geq m$. If $\{S_n\}$ diverges to minus infinity then we write $S_n\to-\infty$ as $n\to\infty$.

Definition: A sequence $\{S_n\}$ is said to properly divergent if either $\lim S_n=+\infty$ or $\lim S_n=-\infty$

Definition: A sequence $\{S_n\}$ is said to oscillate if $\{S_n\}$ is bounded and not convergent.

Monotone sequence:

Definition: A sequence $\{S_n\}$ is said to be increasing (or) non-decreasing if $S_1\leq S_2\leq\cdots\leq S_n\leq S_{n+1}\leq\cdots$ i.e. $\{S_n\}$ is increasing $\Leftrightarrow$ $S_n\leq S_{n+1}$ $\forall\, n\in Z^+$ or $S_n\leq S_m$ $\forall\, n<m$.

Definition: A sequence $\{S_n\}$ is said to be strictly increasing if $S_n<S_{n+1}$ $\forall\, n\in Z^+$ or $S_n<S_m$ $\forall\, n<m$.

Definition: A sequence $\{S_n\}$ is said to be decreasing or non-increasing if $S_1\geq S_2\geq\cdots\geq S_n\geq S_{n+1}\geq\cdots$ i.e $\{S_n\}$ is decreasing $\Leftrightarrow$ $S_n\geq S_{n+1}$ $\forall\, n\in Z^+$ or $S_n\geq S_m$ $\forall\, n<m$.

Definition: A sequence $\{S_n\}$ is said to be strictly decreasing if $S_n>S_{n+1}$ for all $n\in Z^+$

Definition: A sequence $\{S_n\}$ which is either increasing or decreasing is called Monotone sequence.

Theorem 1: An increasing sequence is bounded below

Proof: Let $\{S_n\}$ be an increasing sequence

$\therefore S_1\leq S_2\leq S_3\leq\cdots\leq S_n\leq S_{n+1}\cdots$

$\Rightarrow S_1\leq S_n$ for all $n\in Z^+$

$\therefore \{S_n\}$ is bounded below and S_1 is infimum

Theorem 2: A decreasing sequence is bounded above

Proof: Let $\{S_n\}$ be a decreasing sequence

$\therefore S_1 \geq S_2 \geq S_3 \geq \cdots \geq S_n \geq S_{n+1} \geq \cdots$

$\Rightarrow S_1 \geq S_n$ for all $n \in Z^+$

$\therefore \{S_n\}$ is bounded above and S_1 is infimum.

Theorem: Monotone convergence theorem:

Statement: A monotone sequence is convergent if and only if it is bounded (or)

a) $\{S_n\}$ is bounded increasing sequence $\Leftrightarrow \lim S_n = \sup\{S_n / n \in N\}$

b) $\{S_n\}$ is bounded decreasing sequence $\Leftrightarrow \lim S_n = \inf\{S_n / n \in N\}$

Proof: $\{S_n\}$ is Monotone sequence

Suppose sequence $\{S_n\}$ is convergent

$\lim S_n = l$

i.e $S_n \to l$ as $n \to \infty$

for $\epsilon = 1$ there exists $m \in Z^+$ such that $|S_n - l| < 1$

for all $n \geq m$.

$\Rightarrow l - 1 < S_n < l + 1$ for all $n \geq m$.

Let $k_1 = \min\{S_1, S_2, \ldots, S_{m-1}, l-1\}$

and $k_2 = \max\{S_1, S_2, \ldots, S_{m-1}, l+1\}$

Therefore $k_1 < S_n < k_2$ for all $n \in Z^+$

Therefore the sequence $\{S_n\}$ is bounded.

Case a: Suppose $\{S_n\}$ is a bounded above sequence

To prove that sequence $\{S_n\}$ is convergent

Since sequence $\{S_n\}$ is bounded above.

$\{S_n\}$ has supremum k (say)

$\sup\{S_n\} = l \Rightarrow l - \epsilon$ is not an lower bound of $\{S_n\}$

$\Rightarrow$ There exists $m \in Z^+$ such that $S_m < l - \epsilon$ sequence

$\{S_n\}$ is decreasing $\Rightarrow S_n \le S_m$.

$\Rightarrow S_n \le S_m < l+\epsilon$ for $n>m$

$\Rightarrow S_n < l+\epsilon$ for $n \ge m$

$\Rightarrow \inf \{S_n\} = l$

$\Rightarrow S_n > l \;\forall\; n>m$

$\Rightarrow S_n > l-\epsilon \;\forall\; n$

$l-\epsilon < S_n < l+\epsilon \;\forall\; n \ge m$

$\Rightarrow |S_n - l| < \epsilon \;\forall\; n \ge m$

$\therefore \{S_n\}$ Converges to the Supremum to l

i.e $\lim S_n = \sup \{S_n / n \in N\}$

Case b: Suppose sequence $\{S_n\}$ is bounded below sequence.

To Prove that $\{S_n\}$ is convergent

Similarly we can prove that sequence $\{S_n\}$ converges to infimum to l

i.e, $\lim S_n = \inf \{S_n / n \in N\}$.

Problem1: Prove that the sequence $S_n = \frac{3n+4}{2n+1}$ is decreasing and bounded below.

Solution: Let $S_n = \frac{3n+4}{2n+1} \Rightarrow S_{n+1} = \frac{3(n+1)+4}{2(n+1)+1}$

$$= \frac{3n+7}{2n+3}$$

$$S_n - S_{n+1} = \frac{3n+4}{2n+1} - \frac{3n+7}{2n+3} = \frac{5}{(2n+1)(2n+3)} > 0 \;\forall\; n \in Z^+$$

$\Rightarrow S_n > S_{n+1} \;\forall\; n \in Z^+$

$\Rightarrow \{S_n\}$ is decreasing sequence.

Also $S_n = \frac{(3/2)(2n+1)+5/2}{2n+1} = \frac{3}{2} + \frac{5/2}{2n+1} > \frac{3}{2}$

$\left(\because \frac{5/2}{2n+1} > 0\right)$ for all $n \in \mathbb{Z}^+$

$\therefore \{S_n\}$ is decreasing and bounded below.

Problem 2: Prove that the sequence $S_n = \frac{3n-1}{n+2}$ is increasing and bounded above.

Solution: $S_n = \frac{3n-1}{n+2} \Rightarrow S_{n+1} = \frac{3(n+1)-1}{(n+1)+2} = \frac{3n+2}{n+3}$

$$S_n - S_{n+1} = \frac{3n-1}{n+2} - \frac{3n+2}{n+3}$$

$$= \frac{-7}{(n+2)(n+3)} < 0 \ \forall \ n \in \mathbb{Z}^+$$

$\Rightarrow S_n < S_{n+1} \ \forall \ n \in \mathbb{Z}^+$

$\Rightarrow \{S_n\}$ is increasing

Also, $S_n = \frac{3(n+2)-7}{n+2} = 3 - \frac{7}{n+2} < 3$

$\left(\because \frac{7}{n+2} > 0\right)$ for all $n \in \mathbb{Z}^+$

$\therefore \{S_n\}$ is increasing and bounded above.

Problem 3: Prove that $S_n = 2 - \frac{1}{2^{n-1}}$ is convergent.

Solution: We have $S_{n+1} = 2 - \frac{1}{2^n}$.

For all n, $2^n > 2^{n-1}$.

i.e $\frac{1}{2^n} < \frac{1}{2^{n-1}}$

i.e $2 - \frac{1}{2^n} > 2 - \frac{1}{2^{n-1}}$

$\Rightarrow S_{n+1} > S_n$ for all $n \in \mathbb{N}$

$\therefore \{S_n\}$ is an increasing sequence.

Also, $S_n = 2 - \frac{1}{2^{n-1}} < 2$ for all n $\left(\because \frac{1}{2^{n-1}} > 0\right)$

$\Rightarrow S_n$ is bounded above.

$\therefore \{S_n\}$ is increasing and bounded above.

Hence $\{S_n\}$ is convergent.

Problem 4: If $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$, Prove that $\{S_n\}$ is convergent.

Solution: Let $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$

$$S_{n+1} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)}$$

$$\therefore S_{n+1} - S_n = \frac{1}{(n+1)(n+2)} > 0 \ \forall \ n \in Z^+$$

$\Rightarrow S_{n+1} > S_n \ \forall \ n \in Z^+$

$\Rightarrow \{S_n\}$ is increasing

Also $S_n = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+1}\right)$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1} < 1 \ \forall \ n \in Z.$$

$\therefore \{S_n\}$ is bounded above

Hence $\{S_n\}$ is convergent.

Problem 5: prove that the sequence $\{S_n\}$ where $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ is convergent.

Solution: We have $S_{n+1} = \frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{(n+1)+(n+1)}$

$$= \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+2}$$

$$\therefore S_{n+1} - S_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{(2n+1)(2n+2)} > 0 \text{ for all } n \in Z^+$$

$\therefore S_{n+1} > S_n$ for all $n \in Z^+$

$\Rightarrow \{S_n\}$ is an increasing sequence

Also $|S_n| = \left|\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$

$< \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = \frac{n}{n} = 1$ for all $n \in Z^+$

$\therefore \{S_n\}$ is increasing and bounded above and

Hence $\{S_n\}$ is convergent.

Problem 6: Prove that the sequence $\{S_n\}$ defined by $S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ is convergent.

Solution: We have $S_{n+1} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!}$

$$S_{n+1} - S_n = \frac{1}{(n+1)!} > 0 \text{ for all } n \in Z^+$$

$\therefore S_{n+1} > S_n$ for all $n \in Z^+$

$\Rightarrow \{S_n\}$ is an increasing sequence.

Also $S_n = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots + \frac{1}{1\cdot 2\cdot 3\cdots n}$

$$\leq 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2\cdot 2} + \cdots + \frac{1}{2\cdot 2\cdots (n-1)\text{ times}}$$

$$\leq 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right)$$

$$= 1 + \frac{1\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}$$

$$\leq 1 + 2\left(1 - \frac{1}{2^n}\right)$$

$$= 3 - \frac{1}{2^{n-1}} < 3 \text{ for all } n$$

$$\left(\because \frac{1}{2^{n-1}} > 0\right)$$

$\therefore \{S_n\}$ is bounded above

$\therefore \{S_n\}$ is increasing and bounded above and hence $\{S_n\}$ is convergent.

Further, we have $2 \leq S_n < 3$ for all $n \in Z^+$

$\lim S_n$ is a number lying between 2 and 3

It is denoted by the symbol e and hence, $\lim\left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!}\right) = e$

Problem 7: Prove that the sequence $\{s_n\}$ defined by $s_n = \left(1+\frac{1}{n}\right)^n$ is convergent.

Solution: By binomial theorem,

$$s_n = \left(1+\frac{1}{n}\right)^n = 1+n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots 2\cdot 1}{n!}\frac{1}{n^n}$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

similarly, $s_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \cdots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$$

But $1-\frac{1}{n} < 1-\frac{1}{n+1}$, $1-\frac{2}{n} < 1-\frac{2}{n+1}, \ldots,$

$1-\frac{n-1}{n} < 1-\frac{n}{n+1}$

$\therefore$ The first 2 terms of s_n = the first two terms of s_{n+1}, $s_n < s_{n+1}$ for $n = 3, 4, \ldots, n+1$ and s_{n+1} has one more term, namely, $(n+2)^{th}$ term > 0

$\Rightarrow s_n < s_{n+1}$ for all $n \in \mathbb{Z}^+$

$\therefore \{s_n\}$ is an increasing sequence.

Limit Point of a Sequence:

Definition: $l \in \mathbb{R}$ is said to be a limit point of the sequence $\{s_n\}$ if every neighbourhood of l contains infinite number of terms of the sequence.

Theorem: Every bounded sequence has atleast one limit Point.

Proof: Let $\{S_n\}$ be a bounded sequence and T be its range set.

$\therefore$ The set T is bounded.

Case I: Let T be a finite set and $T = \{a_1, a_2, \ldots a_m\}$

Let $N_i = \{n \in Z^+ \mid S_n - a_i \mid$ for $1 \le i \le m$.

Then N_i is the set of indices n for which the corresponding term is S_n.

Since $\bigcup_{i=1}^{m} N_i = Z^+$ is an infinite set atleast one of N_i must be an infinite set.

If N_j is an infinite set then a_j is equal to infinite number of terms of the sequence.

$\therefore$ a_j is a limit point of the sequence $\{S_n\}$

Case II: Let T be an infinite set

Since T is bounded, by Bolzano - Weierstrass theorem on aggregates, the aggregate T has a limit point, say, l.

$\therefore$ l is a limit point of the sequence.

Theorem: Bolzano - Weierstrass:

Every bounded sequence has a convergent subsequence.

Proof: Let $\{S_n\}$ be a bounded sequence.

by known theorem $\{S_n\}$ has a limit point say l

since $l \in R$ is a limit point of $\{S_n\}$

$\Rightarrow$ every neighbourhood of l contains infinite number of terms of $\{S_n\}$.

for each $k \in \mathbb{N}$, $(l-\frac{1}{k}, l+\frac{1}{k})$ is a neighbourhood of l

therefore the exists a term S_{n_k} of $\{S_n\}$ such that

$l-\frac{1}{k} < S_{n_k} < l+\frac{1}{k}$

i.e $|S_{n_k} - l| < \frac{1}{k}$

Thus we have a subsequence $\{S_{n_k}\}$ of $\{S_n\}$

Let $\epsilon > 0$

by Archimedian property, there exists $m \in \mathbb{Z}^+$ such that

$\frac{1}{k} < \epsilon \ \forall \ n \geq m$

$\Rightarrow |S_{n_k} - l| < \frac{1}{k} < \epsilon$ for all $k \geq m$.

$\lim\limits_{k\to\infty} S_{n_k} = l$ and hence $\{S_{n_k}\}$ converges to l.

i.e there exists a subsequence $\{S_{n_k}\}$ of $\{S_n\}$ that converges to l.

Therefore the bounded sequence $\{S_n\}$ has a convergent subsequence.

<u>Cauchy Sequences</u>

<u>Definition</u>: A sequence $\{S_n\}$ is called a cauchy sequence if, for each $\epsilon > 0$ there exists $m \in \mathbb{Z}^+$ such that $|S_p - S_q| < \epsilon$ for all $p, q \geq m$.

<u>Theorem</u>: If the sequence $\{S_n\}$ is convergent, then $\{S_n\}$ is a cauchy sequence. (or) Every convergent sequence is a cauchy sequence.

<u>Proof</u>: Let $\{S_n\}$ converges to l.

for $\epsilon > 0$ there exists $m \in \mathbb{Z}^+$ such that $|S_n - l| < \frac{\epsilon}{2}$ for all $n \geq m$.

If $p, q \geq m$, then $|s_p - l| < \frac{\epsilon}{2}$, $|s_q - l| < \frac{\epsilon}{2}$

$\therefore s_p - s_q = |(s_p - l) + (l - s_q)|$

$\leq |s_p - l| + |s_q - l|$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$

$= \epsilon$ for all $p, q \geq m$.

$\therefore$ for each $\epsilon > 0$ there exists $m \in \mathbb{Z}^+$ such that $|s_p - s_q| < \epsilon$ for all $p, q \geq m$

$\therefore \{s_n\}$ is a Cauchy sequence.

Theorem: If $\{s_n\}$ is a cauchy sequence, then $\{s_n\}$ is bounded.

Proof: Let $\{s_n\}$ be a cauchy sequence

$\Rightarrow$ For $\epsilon = 1$ there exists $m \in \mathbb{Z}^+$ such that $|s_p - s_q| < 1$ for all $p, q \geq m$.

$\Rightarrow |s_p - s_m| < 1$ for all $p \geq m$

$\Rightarrow s_m - 1 < s_p < s_m + 1$ for all $p \geq m$.

Let $k_1 = \min\{s_1, s_2, \ldots s_{m-1}, s_m - 1\}$

$k_2 = \max\{s_1, s_2, \ldots s_{m-1}, s_m + 1\}$

$\therefore k_1 \leq s_n \leq k_2$ for all $n \in \mathbb{Z}^+$.

$\therefore \{s_n\}$ is bounded.

Theorem 3: If $\{s_n\}$ is a cauchy sequence then $\{s_n\}$ is Convergent.

Proof: Let $\{s_n\}$ be a cauchy sequence

To prove that sequence $\{s_n\}$ is Convergent

Since $\{S_n\}$ is a cauchy sequence $\{S_n\}$ is bounded.
Since $\{S_n\}$ bounded, by Bolzano - weierstrass theorm, $\{S_n\}$ has atleast one limit point say l.
If possible, Let l' be another limit point of $\{S_n\}$
Take $\epsilon = |l - l'| > 0$
Since $\{S_n\}$ is cauchy sequence, for $\epsilon > 0$ there Exists $m \in Z^+$ such that $|S_p - S_q| < \frac{\epsilon}{3}$ for all $p, q \geq m$.
Since l, l' are limit points, there Exists Positive integers $p \geq m$, $q \geq m$ Such that $|S_p - l| < \frac{\epsilon}{3}$ and $|S_q - l'| < \frac{\epsilon}{3}$

$$\therefore |l - l'| = |(l - S_p) + (S_p - S_p) + (S_q - l')|$$
$$\leq |S_p - l| + |S_p - S_q| + |S_p - l'|$$
$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$
$$= |l - l'|$$

This is absurd.
$\therefore$ $\{S_n\}$ has unique limit point l.
$\therefore$ $\{S_n\}$ bounded and has unique limit point and Hence $\{S_n\}$ is convergent.

Theorem: Cauchy Convergence Criterion:
A sequence is convergent iff it is a cauchy sequence
Proof: Let $\{S_n\}$ be convergent sequence.
for $\epsilon > 0$ there Exists $m \in Z^+$ such that $|S_n - l| < \frac{\epsilon}{2}$ for all $n \geq m$.
for $p, q \geq m$ then $|S_p - l| < \frac{\epsilon}{2}$, $|S_q - l| < \frac{\epsilon}{2}$

$$|S_p - S_q| = |S_p - l + l - S_q|$$
$$\leq |S_p - l| + |S_q - l|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore$ for each $\epsilon > 0$ there exists $m \in Z^+$ such that

$|S_p - S_q| < \epsilon$ for all $p, q \geq m$

Therefore $\{S_n\}$ is a cauchy sequence.

Let $\{S_n\}$ be a cauchy sequence

To prove that sequence $\{S_n\}$ is convergent.

Since $\{S_n\}$ is a cauchy sequence $\{S_n\}$ is bounded.

Since $\{S_n\}$ bounded, by Bolzano - weierstrass theorem

$\{S_n\}$ has atleast one limit point.

If possible, let l' be another limit point of $\{S_n\}$.

Take $\epsilon = |l - l'| > 0$.

Since $\{S_n\}$ is cauchy sequence, for $\epsilon > 0$ there exists $m \in Z^+$ such that $|S_p - S_q| < \frac{\epsilon}{3}$ for all $p, q \geq m$.

Since l, l' are limit points, there exists positive integers $p \geq m$, $q \geq m$ such that $|S_p - l| < \frac{\epsilon}{3}$ and $|S_q - l'| < \frac{\epsilon}{3}$

$$\therefore |l - l'| = |(l - S_p) + (S_p - S_q) + (S_q - l')|$$
$$\leq |S_p - l| + |S_p - S_q| + |S_q - l'|$$
$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon = |l - l'|$$

This is absurd.

$\therefore$ $\{S_n\}$ has unique limit point l.

$\therefore$ $\{S_n\}$ is bounded. hence $\{S_n\}$ is convergent.

Cauchy's general principle of convergence or Cauchy's theorem

Statement:

A sequence $\{S_n\}$ is convergent iff for each $\epsilon>0$ there exists $m\in Z^+$ such that $|S_{n+p}-S_n|<\epsilon$ for all $n\geq m$ and $p>0$ (or)

A sequence $\{S_n\}$ is convergent iff $\{S_n\}$ is a cauchy sequence.

Proof: Let $\{S_n\}$ converge to l.

$\therefore$ for $\epsilon>0$ there exists $m\in Z^+$ such that $|S_n-l|<\frac{\epsilon}{2}$ for all $n\geq m$.

If $p,q\geq m$. then $|S_p-l|<\frac{\epsilon}{2}$, $|S_q-l|<\frac{\epsilon}{2}$.

$$\therefore S_p-S_q=|(S_p-l)+(l-S_q)|$$

$$\leq |S_p-l|+|S_q-l|$$

$$<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon \text{ for all } p,q\geq m.$$

$\therefore$ for each $\epsilon>0$ there exists $m\in Z^+$ such that $|S_p-S_q|<\epsilon$ for all $p,q\geq m$.

$\therefore \{S_n\}$ is a cauchy sequence.

Let $\{S_n\}$ be a cauchy sequence

Since $\{S_n\}$ is bounded by bolzano-weierstrass theorem $\{S_n\}$ has at least one limit point, say l.

If possible, let l' be another limit point of $\{S_n\}$.

Take $\epsilon=|l-l'|>0$

Since $\{S_n\}$ is cauchy sequence, for $\epsilon>0$ there exists $m\in Z^+$ such that $|S_p-S_q|<\frac{\epsilon}{3}$ for all $p,q\geq m$.

Since l, l' are limit points, there exist positive integers

$p \geq m, q \geq m$ such that $|S_p - l| < \frac{\epsilon}{3}$ and $|S_q - l'| < \frac{\epsilon}{3}$

$\therefore |l - l'| = |(l - S_p) + (S_p - S_q) + (S_q - l')|$

$\leq |S_p - l| + |S_p - S_q| + |S_p - l'|$

$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon = |l - l'|$

This is absurd.

$\therefore \{S_n\}$ has unique limit point l.

$\therefore \{S_n\}$ is bounded and has unique limit point l.

and Hence $\{S_n\}$ is convergent.

<u>Problem 1</u>: Show directly from definition that the sequence

$S_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ is cauchy sequence.

<u>Solution</u>: Let $m > n$. Then $S_m = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} + \cdots + \frac{1}{m!}$

$$|S_m - S_n| = \left| \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!} \right|$$

$$\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}}$$

$$\leq \frac{1}{2^n} \frac{\left(1 - \left(\frac{1}{2}\right)^{m-n}\right)}{1 - \frac{1}{2}} < \frac{1}{2^{n-1}}$$ (sum to $(m-n)$ terms of G.P)

$$< \epsilon$$

$\because \frac{1}{2^{n-1}}$ is convergent sequence

$\{S_n\}$ is a cauchy sequence.

<u>Problem 2</u>: Show directly from definition that the sequence

$\{n + \frac{(-1)^n}{n}\}$ is not Cauchy sequence

<u>Solution</u>: Let $S_n = n + \frac{(-1)^n}{n}$

Then $S_{n+1} = (n+1) + \frac{(-1)^{n+1}}{n+1}$

$$|S_{n+1} - S_n| = \left| n+1 + \frac{(-1)^{n+1}}{n+1} - n - \frac{(-1)^n}{n} \right|$$

$$= \left| 1 + \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+1}}{n} \right|$$

$$\leq 1 + \frac{1}{n+1} + \frac{1}{n} < 1 + \frac{2}{n}$$

$\therefore \{S_n\}$ is not Cauchy sequence.

Problem 3: prove that the sequence $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not convergent.

Solution: suppose that $\{S_n\}$ is convergent.

$\therefore$ By Cauchy's theorem, for $\epsilon = \frac{1}{2}$ there exists $m \in \mathbb{Z}^+$ such that $|S_{n+p} - S_n| < \frac{1}{2} \ \forall \ n \geq m, \ p > 0$

$$\Rightarrow |S_{m+p} - S_m| < \frac{1}{2} \ \forall \ p > 0$$

$$\Rightarrow \left| \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+p} \right| < \frac{1}{2} \ \forall \ p > 0$$

$$\Rightarrow \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+p} < \frac{1}{2} \ \forall \ p > 0$$

But $\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+p} > \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+p} = \frac{p}{m+p}$

For $p = 2m$, $\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+p} > \frac{2m}{3m} = \frac{2}{3}$

$\therefore$ for $p = 2m$, $\frac{2}{3} < \frac{1}{m+1} + \frac{1}{m+2} \cdots + \frac{1}{m+p} < \frac{1}{2}$

$\Rightarrow \frac{2}{3} < \frac{1}{2}$ which is absurd.

$\therefore \{S_n\}$ is not convergent.

Theorem: Cauchy's first theorem on limits

If $\lim S_n = S$ then $\lim\left(\frac{S_1+S_2+\dots+S_n}{n}\right) = S$

Proof: $\lim S_n = S$

To prove that $\lim\left(\frac{S_1+S_2+\dots+S_n}{n}\right) = S$

Definition the sequence $\{t_n\}$ such that $t_n = S_n - S$

for all $n \in Z^+$

$$\lim t_n = \lim |S_n - S|$$
$$= \lim S_n - \lim S$$
$$= S - S$$
$$= 0$$

$\lim t_n = 0 \Rightarrow$ for $\epsilon > 0$ there exists $r \in Z^+$ such that

$|t_n - 0| = |t_n| < \frac{\epsilon}{2}$ for all $n > r$

$\lim t_n = 0 \Rightarrow \{t_n\}$ is convergent.

$\Rightarrow \{t_n\}$ is bounded.

There exists $K \in R$ such that $|t_n| < K$ for all $n \in Z^+$.

$$\left|\frac{t_1+t_2+\dots+t_n}{n}\right| = \left|\frac{t_1+t_2+\dots+t_r}{n} + \frac{t_{r+1}+t_{r+1}+\dots+t_n}{n}\right|$$

$$\leq \frac{|t_1|+|t_2|+\dots+|t_r|}{n} + \frac{(|t_{r+1}|+\dots+|t_n|)}{n}$$

$$< \frac{K+\dots+K}{n} (r \text{ times}) + \frac{\epsilon/2+\dots+\frac{\epsilon}{2}(n-r) \text{ times}}{n}$$

$$= \frac{rK}{n} + \frac{(n-r)\epsilon}{2n}$$

$$= \frac{rK}{n} + \frac{\epsilon}{2} - \frac{r\epsilon}{2n}$$

$$< \frac{rK}{n} + \frac{\epsilon}{2} \text{ for all } n > r$$

$$\left|\frac{t_1+t_2+\cdots+t_n}{n}-0\right| < \frac{\epsilon}{2}+\frac{\epsilon}{2} \text{ for all } n>m$$

$$< \epsilon$$

$$\Rightarrow \lim\left(\frac{t_1+t_2+\cdots+t_n}{n}\right)=0$$

$$\frac{t_1+t_2+\cdots+t_n}{n} = \frac{(s_1-s)+(s_2-s)+\cdots+(s_n-s)}{n}$$

$$= \frac{(s_1+\cdots+s_n)-ns}{n}$$

$$= \frac{s_1+\cdots+s_n}{n} - s$$

$$= \lim\left(\frac{s_1+\cdots+s_n}{n}\right) = \lim\left(\frac{t_1+\cdots+t_n}{n}+s\right)$$

$$= \lim\left(\frac{s_1+\cdots+s_n}{n}\right) = \lim\left(\frac{t_1+t_2+\cdots t_n}{n}+\lim s\right)$$

$$= \lim\left(\frac{s_1+\cdots+s_n}{n}\right) = 0+s$$

$$= \lim\left(\frac{s_1+\cdots+s_n}{n}\right) = s$$

Problem: $\lim \frac{1}{n}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)=0$

Solution: Let $s_n = \frac{1}{2n-1}$. Then $s_1=1$, $s_2=\frac{1}{3}$

we know that $\lim s_n = \lim \frac{1}{2n-1} = 0$

$$\therefore \lim \frac{s_1+s_2+s_3}{n} = 0$$

$$\Rightarrow \lim \frac{1}{n}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right)=0$$

Problem: $\lim \frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)=0$

Solution: If $s_n=\frac{1}{n}$, then $\lim s_n = 0$ and $\lim \frac{1}{n}(s_1+s_2+\cdots+s_n)=0$

Cauchy's Second theorem: If $\{S_n\}$ is a sequence such that $S_n > 0 \ \forall \ n \in \mathbb{Z}^+$ and $\lim \frac{S_{n+1}}{S_n} = 1$ then $\lim \sqrt[n]{S_n} = 1$

proof: Let $\{S_n\}$ be a sequence such that $S_n > 0 \ \forall \ n \in \mathbb{Z}^+$

and $\lim \frac{S_{n+1}}{S_n} = 1$

Take the sequence $\{t_n\}$ defined by $t_1 = S_1$, $t_2 = \frac{S_2}{S_1}$, $t_3 = \frac{S_3}{S_2}, \dots, t_n = \frac{S_n}{S_{n-1}}, \dots$ so that $t_1, t_2, \dots t_n = S_n$

$S_n > 0$ for all $n \Rightarrow t_n > 0$ for all n

Now we have a sequence $\{t_n\}$ such that $t_n > 0$ for all n and $\lim t_n = 1$

$\therefore$ By theorem (2), $(t_1 t_2, \dots t_n)^{1/n} = 1$

$\Rightarrow \lim (S_n)^{1/n} = 1$ e.g. $\sqrt[n]{n} = 1$

Let $S_n = n$. Then $\frac{S_{n+1}}{S_n} = \frac{n+1}{n} = 1 + \frac{1}{n}$

$\therefore \lim \frac{S_{n+1}}{S_n} = \lim \left(1 + \frac{1}{n}\right) = 1$

$\therefore$ By Cauchy's second theorem,

$\lim (S_n)^{1/n} = 1$

$\Rightarrow \lim n^{1/n} = 1$

Problem: prove that $\lim\limits_{n\to\infty} \frac{1}{n}\left[1 + 2^{1/2} + 3^{1/3} + \dots + n^{1/n}\right] = 1$

Solution: Let $S_n = n^{1/n} > 0$ so that $S_1 = 1$, $S_2 = 2^{1/2}$, $S_3 = 3^{1/3}, \dots$

We know that $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

By Cauchy's first theorem

$\lim\limits_{n\to\infty} \frac{1}{n}\left[S_1 + S_2 + S_3 + \dots + S_n\right] = \lim\limits_{n\to\infty} S_n = 1$

problem 2: Prove that $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \ (x>0)$

Solution: Let $S_n = \frac{x^n}{n!} > 0$ for all $n \in N$.

Then $S_{n+1} = \frac{x^{n+1}}{(n+1)!}$ and

$$\frac{S_{n+1}}{S_n} = \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}$$

$$= \frac{x}{n+1}$$

$$\therefore \lim_{n\to\infty} \frac{S_{n+1}}{S_n} = \lim_{n\to\infty} \frac{x}{n+1} = x \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

$$\therefore \lim_{n\to\infty} S_n = 0 \quad \text{i.e;} \quad \lim_{n\to\infty} \frac{x^n}{n!} = 0$$

UNIT : II

INFINITE SERIES

Definition: If $\{u_n\}$ is a sequence of real number and $S_n = u_1 + u_2 + \cdots + u_n$, $n \in Z^+$, then the sequence $\{S_n\}$ is called an infinite Series. The number u_n is called the n^{th} term of the series. The number S_n is called the n^{th} partial sum of the series. The infinite series $\{S_n\}$ is denoted by $\sum_{n=1}^{\infty} u_n$ (or) $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ or $\sum u_n$ or $\{\sum u_n\}$

Convergence of Series:

Definition: Let $\sum_{n=1}^{\infty} u_n$ be a Series of real numbers with partial sums $S_n = u_1 + u_2 + \cdots + u_n$, $n \in Z^+$. If the sequence $\{S_n\}$ Converges to S, we say that the series $\sum_{n=1}^{\infty} u_n$ converges to S. The number is called the sum of the series and we write $\sum_{n=1}^{\infty} u_n = S$. If the limit of the sequence $\{S_n\}$ does not exist we say that the series $\sum u_n$ diverges.

Theorem: A Necessary condition for convergence (or) nth term test

Statement: If $\sum u_n$ converges, then $\lim u_n = 0$

Proof: Let $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \cdots + u_n + \cdots$

Let $S_n = u_1 + u_2 + \cdots + u_n$

Given $\sum u_n$ convergence to $l \in R$

$\Rightarrow \lim_{n\to\infty} S_n = l$ and $\lim_{n\to\infty} S_{n-1} = l$

claim: $\lim_{n\to\infty} u_n = 0$

Now $S_n = u_1 + u_2 + \cdots + u_{n-1} + u_n$

$S_n = S_{n-1} + u_n$

$u_n = S_n - S_{n-1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (S_n - S_{n-1})$$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= l - l$$

$$= 0$$

Problem: If $\lim u_n \neq 0$ then Σu_n diverges

solution: The series $\Sigma \frac{n}{n+1}$ is divergent.

Here $u_n = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$ and

$$\lim u_n = \lim \frac{1}{1+\frac{1}{n}} = 1 \neq 0.$$

Cauchy's General principle of Convergence:

Theorem: (Cauchy criterion): A series Σu_n converges if and only if for each $\epsilon > 0$ there exists $m \in Z^+$ such that $|u_{p+1} + u_{p+2} + \cdots + u_q| < \epsilon$ for all $q \geq p \geq m$.

Proof: Let S_n be the n^{th} partial sum of Σu_n

$\therefore S_q = u_1 + u_2 + \cdots + u_p + u_{p+1} + \cdots + u_q$

$S_p = u_1 + u_2 + \cdots + u_p$ so that

$S_q - S_p = u_{p+1} + u_{p+2} + \cdots + u_q$

The series Σu_n converges $\Leftrightarrow$ the sequence $\{S_n\}$ converges.

$\Leftrightarrow$ for each $\epsilon > 0$, there exists $m \in Z^+$ such that $|S_q - S_p| < \epsilon$ for all $q \geq p \geq m$.

(using Cauchy's general principle of convergence of sequences)

$\Leftrightarrow$ for each $\epsilon > 0$ there exists $m \in Z^+$ such that $|u_{p+1} + u_{p+2} + \cdots + u_q| < \epsilon$ for all $q \geq p \geq m$.

Problem 1: Prove that $\sum \frac{n^n}{n!}$ is not convergent.

Solution: Here $u_n = \frac{n^n}{n!}$

$$= \left(\frac{n}{1}\right)\left(\frac{n}{2}\right)\left(\frac{n}{3}\right)\cdots\left(\frac{n}{n}\right) \geq 1$$

when $n \geq 1$

$\therefore \lim_{n\to\infty} u_n \neq 0$ and consequently $\sum u_n$ is not convergent.

Problem 2: Using Cauchy's principle prove that $\sum \frac{1}{n}$ is divergent.

Solution: Suppose that $\sum \frac{1}{n}$ is convergent.

Let $u_n = \frac{1}{n}$ and S_n be the n^{th} partial sum of the series.

Since $\sum u_n$ is convergent. By Cauchy's general principle of convergence for each $\epsilon > 0$ there exists $m \in \mathbb{Z}^+$ such that $|S_q - S_p| < \epsilon$ for all $q \geq p \geq m$. Take $\varepsilon = \frac{1}{2}$ and $q = 2p$

$\therefore |S_{2p} - S_p| < \frac{1}{2}$ for all $p \geq m$

$\Rightarrow |u_{p+1} - u_{p+2} + \cdots + u_{2p}| < \frac{1}{2}$ for $p \geq m$.

$\Rightarrow \left| \frac{1}{p+1} + \frac{1}{p+2} + \cdots + \frac{1}{2p} \right| < \frac{1}{2}$ for $p \geq m$.

$\Rightarrow \frac{1}{p+1} + \frac{1}{p+2} + \cdots + \frac{1}{2p} < \frac{1}{2}$ for $p \geq m$.

But $\frac{1}{p+1} + \frac{1}{p+2} + \cdots + \frac{1}{2p} > \frac{1}{2p} + \frac{1}{2p} + \cdots + \frac{1}{2p} > \frac{p}{2p} = \frac{1}{2}$

This is a contradiction.

$\therefore \sum \frac{1}{n}$ is not convergent and hence divergent.

Geometric Series: If $|r|<1$ or $-1<r<1$ the series $\sum_{n=0}^{\infty} r^n$ $(r\in R)$ converges to $\frac{1}{1-r}$ and if $|r|\geq 1$ the series $\sum_{n=0}^{\infty} r^n$ diverges.

Proof: Let $-1<r<1$ or $|r|<1$.

We know that $\{|r|^n\}$ converges to 0 if $|r|<1$.

$\therefore \lim |r|^n = 0$.

Let $S_n = 1+r+r^2+\cdots+r^n$ for $n\geq 0$.

Then $r\cdot S_n = r+r^2+r^3+\cdots+r^n+r^{n+1}$.

$\therefore S_n(1-r) = 1-r^{n+1}$

$$\Rightarrow S_n = \frac{1}{1-r} \quad \frac{r^{n+1}}{1-r}$$

$$\Rightarrow S_n - \frac{1}{1-r} = \frac{-r^{n+1}}{1-r}$$

$$\Rightarrow \left|S_n - \frac{1}{1-r}\right| \leq \frac{|r|^{n+1}}{1-r}$$

From the corollary of squeeze theorem of sequences

$\lim \left|S_n - \frac{1}{1-r}\right| = 0$ as $\lim |r|^{n+1} = 0$

$\therefore$ the sequence of partial sums $\{S_n\}$ converges to $\frac{1}{(1-r)}$.

Hence $\sum r^n$ converges to $\frac{1}{1-r}$ when $|r|<1$.

Let $|r|\geq 1$ (i.e) $r\leq -1$ or $r\geq 1$.

r^n does not approach to zero either when $r\leq -1$ or $r\geq 1$.

By n^{th} term test, the series $\sum r^n$ diverges.

Auxiliary Series or P-series test:

The series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots, p\in R$ (a) converges if $p>1$ (b) diverges, if $0<p\leq 1$ and (c) diverges if $p\leq 0$

Proof: Let S_n be the n^{th} partial sum of $\sum \frac{1}{n^p}$.

$\therefore S_n = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p}$

a) Let $p>1$

$\frac{1}{1^p}=1$

$\frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p}=\frac{2}{2^p}=\frac{1}{2^{p-1}}$

$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}=\frac{4}{4^p}=\frac{1}{2^{2(p-1)}}$

$\frac{1}{(2^n)^p}+\frac{1}{(2^n+1)^p}+\cdots+\frac{1}{(2^{n+1}-p)^p}<\frac{1}{2^{n(p-1)}}$

We have $1+2+2^2+\cdots+2^n=\frac{2^{n+1}-1}{2-1}=2^{n+1}-1$

Adding the above inequalities, sum of first $(2^{n+1}-1)$ terms of the series.

$=S_{2^{n+1}-1}<1+\frac{1}{2^{p-1}}+\frac{1}{2^{2(p-1)}}+\cdots+\frac{1}{2^{n(p-1)}}$ for all $n\in\mathbb{Z}^+$.

$<\frac{1-\left(\frac{1}{2^{p-1}}\right)^{n+1}}{1-\frac{1}{2^{p-1}}}<\frac{1}{1-\frac{1}{2^{p-1}}}$, for all n.

For each $n\in\mathbb{Z}^+$, $2^{n+1}-1>2^n>n$.

$\therefore S_n<S_{2^n}<S_{2^{n+1}-1}<\frac{1}{1-2^{1-p}}$ for all $n\in\mathbb{Z}^+$.

$\Rightarrow \{S_n\}$ is bounded above.

Since $\frac{1}{n^p}>0$, for all n, $\{S_n\}$ is an increasing sequence.

$\therefore \{S_n\}$ converges and hence $\sum\frac{1}{n^p}$ converges.

(b) Let $0<p\le 1$ $\therefore n^p\le n^1$ and hence $\frac{1}{n^p}\ge\frac{1}{n}$, for all $n\in\mathbb{Z}^+$.

$\frac{1}{1^p}+\frac{1}{2^p}\ge 1+\frac{1}{2}$

$\frac{1}{3^p}+\frac{1}{4^p}\ge\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$

$$\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}+\frac{1}{8^p} \geq \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{4}{8}=\frac{1}{2}$$

$$\frac{1}{(2^k+1)^p}+\frac{1}{(2^k+2)^p}+\dots+\frac{1}{(2^k+2^k)^p} > \frac{1}{2}$$

But $2+2+2^2+\dots+2^k = 2+\frac{2(2^k-1)}{2-1} = 2^{k+1}$

Adding the above inequalities, the sum of the first 2^{k+1} terms of the series $= S_{2^{k+1}} > 1+\frac{1}{2}+\dots+\frac{1}{2}$

$$= 1+\frac{k+1}{2} = \frac{k+3}{2}$$

we can find $n \in \mathbb{Z}^+$ such that $n > 2^{k+1} (=m)$

$\therefore S_n > S_{2^{k+1}} > \frac{k+3}{2}$ for $n > m$

$\Rightarrow S_n > \frac{k+3}{2}$ for $n > m$.

$\therefore \{S_n\}$ diverges to infinity and

Hence $\sum \frac{1}{n^p}$ is divergent.

c) Let $p \leq 0$. when $p=0$, $u_n=1$ and $\lim u_n = 1 (\neq 0)$

By n^{th} term test, $\sum \frac{1}{n^p}$ is divergent.

When $p<0$, $u_n = n^q$ where $p=-q$ and $q>0$.

$\therefore u_n = n^q \to \infty$ as $n \to \infty$ and hence $\lim u_n \neq 0$.

By n^{th} term test, $\sum \frac{1}{n^p}$ is divergent.

Comparison test: If $\sum u_n$ and $\sum v_n$ are two series of non negative terms such that (a) there is a positive integer m and $k \in \mathbb{R}^+$ such that $0 \leq u_n \leq k v_n$ for all $n \geq m$, and b) $\sum v_n$ is convergent, then $\sum u_n$ is convergent.

Proof: Let $S_n = u_1+u_2+\dots+u_n$ and $t_n = v_1+v_2+\dots+v_n$

for all $n \geq m$, $S_n = (u_1+u_2+\dots+u_{n-1})+u_m+u_{m+1}+\dots+u_n$

$\leq (u_1+u_2+\cdots+u_{m-1}) + k(v_m + v_{m-1}+\cdots+v_n)$

$\leq a + k\{t_n - (v_1+v_2+\cdots+v_{m-1})\}$, where $a = u_1+u_2+\cdots+u_{m-1}$

$\leq a + k(t_n - b) = a - kb + kt_n$, where $b = v_1+v_2+\cdots+v_{m-1}$

Since $\sum v_n$ is convergent,

$\{t_n\}$ is convergent and hence $\{t_n\}$ is bounded.

$\therefore$ There exists $l \in R$ such that $t_n \leq l$ for all n.

$\therefore$ $0 \leq s_n \leq a - kb + kl$ for all $n \geq m$.

If $l' = \max\{s_1, s_2, \ldots, s_{m-1}, a-kb+kl\}$, then $s_n \leq l'$

for all n.

$\therefore$ $\{s_n\}$ is bounded above.

Also $\{s_n\}$ is increasing as $\sum u_n$ is a series of non-negative terms.

$\therefore$ $\{s_n\}$ is convergent and hence $\sum u_n$ is convergent.

<u>Theorem</u>: If $\sum u_n$ and $\sum v_n$ are two series of non negative terms such that a) there is a positive integer m and $k \in R^+$ such that $u_n \geq k v_n \geq 0$ for all $n \geq m$ and

b) $\sum v_n$ is divergent, then $\sum u_n$ is divergent.

<u>Proof</u>: Let $s_n = u_1+u_2+\cdots+u_n$ and

$t_n = v_1+v_2+\cdots+v_n$ for all $n \geq m$,

$s_n = (u_1+u_2+\cdots+u_n) + (u_m + u_{m+1} + \cdots + u_n)$

$\geq a + k(v_m + v_{m-1} + \cdots + v_n)$

where $a = u_1+u_2+\cdots+u_{m-1}$

$\geq a + k\{t_n - (v_1+v_2+\cdots+v_{m-1}\}$,

$\geq a + kt_n - kb$, where $b = v_1+v_2+\cdots+v_{m-1}$

Since $\sum v_n$ is divergent, $\{t_n\}$ is divergent,

$\therefore$ for $G > 0$ there exists $m_1 \in Z^+$ such that

$t_n > \frac{G+kb-a}{k}$ for all $n \geq m_1$

If $M = \max\{m, m_1\}$, then $S_n > a - kb + k\left(\frac{G+kb-a}{k}\right)$
$= G$ for all $n \geq M$.

$\therefore \{S_n\}$ diverges and hence $\sum u_n$ diverges.

Problem 1: Test the convergence of $\sum_{n=1}^{\infty} \log\left(\frac{1}{n}\right)$

Solution: Let $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \log\left(\frac{1}{n}\right)$

$= \sum_{n=1}^{\infty} -\log n$ so that $u_n = -\log n$

for $n \geq 1$, $\log n < n \Rightarrow -\log n > -n$

By comparison test, $\sum u_n$ is divergent.

Problem 2: Test the convergence of $\sum_{n=2}^{\infty} \frac{\log n}{2n^3-1}$

Solution: For $n \geq 2$; $\log n < n$ and

$$\frac{1}{2n^3-1} \leq \frac{1}{n^3} \Rightarrow \frac{\log n}{2n^3-1} < \frac{n}{n^3} = \frac{1}{n^2}$$

Let $u_n = \frac{\log n}{2n^3-1}$ and $v = \frac{1}{n^2}$ So that $u_n < v_n$ for all $n \geq 2$.

Since $\sum v_n = \sum \frac{1}{n^2}$ is convergent,

by comparison test $\sum u_n$ is also convergent.

Limit comparison test: $\sum u_n$ and $\sum v_n$ be two series of positive terms such that $\lim \frac{u_n}{v_n} = l \in \mathbb{R}$, a) If $l \neq 0$ then the series $\sum u_n$, $\sum v_n$ converge or diverge together.

(b) If $l = 0$ and If $\sum v_n$ is convergent then $\sum u_n$ is convergent.

Proof: a) Let $l \neq 0$, Since $u_n > 0$, $v_n > 0$ for all n,

$\lim \frac{u_n}{v_n} = l \neq 0 \Rightarrow l > 0$

Let $\epsilon > 0$ be such that $0 < \epsilon < 1$

Then $k_1\ k_2 \in R$ are such that $k_1 = l - \epsilon > 0$, $k_2 = l + \epsilon > 0$

$\lim \frac{u_n}{v_n} = l \Rightarrow$ for $\epsilon > 0$ there exists $m \in Z^+$ such that

$\left|\frac{u_n}{v_n} - l\right| < \epsilon$ for all $n \geq m$.

$\Rightarrow l - \epsilon < \frac{u_n}{v_n} < l + \epsilon$ for all $n \geq m$

$\Rightarrow k_1 v_n < u_m < k_2 v_n$ for all $n \geq m$.

Case I: Let $\sum u_n$ be convergent.

Consider $k_1 v_n < u_n\ \forall\ n \geq m$.

$\Rightarrow v_n < \frac{1}{k_1}\ \forall\ n \geq m$.

By known theorem $\sum v_n$ is convergent.

Case II: Let $\sum v_n$ be convergent.

Consider $u_n < k_2 v_n\ \forall\ n \geq m$.

By known theorem $\sum u_n$ is convergent.

Case III: Let $\sum u_n$ be divergent.

Consider $u_n < k_2 v_n\ \forall\ n \geq m$

$\Rightarrow v_n < \frac{1}{k_2} u_n\ \forall\ n \geq m$.

$\therefore$ By known theorem $\sum v_n$ is divergent.

Case IV: Let $\sum v_n$ be divergent.

Consider $k_1 v_n < u_n\ \forall\ n \geq m$.

$\Rightarrow u_n > k_1 v_1\ \forall\ n \geq m$.

By known theorem $\sum u_n$ is divergent.

Here $\sum u_n$ and $\sum v_n$ converge or diverge together.

b) Let $l = 0$. Then $\lim\left(\frac{u_n}{v_n}\right) = 0$

Given $\epsilon > 0$ there exists $m \in N$ such that $\left|\frac{u_n}{v_n}\right| < \epsilon$ $\forall\ n \geq m$.

$\Rightarrow 0 < \frac{u_n}{v_n} < \epsilon \quad \forall\, n \geq m.$

$\Rightarrow 0 < u_n < \epsilon v_n \quad \forall\, n \geq m.$

By known theorem $\sum u_n$ is Convergent.

Problem: Test for convergence $\sum_{n=1}^{\infty} \frac{1}{2^n+3^n}$

Solution: This is a series of positive terms.

Let $u_n = \frac{1}{2^n+3^n} = \frac{1}{3^n\left[1+\frac{2^n}{3^n}\right]} = \frac{1}{3^n\left[1+\left(\frac{2}{3}\right)^n\right]}$

Take $v_n = \frac{1}{3^n}$. we know that $\sum v_n = \sum \frac{1}{3^n}$ is Convergent.

$\therefore \lim \frac{u_n}{v_n} = \lim \frac{1}{3^n\left[1+\left(\frac{2}{3}\right)^n\right]} \times 3^n = \lim \frac{1}{1+\left(\frac{2}{3}\right)^n} = 1 \neq 0.$

$\therefore$ By comparison test, $\sum u_n$ is convergent.

Problem: Test for convergence a) $\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$

b) $\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)$

Solution: a) Let $u_n = \sqrt{n+1}-\sqrt{n}$. This is a series of positive terms.

$$u_n = \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

Take $v_n = \frac{1}{\sqrt{n}}$ So that $\sum v_n = \sum \frac{1}{n^{1/2}}$ is divergent

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n)}+1} = \frac{1}{1+1} = \frac{1}{2} \neq 0$$

By Comparison test, $\sum u_n$ is divergent.

b) Let $u_n = \sqrt{n^2+1} - n\ (>0)$. This is a series of positive terms.

$$u_n = \frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)}{\sqrt{n^2+1} + n}$$

$$= \frac{1}{\sqrt{n^2+1} + n}$$

Take $v_n = \frac{1}{n}$ so that $\sum v_n = \sum \frac{1}{n}$ is divergent.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1} + n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}+1} = \frac{1}{1+1} = \frac{1}{2} \neq 0$$

By comparison test, $\sum u_n$ is divergent.

Problem: Test for convergence:

(a) $\sum_{n=1}^{\infty} (\sqrt{n^3+1} - \sqrt{n^3})$

(b) $\sum_{n=1}^{\infty} (\sqrt{n^4+1} - \sqrt{n^4-1})$

Solution: a) Let $u_n = \sqrt{n^3+1} - \sqrt{n^3}$

This is a series of positive terms.

$$u_n = \frac{(\sqrt{n^3+1} - \sqrt{n^3})\sqrt{n^3+1} + \sqrt{n^3}}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$= \frac{1}{\sqrt{n^3+1}\sqrt{n^3}}$$

Take $v_n = \frac{1}{\sqrt{n^3}}$ so that $\sum v_n = \sum \frac{1}{n^{3/2}}$ is convergent.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n^3}}{\sqrt{n^3+1} + \sqrt{n^3}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^3}} + 1} = \frac{1}{1+1} = \frac{1}{2} \neq 0$$

By comparison test $\sum u_n$ is convergent.

b) Let $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$ (>0)

This is a series of positive terms.

$$u_n = \frac{\left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)\left(\sqrt{n^4+1} + \sqrt{n^4-1}\right)}{\left(\sqrt{n^4+1} + \sqrt{n^4-1}\right)}$$

$$= \frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

Take $V_n = \frac{1}{n^2}$ so that $\sum V_n = \sum \frac{1}{n^2}$ $(P = 2 > 1)$ is convergent.

$$\lim_{n\to\infty} \frac{u_n}{V_n} = \lim_{n\to\infty} \frac{2n^2}{\sqrt{n^4+1}\ \sqrt{n^4-1}} = \lim_{n\to\infty} \frac{2}{\sqrt{1+(1/n^4)} + \sqrt{1-(1/n^4)}}$$

$$= \frac{2}{1+1} = 1 \neq 0$$

By comparison test, $\sum u_n$ is convergent.

Problem: Test for convergence $\sum_{n=1}^{\infty} \left(\sqrt[3]{n^3+1} - n\right)$

Sol: Let $u_n = \sqrt[3]{n^3+1} - n$

$$= \sqrt[3]{n^3+1} - \sqrt[3]{n^3} > 0 \text{ for all } n.$$

This is a series a positive terms.

$$u_n = \frac{\left[(n^3+1)^{1/3} - (n^3)^{1/3}\right]\left[(n^3+1)^{2/3} + (n^3+1)^{1/3}(n^3)^{1/3} + (n^3)^{2/3}\right]}{(n^3+1)^{2/3} + (n^3+1)^{1/3}(n^3)^{1/3} + (n^3)^{2/3}}$$

$$= \frac{(n^3+1) - n^3}{(n^3+1)^{2/3} + (n^3+1)^{1/3} n + n^2}$$

Take $V_n = \frac{1}{(n^3)^{2/3}} = \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ is convergent

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(n^3)^{2/3}}{(n^3+1)^{2/3} + n(n^3+1)^{1/3} + n^2}$$

$$= \lim_{n\to\infty} \frac{1}{(1+(1/n^3))^{2/3} + (1+(1/n^3))^{1/3} + 1}$$

$$= \frac{1}{1+1+1} = \frac{1}{3} \neq 0$$

By Comparison test $\sum u_n$ is Convergent.

<u>Cauchy's nth root test (or) Root test</u>:

<u>Theorem</u>: If $\sum u_n$ is a Series of positive terms such that $\lim u_n^{1/n} = l$ then (a) $\sum u_n$ converges if $l<1$ and

(b) $\sum u_n$ diverges if $l>1$

<u>Proof</u>: Given $\lim_{n\to\infty} u_n^{1/n} = l$

$\forall\ \epsilon>0\ \exists\ m\in Z^+ \ni |u_n^{1/n} - l| < \epsilon\ \forall\, n\geq m.$

$\Rightarrow l-\epsilon < u_n^{1/n} < l+\epsilon\ \forall\ n\geq m.$

$\Rightarrow (l-\epsilon)^n < u_n < (l+\epsilon)^n$ for all $n\geq m$

(a) Let $l<1$

Choose $\epsilon>0$ such that $k = l+\epsilon<1$. Then $0\leq l<k<1$.

From (1) we have $u_n < (l+\epsilon)^n$ for all $n\geq m$.

$\Rightarrow u_n < k_1^n$ for all $n\geq m$.

But $\sum k_1^n$ is a Geometric Series with Common ratio

$\Rightarrow \sum k_1^n$ is Convergent.

$\therefore$ By comparison test Σu_n converges.

(b) Let $l > 1$

Choose $\epsilon > 0$, such that $l - \epsilon > 1$.

from (1) we have, $(l-\epsilon)^n < u_n$ for all $n \geq m$.

$\Rightarrow k_2^n < u_n$ for all $n \geq m$.

$\Rightarrow u_n > k_2^n \Rightarrow \Sigma u_n > \Sigma k_2^n$ Since $k_2 \geq 1$.

By Geometric Series

Σk_2^n is diverges

By Comparison test.

Σu_n is diverges.

Problem: Test for convergence $\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \cdots$

Solution: This is a series of positive terms.

Here $u_n = \left(\frac{n}{2n+1}\right)^n$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(\left(\frac{n}{2n+1}\right)^n\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{n}{n(2+1/n)} = \frac{1}{2+0} = \frac{1}{2} < 1$$

By root test Σu_n is converges.

Problem: Test for convergence $\Sigma \frac{2^n}{3^n}$

Solution: This is a series of positive terms.

Let $u_n = \frac{2^n}{n^3}$

$$u_n^{1/n} = \left(\frac{2^n}{n^3}\right)^{1/n} = \frac{2}{(n^{1/n})^3}$$

$\therefore \lim u_n^{1/n} = \lim \frac{2}{(n^{1/n})^3} = \frac{2}{1^3} = 2$

$\lim \sqrt[n]{n} = 1$

$\therefore$ By Root test Σu_n diverges.

<u>Problem</u>: Test for converges $\sum \frac{x^n}{n^n}$ $(x>0)$

<u>Solution</u>: Since $(x>0)$, this is a series of positive terms

Here $u_n = \frac{x^n}{n^n} = u_n^{1/n} = \frac{x}{n}$

$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{x}{n} = 0$

$\therefore \Sigma u_n$ converges.

<u>Problem</u>: Test for convergence $\Sigma \left(1+\frac{1}{n}\right)^{-n^2}$

<u>Solution</u>: This is a series of positive terms.

Let $u_n = \left(1+\frac{1}{n}\right)^{-n^2}$

$u_n^{1/n} = \left[\left(1+\frac{1}{n}\right)^{-n^2}\right]^{1/n}$

$= \left(1+\frac{1}{n}\right)^{-n}$

$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$

$\therefore \lim u_n^{1/n} = \frac{1}{e} < 1 \quad (\because 2 < e < 3)$

$\therefore$ By root test, Σu_n converges

<u>Problem</u>: Test for convergence $\sum \frac{n^{n^2}}{(n+1)^{n^2}}$

<u>Solution</u>: This is a series of positive terms.

Here $u_n = \frac{n^{n^2}}{(n+1)^{n^2}}$ so that $u_n^{1/n} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+1/n)^n}$

$\therefore \lim_{n\to\infty} u_n^{1/n} = \frac{1}{e} < 1$

$\therefore \sum u_n$ is convergent.

<u>Problem:</u> Test for convergence of $\sum_{n=1}^{\infty} \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n x^n \ \forall \ x>0$

<u>Solution:</u> Let $u_n = \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n x^n$

$$u_n^{1/n} = \frac{1}{(n^{1/n})^3}\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)x$$

$$\therefore \lim u_n^{1/n} = \frac{1}{1^3}\times\left(\frac{1+0}{1+0}\right)x = x$$

By root test $\therefore \sum u_n$ is convergent if $x<1$

and divergent if $x>1$.

when $x=1$; $u_n = \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n$.

Take $v_n = \frac{1}{n^3}$

$$\lim \frac{u_n}{v_n} = \lim\left(\frac{n+2}{n+3}\right)^n = \lim\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)^n = \frac{e^2}{e^3} = \frac{1}{e} \neq 0$$

By limit Comparison test

$\sum u_n$ is convergent.

<u>D'Alembert's test or ratio test:</u>

<u>Theorem:</u> If $\sum u_n$ is a series of positive terms such that $\lim \frac{u_{n+1}}{u_n} = l$, then (a) $\sum u_n$ converges if $l<1$ and

(b) $\sum u_n$ diverges if $l>1$

<u>Proof:</u> Given that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$

$\forall \ \epsilon>0 \ \exists \ m \in \mathbb{Z}^+ \ \ni \left|\frac{u_{n+1}}{u_n} - l\right| < \epsilon \ \forall \ n \geq m$.

$$l-\epsilon < \frac{u_{n+1}}{u_n} < l+\epsilon \quad \forall\ n \geq m.$$

Putting $n = m, m+1, \dots, n-1$ in the above and multiplying the $(n-m)$ inequalities, we get

$$(l-\epsilon)^{n-m} < \frac{u_n}{u_m} < (l+\epsilon)^{n-m} \text{ for all } n > m.$$

$$\Rightarrow U_m(l-\epsilon)^{n-m} < U_n < U_m(l+\epsilon)^{n-m} \quad \forall\ n \geq m.$$

a) Let $l < 1$

choose $\epsilon > 0$ such that $k = l+\epsilon < 1.$

Since $l \geq 0$, $0 < k < 1$

from (1) we have, $u_n < \left(\frac{u_m}{k^m}\right)k^n$ for all $n > m$

$\Rightarrow u_n < \alpha k^n$, $\alpha = \frac{u_m}{k^m} \in R$, for all $n > m.$

Since $0 < k < 1$, the geometric series $\sum k^n$ Converges.

$\therefore$ By comparison test, $\sum u_n$ converges.

b) Let $l > 1$. choose $\epsilon > 0$ such that $k = l-\epsilon > 1.$

from (1) we have, $\left(\frac{u_m}{k^m}\right)k^n < u_n$ for all $n > m$

$\Rightarrow u_n < \beta k^n$, $\beta = \frac{u_m}{k^m} \in R^+$ for all $n > m.$

Since $k > 1$, the geometric series $\sum k^n$ diverges.

By comparison test $\sum U_n$ diverges.

<u>Problem</u>: Test for convergence $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

<u>Solution</u>: Here $u_0 = 1$; and $u_n = \frac{1}{n!} > 0$ for all $n \in Z^+$.

$$u_{n+1} = \frac{1}{(n+1)!} \text{ and } \frac{u_{n+1}}{u_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

$$\therefore \lim \frac{u_{n+1}}{u_n} = \lim \frac{1}{n+1} = 0 < 1$$

By D'Alembert's test Σu_n converges.

The sum of this convergent series is denoted by e.

$$\therefore 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots=\sum_{n=0}^{\infty}\frac{1}{n!}=e.$$

Problem: Test for convergence of $\frac{1}{2}+\frac{1\cdot 3}{2\cdot 5}+\frac{1\cdot 3\cdot 5}{2\cdot 5\cdot 8}+\cdots$

Solution: This is a series of positive integers.

Here $u_n=\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 5\cdot 8\cdots(3n-1)}$

$$u_{n+1}=\frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{2\cdot 5\cdot 8\cdots(3n-1)(3n+2)}$$

$$\frac{u_{n+1}}{u_n}=\frac{2n+1}{3n+2}=\frac{2(1+1/n)}{3(1+2/n)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\frac{2}{3}$$

By ratio test Σu_n is convergent

Problem: Test for convergence $\sum_{n=1}^{\infty}\frac{(n+1)!}{3^n}$

Solution: This is a series of positive terms

Here $u_n=\dfrac{(n+1)!}{3^n}$ and $u_{n+1}=\dfrac{(n+2)!}{3^{n+1}}$

$$\therefore \frac{u_{n+1}}{u_n}=\frac{(n+2)!}{3^{n+1}}\cdot\frac{3^n}{(n+1)}=\frac{n+2}{3}.$$

$$\Rightarrow \frac{u_{n+1}}{u_n}\to\infty \text{ as } n\to\infty$$

By ratio test Σu_n is divergent.

Problem: Test for Convergence $\sum_{n=1}^{\infty} \frac{n^4}{n!}$

Solution: Let $u_n = \frac{n^4}{n!}$

$$u_{n+1} = \frac{(n+1)^4}{(n+1)!}$$

$$= \frac{[n(1+1/n)]^4}{n(1+1/n)!}$$

$$u_{n+1} = \frac{n^4(1+1/n)^4}{n(1+1/n)!}$$

$$\frac{u_{n+1}}{u_n} = \frac{n^4((1+\frac{1}{n})^4}{n(1+1/n)n!} \times \frac{n!}{n^4}$$

$$= \frac{1}{n}(1+1/n)^3$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{n}(1+1/n)^3 = 0 < 1$$

By ratio test $\sum u_n$ converges.

Problem: Test for Convergence $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Solution: Let $u_n = \frac{n!}{n^n}$

$$u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!}$$

$$= \frac{(n+1)\,n!}{(n+1)^n(n+1)} \times \frac{n^n}{n!}$$

$$= \frac{n^n}{n^n(1+1/n)^n}$$

$$= \frac{1}{(1+1/n)^n}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{1+0} = \frac{1}{e} < 1$$

By ratio test.

Σu_n is converges.

Problem: Test for convergence $\Sigma \frac{2^n-2}{2^n+1} x^n \; (x>0)$

Solution: Since $x>0$, this is a series of positive terms

Here $u_n = \frac{2^n-2}{2^n+1} x^n$

$$u_{n+1} = \frac{2^{n+1}-2}{2^{n+1}+1} x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}-2}{2^{n+1}+1} x^{n+1} \times \frac{2^n+1}{2^n-2\,x^n}$$

$$\frac{2^{n+1}-2}{2^{n+1}+1} \cdot \frac{2^n+1}{2^n-2} \cdot x = \frac{2-(2/2^n)}{2+(1/2^n)} \cdot \frac{1+(1/2^n)}{1-(2/2^n)} \cdot x$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{2}{1} \cdot \frac{1}{1} \cdot x = x$$

$\therefore$ by ratio test Σu_n converges if $x<1$

and Σu_n diverges if $x>1$.

when $x=1$

$$u_n = \frac{2^n-2}{2^n+1} = \frac{1-(2/2^n)}{1+(1/2^n)} = 1$$

$\Rightarrow \lim u_n = 1 \neq 0$

$\therefore \sum u_n$ diverges.

<u>Alternating</u> <u>Series</u>: A series whose terms are alternately positive and negative is called an alternating series.

An alternating series may be written as

$u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1} u_n + \cdots$ where each u_n is positive integer. (or) negative.

It is denoted as $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$, $u_n > 0$ or $u_n < 0$.

<u>Theorem</u>: <u>(Leibnitz test)</u>:

<u>Statement</u>: If $\{u_n\}$ is a sequence of positive terms such that (a) $u_1 \geq u_2 \geq u_3 \geq \cdots \geq u_n \geq u_{n+1}, \cdots$ and

(b) $\lim u_n = 0$ then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent.

<u>Proof</u>: Let $u_n \geq u_{n+1}$ $\forall$ $n \in Z^+$ and $\lim_{n\to\infty} u_n = 0$

<u>claim</u>: The alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent.

Let $S_n = u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1} u_n$

$$S_{2n} = u_1 - u_2 + u_3 - u_4 + \cdots + u_{2n-1} - u_{2n} \rightarrow (1)$$

$$S_{2n+2} = u_1 - u_2 + u_3 - u_4 + \cdots + u_n - u_{2n} + u_{2n+1} - u_{2n+2}$$

Now $S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} \geq 0$

$$S_{2n+2} - S_{2n} \geq 0$$

$$S_{2n+2} \geq S_{2n}$$

$S_{2n} \leq S_{2n+2}$

$\therefore \{S_{2n}\}$ is an increasing sequence

from ① $S_{2n} = u_1 - [(u_2 - u_3) + (u_4 - u_5) + \cdots + (u_{2n-1} - u_{2n})]$

$S_{2n} < u_1$

$\therefore \{S_{2n}\}$ is bounded above.

By Monotonic Convergence theorem $\{S_n\}$ is convergent.

$\therefore \lim_{n\to\infty} S_n = l$

from ① $S_{2n} = S_{2n-1} - u_{2n}$

$S_{2n-1} = S_{2n} + u_{2n}$

$\lim_{n\to\infty} S_{2n-1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_n$

$= l + 0 = l$

$\lim_{n\to\infty} S_{2n-1} = l$

$\lim_{n\to\infty} S_n = l$

$\Rightarrow \{S_n\}$ is convergent.

$\therefore$ The alternating series

$\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent

<u>Problem</u>: prove that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges.

<u>Solution</u>: The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is alternating

Here $u_n = \frac{1}{n}$

Clearly, $u_n > 0$ and $u_n > u_{n+1}$ for all n.

Also $\lim u_n = \lim \frac{1}{n} = 0$

$\therefore$ By Leibnitz test $\sum \frac{(-1)^{n-1}}{n}$ Converges.

Problem: Examine the character of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2n-1}$

Solution: Let $u_n = \frac{n}{2n-1}$

Then $u_n - u_{n+1} = \frac{n}{2n-1} - \frac{n+1}{2n+1} = \frac{1}{(2n-1)(2n+1)} > 0$

$\Rightarrow u_n > u_{n+1}$ for all $n \in N$.

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-(1/n)} = \frac{1}{2} \neq 0$

By Leibnitz's test, $\sum (-1)^{n-1} u_n$ is not convergent.

Problem: Prove that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$ Converges

Solution: This is an alternating series

and $u_n = \frac{1}{n!}$

Clearly $u_n > 0$ and $u_n > u_{n+1}$ for all n.

Also, $\lim u_n = \lim \frac{1}{n!} = 0$

$\therefore$ By Leibnitz's test $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ Converges

The sum of this series is denoted by $e^{-1} = 0.3679$ approx.

Problem: Examine the Convergence of $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)$

Solution: Let $u_n = \frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right) > 0$ for all n.

$u_n - u_{n+1} = \frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right) - \left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}+\frac{1}{n+1}\right)\frac{1}{n+1}$

$$= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1}{n+1}\cdot\frac{1}{n+1};$$

$$= \frac{1}{n(n+1)}\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\frac{n}{n+1}\right]$$

$$= \frac{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\left(\frac{1}{n+1}+\frac{1}{n+1}+\cdots n \text{ terms}\right)}{n(n+1)}$$

$$= \frac{\left(1-\frac{1}{n+1}\right)+\left(\frac{1}{2}-\frac{1}{n+1}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)}{n(n+1)}$$

$\therefore\ u_n - u_{n+1} > 0 \Rightarrow u_n > u_{n+1}$ for all n.

Since $\lim \frac{1}{n} = 0$, by Cauchy's first theorem on limits, $\lim \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$ i.e $\lim u_n = 0$.

By Leibnitz's test the given series is converges.

<u>Definition:-</u> The series Σu_n is said to be absolutely convergent if $\Sigma |u_n|$ is convergent

<u>Theorem:-</u> If Σu_n converges absolutely then Σu_n converges

(OR) An absolutely convergent series is always convergent.

<u>Proof:-</u> For each $n \in z^+$, $-|u_n| \le u_n \le |u_n|$

$\Rightarrow -|u_n| + |u_n| \le u_n + |u_n| \le 2|u_n|$

$\Rightarrow 0 \le u_n + |u_n| \le 2|u_n|$

Σu_n converges absolutely

$\Rightarrow \Sigma |u_n|$ converges

$\Rightarrow \Sigma 2|u_n|$ converges

$\therefore$ By comparision test, $\Sigma (u_n + |u_n|)$ converges.

Since $\Sigma (u_n + |u_n|)$ converges and

$\Sigma |u_n|$ converges

We have $\Sigma [u_n + |u_n| - |u_n|] = \Sigma u_n$ converges.

Problem:- Show that the series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ converges absolutely for all values of x.

Solution:- Let $\Sigma u_n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ so that $u_n = \frac{x^n}{n!}$

$$\therefore \Sigma |u_n| = \Sigma \frac{|x|^n}{n!}$$

$$\therefore \left| \frac{u_{n+1}}{u_n} \right| = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1}$$

$$\therefore \lim \left| \frac{u_{n+1}}{u_n} \right| = \lim \frac{|x|}{n+1} = 0 < 1 \text{ for all } x \in R$$

$\therefore$ By ratio test, $\Sigma |u_n|$ converges for all $x \in R$.

Hence Σu_n converges absolutely for all $x \in R$.

Problem: prove that $\sum (-1)^{n-1} \frac{x^n}{n}$ is convergent for

$-1 < x \le 1$

Solution: let $x = 1$

Then $\sum (-1)^{n-1} \frac{x^n}{n} = \sum (-1)^{n-1} \frac{1}{n}$ is convergent.

Let $-1 < x < 1$ i.e $|x| < 1$

let $u_n = (-1)^{n-1} \frac{x^n}{n}$

Then $|u_n| = \frac{|x|^n}{n}$ and

$$|u_{n+1}| = \frac{|x|^{n+1}}{n+1}$$

$$\Rightarrow \left|\frac{u_{n+1}}{u_n}\right| = \frac{n}{n+1} |x|$$

$$= \frac{1}{1+(1/n)} |x|$$

$$\therefore \lim \left|\frac{u_{n+1}}{u_n}\right| = |x|$$

$\therefore$ By Ratio test $\sum |u_n|$ is convergent for $|x| < 1$

$\therefore$ $\sum u_n$ is convergent for $-1 < x < 1$

Hence $\sum u_n$ is convergent for $-1 < x \le 1$

UNIT: III

LIMITS AND CONTINUITY

Limit of a Function

Definition:- Let $f: S \to R$ be a function, 'a' be a limit point of aggregate S and $l \in R$

(i) The function f tends to limit 'l' as 'x' tends to 'a' from left, if, for each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in S$ and $a - \delta < x < a \Rightarrow |f(x) - l| < \epsilon$

We write $f(x) \to l$ as $x \to a^-$ or $\lim\limits_{x \to a^-} f(x) = l$ or $f(a-0) = l$

(ii) The function f tends to limit 'l' as 'x' tends to 'a' from right, if, for each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in S$ and $a < x < a + \delta \Rightarrow |f(x) - l| < \epsilon$

We write $f(x) \to l$ as $x \to a^+$ or $\lim\limits_{x \to a^+} f(x) = l$ or $f(a+0) = l$

(iii) The function f tends to limit 'l' as 'x' tends to 'a' if, for each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in S$ and $0 < |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon$

We write $f(x) \to l$ as $x \to a$ or $\lim\limits_{x \to a} f(x) = l$

Note:-

* $\lim\limits_{x \to a^-} f(x) = l$ is called limit from below or left hand limit of the function
* $\lim\limits_{x \to a^-} f(x) = l$ is called limit from above or right hand limit of the function
* $\lim\limits_{x \to a} f(x) = l$ is called limit of the function

Problem :- If $f: \mathbb{R}-\{a\} \to \mathbb{R}$ is defined by $f(x) = \frac{x^2-a^2}{x-a}$, using the definition of limit prove that $\lim_{x\to a} f(x) = 2a$

Solution:- $f: \mathbb{R}-\{a\} \to \mathbb{R}$ is defined by $f(x) = \frac{x^2-a^2}{x-a}$

let, $\epsilon > 0$

$$|f(x)-2a| = \left|\frac{x^2-a^2}{x-a} - 2a\right|$$

$$\Rightarrow |f(x)-2a| = \left|\frac{(x-a)(x+a)}{x-a} - 2a\right|$$

$$\Rightarrow |f(x)-2a| = |x+a-2a|$$

$$\Rightarrow |f(x)-2a| = |x-a|$$

$|f(x)-2a| < \epsilon$ whenever $|x-a| < \epsilon$

for, each $\epsilon>0$ if we take, $\delta = \epsilon > 0$, then $0<|x-a|<\delta$

$\Rightarrow |f(x)-2a| < \epsilon$

by the definition, $\lim_{x\to a} f(x) = 2a$.

Problem :- let $S = \mathbb{R}-\{0\}$. Define $f: S \to \mathbb{R}$ such that $f(x) = x\sin(1/x)$. using the definition of limit prove that $\lim_{x\to 0} f(x) = 0$

Solution:- let $S = \mathbb{R}-\{0\}$ Define $f: S\to\mathbb{R}$ such that

$$f(x) = x\sin(1/x)$$

let $\epsilon>0$

$$|f(x)-0| = \left|x\sin\tfrac{1}{x} - 0\right|$$

$$\Rightarrow |f(x)-0| = \left|x\sin\tfrac{1}{x}\right|$$

$$\Rightarrow |f(x)-0| = |x|\left|\sin\tfrac{1}{x}\right|$$

$$\Rightarrow |f(x)-0| \le |x-0|$$

$|f(x)-0| < \epsilon$, whenever $|x-0| < \epsilon$

choosing $\delta = \epsilon$, we have $|f(x)-0| < \epsilon$ whenever $0 < |x-0| < \delta$.

by definition of limit,

$\lim_{x\to 0} f(x) = 0$

Problem: If $f(x) = \sin\frac{1}{x}$ $\forall$ $x \in \mathbb{R}-\{0\}$. prove that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist

Solution: $f(x) = \sin\frac{1}{x}$ for all $x \in \mathbb{R}-\{0\}$

If possible suppose that $\lim_{x\to 0} f(x) = l$

case(i) Let $l \neq 1$

for $\epsilon = |l-1| > 0$ there exists $\delta > 0$ such that

$0 < |x| < \delta$

$\Rightarrow |\sin\frac{1}{x} - l| < |l-1|$

for $0 < x = \dfrac{1}{2n\pi + \pi/2} < \delta$, $|\sin(2n\pi + \frac{\pi}{2}) - l| < |l-1|$

$\Rightarrow |1-l| < |l-1|$

It is impossible and hence $l \neq 1$ is not true

case(ii):- Let $l = 1$

for $\epsilon = 1 > 0$ there exists $\delta > 0$ such that $0 < |x| < \delta$

$\Rightarrow |\sin\frac{1}{x} - 1| < 1$

for $0 < x = \frac{1}{n\pi} < \delta$, $|\sin n\pi - 1| < 1 \Rightarrow 1 < 1$

this is impossible and hence $l = 1$ is not true

$\therefore \lim_{x\to 0} \sin\frac{1}{x}$ does not exist

Problem:- Prove that $\lim_{x\to 0} \frac{3x+|x|}{7x-5|x|}$ does not exist

Solution:- $\lim_{x\to 0^-} \frac{3x+|x|}{7x-5|x|} = \lim_{x\to 0^-} \frac{3x+|-x|}{7x-5|-x|}$

$= \lim_{x\to 0^-} \frac{3x-x}{7x+5x}$

$= \lim_{x\to 0^-} \frac{2x}{12x} = \frac{1}{6}$

$\lim_{x\to 0^+} \frac{3x+|x|}{7x-5|x|} = \lim_{x\to 0^+} \frac{3x+x}{7x-5x}$

$= \lim_{x\to 0^+} \frac{4x}{2x}$

$= \lim_{x\to 0^+} 2 = 2$

$\lim_{x\to 0^-} \frac{3x+|x|}{7x-5|x|} \neq \lim_{x\to 0^+} \frac{3x+|x|}{7x-5|x|}$

Therefore $\lim_{x\to 0} \frac{3x+|x|}{7x-5|x|}$ does not exist

Problem:- If $f: \mathbb{R}\to\mathbb{R}$ is a function defined by $f(x)=\frac{|x-2|}{x-2}$ when $x\neq 2$ and $f(x)=0$ when $x=2$, then prove that $\lim_{x\to 2} f(x)$ does not exist.

Solution:- $f: \mathbb{R}\to\mathbb{R}$ is a function defined by

$f(x) = \frac{|x-2|}{x-2}$ when $x\neq 2$ and $f(x)=0$ when $x=2$

when $x<2$, $|x-2|<0$,

$|x-2| = -(x-2)$

when $x>2$, $|x-2|>0$,

$|x-2| = x-2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2^-} -\frac{(x-2)}{x-2} = -1$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{x\to 2^+} \frac{(x-2)}{x-2} = 1$$

$$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$$

$\therefore \lim_{x\to 2} f(x)$ does not exist.

Infinite Limits

Definition:- Let S be an aggregate, 'a' be a limit point of S and $f: S \to R$ be a function. Given any $\epsilon>0$ if there exists $\delta>0$ such that $x\in S, 0<|x-a|<\delta \Rightarrow f(x)>\epsilon$, then we say that $f(x)$ tends to ∞ as 'x' tends to a or $\lim_{x\to a} f(x) = \infty$

Definition:- Let 'S' be an aggregate, 'a' be a limit point of S and $f: S\to R$ be a function. Given any $\epsilon>0$ if there exists $\delta>0$ such that $x\in S$, $0<|x-a|<\delta \Rightarrow f(x)<-\epsilon$, then we say that $f(x)$ tends to $-\infty$ as x tends to 'a'

or $\lim_{x\to a} f(x) = -\infty$

Limits at infinity

Definition:- Let S be an aggregate having the property that given any $b>0$, there exists $x\in S$ such that $x\geq b$.
Let $f: S\to R$ be a function and $L\in R$. Given $\epsilon>0$ there exists $\delta>0$ such that $x\in S, x\geq\delta \Rightarrow |f(x)-L|<\epsilon$ then we say that $f(x)\to l$ as $x\to\infty$.
Then we write $\lim_{x\to\infty} f(x) = l$

Definition:- Let S be an aggregate having the property that given any $b>0$. there exists $x\in S$ such that $x\le -b$. Let $f: S\to R$ be a function and $l\in R$. Then we say that the limit of as $x\to -\infty$ exists if for each $\epsilon>0$ there exists $\delta>0$ such that $x\in S$, $x\le -\delta \Rightarrow |f(x)-l|<\epsilon$ then we write $\lim_{x\to -\infty} f(x)=l$ or $f(x)\to l$ as $x\to -\infty$

Problem:- If $f(x)=\dfrac{e^{1/x}}{1+e^{1/x}}$, find whether $\lim_{x\to 0} f(x)$ exists or not

Solution:- $f(x)=\dfrac{e^{1/x}}{1+e^{1/x}}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{e^{1/x}}{1+e^{1/x}} = \frac{0}{1+0} = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{e^{1/x}}{1+e^{1/x}} = \lim_{x\to 0^+}\frac{e^{1/x}}{e^{1/x}(e^{-1/x}+1)}$$

$$= \lim_{x\to 0^+}\frac{1}{e^{-1/x}+1} = \frac{1}{0+1} = 1$$

$$\lim_{x\to 0^-} f(x) \ne \lim_{x\to 0^+} f(x)$$

$\therefore \lim_{x\to 0} f(x)$ does not exist

Problem:- Prove that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

Solution:- For $x>1$, there exists $n\in Z^+$, such that $n\le x<n+1$

$$\Rightarrow \frac{1}{n}+\frac{1}{x} > \frac{1}{n+1}$$

$$\Rightarrow 1+\frac{1}{n} \geq 1+\frac{1}{x} > 1+\frac{1}{n+1}$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^{n+1} \geq \left(1+\frac{1}{x}\right)^{n} > \left(1+\frac{1}{n+1}\right)^{n}$$

$$\Rightarrow \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n+1} \geq \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{n} \geq \lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{n}$$

By known theorem from sequence $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e$

$$\lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n+1} = e$$

By squeeze theorem, $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x} = e$

<u>Continuity of a function at a point</u>:-

Let S be an aggregate, $f: S\to R$ be a function and $a\in S$. f is said to be continuous at 'a' from left, if given $\epsilon>0$, there exists $\delta>0$ such that

$x\in S$, $a-\delta < x < a \Rightarrow |f(x)-f(a)| < \epsilon$

f is said to be continuous at 'a', from right, if given $\epsilon>0$, there exists $\delta>0$ such that

$x\in S$, $a < x < a+\delta \Rightarrow |f(x)-f(a)| < \epsilon$.

f is said to be continuous at 'a' if given $\epsilon>0$, there exists $\delta>0$ such that

$x\in S$, $|x-a| < \delta \Rightarrow |f(x)-f(a)| < \epsilon$

Definition:- (Limit Notation of continuity at a point)

Let $f: S \to \mathbb{R}$ be a function and $a \in S$ be a limit point of S.

f is said to be continuous at 'a' from left,

if $\lim_{x \to a^-} f(x) = f(a)$ (or) $f(a-0) = f(a)$.

f is said to be continuous at 'a' from right,

if $\lim_{x \to a^+} f(x) = f(a)$ (or) $f(a+0) = f(a)$.

f is said to be continuous at 'a' if $\lim_{x \to a} f(x) = f(a)$

Continuity interval:-

Definition:- A function $f: S \to \mathbb{R}$ is said to be continuous on the domain S, if f is continuous at every point $a \in S$.

Definition (continuity on a closed interval)

$f: [a,b] \to \mathbb{R}$ is said to be continuous on $[a,b]$ if (i) $\lim_{x \to c} f(x) = f(c)$ for $c \in (a,b)$

(ii) $\lim_{x \to a^+} f(x) = f(a)$ and (iii) $\lim_{x \to b^-} f(x) = f(b)$

Definition:- If $f: S \to \mathbb{R}$ is not continuous at $a \in S$, then we say that f is Discontinuous at 'a'. $a \in S$ is called a point of Discontinuity.

Theorem: (Sequence criterion or Heine's theorem)

A function $f: S \to \mathbb{R}$ is continuous at a point $a \in S$ if and only if $f(x_n) \to f(a)$ for every sequence $\{x_n\}$ in S converging to 'a'.

Proof: Let f be continuous at $a \in S$

We have that $f(x_n) \to f(a)$ for every sequence $\{x_n\}$ in S converging to a.

Since f is continuous at 'a', for a given $\epsilon > 0$ there exist $\delta > 0$ such that $x \in S$,

$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$ —①

Since sequence $\{x_n\}$ converges to 'a'.

for $\delta > 0$ there exists $m \in N$ such that

$n \geq m \Rightarrow |x_n - a| < \delta$

using ①, $n \geq m \Rightarrow |x_n - a| < \delta$

$\Rightarrow |f(x_n) - f(a)| < \epsilon$

$\therefore \{f(x_n)\}$ sequence converges to $f(a)$

Let $f(x_n) \to f(a)$ for every sequence $\{x_n\}$ in S converging to 'a'.

We prove that f is continuous at 'a'.

If possible suppose that f is not continuous at 'a'.

$\therefore$ for some $\epsilon > 0$ and any $\delta > 0$ there exists,

$x \in S$, $|x - a| < \delta$

so that $|f(x_n) - f(a)| \geq \epsilon$

Take $\delta = \frac{1}{n}$ for $m \in N$

Then for each m there exists an $x_n \in S$

such that $|x_n - a| < \frac{1}{n} \Rightarrow |f(x_n) - f(a)| \geq \epsilon$

$\therefore x_n \to a$ does not simply $f(x_n) \to f(a)$

This is contradiction

Hence f is continuous at 'a'.

Discontinuity criterion :-

Let $f: S \to \mathbb{R}$ be a function and $a \in S$. Then f is not continuous at 'a'. iff there exists a sequence $\{x_n\}$ in S such that $\lim_{n\to\infty} x_n = a$ but not $\lim_{n\to\infty} f(x_n) \neq f(a)$

Discontinuity :-

① Removable Discontinuity :- If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) \neq f(a)$ or $f(a)$ is not defined, then we say that f has removable discontinuity at 'a'.

In such a case, if we define a new function g such that $g(x) = f(x)\ \forall\ x \neq a$ in the domain of f and $g(a) = \lim_{x\to a} f(x)$, then g is continuous at 'a'.

Example :- consider $f(x) = \frac{x^2-4}{x-2}\ \forall\ x \neq 2$ and $f(x) = 0$ for $x = 2$

Since $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} (x+2) = 4$

and $f(2) = 0$, we have $\lim_{x\to 2} f(x) \neq f(2)$ and

hence f has removable discontinuity at 2.

② **Jump Discontinuity:-** If $\lim_{x\to a^-} f(x) = f(a-0)$ and $\lim_{x\to a^+} f(x) = f(a+0)$ both exist and are not equal then we say that f has jump Discontinuity at 'a'.

$f(a+0) - f(a-0)$ is called the height of the jump at $x=a$ or saltus of $f(x)$ at $x=a$.

Examples:- consider $f(x) = x - [x]$ at $x=1$.

$$f(1-0) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x - \lim_{x\to 1^-} [x] = 1-0$$
$$= 1$$

$$f(1+0) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x - \lim_{x\to 1^+} [x]$$
$$= 1-1$$
$$= 0$$

$\therefore \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$ and hence f has jump discontinuity at '1'. Also height of the jump $= 1-0 = 1$.

Here $f(1) = 0$ is also defined and $f(1) = f(1-0)$.

Definition:- A removable discontinuity or jump discontinuity of a function is called Discontinuity of first kind or simple Discontinuity.

③ **Discontinuity of second kind:-** If atleast one of the limits, $\lim_{x\to a^-} f(x) = f(a-0)$, $\lim_{x\to a^+} f(x) = f(a+0)$ is non-existent and infinite, then we say that f has discontinuity of second kind at 'a'.

Example:- consider $f(x) = \frac{1}{(x-a)}$ $\forall$ $x \neq a$ and $f(a) = K$

Here, $f(a-0) = \lim_{x \to a^-} f(x)$

$= -\infty$

and $f(a+0) = \lim_{x \to a^+} f(x)$

$= \infty$

$\Rightarrow$ $f(x)$ has discontinuity of second kind at $x = a$

Problem:- Show that $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x - |x|}{x}$ for $x \neq 0$ and $f(x) = 2$. for $x = 0$ is continuous at all points of $\mathbb{R}$ except at $x = 0$

Solution:- Domain $f = \mathbb{R}$

let $a \in \mathbb{R}$ and $a < 0$, 'a' is a limit point of $\mathbb{R}$ and

$$f(a) = \frac{a - |a|}{a}$$

$$= \frac{a - (-a)}{a}$$

$$= 2$$

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{x - |x|}{x}$$

$$= \frac{a - |a|}{a}$$

$$= \frac{a - (-a)}{a}$$

$$= 2$$

$$= f(a)$$

$\therefore$ $f(x)$ is continuous at $x = a$ and hence continuous at $(-\infty, 0)$

Let $a \in \mathbb{R}$ and $a=0$. '0' is a limit point of $\mathbb{R}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x-|x|}{x} = \lim_{x\to 0^-} \frac{x-(-x)}{x} \lim_{x\to 0^-} 2 = 2$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x-|x|}{x} = \lim_{x\to 0^+} \frac{x-x}{x} \lim_{x\to 0^+} 0 = 0$$

$\therefore$ $f(x)$ is not continuous at $x=0$

$x=0$ is a point of jump discontinuity at $x=0$ and height of the jump $= 2-0 = 2$

Let $a \in \mathbb{R}$ and $a>0$. 'a' is a limit point of $\mathbb{R}$

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{x-|x|}{x} = \frac{a-|a|}{a} = 0 = f(a)$$

$\therefore$ $f(x)$ is continuous at $x=a$ and

hence $f(x)$ is continuous in $(0,\infty)$

<u>Problem</u>:- By ϵ, δ technique, prove that the function defined by $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(x)=0$ for $x=0$ is continuous at $x=0$

<u>Solution</u>:- Let $\epsilon > 0$

$$|f(x)-f(0)| = |x^2 \sin(\tfrac{1}{x}) - 0|$$
$$= |x^2|\,|\sin(1/x)| \le x^2$$

$\therefore$ $|f(x)-f(0)| < \epsilon$ whenever $x^2 < \epsilon$

i.e whenever $|x| < \sqrt{\epsilon}$

i.e whenever $|x-0| < \sqrt{\epsilon}$

If we take $\delta = \sqrt{\epsilon}$ then $|x-0| < \delta$

$\Rightarrow |f(x)-f(0)| < \epsilon$

By the definition of continuity $f(x)$ has continuous at $x=0$

Problem:- Show that $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=1$ if $x\in Q$ and $f(x)=-1$ if $x\in\mathbb{R}-Q$ is discontinuous $\forall\, x\in\mathbb{R}$.

Solution:- $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=1$ if $x\in Q$ and $f(x)=-1$ if $x\in\mathbb{R}-Q$

case(i) Let $a\in\mathbb{R}$ and 'a' be rational number.

Then $f(a)=1$.

For each $n\in N$, $a-\frac{1}{n}<a+\frac{1}{n}$ and hence by denseness of $\mathbb{R}$, there lies an irrational number x_n such that

$$a-\frac{1}{n}<x_n<a+\frac{1}{n} \Rightarrow |x_n-a|<\frac{1}{n} \text{ for } n\in N$$

$\therefore$ There exists a sequence $\{x_n\}$ of irrational number converging 'a'.

Also, $\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty}(-1)=-1$

Since $f(a)=1$, $\lim_{n\to\infty} f(x_n)\neq f(a)$ and hence f is not continuous at $a\in Q$.

case(ii) Let $a\in\mathbb{R}$ and 'a' be irrational number

Then $f(a)=-1$

For each $n\in N$, $a-\frac{1}{n}<a+\frac{1}{n}$ and hence by denseness of $\mathbb{R}$, there lies on rational number x_n such that

$$a-\frac{1}{n}<x_n<a+\frac{1}{n} \Rightarrow |x_n-a|<\frac{1}{n} \text{ for } n\in N$$

$\therefore$ There exists a sequence $\{x_n\}$ of rational number converging to 'a'.

Also $\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty}(1)=1$

Since $f(a) = -1$, $\lim_{n \to \infty} f(x_n) \neq f(a)$ and hence

f is not continuous at $a \in \mathbb{R}-\mathbb{Q}$

Hence f is discontinuous for all $x \in \mathbb{R}$.

Problem: Show that $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ if $x \in \mathbb{R}-\mathbb{Q}$ and $f(x) = -x$ if $x \in \mathbb{Q}$ is continuous only at '0'.

Solution:- $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ if $x \in \mathbb{R}-\mathbb{Q}$ and $f(x) = -x$ if $x \in \mathbb{Q}$

case(i) let $a = 0$

Then $f(a) = f(0) = 0$

let $\epsilon > 0$

$|f(x) - f(0)| = |x-0|$ if $x \in \mathbb{R}-\mathbb{Q}$

or $|f(x) - f(0)| = |-x-0|$ if $x \in \mathbb{Q}$

$\Rightarrow |f(x) - f(0)| = |x-0|$ if $x \in \mathbb{R}$

If we choose $\delta > 0$ so that $\delta = \epsilon$, then $x \in \mathbb{R}$,

$|x-0| < \delta \Rightarrow |f(x) - f(0)| < \epsilon$

By definition of continuity f is continuous at '0'.

case(ii) let 'a' be rational number and $a \neq 0$ then $f(a) = -a$ for each $n \in \mathbb{N}$, $a - \frac{1}{n} < a + \frac{1}{n}$ and hence by denseness of $\mathbb{R}$, there lies an irrational number x_n such that $a - \frac{1}{n} < x_n < a + \frac{1}{n}$

$\Rightarrow |x_n - a| < \frac{1}{n}$ for $n \in \mathbb{N}$

$\therefore$ There exists a sequence $\{x_n\}$ of irrational number converging to 'a'.

Also $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = a$

since $f(a) = -a$, $\lim_{n\to\infty} f(x_n) \neq f(a)$

f is not continuous at $a \neq 0 \in Q$.

<u>case(iii)</u> :- Let 'a' be irrational number then

$f(a) = a$ for each $n \in N$, $a - \frac{1}{n} < a + \frac{1}{n}$ and hence

by denseness of $\mathbb{R}$, there lies a rational number

x_n such that $a - \frac{1}{n} < x_n < a + \frac{1}{n}$

$$\Rightarrow |x_n - a| < \frac{1}{n} \text{ for } n \in N$$

$\therefore$ there exists a sequence $\{x_n\}$ of rational numbers converging to 'a'.

Also $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (-x_n) = -a$

since $f(a) = a$, $\lim_{n\to\infty} f(x_n) \neq f(a)$ and

hence f is not continuous at $a \in \mathbb{R} - Q$

Hence f is continuous only at '0'.

<u>Theorem</u>:- If $f: S \to \mathbb{R}$ is continuous at $a \in S$ then $|f|$ is continuous at $a \in S$.

<u>Proof</u>:- Since f is continuous at $a \in S$

for Each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in S$,

$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

$\therefore x \in S$ such that $|x - a| < \delta$

$\Rightarrow ||f(x)| - |f(a)|| \leq |f(x) - f(a)| < \epsilon$

$\therefore |f|$ is continuous at $a \in S$

Note:- The converse of the theorem need not be true. That is, if $|f|$ is continuous at 'a', then f need not be continuous at a.

consider $f(x) = 1$ if $x \geq 0$ and $f(x) = -1$ if $x < 0$

Then $|f| : \mathbb{R} \to \mathbb{R}$ is defined $|f|(x) = |f(x)| = 1$

since $|f|$ is a constant function, $|f|$ is continuous on $\mathbb{R}$ and hence at $x = 0$

But $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} -1 = -1$ and

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} 1 = 1$

$\therefore \lim\limits_{x \to 0^+} f(x) \neq \lim\limits_{x \to 0^-} f(x)$ and

hence f is not continuous at $x = 0$

Problem:- Examine for continuity the function f defined by $f(x) = |x| + |x-1|$ at $x = 0, 1$

Solution:- The function f defined by $f(x) = |x| + |x-1|$

if $x < 0$, $x - 1 < 0$

then $|x| = -x$, $|x-1| = -(x-1) = -x+1$

$f(x) = -x - x + 1 = -2x + 1$

If $x=0$, $|x| = |0| = 0$, $|x-1| = |0-1| = 1$

$f(x) = 0+1 = 1$

If $0<x<1$, $x>0$, $x-1<0$

Then $|x| = x$, $|x-1| = -(x-1) = -x+1$

$f(x) = x-x+1 = 1$

If $x=1$, $|x| = |1| = 1$, $|x-1| = |1-1| = 0$

$f(x) = 1+0 = 1$

If $x>1$, $x>0$, $x-1>0$

Then $|x| = x$, $|x-1| = x-1$

$f(x) = x+x-1 = 2x-1$

$$\therefore \quad f(x) = \begin{cases} 1-2x & \text{if } x<0 \\ 1 & \text{if } 0 \le x \le 1 \\ 2x-1 & \text{if } x>1 \end{cases}$$

<u>Continuity at $x=0$</u>:- Here $f(0) = 1$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1-2x) = 1-2(0) = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1$$

$$\lim_{x\to 0} f(x) = 1 = f(0)$$

$\therefore$ f is continuous at $x=0$

Scanned with OKEN Scanner

<u>continuity</u> at <u>$x=1$</u> :-

Here $f(1)=1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1 = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x-1 = 2(1)-1 = 1$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 1$$

$$\lim_{x\to 1} f(x) = 1 = f(1)$$

$\therefore$ f is continuous at $x=1$

<u>Problem</u>:- Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f(x) = \dfrac{\sin(a+1)x+\sin x}{x}$ for $x<0$, $f(x)=c$ for $x=0$ and $f(x) = \dfrac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}}$ for $x>0$. Determine the values of a,b,c for which the function is continuous at $x=0$.

<u>Solution</u>:- Let $f:\mathbb{R}\to\mathbb{R}$ be such that

$$f(x) = \frac{\sin(a+1)x+\sin x}{x} \text{ for } x<0, f(x)=c \text{ for } x=0$$

and $$f(x) = \frac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}} \text{ for}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\frac{\sin(a+1)x+\sin x}{x}\right)$$

$$= \lim_{x\to 0^-} \frac{2\sin\left(\frac{a}{2}+1\right)x\cdot\cos\left(\frac{ax}{2}\right)}{x}$$

$= 2((a/2)+1) \cdot 1$

$= a+2$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}$$

$$= \lim_{x\to 0^+} \frac{x^{1/2}\left[(1+bx)^{1/2}-1\right]}{bx^{3/2}}$$

$$= \lim_{x\to 0^+} \frac{(1+bx)^{1/2}-1}{bx}$$

$$= \lim_{x\to 0^+} \frac{1+bx-1}{bx\left[(1+bx)^{1/2}+1\right]}$$

$$= \lim_{x\to 0^+} \frac{1}{(1+bx)^{1/2}+1}$$

$$= \frac{1}{2}$$

This is independent of b. so b may have any real value other than 0.

Since f is continuous at $x=0$, we have

$$\lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$$

$\therefore\ a+2 = c = \frac{1}{2}$

$\Rightarrow\ a+2 = \frac{1}{2},\ c = \frac{1}{2}$

$\Rightarrow\ a = -\frac{3}{2},\ c = \frac{1}{2}$

Problem:- Let $f: \mathbb{R} \to \mathbb{R}$ be such that $f(x) = \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ if $x \neq 0$ and $f(0) = 1$ Discuss the continuity at $x = 0$.

Solution:- Let $f: \mathbb{R} \to \mathbb{R}$ be such that $f(x) = \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$

if $x \neq 0$ and $f(0) = 1$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} \right)$$

$$= \lim_{x \to 0^-} \left(\frac{e^{1/x} - \frac{1}{e^{1/x}}}{e^{1/x} + \frac{1}{e^{1/x}}} \right)$$

$$= \lim_{x \to 0^-} \left(\frac{\frac{(e^{1/x})^2 - 1}{e^{1/x}}}{\frac{(e^{1/x})^2 + 1}{e^{1/x}}} \right)$$

$$= \lim_{x \to 0^-} \left(\frac{(e^{1/x})^2 - 1}{(e^{1/x})^2 + 1} \right) = \frac{0-1}{0+1} = -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} \right) = \lim_{x \to 0^+} \left(\frac{\frac{1}{e^{-1/x}} - e^{-1/x}}{\frac{1}{e^{-1/x}} + e^{-1/x}} \right)$$

$$= \lim_{x \to 0^+} \left(\frac{\frac{1 - (e^{-1/x})^2}{e^{-1/x}}}{\frac{1 + (e^{-1/x})^2}{e^{-1/x}}} \right)$$

$$= \lim_{x \to 0^+} \left(\frac{1 - (e^{-1/x})^2}{1 + (e^{-1/x})^2} \right) = \frac{1-0}{1+0} = 1$$

$\therefore \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$ and

hence f is not continuous at $x = 0$

Borel's theorem :-

Statement:- If a function f is continuous in $[a,b]$ then for each $\epsilon > 0$ there exists a partition P of $[a,b]$ such that $|f(x_1) - f(x_2)| < \epsilon$ for $x_1, x_2 \in I_r$ where I_r is any subinterval of the partition P.

Definition:- A function $f: S \to \mathbb{R}$ is said to be bounded on S if there exists a constant $M > 0$ such that $|f(x)| \leq M \quad \forall x \in S$

A function f is not bounded on S, if given $M > 0$ there exists a point $x_0 \in S$ such that $|f(x_0)| > M$.

Theorem:- If $f: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. then f is bounded on $[a,b]$.

Proof:- f is continuous on $[a,b]$

let $\epsilon = 1$

since f is continuous on $[a,b]$, there exists a partition $\{a = t_0, t_1, t_2, \cdots t_n = b\}$ on $[a,b]$ such that $|f(x_1) - f(x_2)| < 1$ for $x_1, x_2 \in [t_{r-1}, t_r]$

Let $x \in [a,b]$ and $t_1, t_2, \dots, t_r$ be the points lying b/w 'a' and 'x'.

Then $|f(x)-f(a)| = |f(x)-f(t_r)+f(t_r)-f(t_{r-1})+\dots+f(t_1)-f(a)|$

$\leq |f(x)-f(t_r)|+|f(t_r)-f(t_{r-1})|+\dots+|f(t_1)-f(a)|$

$< 1+1+1+\dots+1$ ($r+1$ times)

$= r+1$

For any $x \in [a,b]$, there will not be more than $(n-1)$ points of the partition between 'a' and x.

$\therefore$ For every $x \in [a,b]$, $|f(x)-f(a)| < n-1+1 = n$

$\Rightarrow f(a)-n < f(x) < f(a)+n$

Hence f is bounded on $[a,b]$

Definition:- A function $f: S \to \mathbb{R}$ is said to have an absolute maximum on S, if there exists a point $x_0 \in S$ such that $f(x_0) \geq f(x)$ $\forall x \in S$, we say that $x_0 \in S$ is an absolute maximum point of f on S.

A function $f: S \to \mathbb{R}$ is said to have an absolute minimum on S, if there exists a point $x_0 \in S$ such that $f(x_0) \leq f(x)$ $\forall x \in S$ we say that $x_0 \in S$ is an absolute minimum point of f on S.

Theorem:- (Absolute maximum - minimum theorem):-

If $f: I = [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ then f is bounded on $[a,b]$ and attains its bounds or infimum and supremum

Proof: $f: I = [a,b] \to \mathbb{R}$ is continuous on $[a,b]$

$\Rightarrow f$ is bounded on $[a,b]$

Let $M = \sup\{f(x) / x \in [a,b]\} = \sup f(I)$ and

$m = \inf\{f(x) / x \in [a,b]\} = \inf f(I)$

Then $m \le f(x) \le M \quad \forall x \in [a,b]$

In order to prove that f attains its infimum and supremum

we have to show that there exists $c, d \in [a,b]$ such that $f(c) = M$ and $f(d) = m$

(1) If possible, let $f(x) < M$ for all $x \in [a,b]$ then $M - f(x) > 0$ for all $x \in [a,b]$

M is a constant function, f is continuous on $[a,b]$

$\Rightarrow \dfrac{1}{M-f}$ is continuous on $[a,b]$

$\Rightarrow \dfrac{1}{M-f}$ is bounded on $[a,b]$

$\therefore$ there exists $K > 0$ such that $\dfrac{1}{M - f(x)} \le K \quad \forall x \in [a,b]$

$\Rightarrow M - f(x) \ge \dfrac{1}{K}$ for all $x \in [a,b]$

$\Rightarrow f(x) \le M - \dfrac{1}{K}$ for all $x \in [a,b]$

$\therefore M - \dfrac{1}{K} < M = \sup f$ is an upperbound of f on $[a,b]$

This is a contradiction

Hence there exists $c \in [a,b]$ such that $f(c) = M = \sup(f[a,b])$

Similarly we can prove that

(ii) If possible, let $f(x) > m$ for all $x \in [a,b]$

then $f(x) - m > 0$ for all $x \in [a,b]$

m is a constant function, f is continuous on $[a,b]$

$\Rightarrow f - m$ is continuous on $[a,b]$

$\Rightarrow \frac{1}{f-m}$ is continuous on $[a,b]$

$\Rightarrow \frac{1}{f-m}$ is bounded on $[a,b]$

$\therefore$ there exist $l > 0$ such that $\frac{1}{f(x)-m} \le l \;\forall\; x \in [a,b]$

$\Rightarrow f(x) - m \ge \frac{1}{l}$ for all $x \in [a,b]$

$\Rightarrow f(x) \ge m + \frac{1}{l}$ for all $x \in [a,b]$

$\therefore m + \frac{1}{l} > m = \inf$ is a lower bound of f on $[a,b]$

This is a contradiction

Hence there exists $d \in [a,b]$ such that $f(d) = m = \inf [f[a,$

Neighbourhood property (or) sign of continuous function

Theorem:- If $f:[a,b] \to \mathbb{R}$ is continuous at 'a' and $f(a) \ne 0$

then there exists $\delta > 0$ such that $x \in [a, a+\delta] = f(x)$ has

the same sign as $f(a)$

Proof:- $f:[a,b] \to \mathbb{R}$ is continuous at 'a' and $f(a) \ne 0$

f is continuous at 'a'

$\Rightarrow$ for a given $\epsilon > 0$ there exists $\delta > 0$ such that

$|f(x) - f(a)| < \epsilon$ for all $x \in (a, a+\delta) \subset [a,b]$

$\Rightarrow f(a) - \epsilon < f(x) < f(a) + \epsilon \quad \forall x \in (a, a+\delta)$ — ①

Let $f(a) > 0$

Take $\epsilon = \frac{1}{2} f(a) > 0$

Then ① $\Rightarrow 0 < \frac{1}{2} f(a) < f(x) < \frac{3}{2} f(a) \quad \forall x \in (a, a+\delta)$

$\Rightarrow f(x) > 0 \quad \forall x \in (a, a+\delta)$

Let $f(a) < 0$

Take $\epsilon = -\frac{1}{2} f(a) > 0$

Then ① $\Rightarrow \frac{3}{2} f(a) < f(x) < \frac{1}{2} f(a) < 0 \quad \forall x \in (a, a+\delta)$

$\Rightarrow f(x) < 0 \quad \forall x \in (a, a+\delta)$

Hence $f(x)$ has the same sign as $f(a)$.

Theorem:- If $f : [a,b] \to \mathbb{R}$ is continuous at $c \in (a,b)$ and $f(c) \neq 0$ then there exists $\delta > 0$ such that $x \in (c-\delta, c+\delta) \Rightarrow f(x)$ has the same sign as $f(c)$.

Proof: f is continuous at c

$\Rightarrow$ for a given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $x \in (c-\delta, c+\delta) \subset [a,b]$

$\Rightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon \quad \forall x \in (c-\delta, c+\delta)$ — ①

Let $f(c) > 0$

Take $\epsilon = \frac{1}{2} f(c) > 0$

Then ① $\Rightarrow 0 < \frac{1}{2} f(c) < f(x) < \frac{3}{2} f(c)$

$\Rightarrow f(x) > 0 \quad \forall x \in (c-\delta, c+\delta)$

Let $f(c) < 0$, Take $\epsilon = -\frac{1}{2} f(c) > 0$

Then ① $\Rightarrow \frac{3}{2} f(c) < f(x) < \frac{1}{2} f(c) < 0$

$\Rightarrow f(x) < 0 \quad \forall x \in (c-\delta, c+\delta)$

Hence $f(x)$ has the same sign as $f(c)$.

Note:- If $f:(a,b] \to \mathbb{R}$ is continuous at 'b' and $f(b) \neq 0$ then there exists $\delta > 0$ such that $x \in (b-\delta, b] \Rightarrow f(x)$ has the same sign as $f(b)$.

Intermediate value property:-

Theorem:- (Bolzano's or location of roots theorem)

If f is continuous on $[a,b]$ and $f(a), f(b)$ have opposite signs then there exists $c \in (a,b)$ such that $f(c) = 0$

Proof:- f is continuous on $[a,b]$ and $f(a), f(b)$ have opposite signs

To prove that there exists $c \in (a,b)$ such that $f(c) = 0$

Let $f(a) < 0$, $f(b) > 0$ such that $f(a) < 0 < f(b)$

Let $S = \{x \in [a,b] / f(x) < 0\}$

$a \in [a,b]$ and $f(a) < 0 \Rightarrow a \in S$

$b \in [a,b]$ and $f(b) > 0 \Rightarrow b \notin S$

$\therefore S \neq \phi$ and b is an upper bound of S.

By completeness axiom, S has supremum, say, c

Since f is continuous at 'a' and $f(a) < 0$, there exists $\delta > 0$ such that $f(x) < 0 \ \forall x \in [a, a+\delta) \subset [a,b]$

$x \in [a, a+\delta) \Rightarrow x \in S \Rightarrow a + \frac{\delta}{2} \leq \sup S = c \Rightarrow a \neq c$

Since f is continuous at 'b' and $f(b) > 0$ there exists $\delta_1 > 0$ such that $f(x) > 0 \ \forall x \in (b-\delta_1, b] \subset [a,b]$

$x \in (b-\delta_1, b] \Rightarrow x \notin S \Rightarrow b - \frac{\delta_1}{2} \geq c \Rightarrow b \neq c$

$\therefore c \in (a,b)$ or $a < c < b$

If possible, let $f(c) < 0$

then by nbd property there exists $\delta > 0$ such that

$f(x) < 0 \ \forall x \in (c-\delta, c+\delta)$

For some $x \in (c, c+\delta)$

i.e for $x > c$ we have $f(x) < 0$

This is a contradiction, as $c = \sup S$

$\therefore f(c) \nless 0$ —— ①

If possible let $f(c) > 0$

Then by nbd property there exists $\delta > 0$ such that

$f(x) > 0 \ \forall x \in (c-\delta, c+\delta)$

Since $c = \sup S$ there exists $d \in S$ such that

$c - \delta < d \leq c$

But $d \in S \Rightarrow f(d) < 0$

This is a contradiction as $f(d) > 0$

$\therefore f(c) \ngtr 0$ —— ②

from ① and ②, $f(c) = 0$

case(ii) Let $f(a) > 0$, $f(b) < 0$ such that $f(b) < 0 < f(a)$

Define $F: [a,b] \to \mathbb{R}$ such that $F(x) = -f(x) \ \forall x \in [a,b]$

Then $F(a) = -f(a)$, $F(b) = -f(b)$ so that $F(a) < 0 < F(b)$

f is continuous on $[a,b] \Rightarrow F$ is continuous on $[a,b]$

By case(i) there exists $c \in (a,b)$ such that

$$F(c) = 0$$

i.e $-f(c) = 0$

i.e $f(c) = 0$

Theorem (Bolzano intermediate value theorem)

If f is continuous on $[a,b]$ and $f(a) \neq f(b)$ then f takes every value between $f(a)$ and $f(b)$ atleast once

OR

If f is continuous on $[a,b]$ and K be any real numbers between $f(a)$ and $f(b)$ then there exists $c \in (a,b)$ such that $f(c) = K$.

Proof: f is continuous on $[a,b]$ and K be any real number between $f(a)$ and $f(b)$

Define $g : [a,b] \to \mathbb{R}$ by $g(x) = f(x) - K$

f is continuous on $[a,b]$, K is a constant function

$\Rightarrow g = f - K$ is continuous on $[a,b]$

$f(a) < K < f(b) \Rightarrow f(a) - K < 0 < f(b) - K$

$\Rightarrow g(a) < 0 < g(b)$

Now the function is continuous on $[a,b]$ and $g(a), g(b)$ have opposite signs.

By Bolzano's theorem, there exists $c \in (a,b)$ such that

$g(c) = 0$

$\Rightarrow f(c) - K = 0$

$\Rightarrow f(c) = K$.

Uniform continuity

Definition:- Let S be an aggregate and $f : S \to \mathbb{R}$ be a function we say that f is uniformly continuous on S if given $\epsilon > 0$ there exists $\delta > 0$ such that

$x_1, x_2 \in S$, $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon$

Theorem:- If $f: S \to \mathbb{R}$ is uniformly continuous, then f is continuous in S.

Proof:- $f: S \to \mathbb{R}$ is uniformly continuous

To prove that f is continuous in S.

f is uniformly continuous in $S \Rightarrow$ for $\epsilon > 0$ there exist $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ for x_1, x_2 being any pair of arbitrary points of S such that $|x_1 - x_2| < \delta$

let $c \in S$

on taking $x_1 = x$ and $x_2 = c$ we have for $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for $|x - c| < \delta$

$\Rightarrow$ f is continuous at any point 'c' of S

Since c is arbitrary, f is continuous at every point of S.

$\therefore$ f is continuous in S.

Problem:- Prove that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is a continuous function on $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$

Solution:- $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

clearly, f is continuous on $\mathbb{R}$

so, we show that f is not uniformly continuous on $\mathbb{R}$

Given $\epsilon > 0$

let us assume that there exist a number $\delta > 0$

then for x_1 and $x_2 = x_1 + \frac{\delta}{2}$, $|x_1 - x_2| = |x_1 - (x_1 + \frac{\delta}{2})|$

$$= \frac{\delta}{2} < \delta$$

$$\therefore |f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |(x_1 - x_2)(x_1 + x_2)|$$

$$= |x_1 - x_2|\,|x_1 + x_2|$$

$= \frac{\delta}{2} \; |x_1 + x_1 + \frac{\delta}{2}|$

$= \frac{\delta}{2} \; |2x_1 + \frac{\delta}{2}|$

$= x_1\delta + \frac{\delta^2}{4} < \epsilon$ if $x_1 > 0$

since $\frac{\delta^2}{4} > 0$, we must have $x_1\delta < \epsilon$ for all $x_1 \in \mathbb{R}, x_1 > 0$

But this is impossible

$\therefore$ δ depends on ϵ and x_1 and hence the function f is not uniformly continuous on $\mathbb{R}$.

Theorem: If a function f is continuous on $[a,b]$, then it is uniformly continuous on $[a,b]$

Proof: let $\epsilon > 0$

f is continuous on $[a,b]$

$\Rightarrow$ for $\epsilon > 0$, we can divide $[a,b]$ into a finite number say n, of subintervals $[a = t_0, t_1], [t_1, t_2], \dots, [t_{r-1}, t_r], [t_r, t_{r+1}], \dots, [t_{n-1}, t_n = b]$

such that $|f(x_1) - f(x_2)| < \frac{\epsilon}{2}$ for x_1, x_2 belonging to the same sub-interval.

let $\delta = \frac{1}{2} \min \{ |t_r - t_{r-1}| > 0, \; 1 \le r \le n \}$

Let x_1, x_2 be any two points of $[a,b]$ such that $|x_1 - x_2| < \delta$

Then x_1, x_2 either belong to the same sub-interval or to two consecutive sub-intervals with a common end point.

case(i) let x_1, x_2 belong to the same sub-intervals

we have from ①

$$|f(x_1)-f(x_2)| < \frac{\epsilon}{2} < \epsilon \quad \text{for } |x_1-x_2| < \delta$$

case(ii) Let x_1, x_2 belong to two consecutive sub-intervals with a common end point, say t_r.

we have from ①,

$$|f(x_1)-f(t_r)| < \frac{\epsilon}{2} \text{ and } |f(t_r)-f(x_2)| < \frac{\epsilon}{2}$$

$$\therefore |f(x_1)-f(x_2)| = |\{f(x_1)-f(t_r)\}+\{f(t_r)-f(x_2)\}|$$

$$\leq |f(x_1)-f(t_r) + f(t_r)-f(x_2)|$$

$$< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon \quad \text{for } |x_1-x_2| < \delta$$

Thus in either case, we have for any $\epsilon>0$ there exists $\delta>0$ such that

$|f(x_1)-f(x_2)| < \epsilon$ for any arbitrary points x_1, x_2 of $[a,b]$ such that $|x_1-x_2| < \delta$

$\therefore f$ is uniformly continuous in $[a,b]$

Theorem:- A function defined on $[a,b]$ is uniformly continuous on $[a,b]$ iff it is continuous on $[a,b]$

proof: f is uniformly continuous

To prove that f is continuous in S

f is uniformly continuous in $S \Rightarrow$ for $\epsilon>0 \; \exists \; \delta>0$ such that $|f(x_1)-f(x_2)| < \epsilon$ for x_1, x_2 being any pair of arbitrary points of S such that $|x_1-x_2| < \delta$

Let $c \in S$

on taking $x_1 = x$ and $x_2 = c$ we have for $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for $|x - c| < \delta$

$\Rightarrow$ f is continuous at any point 'c' of S

since c is arbitrary, f is continuous at every point of S.

$\therefore$ f is continuous in S

f is continuous on $[a,b]$

To prove that f is uniformly continuous on $[a,b]$

let $\epsilon > 0$

f is continuous on $[a,b]$

$\Rightarrow$ for $\epsilon > 0$, we can divide $[a,b]$ into a finite number say n, of subintervals $[a = t_0, t_1], [t_1, t_2], \ldots, [t_{r-1}, t_r], [t_r, t_{r+1}], \ldots, [t_{n-1}, t_n = b]$

such that $|f(x_1) - f(x_2)| < \frac{\epsilon}{2}$ for x_1, x_2 belonging to the same sub-interval.

Let $\delta = \frac{1}{2} \min \{ |t_r - t_{r-1}| > 0, \ 1 \le r \le n \}$

let x_1, x_2 be any two points of $[a,b]$ such that $|x_1 - x_2| < \delta$ Then x_1, x_2 either belong to the same sub-interval or to two consecutive sub-intervals with a common end point.

case(i) let x_1, x_2 belong to the same sub-intervals

we have from (i)

$|f(x_1) - f(x_2)| < \frac{\epsilon}{2} < \epsilon$ for $|x_1 - x_2| < \delta$

case(ii) Let x_1, x_2 belong to two consecutive sub-intervals with a common end point, say t_r

we have from (i),

$|f(x_1) - f(t_r)| < \frac{\epsilon}{2}$ and $|f(t_r) - f(x_2)| < \frac{\epsilon}{2}$

$\therefore |f(x_1) - f(x_2)| = |\{f(x_1) - f(t_r)\} + \{f(t_r) - f(x_2)\}|$

$\leq |f(x_1) - f(t_r) + f(t_r) - f(x_2)|$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for $|x_1 - x_2| < \delta$

Thus in either case, we have for any $\epsilon > 0$ there exists $\delta > 0$ such that

$|f(x_1) - f(x_2)| < \epsilon$ for any arbitrary points x_1, x_2 of $[a,b]$

such that $|x_1 - x_2| < \delta$

$\therefore f$ is uniformly continuous in $[a,b]$

Problem:- Show that the function f defined by $f(x) = x^3$, is uniformly continuous in $[-2,2]$

Solution:- The function f defined by $f(x) = x^3$

Let $x_1, x_2 \in [-2,2]$

Then $|x_1| \leq 2$, $|x_2| \leq 2$

$|f(x_1) - f(x_2)| = |x_1^3 - x_2^3|$

$= |(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2)|$

$= |x_1 - x_2|\,|x_1^2 + x_1x_2 + x_2^2|$

$= |x_1 - x_2|\,[|x_1|^2 + |x_1||x_2| + |x_2|^2]$

$\leq |x_1 - x_2|\,(|2|^2 + |2||2| + |2|^2)$

$= |x_1 - x_2|\,[4 + 4 + 4]$

$= 12\,|x_1 - x_2|$

$\therefore\ |f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \frac{\epsilon}{12}$

$\therefore$ Given $\epsilon > 0$ there exists $\delta = \frac{\epsilon}{12}$ such that

$|f(x_1) - f(x_2)| < \epsilon$ when $|x_1 - x_2| < \delta$ for every $x_1, x_2 \in [-2,2]$

Hence $f(x)$ is uniformly continuous in $[-2,2]$

Problem:- From the definition, show that $f(x) = x^2 + 3x$ is uniformly continuous on $[-1,1]$

Solution:- $f(x) = x^2 + 3x$

let $\epsilon > 0$ and $x_1, x_2 \in [-1,1]$

Then $|x_1| \le 1$, $|x_2| \le 1$

$$
\begin{aligned}
|f(x_1) - f(x_2)| &= |(x_1^2 + 3x_1) - (x_2^2 + 3x_2)| \\
&= |(x_1^2 - x_2^2) + 3(x_1 - x_2)| \\
&= |(x_1 - x_2)(x_1 + x_2) + 3(x_1 - x_2)| \\
&= |(x_1 - x_2)(x_1 + x_2 + 3)| \\
&= |x_1 - x_2|\,|x_1 + x_2 + 3| \\
&\le |x_1 - x_2|\,[|x_1| + |x_2| + 3] \\
&\le |x_1 - x_2|\,[1 + 1 + 3] \\
&< 5\,|x_1 - x_2|
\end{aligned}
$$

$\therefore\ |f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \frac{\epsilon}{5}$

For a given $\epsilon > 0$ there exists $\delta\left(\frac{\epsilon}{5}\right) > 0$ such that $x_1, x_2 \in [-1,1]$, $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon$

$\therefore\ f(x) = x^2 + 3x$ is uniformly continuous on $[-1,1]$

UNIT : IV

DIFFERENTIATION

Derivability of a function at a point:

Definition: Let S be an aggregate and $f: S \to \mathbb{R}$ be a function. Let $c \in S$ be a limit point of S or $c \in S \cap \bar{S}$

(i) If $\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c}$ $(x \neq c)$ exists, then we say that f is derivable (differentiable) from left at 'c'. The limit is called the left-derivative of f at 'c' and is denoted by $f'(c-0)$ or $Lf'(c)$.

(ii) If $\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c}$ $(x \neq c)$ exists, then we say that f is derivable (differentiable) from right at 'c'. The limit is called the right-derivable of f at 'c' and is denoted by $f'(c+0)$ or $Rf'(c)$.

(iii) If $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ $(x \neq c)$ exists, then we say that f is derivable (differentiable) at 'c'. The limit is called the derivative of f at c and is denoted by $f'(c)$.

Definition: Let S be an aggregate and $f: S \to \mathbb{R}$ be a function. Let $c \in S$ be a limit point of S and $L \in \mathbb{R}$. f is said to be derivable at 'c' if for a given $\epsilon > 0$ there exists $\delta > 0 \ni 0 < |x-c| < \delta \Rightarrow \left| \frac{f(x)-f(c)}{x-c} - l \right| < \epsilon$. The number l is called the derivative of f at c is denoted by $f'(c)$.

Definition: Let $f:[a,b]\to\mathbb{R}$ be a function and $a<c<b$

(i) If $\lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c}$ $(x\neq c)$ exists, then we say that f is derivable from left at c. The limit is called the left derivative of f at 'c' and is denoted by $f'(c-0)$ or $Lf'(c)$.

(ii) If $\lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c}$ $(x\neq c)$ exists, then we say that f is derivable from right at c. The limit is called the right derivative of f at 'c' and is denoted by $f'(c+0)$ or $Rf'(c)$.

(iii) If $\lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c}$ $(x\neq c)$ exists, then we say that f is derivable of f at 'c' The limit is called the derivative of f at 'c' and is denoted by $f'(c)$.

Derivability of an interval:-

Let $f:[a,b]\to\mathbb{R}$ be a function. f is said to be derivable on $[a,b]$ if (i) $f'(c)$ exists for each $c\in(a,b)$ (ii) $Rf'(a)=f'(a+0)$ exists and (iii) $Lf'(b)=f'(b-0)$ exists.

Derivability and continuity of a function:-

Theorem:- If $f:[a,b]\to\mathbb{R}$ is derivable at $c\in(a,b)$, then f is continuous at c.

proof:- $f:[a,b]\to\mathbb{R}$ is derivable at $c\in[a,b]$

let $c\in(a,b)$

f is derivable at $c \Rightarrow \lim_{x\to c}\dfrac{f(x)-f(c)}{x-c}=f'(c)$

for $x\neq c$, $f(x)-f(c)=\left(\dfrac{f(x)-f(c)}{x-c}\right)(x-c)$

$$\lim_{x\to c}[f(x)-f(c)]=\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right](x-c)$$

$$=\left[\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right]\left[\lim_{x\to c}(x-c)\right]$$

$$\lim_{x\to c}[f(x)-f(c)]=f'(c)\cdot 0$$

$$\lim_{x\to c}[f(x)-f(c)]=0$$

$$\Rightarrow \lim_{x\to c}f(x)-\lim_{x\to c}f(c)=0$$

$$\Rightarrow \lim_{x\to c}f(x)-f(c)=0$$

$$\Rightarrow \lim_{x\to c}f(x)=f(c)$$

$\therefore$ f is continuous at $c\in[a,b]$

let $c=a$

f is right derivable at $a \Rightarrow \lim_{x\to a^+}\dfrac{f(x)-f(c)}{x-a}=Rf'(a)$

For $x\neq a$, $f(x)-f(a)=\left[\dfrac{f(x)-f(a)}{x-a}\right](x-a)$

$$\lim_{x\to a^+}[f(x)-f(a)]=\lim_{x\to a^+}\left(\frac{f(x)-f(a)}{x-a}\right)(x-a)$$

$$= \lim_{x \to a^+} \left(\frac{f(x)-f(a)}{x-a} \right) \lim_{x \to a^+} (x-a)$$

$$= Rf'(a) \cdot (a-a) = Rf'(a) \cdot 0$$

$$\lim_{x \to a^+} [f(x) - f(a)] = 0$$

$$\Rightarrow \lim_{x \to a^+} f(x) - \lim_{x \to a^+} f(a) = 0$$

$$\Rightarrow \lim_{x \to a^+} f(x) - f(a) = 0$$

$$\Rightarrow \lim_{x \to a^+} f(x) = f(a)$$

$\therefore$ f is right continuous at a

let $c = b$

f is derivable at b $\Rightarrow \lim_{x \to b^-} \frac{f(x)-f(b)}{x-b} = Lf'(b)$

For $x \neq b$, $f(x) - f(b) = \left[\frac{f(x)-f(b)}{x-b} \right] (x-b)$

$$\lim_{x \to b^-} [f(x)-f(b)] = \lim_{x \to b^-} \left[\left(\frac{f(x)-f(b)}{(x-b)} \right) (x-b) \right]$$

$$= \lim_{x \to b^-} \left(\frac{f(x)-f(b)}{x-b} \right) \lim_{x \to b^-} (x-b)$$

$$= Lf'(b) \cdot 0$$

$$\lim_{x \to b^-} [f(x)-f(b)] = 0$$

$$\Rightarrow \lim_{x\to b^-} f(x) - \lim_{x\to b^-} f(b) = 0$$

$$\Rightarrow \lim_{x\to b^-} f(x) - f(b) = 0$$

$$\Rightarrow \lim_{x\to b^-} f(x) = f(b)$$

$\therefore$ f is left continuous at b

Hence f is continuous at $c \in [a,b]$.

Note:- The converse of this theorem need not be true. That is, if f is continuous at c then f need not be derivable at c.

We know that $f(x) = |x|$ is continuous at $x = 0$. We have $f(x) = -x$ for $x < 0$ and $f(x) = x$ for $x \geq 0$.

But $\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0}$

$$= \lim_{x\to 0^-} \frac{-x}{x}$$

$$= \lim_{x\to 0^-} (-1) = -1$$

and $\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0}$

$$= \lim_{x\to 0^+} \frac{x}{x}$$

$$= \lim_{x\to 0^+} 1 = 1$$

So that $Lf'(0) \neq Rf'(0)$.

Hence $f(x) = |x|$ is not derivable at $x = 0$

Problem:- show that $f(x) = \sin x$ is derivable at every $a \in R$.

Solution:- $f(x) = \sin x$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x-a} = \lim_{x \to a} \left(\frac{\sin x - \sin a}{x-a}\right)$$

$$= \lim_{x \to a} \left[\frac{2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{x-a}\right]$$

$$= \lim_{x \to a} \left[\cos\left(\frac{x+a}{2}\right)\left(\frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}\right)\right]$$

$$= \left[\lim_{x \to a} \cos\left(\frac{x+a}{2}\right)\right]\left[\lim_{\frac{x-a}{2} \to 0} \frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}\right]$$

$$= \cos\left(\frac{a+a}{2}\right) \cdot 1 = \cos\left(\frac{2a}{2}\right)$$

$$= \cos a$$

$\therefore f(x) = \sin x$ is derivable at $a \in R$ and $f'(x) = \cos a$

since $a \in \mathbb{R}$ is arbitrary $f'(x) = \cos x \ \forall x \in \mathbb{R}$

i.e $f(x) = \sin x$ is derivable at every $a \in \mathbb{R}$.

Problem:- Discuss the differentiability of $f(x) = |x-a|$ on $\mathbb{R}$.

Solution:- Let $c \in \mathbb{R}$ and $c < a$

then $c - a < 0$

There exists a deleted nbd of 'c' such that $x \in c$ - deleted nbd $\Rightarrow x < a$

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c}\frac{|x-a|-|c-a|}{x-c}$$

$$= \lim_{x\to c}\frac{-(x-a)-\{-(c-a)\}}{x-c}$$

$$= \lim_{x\to c}\frac{c-x}{x-c} = \lim_{x\to c}(-1) = -1$$

$\therefore$ $f(x)$ is derivable at $c\,(<a)\in\mathbb{R}$ and $f'(c)=-1$

let $c\in\mathbb{R}$ and $c>a$. Then $c-a>0$

There exists a deleted nbd of 'c' such that

$x\in c$ - deleted nbd $\Rightarrow x>a$

$$\lim_{x\to c}\frac{f(x)-f(a)}{x-c} = \lim_{x\to c}\frac{(x-a)-(c-a)}{(x-a)}$$

$$= \lim_{x\to c}\frac{x-c}{x-c}$$

$$= \lim_{x\to c} 1 = 1$$

$\therefore$ $f(x)$ is derivable at $c\,(>a)\in\mathbb{R}$ and $f'(c)=1$

let $c\in\mathbb{R}$ and $c=a$. Then $f(c)=c-a=0$

For $x\in c$ - left nbd $\Rightarrow x<a$ so that $Lf'(c)=1$

For $x\in c$ - right nbd $\Rightarrow x>a$ so that $Rf'(c)=1$

$\therefore$ $f(x)$ is not derivable at $c\,(=a)\in\mathbb{R}$.

Hence $f(x)$ is derivable in $\mathbb{R}-\{a\}$.

<u>Problem</u>: Discuss the derivability of $f(x)=|x|+|x-1|$ in $\mathbb{R}$.

<u>Solution</u>:- $f(x)=\begin{cases} 1-2x & \text{if } x<0 \\ 1 & \text{if } 0\le x\le 1 \\ 2x-1 & \text{if } x>1 \end{cases}$

$$Lf'(0) = \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^-} \left[\frac{(1-2x)-1}{x-0}\right]$$

$$= \lim_{x \to 0^-} \frac{-2x}{x} = \lim_{x \to 0^-} (-2) = -2$$

$$Rf'(0) = \lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^+} \left[\frac{1-1}{0}\right]$$

$$= \lim_{x \to 0^+} \left(\frac{0}{x}\right)$$

$$= \lim_{x \to 0^+} (0) = 0$$

$Lf'(0) \neq Rf'(0)$ and hence $f'(0)$ does not exist

$$Lf'(1) = \lim_{x \to 1^-} \frac{f(x)-f(0)}{x-1} = \lim_{x \to 1^-} \left(\frac{1-1}{x-1}\right)$$

$$= \lim_{x \to 1^-} \left(\frac{0}{x-1}\right)$$

$$= \lim_{x \to 1^-} (0)$$

$$= 0$$

$$Rf'(1) = \lim_{x \to 1^+} \frac{f(x)-f(0)}{x-1} = \lim_{x \to 1^+} \left[\frac{(2x-1)-1}{x-1}\right]$$

$$= \lim_{x \to 1^+} \left(\frac{2x-2}{x-1}\right)$$

$$= \lim_{x \to 1^+} \frac{2(x-1)}{(x-1)}$$

$$= \lim_{x \to 1^+} (2)$$

$$= 2$$

$Lf'(0) \neq Rf'(0)$ and hence $f'(1)$ does not exist

Hence f is derivable at every $\mathbb{R} - \{0, 1\}$.

Problem:- Show that $f(x) = x\sin(1/x)$, $x \neq 0$, $f(x) = 0$, $x = 0$ is continuous but not derivable at $x = 0$.

Solution: $f(x) = x\sin\frac{1}{x}$, $x \neq 0$, $f(x) = 0$, $x = 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$$

$$= \lim_{x\to 0} (x) \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$

$$= 0$$

$$\lim_{x\to 0} f(x) = f(0)$$

$\Rightarrow f$ is continuous at $x = 0$

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\left(\frac{x\sin\left(\frac{1}{x}\right)-0}{x-0}\right)$$

$$= \lim_{x\to 0}\left(\frac{x\sin\left(\frac{1}{x}\right)}{x}\right)$$

$$= \lim_{x\to 0} \sin\left(\frac{1}{x}\right), \text{ does not exist}$$

Hence f is not derivable at $x = 0$

Increasing and decreasing functions:-

Definition:- Let $f: I \to \mathbb{R}$ be a function and $c \in I$. f is said to be locally increasing at 'c' if there exists $\delta > 0$ such that $f(x) < f(c)$ for $x \in (c-\delta, c) \subset I$ and $f(x) > f(c)$ for $x \in (c, c+\delta) \subset I$

f is said to be locally decreasing at 'c' if $(-f)$ function is locally increasing at 'c'.

Definition: Let $f: I \to \mathbb{R}$ be a function. f is said to be increasing on the interval I, if $x_1, x_2 \in I$, $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$. f is said to be strictly increasing on I if $x_1, x_2 \in I$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. f is said to decreasing on I if the function $(-f)$ is increasing on I.

Note: If $f: I \to \mathbb{R}$ is derivable at $c \in I$ and $f'(c) > 0$ then f is locally increasing at c.

Theorem (Darboux's theorem)

If $f: [a,b] \to \mathbb{R}$ is such that (i) f is derivable on $[a,b]$ and (ii) $f'(a), f'(b)$ have opposite signs (i.e $f'(a)f'(b) < 0$), then there exists $c \in [a,b]$ such that $f'(c) = 0$

proof: $f: [a,b] \to \mathbb{R}$ is such that

(i) f is derivable on $[a,b]$ and

(ii) $f'(a), f'(b)$ have opposite signs

Let $f'(a) < 0$ and $f'(b) > 0$ so that $f'(a)f'(b) < 0$

$f'(a) < 0 \Rightarrow$ there exists $\delta_1 > 0$ such that $f(x) < f(a)$ for $x \in (a, a+\delta_1) \subset [a,b]$ — (1)

f is derivable on $[a,b] \Rightarrow f$ is continuous on $[a,b]$ and attains its bounds on $[a,b]$

$\Rightarrow$ there exists $\alpha, \beta \in [a,b]$ such that

$f(\alpha) = \inf f$ and $f(\beta) = \sup f$ in $[a,b]$

If possible, let $\alpha = a$ then $f(a) = \inf f$.

from ① . $f(x) < f(a) = \inf f$ for $x \in [a, a+\delta) \subset [a,b]$

This is a contradiction

$\therefore \alpha \neq a$

Similarly we can prove that $\alpha \neq b$

$\therefore \alpha \in (a,b)$

Now, we prove that $f'(\alpha) = 0$

If possible, let $f'(\alpha) < 0$

$\therefore$ There exists $\delta_2 > 0$ such that $f(x) < f(\alpha) = \inf f$

for $x \in (\alpha, \alpha+\delta_2) \subset [a,b]$

This is a contradiction and hence $f'(\alpha) \nless 0$

If possible, let $f'(\alpha) > 0$

$\therefore$ There exists $\delta_3 > 0$ such that $f(x) < f(\alpha) = \inf$

for $x \in (\alpha-\delta_3, \alpha) \subset [a,b]$

This is a contradiction and hence $f'(\alpha) \ngtr 0$

Hence $f'(\alpha) = 0$

If $f'(a) > 0$ and $f'(b) < 0$, then we can prove that

$f'(\beta) = 0$ where $\beta \in (a,b)$

If $f'(a)f'(b) < 0$ then there exist $c \in [a,b]$

such that $f'(c) = 0$

Problem: show that $\log(1+x) - \frac{2x}{2+x}$ is increasing when $x > 0$

Solution:- let $f(x) = \log(1+x) - \frac{2x}{2+x}$ for $x > 0$

we know that $\log(1+x)$ and $\frac{2x}{2+x}$ are derivable for $x > 0$

$$f'(x) = \frac{1}{1+x} - \left[\frac{(2+x)(2) - (2x)(1)}{(2+x)^2}\right]$$

$$= \frac{1}{1+x} - \left[\frac{x+2x-2x}{(2+x)^2}\right]$$

$$= \frac{1}{1+x} - \frac{x}{(2+x)^2}$$

$$= \frac{(2+x)^2 - x(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{4+x^2+4x-4-4x}{(1+x)(2+x)^2}$$

$$= \frac{x^2}{(1+x)(2+x)^2} > 0 \quad \text{for } x>0$$

Hence $f(x)$ is increasing for $x>0$

<u>Problem</u>:- Prove that $\tan x > x > \sin x \ \forall \ x \in (0, \frac{\pi}{2})$

<u>Solution</u>:- Let $f(x) = \tan x - x$

f is derivable on $(0, \pi/2)$ and

$f'(x) = \sec^2 x - 1$ for all $x \in (0, \pi/2)$

$f'(x) > 0$ for all $x \in (0, \pi/2)$

$\Rightarrow f$ is increasing on $(0, \pi/2)$

Also $f(0) = \tan 0 - 0 = 0$

$\therefore f(x) > 0$ for all $x \in (0, \pi/2)$

$\Rightarrow \tan x - x > 0$ for all $x \in (0, \pi/2)$

$\Rightarrow \tan x > x$ for all $x \in (0, \pi/2)$ — (1)

let $g(x) = x - \sin x$

g is derivable on $(0, \pi/2)$ and

$g'(x) = 1 - \cos x$ for all $x \in (0, \pi/2)$

$g'(x) > 0$ for all $x \in (0, \pi/2)$

$\Rightarrow$ g is increasing on $(0, \pi/2)$

Also $g(0) = 0 - \sin 0 = 0$

$\therefore g(x) > 0$ for all $x \in (0, \pi/2)$

$\Rightarrow x - \sin x > 0$ for all $x \in (0, \pi/2)$

$\Rightarrow x > \sin x$ for all $x \in (0, \pi/2)$ — (2)

from (1) and (2), $\tan x > x > \sin x \ \forall \ x \in (0, \pi/2)$

<u>problem</u>: prove that $f(x) = \dfrac{x}{\sin x}$ is increasing in $(0, \frac{\pi}{2})$

<u>solution</u>: $f(x) = \dfrac{x}{\sin x}$

$x, \sin x$ are derivable in $(0, \pi/2)$ and

$\sin x \neq 0$ for all $x \in (0, \pi/2)$

$f'(x)$ exists for all $x \in (0, \pi/2)$ and

$$f'(x) = \frac{\sin x(1) - (x)(\cos x)}{(\sin x)^2}$$

$$= \frac{\sin x - x\cos x}{\sin^2 x}$$

$$= \frac{\dfrac{\sin x}{\cos x} - \dfrac{x\cos x}{\cos x}}{\left(\dfrac{1}{\cos x}\right)(\sin^2 x)} = \frac{\tan x - x}{\sec x \sin^2 x}$$

we have $\tan x > x$, $\cos x > 0$, $\sin^2 x > 0 \ \forall \ x \in (0, \pi/2)$

$\Rightarrow f'(x) > 0$ for all $x \in (0, \pi/2)$

Hence $f(x) = \dfrac{x}{\sin x}$ is increasing in $(0, \pi/2)$

Mean value theorems

Theorem (Rolle's theorem)

If a function $f: I=[a,b]\to\mathbb{R}$ is such

(i) f is continuous on $[a,b]$

(ii) f is derivable on (a,b) and

(iii) $f(a)=f(b)$, then there exists $c\in(a,b)$ such that $f'(c)=0$

Proof: $f: I=[a,b]\to\mathbb{R}$ is such that

(i) f is continuous on $[a,b]$

(ii) f is derivable on (a,b) and

(iii) $f(a)=f(b)$

f is continuous on $[a,b]$
$\Rightarrow f$ is bounded on $[a,b]$ and attains the infimum

and supremum

$\Rightarrow$ there exist $\alpha,\beta\in[a,b]$ such that

$f(\alpha)=m=\inf$ and $f(\beta)=M=\sup f$ in $[a,b]$

case(i):- let $m=M$

Then $f(x)=m$ for $x\in[a,b]$

$\therefore f$ is a constant function in $[a,b]$

and hence $f'(x)=0$ for every $x\in[a,b]$

Thus the theorem is true.

case(ii):- let $m\neq M$

Since $f(a)=f(b)$ and $m\neq M$ we have either $M\neq f(a)$ and hence $M\neq f(b)$ or $m\neq f(a)$ and hence $m\neq f(b)$

let us suppose that $M\neq f(a)$, $M\neq f(b)$

$f(\beta) = M \neq f(a) \Rightarrow \beta \neq a$ and

$f(\beta) = M \neq f(b) \Rightarrow \beta \neq b$

$\therefore a < \beta < b$ or $\beta \in [a, b]$

f is derivable on (a, b) and $\beta \in (a, b)$

$\Rightarrow f$ is derivable at β

Now, we prove that $f'(\beta) = 0$

If possible, let $f'(\beta) < 0$

$\therefore$ There exists $\delta_1 > 0$ such that $f(x) > f(\beta) = M$

for all $x \in (\beta - \delta_1, \beta) \subset [a, b]$

This is a contradiction as M is the supremum

$\therefore f'(\beta) \nless 0$

Similarly we can prove that $f'(\beta) \ngtr 0$ Hence $f'(\beta) = 0$

If $m \neq f(a)$, $m \neq f(b)$ then we can prove that there exists $\alpha \in (a, b)$

such that $f'(\alpha) = 0$

Hence there exists $c \in (a, b)$ such that $f'(c) = 0$

Problem:- Verify Rolle's theorem in the interval $[a, b]$ for the function $f(x) = (x-a)^m (x-b)^n$; m, n being +ve integers

Solution:- $f(x) = (x-a)^m (x-b)^n$

Since $f(x) = (x-a)^m (x-b)^n$ is a polynomial function of $(m+n)^{th}$ degree

(i) f is continuous on $[a, b]$ and

(ii) f is derivable on (a, b)

Also $f(a) = 0 = f(b)$

f satisfies all the conditions of Rolle's theorem on $[a,b]$

for all $x \in \mathbb{R}$.

$$f'(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m n(x-b)^{n-1}$$

$$= m(x-a)^{m-1}(x-b)^{n-1} + n(x-a)^{m-1}(x-a)(x-b)^{n-1}$$

$$= (x-a)^{m-1}(x-b)^{n-1}[m(x-b)+n(x-a)]$$

$$= (x-a)^{m-1}(x-b)^{n-1}[mx-mb+nx-na]$$

$$= (x-a)^{m-1}(x-b)^{n-1}[x(m+n)-(mb+na)]$$

clearly, $a < \dfrac{mb+na}{m+n} < b$ for all $m,n \in Z^+$

$\therefore$ for $c = \dfrac{mb+na}{m+n} \in (a,b)$ we have $f'(c) = 0$

Hence Rolle's theorem is verified.

Problem:- Examine the applicability of Rolle's theorem for $f(x) = 1-(x-1)^{2/3}$ on $[0,2]$

Solution:- $f(x) = 1-(x-1)^{2/3}$

Since f is an algebraic function, f is continuous on $[0,2]$

If f is derivable at $x \in \mathbb{R}$, then

$$f'(x) = 0 - \frac{2}{3}(x-1)^{2/3-1} = -\frac{2}{3}(x-1)^{-1/3}$$

$$= \frac{-2}{3(x-1)^{1/3}}$$

The function f' is not defined at $x = 1$

$\therefore$ f is not derivable at $1 \in [0,2]$

$\therefore$ f does not satisfy the condition (ii) of Rolle's theorem

Hence Rolle's theorem is not applicable for for f on $[0,2]$.

Theorem:- (Lagranges mean value theorem or first mean value theorem)

If $f:[a,b]\to\mathbb{R}$ is a function such that

(i) f is continuous on $[a,b]$ and

(ii) f is derivable on (a,b) then there exists a point $c\in(a,b)$ such that

$$\frac{f(b)-f(a)}{b-a}=f'(c) \quad (\text{OR}) \quad f(b)-f(a)=f'(c)(b-a)$$

Proof:- $f:[a,b]\to\mathbb{R}$ is a function such that

(i) f is continuous on $[a,b]$ and

(ii) f is derivable on (a,b)

Define the function $\phi:[a,b]\to\mathbb{R}$ such that $\phi(x)=f(x)+kx$ where $k\in\mathbb{R}$ is given by $\phi(a)=\phi(b)$

$$\phi(a)=\phi(b) \Rightarrow f(a)+ka=f(b)+kb$$

$$\Rightarrow f(b)-f(a)=ka-kb$$

$$\Rightarrow f(b)-f(a)=-k(b-a)$$

$$\Rightarrow -k=-\frac{f(b)-f(a)}{b-a}$$

$k\in\mathbb{R}$, x is continuous on $\mathbb{R}$.

$\Rightarrow kx$ is continuous and derivable on $\mathbb{R}$

f is continuous on $[a,b]$ and kx is continuous on $\mathbb{R}$

$\Rightarrow \phi(x)=f(x)+kx$ is continuous on $\mathbb{R}$

f is derivable on (a,b) and kx is derivable on $\mathbb{R}$

$\Rightarrow \phi(x)=f(x)+kx$ is derivable on (a,b)

Further from the definition of ϕ, $\phi(a)=\phi(b)$

$\therefore$ The function ϕ satisfies all the conditions of Rolle's theorem

Thus there exists $c \in (a,b)$ such that $\phi'(c) = 0$

Since $\phi(x) = f(x) + kx$ for all $x \in [a,b]$,

$$\phi'(x) = f'(x) + K$$

for $c \in (a,b)$ and $\phi'(c) = 0$

$$\Rightarrow f'(c) + K = 0$$

$$\Rightarrow f'(c) = -K$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

Hence there exists $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c) \text{ (or) } f(b)-f(a) = f'(c)(b-a).$$

Problem:- using lagranges theorem, show that

$x > \log(1+x) > \frac{x}{1+x}$ if $f(x) = \log(1+x)\ \forall\ x > 0$

Solution:- consider $f(x) = \log(1+x)$ defined on $(0,t)$ where $t > 0$ clearly f is continuous on $[0,t]$ and derivable on $(0,t)$

Also $f'(x) = \frac{1}{1+x}$ for all $x \in (0,t)$

$\therefore$ By lagranges theorem, there exists $c \in (0,t)$

such that $\frac{f(t)-f(0)}{t-0} = f'(c)$

$$\Rightarrow \frac{\log(1+t) - \log 1}{t-0} = \frac{1}{1+c}$$

$$\Rightarrow \frac{\log(1+t)}{t} = \frac{1}{1+c} \quad \text{—①} \quad \forall \; 0<c<t$$

But for $0<c<t \Rightarrow 1+0<1+c<1+t$

$$\Rightarrow \frac{1}{1+0} > \frac{1}{1+c} > \frac{1}{1+t} \quad \text{—②}$$

$$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+t}$$

from ① and ② $\quad 1 > \frac{\log(1+t)}{t} > \frac{1}{1+t} \quad \forall \; t>0$

$$\Rightarrow t > \log(1+t) > \frac{t}{1+t} \quad \forall \; t>0$$

Hence $x > \log(1+x) > \frac{x}{1+x}$ for all $x>0$

<u>Problem:-</u> prove that inequality $\tan^{-1}v - \tan^{-1}u < v-u$ where $v>u\geq 0$. Hence prove that $\tan^{-1}x < x$ for $x>0$

<u>solution:-</u> consider $f(x) = \tan^{-1}x$ on (u,v)

we know that $f(x)$ is continuous and derivable on $[u,v]$ and $f'(x) = \frac{1}{1+x^2} \; \forall \; x \in (u,v)$

By lagranges theorem, there exists $c \in (u,v)$ such that $\frac{f(v)-f(u)}{v-u} = f'(c)$

$$\Rightarrow \frac{\tan^{-1}v - \tan^{-1}u}{v-u} = \frac{1}{1+c^2}$$

$$\Rightarrow \tan^{-1}v - \tan^{-1}u = \frac{1}{1+c^2}(v-u) \text{ for } u<c<v$$

Since $(v-u)>0$, $\quad 0 < \frac{1}{1+c^2} < 1$

$$\Rightarrow 0 < \frac{1}{1+c^2}(v-u) < v-u$$

$\Rightarrow 0 < \mathrm{Tan}^{-1}v - \mathrm{Tan}^{-1}u < v-u$

Then $\mathrm{Tan}^{-1}v - \mathrm{Tan}^{-1}u < v-u$ where $v>u>0$

In particular take $u=0$, $v=x$ so that $v-u=x>0$

$$\mathrm{Tan}^{-1}x - \mathrm{Tan}^{-1}0 < x$$

$$\Rightarrow \mathrm{Tan}^{-1}x < x \text{ for } x>0$$

Problem:- Show that $\dfrac{v-u}{1+v^2} < \mathrm{Tan}^{-1}v - \mathrm{Tan}^{-1}u < \dfrac{v-u}{1+u^2}$ for $0<u<v$. Hence deduce that $\dfrac{\pi}{4} + \dfrac{3}{25} < \mathrm{Tan}^{-1}\dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$

Solution: consider $f(x) = \mathrm{Tan}^{-1}x$ on (u,v)

clearly, $f(x)$ is continuous and derivable on (u,v)

and $f'(x) = \dfrac{1}{1+x^2}$ for all $x \in (u,v)$

By lagranges theorem, there exists $c \in (u,v)$

such that $\dfrac{f(v)-f(u)}{v-u} = f'(c)$.

$$\Rightarrow \frac{\mathrm{Tan}^{-1}v - \mathrm{Tan}^{-1}u}{v-u} = \frac{1}{1+c^2}$$

$$\Rightarrow \mathrm{Tan}^{-1}v - \mathrm{Tan}^{-1}u = \frac{1}{1+c^2}(v-u) \quad \text{for } u<c<v$$

$$\Rightarrow u^2 < c^2 < v^2$$

$$\Rightarrow 1+u^2 < 1+c^2 < 1+v^2$$

$$\Rightarrow \frac{1}{1+u^2} < \frac{1}{1+c^2} < \frac{1}{1+v^2}$$

$$\Rightarrow \frac{v-u}{1+u^2} < \frac{v-u}{1+c^2} < \frac{v-u}{1+v^2}$$

Thus $\frac{v-u}{1+v^2} < \frac{v-u}{1+c^2} < \frac{v-u}{1+u^2}$ for $0<u<v$

putting $u=1$, $v=\frac{4}{3}$, $\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}$

$\forall\ 0<u<v$

$$\Rightarrow \frac{4/3-1}{1+(4/3)^2} < \tan^{-1}(4/3) - \tan^{-1}(1) < \frac{(4/3)-(1)}{1+(1)^2}$$

$$\Rightarrow \frac{1/3}{1+16/9} < \tan^{-1}(4/3) - \tan^{-1}(1) < \frac{1/3}{1+1}$$

$$\Rightarrow \frac{1/3}{25/9} < \tan^{-1}(4/3) - \tan^{-1}(1) < \frac{1/3}{2}$$

$$\Rightarrow 3/25 < \tan^{-1}(4/3) - \tan^{-1}(1) < 1/6$$

$$\Rightarrow 3/25 < \tan^{-1}(4/3) - \pi/4 < 1/6$$

$$\Rightarrow \pi/4 + 3/25 < \tan^{-1}(4/3) < \pi/4 + \frac{1}{6}$$

Problem:- prove that $\frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}0.6 < \frac{\pi}{6} + \frac{1}{8}$

Solution:- let $f(x) = \sin^{-1}x$ for $[a,b]$, where $a>0$, $b<1$

f is continuous and derivable on $[a,b]$

Also $f'(x) = \frac{1}{\sqrt{1-x^2}}$ for all $x \in (a,b)$

By Lagrange's theorem there exists $c \in (a,b)$.

such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

$$\Rightarrow \frac{\sin^{-1}b - \sin^{-1}a}{b-a} = \frac{1}{\sqrt{1-c^2}}, \quad a<c<b$$

But $a < c < b \Rightarrow a^2 < c^2 < b^2$

$$\Rightarrow -a^2 > -c^2 > -b^2$$

$$\Rightarrow 1-a^2 > 1-c^2 > 1-b^2$$

$$\Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}b - \sin^{-1}a}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

put $b=0$, $c = \frac{3}{5}$, $a = \frac{1}{2}$

Then $$\frac{1}{\sqrt{1-(\frac{1}{2})^2}} \leq \frac{\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2}}{\frac{3}{5}-\frac{1}{2}} < \frac{1}{\sqrt{1-(\frac{3}{5})^2}}$$

$$\Rightarrow \frac{1}{\sqrt{1-\frac{1}{4}}} < \frac{\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2}}{\frac{1}{10}} < \frac{1}{\sqrt{1-\frac{9}{25}}}$$

$$\Rightarrow \frac{1}{\sqrt{\frac{3}{4}}} < \frac{\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2}}{\frac{1}{10}} < \frac{1}{\frac{4}{5}}$$

$$\Rightarrow \frac{1}{\frac{\sqrt{3}}{2}} < \frac{\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2}}{\frac{1}{10}} < \frac{1}{\frac{4}{5}}$$

$$\Rightarrow \frac{2}{\sqrt{3}} < 10\left[\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2}\right] < \frac{5}{4}$$

$$\Rightarrow \frac{2\sqrt{3}}{3} < 10\left(\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2}\right] < \frac{5}{4}$$

$$\Rightarrow \frac{2\sqrt{3}}{30} < \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2} < \frac{5}{40}$$

$\Rightarrow \frac{\sqrt{3}}{15} < \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2} < \frac{1}{8}$

$\Rightarrow \frac{\sqrt{3}}{15} < \sin^{-1}0.6 - \frac{\pi}{6} < \frac{1}{8}$

$\Rightarrow \frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}0.6 < \frac{\pi}{6} + \frac{1}{8}$

Cauchy's mean value theorem :-

Theorem:- If $f : [a,b] \to \mathbb{R}$ are such that

(i) f, g are continuous on $[a,b]$

(ii) f, g are derivable on (a,b) and

(iii) $g'(x) \neq 0 \ \forall \ x \in (a,b)$, then there exists point $c \in (a,b)$

such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

Proof:- $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ are such that

(i) f, g are continuous on $[a,b]$

(ii) f, g are derivable on (a,b)

(iii) $g'(x) \neq 0$ for all $x \in (a,b)$

Define $\phi : [a,b] \to \mathbb{R}$ such that $\phi(x) = f(x) + kg(x) \ \forall \ c \in (a,b)$

with $\phi(a) = \phi(b)$

$\phi(a) = \phi(b) \Rightarrow f(a) + kg(a) = f(b) + kg(b)$

$\Rightarrow kg(a) - kg(b) = f(b) - f(a)$

$\Rightarrow k[g(a) - g(b)] = f(b) - f(a)$ —— (1)

If possible, let $g(b) - g(a) = 0$

$\Rightarrow g(b) = g(a)$

Then g satisfied all the conditions of Rolle's theorem.

Thus there exists $d \in (a,b)$ such that $g'(d)=0$

But by condition (iii) $g'(x) \neq 0$ for all $x \in (a,b)$

This is a contradiction, and hence $g(b)-g(a) \neq 0$

$\therefore$ ① implies that $K = -\left[\dfrac{f(b)-f(a)}{g(b)-g(a)}\right]$ —— ②

f is continuous on $[a,b]$, g is continuous on $[a,b]$

$\Rightarrow \phi = f+Kg$ is continuous on $[a,b]$

f is derivable on (a,b), g is derivable on (a,b)

$\Rightarrow \phi = f+Kg$ is derivable on (a,b)

Also $\phi(a) = \phi(b)$

Thus ϕ satisfies all the conditions of Rolle's theorem

$\therefore$ There exists $c \in (a,b)$ such that $\phi'(c) = 0$

Since ϕ is derivable on (a,b) and $c \in (a,b)$,

we have $\phi'(c) = f'(c) + K g'(c)$

$$\phi'(c) = 0 \Rightarrow f'(c) + K g'(c) = 0$$

$$\Rightarrow K g'(c) = -f'(c)$$

$$\Rightarrow K = -\frac{f'(c)}{g'(c)}$$

$$\Rightarrow -\left[\frac{f(b)-f(a)}{g(b)-g(a)}\right] = -\frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Hence, there exists $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Problem:- Find 'c' of cauchys mean value theorem for $f(x) = 1/x^2$ and $g(x) = 1/x$ on $[a,b]$; $a, b > 0$

Solution: We know that $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$ are continuous and derivable on $[a,b]$

Also $f'(x) = -\frac{2}{x^2}$ and $g'(x) = -\frac{1}{x^2} \neq 0 \;\forall\, x \in [a,b]$

By cauchys mean value theorem, there exist $c \in [a,b]$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}} = \frac{\frac{-2}{c^3}}{\frac{-1}{c^2}}$$

$$\Rightarrow \frac{\left(\frac{a^2-b^2}{a^2b^2}\right)}{\left(\frac{a-b}{ab}\right)} = \left(\frac{2}{c^3}\right)\left(\frac{c^2}{1}\right)$$

$$\left(\frac{(a+b)(a-b)}{a^2-b^2}\right)\left(\frac{ab}{a-b}\right) = \left(\frac{2}{c^3}\right)\left(\frac{c^2}{1}\right)$$

$$\Rightarrow \frac{a+b}{ab} = \frac{2}{c}$$

$$\Rightarrow c = \frac{2ab}{a+b} \in (a,b)$$

problem: Find c of cauchy's mean-value theorem for $f(x) = \sqrt{x}$ and $g(x) = 1/\sqrt{x}$ in $[a, b]$ where $0<a<b$

Solution: $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ in $[a, b]$ where $0<a<b$

Since $x^n, n \in \mathbb{R}$ is continuous on $\mathbb{R}^-$.

f, g are continuous on $[a, b] \subset \mathbb{R}^+$

$f'(x) = \frac{1}{2\sqrt{x}}$, $g'(x) = \frac{-1}{2x\sqrt{x}}$ exist for all $x>0$

$\Rightarrow f, g$ are derivable on $(a, b) \subset \mathbb{R}^+$ further $g'(x) \neq 0$ for all $x \in (a, b) \subset \mathbb{R}$

cauchy mean value theorem is applicable for the functions f, g on $[a, b]$

$\therefore$ there exists $c \in (a, b)$ such that $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$

$$\Rightarrow \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}}$$

$$\Rightarrow \frac{\sqrt{b} - \sqrt{a}}{\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a}\sqrt{b}}} = \frac{1}{2\sqrt{c}}\left(\frac{-2c\sqrt{c}}{1}\right)$$

$$\Rightarrow \frac{-(\sqrt{a} - \sqrt{b}]}{\sqrt{a} - \sqrt{b}} = -c$$

$$\Rightarrow c = \sqrt{ab}$$

Since $a, b > 0$ and $\sqrt{ab}$ is their geometric mean,

we have $a < \sqrt{ab} < b$.

UNIT : V

RIEMANN INTEGRATION

Let $[a,b]$ be a closed interval. If $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, then the finite set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is called a partition of $[a,b]$.

* The $n+1$ points $x_0, x_1, x_2, \ldots, x_n$ are called partition points of P.
* The n sub-intervals $[x_0, x_1], \ldots, [x_{r-1}, x_r], \cdots [x_{n-1}, x_n]$ are called the segments of the partition P or components of the partition P.
* The r^{th} subinterval $[x_{r-1}, x_r]$ is denoted by I_r, its length $= x_r - x_{r-1}$ is denoted by δ_r (or) Δ_r
* The maximum of the lengths of subintervals of a partition P i.e $\max(\delta_1, \delta_2, \ldots \delta_r, \ldots \delta_n)$ is called the norm of the partition P and is denoted by $\|P\|$ or $\Delta(P)$

Upper and lower Reimann sums :-

Let $f: [a,b] \to \mathbb{R}$ be a bounded function and $P = \{a = x_0, x_1, x_2, \cdots, x_n = b\}$ be a partition of $[a,b]$.

Since f is bounded on $[a,b]$, f is also bounded on each of the subintervals. Let M, m be the supremum and infimum of f in $[a,b]$ and M_r, m_r be the supremum and infimum of f in the r^{th} subinterval $I_r = [x_{r-1}, x_r]$; $r = 1, 2, \ldots, n$

The sum $M_1\delta_1 + M_2\delta_2 + \cdots + M_r\delta_r + \cdots + M_n\delta_n = \sum_{r=1}^{n} M_r\delta_r$ is defined as the upper Riemann sum or upper Darboux sum of f corresponding to the partition P and is denoted by $U(P,f)$.

The sum $m_1\delta_1 + m_2\delta_2 + \cdots + m_r\delta_r + \cdots + m_n\delta_n = \sum_{r=1}^{n} m_r\delta_r$ is defined as the lower Riemann sum or lower Darboux sum of f corresponding to the partition P and is denoted by $L(P,f)$.

Thus we have $U(P,f) = \sum_{r=1}^{n} M_r\delta_r$ and

$$L(P,f) = \sum_{r=1}^{n} m_r\delta_r$$

<u>Problem</u>:- If $f(x) = x$ on $[0,1]$ and $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, find $U(P,f)$ and $L(P,f)$

<u>Solution</u>:- The subintervals of P are

$I_1 = [0, \frac{1}{3}]$, $I_2 = [\frac{1}{3}, \frac{2}{3}]$, $I_3 = [\frac{2}{3}, 1]$

The lengths of the subintervals are

$\delta_1 = \frac{1}{3} - 0$, $\delta_2 = \frac{2}{3} - \frac{1}{3}$, $\delta_3 = 1 - \frac{2}{3}$

$\delta_1 = \frac{1}{3}$ $\qquad \delta_2 = \frac{1}{3}$ $\qquad \delta_3 = \frac{1}{3}$.

$f(x) = x$ is increasing on $[0,1]$

In I_1 $\sup f = M_1$, $\inf = m_1$

$M_1 = \frac{1}{3}$ $m_1 = 0$

In I_2 $M_2 = \frac{2}{3}$, $m_2 = \frac{1}{3}$

In I_3 $M_3 = 1$, $m_3 = \frac{2}{3}$

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r$$

$$= M_1\delta_1 + M_2\delta_2 + M_3\delta_3$$

$$= \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} + 1\cdot\frac{1}{3}$$

$$= \frac{1}{9} + \frac{2}{9} + \frac{1}{3}$$

$$= \frac{1+2+3}{9} = \frac{6}{9} = \frac{2}{3}$$

$\therefore U(P,f) = \frac{2}{3}$

$$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$$

$$= m_1\delta_1 + m_2\delta_2 + m_3\delta_3$$

$$= 0\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3}$$

$$= 0 + \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = \frac{1}{3}$$

$\therefore L(P,f) = \frac{1}{3}$

Theorem:- If $f:[a,b] \to \mathbb{R}$ is a bounded function and $P \in \phi[a,b]$ then (i) $U(P,f) \geq L(P,f)$ and

(ii) $U(P,-f) = -L(P,f)$, $L(P,-f) = -U(P,f)$

proof: let $P = \{a = x_0, x_1, x_2, \dots x_n = b\}$

let M, m be the sup and inf on $[a,b]$ and

M_r, m_r be the sup and inf of f on $I_r = [x_{r-1}, x_r]$,

$r = 1, 2, \dots, n$

By def. $U(P,f) = \sum_{r=1}^{n} M_r \delta_r$ and

$$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$$

(i) We have $M_r \geq m_r$, $r = 1, 2, \dots, n$

$\Rightarrow M_r \delta_r \geq m_r \delta_r$ for $r = 1, 2, \dots, n$ $(\delta_r > 0)$

$$\therefore \sum_{r=1}^{n} M_r \delta_r \geq \sum_{r=1}^{n} m_r \delta_r$$

$\Rightarrow U(P,f) \geq L(P,f)$

(ii) f is bounded on $[a,b]$

$\Rightarrow -f$ is bounded on $[a,b]$

M_r, m_r are sup, inf of f on I_r

$\Rightarrow -m_r, -M_r$ are sup and inf of $-f$ on I_r

By def

$$U(P,-f) = \sum_{r=1}^{n} (-m_r) \delta_r = -\sum_{r=1}^{n} m_r \delta_r = -L(P,f)$$

$$L(P,-f) = \sum_{r=1}^{n} (-M_r) \delta_r = -\sum_{r=1}^{n} M_r \delta_r = -U(P,f)$$

Definition:- If $f:[a,b]\to\mathbb{R}$ is a bounded function and $P\in\wp[a,b]$ then $U(P,f)-L(P,f)$ is called the oscillatory sum of f corresponding to the partition P. $U(P,f)-L(P,f)$ is denoted by $\omega(P,f)$ or $O(P,f)$.

Upper and lower Riemann integrals

Let $f:[a,b]\to\mathbb{R}$ be a bounded function and $P=\{a=x_0,x_1,\dots x_n=b\}$ be a partition of $[a,b]$.

The lower Riemann integral of f on $[a,b]$ is defined as $\sup\{L(P,f)/P\in\wp[a,b]\}$ and is denoted by $\underline{\int_a^b} f$ or $\underline{\int_a^b} f(x)dx$.

The upper Riemann integral of f on $[a,b]$ is defined as $\inf\{U(P,f)/P\in\wp[a,b]\}$ and is denoted by $\overline{\int_a^b} f$ or $\overline{\int_a^b} f(x)dx$.

Note: $\underline{\int_a^b} f(x)\,dx=\sup\{L(P,f)/P\in\wp[a,b]\}$ and

$$\overline{\int_a^b} f(x)\,dx=\inf\{U(P,f)/P\in\wp[a,b]\}$$

Note:- Since $M(b-a)$ is an upper bound of the set $\{L(P,f)/P\in\wp[a,b]\}$ and $\underline{\int_a^b} f(x)\,dx=\sup\{L(P,f)/P\in\wp[a,b]\}$

we have $\underline{\int_a^b} f(x)dx\le M(b-a)$

Since $m(b-a)$ is a lower bound of the set $\{U(P,f)/P\in\wp[a,b]\}$ and $\overline{\int_a^b} f(x)\,dx=\inf\{U(P,f)\,/\,P\in\wp[a,b]\}$ we have

$$\overline{\int_a^b} f(x)\,dx\ge m(b-a).$$

Theorem: If $f:[a,b]\to\mathbb{R}$ is a bounded function then

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx.$$

Proof: Let $P_1, P_2 \in \wp[a,b]$.

Then we have $L(P_1,f) \le U(P_2,f)$

This is true for each $P_1 \in \wp[a,b]$

$\therefore$ The set of all lower sums has an upper bound $U(P_2,f)$

But supremum of the set of all lower sums is $\underline{\int_a^b} f(x)\,dx$

Since supremum $\le$ upper bound, we have

$$\underline{\int_a^b} f(x)\,dx \le U(P_2,f)$$

$$\Rightarrow U(P_2,f) \ge \underline{\int_a^b} f(x)\,dx \quad \forall\, P_2 \in \wp[a,b]$$

$\therefore \underline{\int_a^b} f(x)\,dx$ is a lower bound of the set of all upper sums

But infimum of the set of all upper sums is $\overline{\int_a^b} f(x)\,dx$.

Since infimum $\ge$ lower bound,

we have $\overline{\int_a^b} f(x)\,dx \ge \underline{\int_a^b} f(x)\,dx.$

The Riemann integral:-

Definition:- Let $f:[a,b]\to\mathbb{R}$ be a bounded function and $P=\{a=x_0,x_1,\dots x_n=b\}$ be a partition of $[a,b]$.

If $\underline{\int_a^b} f(x)dx = \sup\{L(P,f) \mid P \in \wp[a,b]\}$ is equal to

$\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) \mid P \in \wp[a,b]\}$ then we say that

f is Riemann integrable over $[a,b]$ and the common value of these integrals is called the Riemann integral of f on $[a,b]$.

The Riemann integral of f on $[a,b]$ is denoted by

$\int_a^b f$ or $\int_a^b f(x)\,dx$

The numbers a, b are called the lower and upper limits of the integral $\int_a^b f(x)\,dx$.

<u>Problem:-</u> A constant function is Riemann integrable on $[a,b]$.

<u>Solution:-</u> Let $f(x) = K \;\forall\; x \in [a,b]$

where $K \in \mathbb{R}$ be a constant function

clearly f is bounded on $[a,b]$ and

$\inf f = K$ and $\sup f = K$

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition on $[a,b]$.

Let m_r, M_r be the inf and sup of f on $I_r = [x_{r-1}, x_r]$

Since $f(x) = K \;\forall\; x \in [a,b]$, $m_r = M_r = K$

$$\therefore L(P,f) = \sum_{r=1}^{n} m_r \delta_r$$

$$= K \sum_{r=1}^{n} \delta_r$$

$L(P,f) = K(b-a)$ and

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r$$

$$= K \sum_{r=1}^{n} \delta_r = K(b-a)$$

Since $L(P,f) = K(b-a)$ is a constant,

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) / P \in \phi[a,b]\}$$

$$= K(b-a)$$

Similarly, $\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) / P \in \phi[a,b]\}$

$$= K(b-a)$$

$$\therefore \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

$$= K(b-a)$$

$\therefore$ f is Riemann integrable on $[a,b]$ and

$$\int_a^b K\,dx = K(b-a).$$

<u>Problem</u>:- The function $f(x)=1$ when $x \in Q$ and $f(x) = -1$ when $x \in \mathbb{R}-Q$ is not Riemann integrable on $[a,b]$.

<u>Solution</u>:- By the definition of the function f,

$-1 \leq f(x) \leq 1 \;\; \forall \; x \in [a,b]$.

$\therefore$ f is bounded on $[a,b]$ and $\inf f = -1$, $\sup f = 1$

Let $P = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition of $[a,b]$ and m_r, M_r be the inf and sup of f on $[x_{r-1}, x_r]$

$\therefore m_r = -1$ and $M_r = 1$ for $r = 1, 2, \cdots, n$

$$L(p,f) = \sum_{r=1}^{n} m_r \delta_r$$

$$= \sum_{r=1}^{n} (-1)\delta_r$$

$$= -(b-a) \quad \text{and}$$

$$U(p,f) = \sum_{r=1}^{n} M_r \delta_r$$

$$= \sum_{r=1}^{n} 1\,\delta_r$$

$$= (b-a)$$

Since $L(p,f) = -(b-a)$ is a constant,

$$\underline{\int_a^b} f(x)\,dx = -(b-a)$$

Since $U(p,f) = b-a$ is a constant,

$$\overline{\int_a^b} f(x)\,dx = b-a$$

$\therefore$ Lower and upper integrals exist but are not equal.

$\Rightarrow$ f is not Riemann integrable on $[a,b]$

Theorem:- Darboux's theorem

If $f : [a,b] \to \mathbb{R}$ is a bounded function, then for each $\epsilon > 0$ there exists $\delta > 0$ such that (1) $U(p,f) < \overline{\int_a^b} f(x)dx + \epsilon$ and $L(p,f) > \underline{\int_a^b} f(x)\,dx - \epsilon$ for each $p \in \wp[a,b]$ with $\|p\| < \delta$.

Proof: f is bounded on $[a,b]$

$\Rightarrow |f(x)| < K \ \forall x \in [a,b], K \in \mathbb{R}$ — ①

By definition $\bar{\int_a^b} f(x)\,dx = \inf \{ U(P,f) / P \in \phi[a,b] \}$

$\therefore$ For $\epsilon > 0$ there exists $P_1 \in \phi[a,b]$ such that

$U(P_1, f) < \bar{\int_a^b} f(x)\,dx + \frac{\epsilon}{2}$ — ②

Let P_1 have $(p-1)$ partition points excluding the end points a, b

choose $\delta > 0$ such that $2K(p-1)\delta = \frac{\epsilon}{2}$ — ③

Let P be any partition of $[a,b]$ with $\|P\| < \delta$ and $P_2 = P \cup P_1$

Then P_2 contains at most $(p-1)$ points more than P.

$\therefore$ By known theorem

$U(P,f) - U(P_2,f) \leq 2K(p-1)\delta$

$\Rightarrow U(P,f) \leq U(P_2,f) + 2K(p-1)\delta$ — ④

Since $P_1 \subset P_2$, $U(P_2,f) \leq U(P_1,f)$ — ⑤

from ②, ④ and ⑤

$$U(P,f) \leq \bar{\int_a^b} f(x)\,dx + \frac{\epsilon}{2} + 2K(p-1)\delta$$

$$< \bar{\int_a^b} f(x)\,dx + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \bar{\int_a^b} f(x)\,dx + \epsilon$$

$\therefore$ For $\epsilon > 0$ there exists $\delta > 0$ such that $U(P,f) < \bar{\int_a^b} f(x)\,dx + \epsilon$ for $P \in \phi[a,b]$ with $\|P\| < \delta$

Similarly the result ② can be proved.

A Necessary and sufficient condition for integrability:

Theorem: A bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ iff for each $\epsilon>0$ there exists a partition of P of $[a,b]$ such that $0\le U(P,f)-L(P,f)<\epsilon$

proof:- Necessary condition:-

let, f be Riemann integrable on $[a,b]$

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

let, $\epsilon>0$

By Darboux thm,

there exist $\delta>0$ such that $U(P,f) < \overline{\int_a^b} f(x)dx + \frac{\epsilon}{2}$

$$L(P,f) > \underline{\int_a^b} f(x)dx - \frac{\epsilon}{2}$$

For each $P\in\wp[a,b]$ with $\|P\|<\delta$.

$$U(P,f) < \int_a^b f(x)\,dx + \frac{\epsilon}{2}$$

$$-L(P,f) < -\int_a^b f(x)dx + \frac{\epsilon}{2}$$

$$\int_a^b f(x)\,dx < L(P,f) + \frac{\epsilon}{2}$$

$$U(P,f) < L(P,f) + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$U(P,f) - L(P,f) < \epsilon$$

Also, $U(P,f) - L(P,f) \geq 0$

$$0 \leq U(P,f) - L(P,f) < \epsilon$$

For each $\epsilon > 0$ there exist partition P of $[a,b]$ such that $0 \leq U(P,f) - L(P,f) < \epsilon$

<u>Sufficient condition</u>

Let, $\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) \mid P \in \phi[a,b]\}$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx \leq U(P,f)$$

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) \mid P \in \phi[a,b]\}$$

$$\Rightarrow \underline{\int_a^b} f(x)\,dx \geq L(P,f)$$

$$\Rightarrow -\underline{\int_a^b} f(x)\,dx \leq -L(P,f)$$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq U(P,f) - L(P,f)$$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \epsilon$$

Also $\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \geq 0$

For each $\epsilon > 0$ we have $0 \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \epsilon - \epsilon < 0$

then $-\epsilon < 0 \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \epsilon$

$$\left| \int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^{b} f(x)\,dx \right| < \epsilon$$

for each $\epsilon > 0$, however smallest it may be

$$\int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^{b} f(x)\,dx = 0.$$

$$\Rightarrow \int_a^{\bar{b}} f(x)\,dx = \int_{\underline{a}}^{b} f(x)\,dx$$

$\therefore$ f is riemann integral on $[a, b]$.

Problem:- Show that $f(x) = 3x+1$ is integrable on $[1, 2]$ and $\int_1^2 (3x+1)\,dx = 11/2$

Solution! $f(x) = 3x+1$ is bounded on $[1, 2]$

Consider the partition, $P = \{1, 1+1/n, \dots, 1+\frac{r}{n}, \dots, 1+\frac{n}{n} \dots\}$

$$I_r = \left[1 + \frac{(r-1)}{n}, 1 + \frac{r}{n}\right]$$

length of the Each subinterval is $\delta_r = \frac{1}{n}$

Since $f(x) = 3x+1$ is increasing on $[1, 2]$

M_r = supremum of f in I_r

$$M_r = 3(1+r/n)+1 = 3 + \frac{3r}{n} + 1$$

$$M_r = 4 + \frac{3r}{n}$$

m_r = infimum of f in I_r

$$m_r = 3\left(1 + \frac{(r-1)}{n}\right) + 1 = 3 + \frac{3(r-1)}{n} + 1$$

$$m_r = 4 + \frac{3(r-1)}{n}$$

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r$$

$$= M_1\delta_1 + M_2\delta_2 + \dots + M_n\delta_n$$

$$= \sum_{r=1}^{n} \left(4 + \frac{3r}{n}\right)\left(\frac{1}{n}\right)$$

$$= \sum_{r=1}^{n} \frac{4}{n} + \frac{3r}{n^2} = \sum_{r=1}^{n} \frac{4}{n} + \frac{3}{n^2} r$$

$$= \sum_{r=1}^{n} \frac{4}{n} + \sum_{r=1}^{n} \frac{3}{n^2} r$$

$$= \frac{4}{n} \sum_{r=1}^{n} 1 + \frac{3}{n^2} \sum_{r=1}^{n} r$$

$$= \frac{4}{n}(n) + \frac{3}{n^2}\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{4n}{n} + \frac{3}{n^2}\left(\frac{n \times n(1+1/n)}{2}\right)$$

$$U(P,f) = 4 + \frac{3}{2}(1+1/n)$$

$$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$$

$$= \sum_{r=1}^{n} \left(4 + \frac{3(r-1)}{n}\right)\left(\frac{1}{n}\right)$$

$$= \sum_{r=1}^{n} \left(\frac{4}{n} + \frac{3(r-1)}{n^2}\right)$$

$$= \sum_{r=1}^{n} \frac{4}{n} + \sum_{r=1}^{n} \frac{3}{n^2}(r-1)$$

$$= \frac{4}{n} \sum_{r=1}^{n} 1 + \frac{3}{n^2} \sum_{r=1}^{n} (r-1)$$

$$L(P,f) = \frac{4}{n}\times n + \frac{3}{n^2}\left(\frac{n(n-1)}{2}\right)$$
$$= 4 + \frac{3}{n^2}\left(\frac{n\times n(1-1/n)}{2}\right)$$
$$= 4 + \frac{3}{2}\left(1-\frac{1}{n}\right)$$

$$\underline{\int_1^2} f(x)\,dx = \lim_{n\to\infty} L(P,f)$$
$$= \lim_{n\to\infty}\left(4+\frac{3}{2}\left(1-\frac{1}{n}\right)\right)$$
$$\underline{\int_1^2} f(x)\,dx = 4+\frac{3}{2}(1-0) = \frac{8+3}{2} = \frac{11}{2}$$

$$\overline{\int_1^2} f(x)\,dx = \lim_{n\to\infty} U(P,f)$$
$$= \lim_{n\to\infty}\left(4+\frac{3}{2}\left(1+\frac{1}{n}\right)\right)$$
$$\overline{\int_1^2} f(x)\,dx = 4+\frac{3}{2}(1+0) = \frac{11}{2}$$
$$\therefore \overline{\int_1^2} f(x)\,dx = \underline{\int_1^2} f(x)\,dx$$

$f(x) = 3x+1$ is integrable on $[1,2]$

$$\int_1^2 f(x)\,dx = \frac{11}{2} \Rightarrow \int_1^2 (3x+1)\,dx = \frac{11}{2}.$$

Problem:- prove that $f(x) = x^2$ is integrable on $[0,a]$ and

$$\int_0^a x^2\,dx = \frac{a^3}{3}$$

Solution:- $f(x) = x^2$ is bounded on $[0,a]$

consider the partition $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \ldots, \frac{ra}{n}, \ldots, \frac{na}{n}\right\}$

I_r = rth subinterval $= \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$

length of each subinterval $= \delta_r = \frac{a}{n}$

since $f(x) = x^2$ is increasing function in $[0, a]$

M_r = sup of f in $I_r = \left(\frac{ra}{n}\right)^2 = \frac{r^2a^2}{n^2}$,

m_r = inf of f in $I_r = \left(\frac{(r-1)a}{n}\right)^2 = \frac{(r-1)^2a^2}{n^2}$

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r$$

$$= \sum_{r=1}^{n} \frac{r^2a^2}{n^2} \times \frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{r=1}^{n} r^2$$

$$= \frac{a^3}{n^3} \times \frac{n(n+1)(2n+1)}{6}$$

$$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$$

$$= \sum_{r=1}^{n} \frac{(r-1)^2a^2}{n^2} \times \frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{r=1}^{n} (r-1)^2$$

$$= \frac{a^3}{n^3} \times \frac{(n-1)n(2n-1)}{6}$$

$$\therefore \int_0^a f(x)\,dx = \lim_{n\to\infty} L(P,f)$$

$$= \lim_{n\to\infty} \frac{a^3}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= \frac{2a^3}{6}$$

$$\therefore \int_0^{\bar{a}} f(x)\,dx = \lim_{n\to\infty} U(P,f)$$

$$= \lim_{n\to\infty} \frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \frac{2a^3}{6}$$

Hence $\int_{\underline{0}}^{a} f(x)\,dx = \int_0^{\bar{a}} f(x)\,dx = \frac{a^3}{3}$

$\therefore f(x) = x^2$ is integrable on $(0,a)$ and

$$\int_0^a x^2\,dx = \frac{a^3}{3}.$$

Problem:- If $f(x) = x^2$ on $(0,1]$ and $P = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ compute $L(P,f)$ and $U(P,f)$

Solution:- $f(x) = x^2$ on $(0,1]$ and

partition $P = \{0, 1/4, 2/4, 3/4, 1\}$

$I_1 = [0, 1/4]$, $I_2 = [1/4, 2/4]$, $I_3 = [2/4, \frac{3}{4}]$, $I_4 = [\frac{3}{4}, 1]$

$\delta_1 = 1/4$, $\delta_2 = 1/4$, $\delta_3 = 1/4$, $\delta_4 = 1/4$

(i) $U(P,f) = \sum_{r=1}^{n} M_r \delta_r$

$$= \sum_{r=1}^{4} M_r \delta_r$$

$$= M_1\delta_1 + M_2\delta_2 + M_3\delta_3 + M_4\delta_4$$

$$= \frac{1}{16}\left(\frac{1}{4}\right) + \frac{4}{16}\left(\frac{1}{4}\right) + \frac{9}{16}\left(\frac{1}{4}\right) + 1\left(\frac{1}{4}\right)$$

$$= \frac{1}{64} + \frac{4}{64} + \frac{9}{64} + \frac{1}{4} = \frac{1+4+9+16}{64} = \frac{30}{64} = \frac{15}{32}$$

$$\therefore U(P,f) = \frac{15}{32}$$

(ii) $L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$= \sum_{r=1}^{4} m_r \delta_r$

$= m_1 \delta_1 + m_2 \delta_2 + m_3 \delta_3 + m_4 \delta_4$

$= 0\,(1/4) + \frac{1}{16}(1/4) + \frac{4}{16}(1/4) + \frac{9}{16}(1/4)$

$= 0 + \frac{1}{64} + \frac{4}{64} + \frac{9}{64}$

$= \frac{14}{64}$

$L(P,f) = \frac{7}{32}$

Problem:- Find upper and lower Riemann sums of $f(x) = 2x-1$ on $[0,1]$ for the partition $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

Solution:- $f(x) = 2x-1$ on $[0,1]$

partition $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

$I_1 = [0, \frac{1}{3}]$, $I_2 = [\frac{1}{3}, \frac{2}{3}]$, $I_3 = [\frac{2}{3}, 1]$

$\delta_1 = 1/3$, $\delta_2 = 1/3$, $\delta_3 = 1/3$

(i) $U(P,f) = \sum_{r=1}^{n} M_r \delta_r$

$= \sum_{r=1}^{3} M_r \delta_r$

$= M_1 \delta_1 + M_2 \delta_2 + M_3 \delta_3$

$= (-1/3)(1/3) + \frac{1}{3}(\frac{1}{3}) + 1(\frac{1}{3})$

$= -\frac{1}{9} + \frac{1}{9} + \frac{1}{3}$

$U(P,f) = \frac{1}{3}$

(ii) $L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$= \sum_{r=1}^{3} m_r \delta_r$

$= m_1\delta_1 + m_2\delta_2 + m_3\delta_3$

$= (-1)\left(\frac{1}{3}\right) + \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$

$= -\frac{1}{3} - \frac{1}{9} + \frac{1}{9}$

$L(P,f) = -\frac{1}{3}$

Another Definition of Riemann integral:-

Definition: Let $f: [a,b] \to \mathbb{R}$ be a function and $P = \{a = x_0, x_1, \dots x_n = b\}$ be a partition of $[a,b]$. Let $\{\xi_1, \xi_2, \dots \xi_n\} \subset [a,b]$ be such $x_{r-1} \le \xi_r \le x_r$ for $r = 1, 2, \dots n$. The function f is said to be Riemann integrable over $[a,b]$, if to each $\epsilon > 0$ there exists $\delta > 0$ and a number such that $\left|\sum_{r=1}^{n} f(\xi_r)\delta_r - \right.$ for $P \in \Phi[a,b]$ with $\|P\| < \delta$ and $\xi_r \in [x_{r-1}, x_r]$
the number I is the Riemann integral of f over $[a,b]$
i.e $\int_a^b f(x)\, dx$.

Theorem: If $f: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Proof:- f is continuous on $[a,b]$
$\Rightarrow f$ is bounded on $[a,b]$
Since, f is continuous on $[a,b]$

By Borels theorem, for each $\epsilon > 0$

there exists partition $P = \{a = x_0, x_1, x_2, \cdots x_n = b\}$ of $[a,b]$

such that $|f(y_r) - f(z_r)| < \frac{\epsilon}{b-a}$ for $y_r, z_r \in I_r, r = 1, 2, \cdots n$

for the partition P,

let m_r, M_r be the infimum, supremum of f in I_r

f is continuous on I_r

$\Rightarrow$ there exists $\alpha_r, \beta_r \in I_r$ such that $m_r = f(\alpha_r)$

$M_r = f(\beta_r)$

$M_r - m_r = |f(\alpha_r) - f(\beta_r)|$

$M_r - m_r < \frac{\epsilon}{b-a}$, for $r = 1, 2, \cdots n$

$$U(P,f) - L(P,f) = \sum_{r=1}^{n} (M_r - m_r)\,\delta_r$$

$$< \sum_{r=1}^{n} \frac{\epsilon}{b-a}\,\delta_r$$

$$= \frac{\epsilon}{b-a} \sum_{r=1}^{n} \delta_r$$

$$= \frac{\epsilon}{(b-a)}(b-a)$$

$\Rightarrow U(P,f) - L(P,f) < \epsilon$

Hence, for each $\epsilon > 0$ there exists partition P of $[a,b]$

such that $0 \le U(P,f) - L(P,f) < \epsilon$

Therefore, f is integrable on $[a,b]$

Theorem: If $f:[a,b]\to\mathbb{R}$ is monotonic on $[a,b]$ then f is integrable on $[a,b]$.

proof: f is Monotonic on $[a,b]$

To prove that, f is integrable on $[a,b]$

either f is increasing or decreasing in $[a,b]$

let f be increasing on $[a,b]$

then $f(a)\le f(x)\le f(b)$ for all $x\in[a,b]$

f is bounded on $[a,b]$

$\inf f = f(a)$, $\sup f = f(b)$

let $\epsilon>0$

$P=\{a=x_0,x_1,x_2,\dots,x_n=b\}$ be a partition of $[a,b]$

such that $\delta_r < \dfrac{\epsilon}{f(b)-f(a)+1}$ for $r=1,2,\dots n$

let m_r, M_r be infimum, supremum of f on I_r

then $m_r = f(x_{r-1})$

$M_r = f(x_r)$

$$U(P,f)-L(P,f) = \sum_{r=1}^{n}(M_r-m_r)\,\delta_r$$

$$< \sum_{r=1}^{n}[f(x_r)-f(x_{r-1})]\,\frac{\epsilon}{f(b)-f(a)+1}$$

$$= \frac{\epsilon}{f(b)-f(a)+1}\sum_{r=1}^{n}[f(x_r)-f(x_{r-1})]$$

$$= \frac{\epsilon}{f(b)-f(a)+1}\left[(f(x_1)-f(x_0)+\dots+f(x_n)-f(x_{n-1})\right]$$

$$U(P,f) - L(P,f) = \frac{\epsilon}{f(b)-f(a)+1} [f(x_n) - f(x_0)]$$

$$= \frac{\epsilon}{f(b)-f(a)+1} [f(b) - f(a)]$$

$U(P,f) - L(P,f) < \epsilon$

Therfore, for each $\epsilon > 0$ there exist partition P such that $0 \le U(P,f) - L(P,f) < \epsilon$

Therfore, f is integrable on $[a,b]$

Similarly, we can prove that, if f is decreasing on $[a,b]$ then f is integrable on $[a,b]$.

Theorem:- If $f \in R[a,b]$, then $|f| \in R[a,b]$

Proof:- $f \in R[a,b]$

$\Rightarrow$ for a given $\epsilon > 0$ there exists a partition $P = \{a = x_0, x_1, \cdots x_n$

such that $0 \le U(P,f) - L(P,f) < \epsilon$

f is bounded on $[a,b]$

$\Rightarrow f(x) < K$, $K \in R^+$ $\forall x \in [a,b]$

$\Rightarrow |f|$ is bounded on $[a,b]$

let m_r, M_r be the inf and sup of f on I_r and

m_r', M_r' be the inf and sup of $|f|$ on I.

For each $\alpha, \beta \in I_r$ $\big||f|(\alpha) - |f|(\beta)\big| = \big||f(\alpha)| - |f(\beta)|\big| \le |f(\alpha) - f(\beta)|$

$\therefore M_r' - m_r' \le M_r - m_r$ for $r = 1, 2, \cdots n$

$$U(P,|f|) - L(P,|f|) = \sum_{r=1}^{n} (M_r' - m_r') \le \sum_{r=1}^{n} (M_r - m_r)\delta_r$$

$$= U(P,f) - L(P,f) < \epsilon$$

$\therefore |f| \in R[a,b]$

Note:- The converse of this theorem is not true. That is if $|f|$ is integrable on $[a,b]$, f need not be integrable on $[a,b]$.

consider $f:[a,b]\to R$ defined as $f(x)=1, x\in Q, f(x)=-1, x\in R-Q$

let $P=\{a=x_0,x_1,\dots x_n=b\}$ be a partition of $[a,b]$

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f)\} = \inf\left[\sum_{r=1}^{n} 1\,\delta_r\right]$$

$$= \inf(b-a) = b-a$$

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f)\} = \sup\left[\sum_{r=1}^{n}(-1)\delta_r\right]$$

$$= \sup\{-(b-a)\}$$

$$= -(b-a)$$

$\therefore\ f\notin R[a,b]$

But $|f|(x)=|f(x)|=1\ \forall\ x\in R$

Since $|f|$ is constant function, $|f|\in R[a,b]$.

Functions Defined by integrals

Definition:- let $f\in R[a,b]$. Then for each $t\in[a,b]$, $[a,t]\subset[a,b]$ and hence $f\in R[a,t]$. Therfore $\int_a^t f(x)\,dx$ is well defined. The function $\phi(t)=\int_a^t f(x)\,dx$, $t\in[a,b]$ is called the integral function of f. ϕ is called indefinite integral of f on $[a,b]$

Definition:- If $f\in R[a,b]$ and if there exists $\phi:[a,b]\to R$ such that $\phi'(x)=f(x)\ \forall\ x\in[a,b]$, then ϕ is called a primitive or antiderivative of f.

Example: let $f: [a,b] \to R$ be defined by $f(x) = \sin x$

$f(x) = \sin x$ is continuous on $[a,b]$.

$\therefore$ primitive of $\sin x$ exists on $[a,b]$.

If $\phi: [a,b] \to R$ is defined by $\phi(x) = -\cos x$, then

we know that $\phi'(x) = (-\cos x)' = \sin x \ \forall x \in [a,b]$.

Therfore by the above def,

$-\cos x$ is a primitive of $\sin x$ on $[a,b]$

Example consider the function ϕ defined on $[0, 4]$ by $\phi(x) = \frac{x^2-4}{x-2}$ if $x \neq 2$ and $\phi(x) = 0$ if $x = 2$. clearly $\phi(x)$ is not continuous at $x=2$. The integral function f of ϕ, namely $f(x) = (x^2/2) + 2x \ \forall x \in [0,4]$ is continuous and derivable in $[0,4]$ including $x = 2$.

Example consider a function ϕ defined on $[0,1]$ by

$\phi(x) = x^2 \sin(1/x), x \neq 0$, $\phi(x) = 0, x = 0$

Then $\phi'(x) = 2x\sin(1/x) - \cos(1/x), x \neq 0$

$\phi'(x) = 0, x = 0$

We know that $\phi'(x)$ is not continuous at $x=0$

If we take $f(x) = \phi'(x)$ in $[0,1]$,

then $f(x)$ is not continuous in $[0,1]$

Eventhough $f(x)$ admits of a primitive $\phi(x)$ in $[0,1]$

it fails to be continuous in $[0,1]$.

Fundamental theorem of integral calculus

Theorem: If $f \in R[a,b]$ and ϕ is a primitive of f, then $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$.

proof:- $f \in R[a,b]$ and ϕ is primitive of f

To prove that, $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$

ϕ is primitive of f on $[a,b]$

$\phi'(x) = f(x)$ for all $x \in [a,b]$

$f \in R[a,b]$, for $P = \{a = x_0, x_1, x_2, \cdots x_n = b\}$ of $[a,b]$.

$x_{r-1} \le \xi_r \le x_r$, $r = 1, 2, 3, \cdots n$

$$\lim_{\|P\| \to 0} \sum_{r=0}^{n} f(\xi_r)\, \delta_r = \int_a^b f(x)\,dx$$

ϕ is derivable on $[a,b]$

$\Rightarrow \phi$ is continuous, derivable on $[x_{r-1}, x_r]$, $r = 1, 2, \cdots n$

by Lagrange theorem

$$\frac{\phi(x_r) - \phi(x_{r-1})}{x_r - x_{r-1}} = \phi'(\xi_r) \text{ for, } \xi_r \in (x_{r-1}, x_r)$$

$\phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1})\, \phi'(\xi_r)$ for, $\xi_r \in (x_{r-1}, x_r)$

$$\sum_{r=1}^{n} (\phi(x_r) - \phi(x_{r-1})) = \sum_{r=1}^{n} \phi'(\xi_r)\, \delta_r$$

$$\Rightarrow \sum_{r=1}^{n} (\phi(x_r) - \phi(x_{r-1})) = \sum_{r=1}^{n} f(\xi_r)\, \delta_r$$

$\phi(x_1) - \phi(x_0) + \phi(x_2) - \phi(x_1) + \cdots + \phi(x_n) - \phi(x_{n-1})$
$$= \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$\Rightarrow \phi(x_n) - \phi(x_0) = \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$\lim_{\|P\| \to 0} (\phi(x_n) - \phi(x_0)) = \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$\Rightarrow \phi(b) - \phi(a) = \int_a^b f(x)\,dx$$

i.e $\int_a^b f(x)\,dx = \phi(b) - \phi(a).$

Problem:- show that $\int_0^1 x^4\,dx = \frac{1}{5}$

Solution:- $f(x) = x^4$ is continuous on R and hence continuous on $[0,1]$

$\Rightarrow \int_0^1 x^4\,dx$ exists.

consider $\phi(x) = \frac{x^5}{5}$ defined on $[0,1]$

clearly ϕ is derivable on $[0,1]$

and $\phi'(x) = x^4 = f(x)\ \forall\ x \in [0,1]$

$\therefore \phi$ is a primitive of f on $[0,1]$

$\therefore$ By fundamental theorem,

$$\int_0^1 x^4\,dx = \phi(1) - \phi(0)$$
$$= \frac{1}{5}.$$

Problem:- show that $\int_a^b \cos x \, dx = \sin b - \sin a$

Solution:- $f(x) = \cos x$ is continuous on $\mathbb{R}$ and in particular on $[a,b]$

$\Rightarrow \int_a^b \cos x \, dx$ exists.

consider $\phi(x) = \sin x$ defined on $[a,b]$.

ϕ is derivable on $[a,b]$ and

$\phi'(x) = \cos x = f(x)$

$\Rightarrow \phi$ is a primitive of f on $[a,b]$.

$\therefore$ By the fundamental theorem,

$\int_a^b \cos x \, dx = \phi(b) - \phi(a) = \sin b - \sin a.$

Problem:- prove that $\int_a^b e^x dx = e^b - e^a$

Solution:- $f(x) = e^x$ is continuous on $\mathbb{R}$ and in particular on $[a,b]$

$\Rightarrow \int_a^b e^x dx$ exists.

$\phi'(x) = e^x = f(x)$

$\Rightarrow \phi$ is primitive of f on $[a,b]$.

$\therefore \int_a^b e^x dx = \phi(b) - \phi(a) = e^b - e^a.$

Problem:- Evaluate $\int_0^{\pi/4} (\sec^4 x - \tan^4 x)\, dx$

Solution:- let $f(x) = \sec^4 x - \tan^4 x$

$= (\sec^2 x - \tan^2 x)(\sec^2 x + \tan^2 x)$

$= \sec^2 x + \tan^2 x$

$= 2\sec^2 x - 1$;

$f(x) = 2\sec^2 x - 1$ is continuous on $[0, \pi/4]$ and hence $\int_0^{\pi/4} f(x)\, dx$ exists.

consider $\phi(x) = 2\tan x - x$ on $[0, \pi/4]$

Then $\phi(x)$ is derivable on $[0, \pi/4]$ and $\phi'(x) = f(x)$

$\therefore \phi$ is a primitive of f on $[0, \pi/4]$

By fundamental theorem

$$\int_0^{\pi/4} (\sec^4 x - \tan^4 x)\, dx = \phi(\pi/4) - \phi(0)$$
$$= (2 - (\pi/4)) - (0 - 0) = 2 - \pi/4.$$

Theorem:- (First mean value - Theorem)

If $f, g \in R[a,b]$ and g keeps the same sign on $[a,b]$, then there exists $\mu \in R$ lying between the inf and sup of f such that $\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx$.

Proof:- let g be non-negative on $[a,b]$

Then $g(x) \geq 0 \ \forall\, x \in [a,b]$

$f \in R[a,b]$

$\Rightarrow f$ is bounded on $[a,b]$

$\Rightarrow m \leq f(x) \leq M \quad \forall x \in [a,b]$

where m, M are the inf and sup of f.

Since $g(x) \geq 0 \;\forall x \in [a,b]$

$mg(x) \leq f(x)\, g(x) \leq Mg(x)$

$\therefore \int_a^b mg(x)\, dx \leq \int_a^b f(x)g(x)\, dx \leq \int_a^b Mg(x)\, dx$

$\Rightarrow m\int_a^b g(x)\, dx \leq \int_a^b f(x)\, g(x)\, dx \leq M\int_a^b g(x)\, dx.$

$\therefore$ There exists $\mu \in [m, M]$ such that

$$\int_a^b f(x)\, g(x)\; dx = \mu\int_a^b g(x)\, dx$$

If g be non-~~positive~~ on $[a,b]$, then $g(x) \leq 0 \;\forall x \in [a,b]$

Then we have $mg(x) \geq f(x)g(x) \geq Mg(x) \;\forall x \in [a,b]$

$$\therefore m\int_a^b g(x)\, dx \geq \int_a^b f(x)\, g(x)\; dx$$
$$\geq M\int_a^b g(x)\; dx$$

$\therefore$ There exists $\mu \in [m, M]$ such that

$$\int_a^b f(x)\; g(x)\; dx = \mu\int_a^b g(x)\, dx.$$

Problem: Prove that there exists $\xi \in (0, \pi/2)$ such that

$$\int_0^{\pi/2} x \sin x \, dx = \xi$$

Solution: Take $f(x) = x$, $g(x) = \sin x$

We know that f is continuous on $(0, \pi/2)$ and

g is integrable on $(0, \pi/2)$

Also $g(x) = \sin x \geq 0 \quad \forall x \in [0, \pi/2]$

$\therefore$ By first mean-value theorem,

$$\int_0^{\pi/2} x \sin x \, dx = \xi \int_0^{\pi/2} \sin x \, dx \quad \text{where } \xi \in (0, \pi/2)$$

$\therefore$ There exist $\xi \in (0, \pi/2)$ such that

$$\int_0^{\pi/2} x \sin x \, dx = \xi$$

Problem: Prove that $\frac{1}{\pi} \leq \int_0^1 \frac{\sin \pi x}{1+x^2} dx \leq \frac{2}{\pi}$

Solution: Take $f(x) = \frac{1}{1+x^2}$ and $g(x) = \sin \pi x$.

clearly f, g are continuous on $[0, 1]$ and

hence integrable on $[0, 1]$.

Also $g(x) = \sin \pi x$ is positive on $(0, 1)$

Since f is decreasing on $(0,1)$, $\inf = f(1) = 1/2$ and

$\sup f = f(0) = 1$

$\therefore$ By first mean-value theorem there exists

$\mu \in [1/2, 1]$ such that

$$\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx = \mu \int_0^1 \sin \pi x\,dx$$

$$= f(\xi)\int_0^1 \sin \pi x\,dx \quad \text{where } \xi \in (0,1)$$

But by fundamental theorem, $\int_0^1 \sin \pi x\,dx = \frac{2}{\pi}$

$$\therefore \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx = f(\xi)\cdot\frac{2}{\pi} \quad \text{where } 0<\xi<1$$

But $0 \le \xi \le 1 \Rightarrow \frac{1}{2} < f(\xi) \le 1$

$$\Rightarrow \frac{1}{2}\cdot\frac{2}{\pi} \le \frac{2}{\pi} f(\xi) \le 1\cdot\frac{2}{\pi}$$

$$\therefore \frac{1}{\pi} \le \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \le \frac{2}{\pi}$$

Problem:- Prove that $\frac{\pi^3}{24} \le \int_0^{\pi} \frac{x^2}{5+3\cos x}\,dx \le \frac{\pi^3}{6}$

Solution:- Take $g(x) = x^2$, $f(x) = \frac{1}{5+3\cos x}$

Since $g(x), f(x)$ are continuous on $[0,\pi]$, g, f are integrable on $[0,\pi]$

Also g keeps the same sign on $[0, \pi]$

for all $x \in [0, \pi]$, $-1 \leq \cos x \leq 1$

$$\Rightarrow 2 \leq 5 + 3\cos x \leq 8$$

$$\Rightarrow \frac{1}{8} \leq \frac{1}{5+3\cos x} \leq \frac{1}{2}$$

$\therefore m = \frac{1}{8}$ and $M = \frac{1}{2}$ for $f(x) = \frac{1}{5+3\cos x}$ on $[0, \pi]$

By first mean-value theorem,

$$m \int_0^\pi g(x)\,dx \leq \int_0^\pi f(x)\,g(x)\,dx \leq M \int_0^\pi g(x)\,dx$$

$$\Rightarrow \frac{1}{8} \int_0^\pi x^2\,dx \leq \int_0^\pi \frac{x^2}{5+3\cos x}\,dx \leq \frac{1}{2}\int_0^\pi x^2\,dx$$

$$\Rightarrow \frac{1}{8} \times \frac{\pi^3}{3} \leq \int_0^\pi \frac{x^2}{5+3\cos x}\,dx \leq \frac{1}{2} \cdot \frac{\pi^3}{3}.$$

Problem:- using first mean-value theorem of integral, prove that $x > \log(1+x) > \frac{x}{1+x}$; $x \geq 0$

Solution:- Let $f(x) = \frac{1}{1+x}$ and $g(x) = 1$ in $[0, t]$

clearly f, g are bounded and integrable on $[0, t]$ and g keeps the same sign in $[0, t]$.

$$0 < x < t \Rightarrow 1 < 1+x < 1+t \Rightarrow 1 > \frac{1}{1+x} > \frac{1}{1+t}$$

By first mean value theorem,

$$\int_0^t f(x)\,g(x)\,dx = u\int_0^1 g(x)\,dx$$

where u is a number between the bounds of f.

$$\therefore \int_0^1 \frac{1}{1+x}\,dx = u\int_0^t 1\,dx$$

$$= ut$$

$$\Rightarrow \log(1+t) = ut \quad \text{where } \frac{1}{1+t} < u < 1$$

For $t \geq 0$; $\frac{1}{1+t} < u < 1$

$$\Rightarrow \frac{1}{1+t} < \frac{\log(1+t)}{t} < 1$$

$$\Rightarrow \frac{t}{1+t} < \log(1+t) < t$$