D.N.R.College (Autonomous) Bhimavaram

Department of Mathematics

Paper: 5 - Linear Algebra

II B.Sc

Semester - IV

Vector Spaces

- **Definition**
- **General properties of vector spaces**
- **Vector Subspaces**
- **Linear combination of vectors**
- **Linear span**
- **Linear sum of two subspaces**
- **Linear dependence and linear independence of vectors**

VECTOR SPACES

Internal Composition: Let A be any set. If $a*b \in A$ $\forall a,b \in A$ and $a*b$ is unique then $*$ is said to be an internal composition in the set A

External composition: Let V and F be any two sets. If $a \circ \alpha \in V$, $\forall a \in F$ and $\forall \alpha \in V$ and $a \circ \alpha$ is unique, then $\circ$ is said to be an external composition in V over F.

Vector Space: Let (F,+,.) be a field. The elements of F will be called scalars. Let V be a non-empty set whose elements will be called vectors. Then V is a vector space over the field F, if it satisfies the following properties.

i) $\alpha+\beta \in V$ for all $\alpha,\beta \in V$

ii) $\alpha+\beta=\beta+\alpha$ for all $\alpha,\beta \in V$

iii) $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$ for all $\alpha,\beta,\gamma \in V$

iv) $\exists$ an element $\bar{o} \in V$ such that $\alpha+\bar{o}=\alpha$ for all $\alpha \in V$

v) To every vector $\alpha \in V$ there exists a vector $-\alpha \in V$ such that $\alpha+(-\alpha)=\bar{o}$

vi) $a\alpha \in V$ for all $a \in F, \alpha \in V$

vii) $a(\alpha+\beta)=a\alpha+a\beta$ for all $a \in F$, $\alpha,\beta \in V$

viii) $(a+b)\alpha=a\alpha+b\alpha$ for all $a,b \in F$ and $\alpha \in V$

ix) $(ab)\alpha=a(b\alpha)$ for all $a,b \in F$ and $\alpha \in V$

x) $1\alpha=\alpha$ for all $\alpha \in V$

Example: Show that a field K can be regarded as a vector space over any subfield F of K

Solution: K is the set of vectors.

Since K is a field, (K,+) is an abelian group.

The elements of the subfield F are scalars.

Since K is a field, $a\alpha \in K$, $\forall a \in F$, $\forall \alpha \in K$ and $a, \alpha \in K$

If 1 is the unity element of K, 1 is the unity element of F

(i) $a(\alpha+\beta)=a\alpha+a\beta$, $\forall$ $a \in F$ and $\forall$ $\alpha \in K$, since K is field

(ii) $(a+b)\alpha=a\alpha+b\alpha$ $\forall$ $a,b \in F$ and $\forall$ $\alpha \in K$, since K is field

(iii) $(ab)\alpha=a(b\alpha)$ $\forall$ $a,b \in F$ and $\forall$ $\alpha \in K$, since K is field

(iv) $1\alpha = \alpha, \forall \alpha \in K$

Hence K(F) is a vector space

Theorem : Let (F, +, ·) be a field. Let $V_n(F) = \{(a_1, a_2, \ldots, a_n): a_1, a_2, \ldots, a_n \in F\}$. Then $V_n(F)$ is a vector space with respect to internal composition defined by $\alpha + \beta = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ and external composition by $a\alpha = (aa_1, aa_2, \ldots, aa_n)$ where $a \in F$, $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_n) \in V_n(F)$.

Proof: Let $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_n)$, $\gamma = (c_1, c_2, \ldots, c_n) \in V_n(F)$, $a, b \in F$

$V_n(F)$ is closed under Vector addition:

Let $\alpha, \beta \in V$. Then $\alpha + \beta = \{(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \in V_n(F)\}$

+ is associative:

Let $\alpha, \beta, \gamma \in V$. Then $(\alpha + \beta) + \gamma = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) + (c_1, c_2, \ldots, c_n)$

$= (\overline{a_1 + b_1} + c_1, \overline{a_2 + b_2} + c_2, \ldots, \overline{a_n + b_n} + c_n)$

$= (a_1 + \overline{b_1 + c_1}, a_2 + \overline{b_2 + c_2}, \ldots, a_n + \overline{b_n + c_n})$ by associative law in F.

$= (a_1, a_2, \ldots, a_n) + (b_1 + c_1, b_2 + c_2, \ldots, b_n + c_n) = \alpha + (\beta + \gamma)$

$\bar{0} = (0, 0, \ldots, 0)$ is Zero element:

Clearly $\bar{0} = (0, 0, \ldots, 0) \in V_n(F)$.

Let $\alpha \in V_n(F)$.

Then $\alpha + \bar{0} = (a_1 + 0, a_2 + 0, \ldots, a_n + 0) = (a_1, a_2, \ldots, a_n) = \alpha$

$-\alpha = (-a_1, -a_2, \ldots, -a_n)$ is the additive inverse of α:

Let $\alpha \in V_n(F)$. Clearly $-\alpha = (-a_1, -a_2, \ldots, -a_n) \in V_n(F)$ and

$\alpha + (-\alpha) = (a_1 + -a_1, a_2 + -a_2, \ldots, a_n + -a_n) = (0, 0, \ldots, 0) = \bar{0}$

+ is commutative:

Let $\alpha, \beta, \gamma \in V_n(F)$.

Then $\alpha + \beta = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) = (b_1 + a_1, b_2 + a_2, \ldots, b_n + a_n) = \beta + \alpha$

$\therefore$ $(V_n(F), +)$ is an abelian group.

$V_n(F)$ is closed under scalar multiplication: Let $a \in F$, $\alpha \in V$. Then

$a\alpha = a(a_1, a_2, \ldots, a_n) = (a\,a_1, a\,a_2, \ldots, a\,a_n)$ is an unique element of $V_n(F)$ $\forall a \in F, \alpha \in V$

To show $a(\alpha + \beta) = a\alpha + a\beta$: Let $a \in F$, $\alpha, \beta \in V$. Then

$a(\alpha + \beta) = a(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) = (a\,a_1 + a\,b_1, a\,a_2 + a\,b_2, \ldots, a\,a_n + a\,b_n)$

$= (a\,a_1, a\,a_2, \ldots, a\,a_n) + (a\,b_1, a\,b_2, \ldots, a\,b_n) = a(a_1, a_2, \ldots, a_n) + a(b_1, b_2, \ldots, b_n)$

$= a\,\alpha + a\beta$.

To show $(a + b)\,\alpha = a\alpha + b\alpha$: Let a b $\in$ F, $\alpha \in V_n(F)$.

Then $(a + b)\,\alpha = (a + b)\,(a_1, a_2, \ldots, a_n) = (\overline{a+b}a_1, \overline{a+b}a_2, \ldots, \overline{a+b}a_n)$

$= (a\,a_1 + b\,a_1, a\,a_2 + b\,a_2, \ldots, a\,a_n + b\,a_n)$

$= (a\,a_1, a\,a_2, \ldots, a\,a_n) + (b\,a_1, b\,a_2, \ldots, b\,a_n) = a(a_1, a_2, \ldots, a_n) + b\,(a_1, a_2, \ldots, a_n) = a\alpha + b\alpha$

To show $a(b\alpha) = (ab)\alpha$: Let a, b $\in$ F, $\alpha \in V_n(F)$. Then

$a(b\alpha) = a(b\,a_1, b\,a_2, \ldots, b\,a_n) = (a\overline{ba_1}, a\overline{ba_2}, \ldots, a\overline{ba_n}) = (\overline{ab}a_1, \overline{ab}a_2, \ldots, \overline{ab}a_n)$

$= ab(a_1, a_2, \ldots, a_n) = (ab)\alpha$

To show that $1\alpha = \alpha$: Let $\alpha \in V_n(F)$

Then $1\,\alpha = 1(a_1, a_2, \ldots, a_n) = (1\,a_1, 1\,a_2, \ldots, 1\,a_n) = (a_1, a_2, \ldots, a_n) = \alpha$

Hence $V_n(F)$ is a vector space.

Example: Prove that the set of all polynomials in an indeterminate x over a field F is a vector space

Solution: Let F[x] be the set of all polynomials over F

Let f(x)=$a_0 + a_1x + a_2x^2 + \cdots$, g(x)=$b_0 + b_1x + b_2x^2 + \cdots \in F[x]$ and $c \in F$

f(x)+g(x)=$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots \in F[x]$

Let f(x)=$a_0 + a_1x + a_2x^2 + \cdots$, g(x)=$b_0 + b_1x + b_2x^2 + \cdots$, h(x)=$c_0 + c_1x + c_2x^2 + \cdots \in F[x]$

[f(x)+g(x)]+h(x)=$[(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots] + (c_0 + c_1x + c_2x^2 + \cdots)$

$= [(a_0 + b_0) + c_0] + [(a_1 + b_1) + c_1]x + [(a_2 + b_2) + c_2]x^2 + \cdots$

$= [a_0 + (b_0 + c_0)] + [a_1 + (b_1 + c_1)]x + [a_2 + (b_2 + c_2)]x^2 + \cdots$

$= (a_0 + a_1x + a_2x^2 + \cdots) + [(b_0 + c_0) + (b_1 + c_1)x + (b_2 + c_2)x^2 + \cdots]$
$= f(x) + [g(x) + h(x)]$

Therefore + is associative in F[x]

f(x)+0(x)=$(a_0 + 0) + (a_1 + 0)x + (a_2 + 0)x^2 + \cdots = a_0 + a_1x + a_2x^2 + \cdots$=f(x)

Similarly 0(x)+f(x)=f(x)

Therefore 0(x) is the additive identity in F[x]

Let f(x)=$a_0 + a_1x + a_2x^2 + \cdots \in F[x]$

(-f)(x)=$(-a_0)+(-a_1)x+(-a_2)x^2+\cdots \in F[x]$

f(x)+(-f)(x)=$[a_0+(-a_0)]+[a_1+(-a_1)]x+[a_2+(-a_2)]x^2+\cdots$

$=0+0x+0x^2+\cdots$= 0(x)

Similarly (-f)(x)+f(x)= 0(x)

$\therefore$ (-f)(x) is the additive inverse of f(x) in F[x]

f(x) + g(x)=$(a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+\cdots$
=$(b_0+a_0)+(b_1+a_1)x+(b_2+a_2)x^2+\cdots=$ g(x) + f(x)

$\therefore$ + is commutative in F[x]

a[f(x)+g(x)]=a$[(a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+\cdots]$

= a$(a_0+b_0)+a(a_1+b_1)x+a(a_2+b_2)x^2+$...

=(a$a_0+ab_0)+(aa_1+ab_1)x+(aa_2+ab_2)x^2+$...

=[a$a_0+aa_1x+aa_2x^2+\cdots]+[ab_0+ab_1x+ab_2x^2+\cdots]$ =af(x)+ag(x)

$\therefore$ a[f(x)+g(x)]= af(x)+ag(x)

(a+b)f(x)=(a+b)($a_0+a_1x+a_2x^2+\cdots$) =(a+b)
$a_0+(a+b)a_1x+(a+b)a_2x^2+\cdots$

=(a$a_0+ba_0)+(aa_1+ba_1)x+(aa_2+ba_2)x^2+$...

=[a$a_0+aa_1x+aa_2x^2+\cdots]+[ba_0+ba_1x+ba_2x^2+\cdots]$ =af(x)+bf(x)

$\therefore$(a+b)f(x)= af(x)+bf(x)

(ab)f(x)= (ab)$a_0+(ab)a_1x+(ab)a_2x^2+\cdots$

=a(b$a_0)+a(ba_1)x+a(ba_2)x^2+\cdots$

= a[b$a_0+ba_1x+ba_2x^2+\cdots$]=a[bf(x)]

$\therefore$(ab)f(x)= a[bf(x)]

1f(x)=$1a_0+1a_1x+1a_2x^2+\cdots$=f(x)

$\therefore$F[x] is a vector space over F

Example: Show that the set V of all matrices with their elements as real numbers is a vector space over the field F of real numbers with respect to addition of matrices as addition of vectors and multiplication of matrices by a scalar as scalar multiplication.

Solution: Let V = $\left\{[a_{ij}]_{m\times n}:\ a_{ij} \in \mathbb{R}\right\}$. Let A = $[a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$, $C = [c_{ij}]_{m\times n}$ ∈ V where $a_{ij},\ b_{ij},\ c_{ij} \in \mathbb{R}$

Addition of matrices "+" is internal composition: Let A, B ∈ V.

Now A + B = $[a_{ij}]_{m\times n}$+ $[b_{ij}]_{m\times n}$ = $[a_{ij}+b_{ij}]_{m\times n}$ ∈ V since $a_{ij}+b_{ij} \in \mathbb{R}$.

"+" is associative: Let A, B, C ∈ V.

Then (A + B) + C = $[a_{ij}+b_{ij}]_{m\times n} + [c_{ij}]_{m\times n}$ = $[(a_{ij}+b_{ij})+c_{ij}]_{m\times n}$ = $[a_{ij}+(b_{ij}+c_{ij})]_{m\times n}$ = $[a_{ij}]_{m\times n} + [b_{ij}+c_{ij}]_{m\times n}$ = A + (B + C)

∴ (A + B) + C = A + (B + C).

O = $[0]_{m\times n}$ is the zero element:

Clearly Let A ∈ V. Then A + O = $[a_{ij}+0]_{m\times n} = [a_{ij}]_{m\times n} = A$.

∴ O = $[0]_{m\times n}$ is the zero element.

– A is the negative of A:

Let A ∈ V. Then – A = $[-a_{ij}]_{m\times n}$ ∈ V and A + (–A) = $[a_{ij}+(-a_{ij})]_{m\times n}$ = $[0]_{m\times n}$ = O

∴ – A is the negative of A

"+" is commutative:

Let A, B ∈ V.

Now A + B = $[a_{ij}]_{m\times n}$+ $[b_{ij}]_{m\times n}$ = $[a_{ij}+b_{ij}]_{m\times n}$ = $[b_{ij}+a_{ij}]_{m\times n}$ = B + A

Scalar multiplication is an external composition:

Let a ∈ F and A ∈ V.

a A =$[a\ a_{ij}]_{m\times n}$ ∈ V since $a\ a_{ij} \in F$

(i) a(A + B) = aA + aB ∀ a ∈ F, A, B ∈ V:

Let a ∈ F and A, B ∈ V.

a(A + B) = $a[a_{ij}+b_{ij}]_{m\times n}$ = $[a\ a_{ij}+a\ b_{ij}]_{m\times n}$ = $[a\ a_{ij}]_{m\times n}$+ $[a\ b_{ij}]_{m\times n}$ =aA + aB

(ii) <u>(a + b) A = aA + bA ∀ a, b ∈ F, A ∈ V:</u>

(a + b) A = (a + b) $\left[a_{ij}\right]_{m\times n}$ = $\left[(a+b)\ a_{ij}\right]_{m\times n}$ = $\left[a\ a_{ij}\right]_{m\times n} + \left[b\ a_{ij}\right]_{m\times n}$ =aA + bA

(iii) <u>a(bA) = (ab)A ∀ a, b ∈ F, A ∈ V :</u>

Let a, b ∈ F and A ∈ V. Then a(bA) = a$\left[b\ a_{ij}\right]_{m\times n}$ = $\left[a(ba_{ij})\right]_{m\times n}$ = $\left[(ab)a_{ij}\right]_{m\times n}$=(ab)A

(iv) <u>1A = A ∀ A ∈ V</u>:

Let A ∈ V. Then 1A = $\left[1\ a_{ij}\right]_{m\times n}$= $\left[a_{ij}\right]_{m\times n} = A$

∴ V is a vector space over F

Properties of vector spaces:

Let V(F) be a vector space and ō be the zero vector of V. Then

i) aō=ō for all a∈F

ii) oα=ō for allα∈V

iii) a(-α)= -(aα)

iv) (-a)α= -(aα)

v) a(α-β)=aα-aβ

vi) aα=ō implies a=o or α=ō

Proof: (i) aō=a(ō+ō)=aō+aō

Therefore ō+aō=aō+aō⇒ō=aō

(ii)0α=(0+0)α=0α+0α ō+0α=0α+0α⇒ō=0α

(iii)a[α+(-α)]=aα+a(-α)⇒aō=aα+a(-α)

⇒ō=aα+a(-α) ⇒a(-α)=-(a α)

(iv)[a+(-a)]α=aα+(-a)α⇒0α=aα+(-a)α⇒ō=aα+(-a)α ⇒(-a)α=-(aα)

(v)a(α-β)=a[α+(-β)]=aα+a(-β) =aα+[-(aβ)]=aα-aβ

Vector subspace: Let V be a vector space over the field F and let $W \subseteq V$. Then W is called a subspace of V if W itself is a vector space over F with respect to the operations of vector addition and scalar multiplication in V

Theorem: The necessary and sufficient condition for a nonempty subset W of a vector space V(F) to be a subspace of V is that W is closed under vector addition and scalar multiplication in V

Proof:

Necessary condition

If W itself is a vector space over F with respect to vector addition and scalar multiplication in V, then W must closed with respect to these compositions.

Sufficient Condition

Suppose W is a nonempty subset of V and W is closed under vector addition and scalar multiplication in V.

Let $x \in W$. If $1 \in F$, then $-1 \in F$.

$-1 \in F, x \in W \Rightarrow (-1)x \in W \Rightarrow -(1x) \in W \Rightarrow -x \in W$

$x \in W, -x \in W \Rightarrow x+(-x) \in W \Rightarrow \bar{o} \in W$

$x,y,z \in W,\ W \subseteq V \Rightarrow x,y,z \in V \Rightarrow (x+y)+z = x+(y+z)$

$x,y \in W,\ W \subseteq V \Rightarrow x,y \in V \Rightarrow x+y = y+x$

$a,\ 0 \in F,\ x,x \in W \Rightarrow ax+0x \in W \Rightarrow ax \in W$

$a,b \in F, x,y \in V \Rightarrow a(x+y) = ax+ay,$

$(a+b)x = ax+bx, (ab)x = a(bx), 1x = x$

$\therefore$ W is a vector space and hence W is a subspace of V(F)

Theorem: The necessary and sufficient condition for a nonempty subset W of a vector space V(F) to be a subspace of V is $a,b \in F$ and $x,y \in W$ $ax+by \in W$

Proof: Necessary condition

Suppose W is a subspace of a vector space V(F)

Let $a,b \in F,\ x,y \in W$

$a \in F, x \in W \Rightarrow ax \in W$

$b \in F, y \in W \Rightarrow by \in W$ ($\because$ W is closed under scalar multiplication)

$ax \in W, by \in W \Rightarrow ax+by \in W$ ($\because$ W is closed under vector addition)

∴ a,b∈F, x,y∈W⇒ax+by∈W

Sufficient condition: Suppose that W is a nonempty subset of V such that a,b∈F, x,y∈W⇒ax+by∈W

1∈F,x,y∈W⇒1x+1y∈W⇒x+y∈W

0∈F, x∈W⇒0x+0x∈W⇒**0**∈W

-1,0∈F, x∈W⇒(-1)x+0x∈W⇒ **-x**∈W

Let x,y,z∈W

x,y,z∈W, W⊆V⇒x,y,z∈V⇒(x+y)+z=x+(y+z)

x,y∈W, W⊆V⇒x,y∈V⇒x+y=y+x

a, 0∈F, x∈W⇒ax+0x∈W⇒ax∈W

a,b∈F,x,y∈V⇒a(x+y)=ax+ay, (a+b)x=ax+bx, (ab)x=a(bx), 1x=x
∴W is a vector space and hence W is a subspace of V(F)

Example: The set W of ordered triads (x,y,0) where x,y∈F is a subspace of V_3(F)

Solution: Let α,β∈W where α=$(x_1,y_1,0)$, β=$(x_2,y_2,0)$ for some x_1,y_1,x_2,y_2∈F.
Let a,b∈F, aα+bβ=a$(x_1,y_1,0)$+b$(x_2,y_2,0)$ = $(ax_1,ay_1,0)+(bx_2,by_2,0)$ = $(ax_1+bx_2,ay_1+by_2,0)$∈F

Hence W is a subspace of V_3(F)

Example: Prove that the set of all solutions (a,b,c) of the equation a+b+2c=0 is a subspace of the vector space $V_3(R)$

Solution: Let W={(a,b,c): a,b,c∈R and a+b+2c=0}

Let $\alpha=(a_1,b_1,c_1),\ \beta=(a_2,b_2,c_2)\in W$

Then $a_1+b_1+2c_1=0$ and $a_2+b_2+2c_2=0$

If a,b∈ R, then aα+bβ=$a(a_1,b_1,c_1)$+b(a_2,b_2,c_2)

=(aa_1,ab_1,ac_1)+(ba_2,bb_2,bc_2)

=$(aa_1+ba_2,ab_1+bb_2,ac_1+bc_2)$

Now $(aa_1+ba_2)+(ab_1+bb_2)+2(ac_1+bc_2)$

=a$(a_1+b_1+2c_1)$+b$(a_2+b_2+2c_2)$=a.0+b.0=0

aα+bβ∈W

Hence W is a subspace of the vector space $V_3(R)$

Example: Let R be the field of real numbers and W={(x,y,z)/x.y,z are rational numbers}. Is W a subspace of V_3(R)?

Solution: Let α=(2,3,4)∈W, a=√7∈R

aα=√7(2,3,4)=(2√7,3√7,4√7)∉W

Hence W is not a subspace of V_3(R)

Example: Show that W={(a,2b,3c):a,b,c∈R} is a subspace of V_3(R)

Solution: Let x = $(a_1, 2b_1, 3c_1)$, $y = (a_2, 2b_2, 3c_2) \in W$ and a,b ∈R

ax+by=a($a_1, 2b_1, 3c_1$)+b($a_2, 2b_2, 3c_2$) =(a$a_1, 2ab_1, 3ac_1$)+ $(ba_2, 2bb_2, 3bc_2)$

=(a$a_1 + ba_2, 2ab_1 + 2bb_2, 3ac_1 + 3bc_2$)

=(a$a_1 + ba_2, 2(ab_1 + bb_2)$,3$(ac_1 + bc_2)$)∈ W

∴ W is a subspace of $V_3(R)$

Example: If a_1, a_2, a_3 are fixed elements of a field F, then the set W of all ordered triads (x_1, x_2, x_3) of elements of F such that $a_1x_1 + a_2 x_2 + a_3x_3$=0 is a subspace of V_3(F).

Solution: Let α= (x_1, x_2, x_3) and β= (y_1, y_2, y_3) ∈W where $x_1, x_2, x_3, y_1, y_2, y_3$ ∈ F

Then $a_1x_1 + a_2 x_2 + a_3x_3$=0, $a_1y_1 + a_2 y_2 + a_3y_3$=0

If a,b∈F, then aα+bβ= a (x_1, x_2, x_3) + b (y_1, y_2, y_3)

= (ax_1, ax_2, ax_3) + (by_1, by_2, by_3) = (ax_1 + by_1, ax_2 + by_2, ax_3 +by_3)

Now a_1(ax_1 + by_1) + a_2(ax_2 + by_2) + a_3(ax_3 +by_3) =a($a_1x_1 + a_2 x_2 + a_3x_3$)+b($a_1y_1 + a_2 y_2 + a_3y_3$)=a0+b0=0

∴ aα+bβ=(ax_1 + by_1, ax_2 + by_2, ax_3 +by_3) ∈W

Hence W is a subspace of $V_3(F)$

Theorem:The intersection of any two subspaces W_1 *and* W_2 of a vector space V(F) is a subspace of V(F)

Proof: Since ō∈W_1 *and* W_2, $W_1 \cap W_2 \neq \emptyset$

Let $\alpha, \beta \epsilon W_1 \cap W_2$ and a,b∈F $\alpha \epsilon W_1 \cap W_2 \Rightarrow \alpha \epsilon W_1$ *and* $\alpha \epsilon W_2$,

$$\beta \epsilon W_1 \cap W_2 \ \Rightarrow \beta \epsilon W_1 and\ \beta \epsilon\ W_2$$

Since $W_1 is\ a\ subspace,$ $\mathrm{a, b} \in \mathrm{F}$ and $\alpha, \beta \in W_1 \Rightarrow \mathrm{a}\alpha + \mathrm{b}\beta \in W_1$

Similarly $W_2 is\ a\ subspace,$ $\mathrm{a, b} \in \mathrm{F}$ and $\alpha, \beta \in W_2 \Rightarrow \mathrm{a}\alpha + \mathrm{b}\beta \in W_2$

Thus a,b∈F, $\alpha, \beta \epsilon W_1 \cap W_2 \Rightarrow \mathrm{a}\alpha + \mathrm{b}\beta \in W_1 \cap W_2$

Hence $W_1 \cap W_2$ is a subspace of V(F)

Note: The union of two subspaces of V(F) may not be a subspace of V(F)

Example: If R be the field of real numbers, then W_1={(0,0,z): z∈R} and W_2={(0,y,0): y∈R} are two subspaces of V_3(R)

(0,0,2)∈ W_1 and (0,3,0)∈ W_2

∴ (0,0,2) and (0,3,0) ∈$W_1 \cup W_2$

But (0,0,2)+(0,3,0)=(0,3,2) ∉ $W_1 \cup W_2$

Hence $W_1 \cup W_2$ *is not a subspace of* V_3(R)

Theorem: The union of two subspaces is a subspace iff one is contained in the other.

Proof: Let $W_1 and\ W_2$ be two subspaces of a vector space V(F)

Suppose $W_1 \cup W_2$ is a subspace of V

If possible suppose that $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$

$$W_1 \not\subseteq W_2 \Rightarrow \exists x \in W_1 \ni x \notin W_2$$

$$W_2 \not\subseteq W_1 \Rightarrow \exists y \in W_2 \ni y \notin W_1$$

x∈W_1 ,y∈W_2 ⇒x,y∈$W_1 \cup W_2$ ⇒x+y∈$W_1 \cup W_2$⇒x+y∈W_1 or $\mathrm{x + y} \in\ W_2$

If x+y∈W_1 then x∈W_1, x+y∈W_1⇒y=(x+y)-x∈W_1 If x+y∈W_2 then y∈W_2, x+y∈W_2⇒x=(x+y)-y∈W_2

It is a contradiction

∴$W_1 \subseteq W_2\ orW_2 \subseteq W_1$. Conversely suppose that $W_1 \subseteq W_2\ orW_2 \subseteq W_1$

If $W_1 \subseteq W_2$ then $W_1 \cup W_2$=W_2 is a subspace of V

If $W_2 \subseteq W_1$ then $W_1 \cup W_2$=W_1 is a subspace of V

∴ $W_1 \cup W_2$is a subspace of V

Smallest subspace containing any subset of V(F): Let V(F) be a vector space and S be any subset of V. If U is a subspace of V containing S and is itself contained in every subspace of V containing S, then U is called the smallest subspace of V containing S.

The smallest subspace of V containing S is also called the subspace of V generated or spanned by S and denote it by {S}

Linear combination of vectors: Let V(F) be a vector space. If $\alpha_1, \alpha_2, \dots, \alpha_n \in$V, then any vector α= $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ where $a_1, a_2, \dots, a_n \in$F is called a linear combination of the vectors $\alpha_1, \alpha_2, \dots, \alpha_n$

Linear span: Let V(F) be a vector space and S be any non-empty subset of V. Then the linear span of S is the set of all linear combinations of finite sets of elements of S and is denoted by L(S).

Example: Express the vector x=(1,-2,5) as a linear combination of the vectors x_1=(1,1,1), x_2=(1,2,3), x_3=(2,-1,1)

Solution: Let x=ax_1+bx_2+cx_3 ⇒(1,-2,5)=a(1,1,1)+b(1,2,3)+c(2,-1,1)

⇒(1,-2,5)=(a+b+2c,a+2b-c,a+3b+c)

⇒a+b+2c=1, a+2b-c=-2, a+3b+c=5

Solving these equations, we get a=-6,b=3,c=2

∴ x=-6x_1+3x_2+2x_3

THEOREM: The linear span L(S) of any subset S of a vector space V(F) is a subspace of V generated by S ie., L(S)={S}

Proof: Let α,β∈S

Then α=$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m$ and β=$b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n$

If a,b∈F then aα+bβ=a($a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m$)+b($b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n$)

=a($a_1\alpha_1$) $+ a(a_2\alpha_2) + \cdots + a(a_m\alpha_m)$+b($b_1\beta_1$) $+ b(b_2\beta_2) + \cdots + b(b_n\beta_n)$

=(aa_1)$\alpha_1 + (aa_2)\alpha_2 + \cdots + (aa_m)\alpha_m$+(b$b_1$)$\beta_1 + (bb_2)\beta_2 + \cdots + (bb_n)\beta_n$∈L(S)

Thus a,b∈F and α,β∈L(S)⇒aα+bβ∈L(S)

Hence L(S) is a subspace of V(F)

If α_r ∈S then $\alpha_r = 1\alpha_r$⇒$\alpha_r \in L(S)$⇒S⊂L(S)

∴ L(S) is a subspace of V and S is contained in L(S)

If W is any subspace of V containing S, then each element of L(S) belongs to W because W is closed under vector addition and scalar multiplication. Therefore L(S) will be contained in W. Hence L(S)={S}

Linear sum of two subspaces: Let W_1 and W_2 be the two subspaces of the vector space V(F). Then the linear sum of the subspaces $W_1 and\ W_2$ denoted by $W_1 + W_2$ is the set of all sums $\alpha_1 + \alpha_2$ such that $\alpha_1 \in W_1$, $\alpha_1 \in W_1$.

Thus $W_1 + W_2 = \{ \alpha_1 + \alpha_2 : \alpha_1 \in W_1 , \alpha_2 \in W_2 \}$

Theorem: If W_1 and W_2 are subspaces of the vector space V(F), then (i) $W_1 + W_2$ is a subspace of V(F) (ii) L($W_1 \cup W_2$)=$W_1 + W_2$

Proof: Let α,β∈ $W_1 + W_2$

Then α = $\alpha_1 + \alpha_2$ and β = $\beta_1 + \beta_2$ where $\alpha_1 , \beta_1 \in W_1\ and\ \alpha_2 , \beta_2 \in W_2$

If a,b∈F, then aα+bβ = a$(\alpha_1 + \alpha_2) + b(\beta_1 + \beta_2)$

= (a$\alpha_1 + b\beta_1) + (a\alpha_2 + b\beta_2)$ Since $W_1 is$ a subspace, a$\alpha_1 + b\beta_1 \in W_1$. Similarly a$\alpha_2 + b\beta_2 \in W_2$

∴ aα+bβ=(a$\alpha_1 + b\beta_1) + (a\alpha_2 + b\beta_2) \in W_1 + W_2$

Hence $W_1 + W_2$ is a subspace of V(F)

(ii) Since ō∈W_2, if $\alpha_1 \in W_1$ we can write $\alpha_1 = \alpha_1 +$ ō∈ $W_1 + W_2$ ⇒ $W_1 \subseteq W_1 + W_2$. Similarly $W_2 \subseteq W_1 + W_2$

∴ $W_1 \cup W_2 \subseteq W_1 + W_2$

Hence $W_1 + W_2$ is a subspace of V containing $W_1 \cup W_2$

Let α=$\alpha_1 + \beta_1 \in W_1 + W_2$. Then $\alpha_1 \in W_1, \beta_1 \in W_2 \Rightarrow \alpha_1, \beta_1 \in W_1 \cup W_2$

Also $\alpha_1 + \beta_1$=1α_1 + 1$\beta_1 \Rightarrow \alpha_1 + \beta_1$ is a linear combination of a finite number of elements $\alpha_1, \beta_1 \in W_1 \cup W_2 \Rightarrow \alpha_1 + \beta_1 \in$ L($W_1 \cup W_2$)

∴$W_1 + W_2 \subseteq L(W_1 \cup W_2)$

$L(W_1 \cup W_2)$ is the smallest subspace containing $W_1 \cup W_2$ and $W_1 + W_2$ is a subspace containing $W_1 \cup W_2 \Rightarrow L(W_1 \cup W_2) \subseteq W_1 + W_2$

Hence $W_1 + W_2 = L(W_1 \cup W_2)$

Example: If S,T are subsets of V(F), then

(i) S⊆ T⇒ L(S)⊆L(T) (ii) L(S∪T)=L(S)+L(T)

(iii) S is a subspace of V⇔ L(S)=S (iv) L(L(S))=L(S)

Solution:(i) Let α∈ L(S)

Then α=$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ where $\alpha_1, \alpha_2, \dots, \alpha_n$∈ S and $a_1, a_2, \dots, a_n$∈F

$\alpha_1, \alpha_2, \dots, \alpha_n$∈ S, S⊆ T ⇒ $\alpha_1, \alpha_2, \dots, \alpha_n$∈ T

$a_1, a_2, \dots, a_n$∈F, $\alpha_1, \alpha_2, \dots, \alpha_n$∈ T ⇒$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ ∈ L(T) ⇒α∈ L(T)

∴ L(S)⊆L(T)

(ii) Let α∈L(S∪T)

Then α=$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + \cdots + b_p\beta_p$ where $\{\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_p\}$ is a finite subset of S∪T such that $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$∈S and $\{\beta_1, \beta_2, \dots, \beta_p\}$∈T $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m$ ∈L(S) and $b_1\beta_1 + b_2\beta_2 + \cdots + b_p\beta_p$ ∈L(T)

∴α∈L(S)+L(T) and L(S∪T)⊆ L(S)+L(T)

Let γ∈L(S)+L(T)

Then γ=β+δ where β∈L(S) and δ∈L(T). Now β will be a linear combination of a finite number of elements of S and δ will be a linear combination of a finite number of elements of T

⇒β+δ will be a linear combination of a finite number of elements of S∪T

∴ β+δ∈L(S∪T) and L(S)+L(T)⊆ L(S∪T)

Hence L(S∪T)=L(S)+L(T)

(iii) Suppose S is a subspace of V

Let α∈ L(S)

Then α=$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ where $\alpha_1, \alpha_2, \dots, \alpha_n$∈ S and $a_1, a_2, \dots, a_n$∈F

Since S is a subspace of V, it is closed with respect to scalar multiplication and vector addition.

∴α∈L(S)⇒α∈S and L(S)⊆S

Also S⊆L(S), we have L(S)=S

Conversely suppose that L(S)=S

Since L(S) is a subspace of V and S=L(S), S is also a subspace of V

(iv) Let $\alpha \in L(S)$. Then $\alpha = 1\alpha \in L(L(S))$

$\therefore L(S) \subseteq L(L(S))$

Let $\alpha \in L(L(S))$. Then $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ where $\alpha_1, \alpha_2, \dots, \alpha_n \in L(S)$ and $a_1, a_2, \dots, a_n \in F$

$\alpha_1, \alpha_2, \dots, \alpha_n \in L(S)$ and L(S) is a subspace of V$\Rightarrow$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in L(S) \Rightarrow \alpha \in L(S)$

$\therefore L(L(S)) \subseteq L(S)$ and hence $L(L(S)) = L(S)$

Linear Dependence: Let V(F) be a vector space. A finite set $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$of vectors of V is said to be linearly dependent if there exist scalars $a_1, a_2, \dots, a_n \in F$ not all zero such that $a_1\alpha_1 + a_2\alpha_2 + a_n\alpha_n = \bar{o}$

Linear independence: Let V(F) be a vector space. A Finite set $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of vectors of V is said to be linearly independent if every relation of the form $a_1\alpha_1 + a_2\alpha_2 + a_n\alpha_n = \bar{o} \Rightarrow a_1=0, a_2=0, \dots, a_n=0$

Example: Show that the three vectors (1,1,-1), (2,-3,5) and (-2,1,4) of R^3 are linearly independent.

Solution: Let a,b,c be the real numbers such that

a(1,1,-1)+b(2,-3,5)+c(-2,1,4)=(0,0,0)

$\Rightarrow$(a+2b-2c, a-3b+c, -a+5b+4c)=(0,0,0)

$\Rightarrow$ a+2b-2c=0 (1)
a-3b+c=0 (2) ----------
a+5b+4c=0 (3)

Multiplying (2) by 2 and adding to (1), we get 3a-4b=0 (4)

Multiplying (1) by 2 and adding to (3), we get a+9b=0 (5)

Multiplying (5) by 3 and subtracting by (4), we get -31b=0 or b=0

Putting b=0 in (5) we get a=0

Putting a=0, b=0 in (1), we get c=0

Thus a=0,b=0,c=0 is the only solution of the above equations

∴ a(1,1,-1)+b(2,-3,5)+c(-2,1,4)=(0,0,0)⇒a=0,b=0,c=0

Hence the given vectors of R^3 are linearly independent.

Example: Show that the system of vectors (1,3,2), (1,-7,-8), (2,1,-1) of V_3(R) is linearly dependent.

Solution: Suppose a(1,3,2) + b(1,-7,-8) + c(2,1,-1) = (0,0,0)

⇒ (a+b+2c, 3a-7b+c, 2a-8b-c) = (0,0,0)

⇒ a+b+2c = 0 (1)

3a-7b+c = 0 (2)

2a-8b-c = 0 (3)

Multiplying (2) by 2, we get 6a-14b+2c = 0 (4)

Subtracting (1) from (2), 5a-15b = 0⇒ a=3b

Adding (2) and (3), 5a-15b = 0⇒ a=3b

Put b=1, then a=3

Putting these values in (1), c = -2

∴ 3(1,3,2) + 1(1,-7,-8) - 2(2,1,-1) = (0,0,0)

Hence the given vectors are linearly dependent.

Example: Show that the vectors (1,1,2,4), (2,-1,-5,2), (1,-1,-4,0) and (2,1,1,6) are linearly dependent in R^4

Solution: Let (1,1,2,4)= a(2,-1,-5,2)+b(1,-1,-4,0)+c(2,1,1,6)

Then 2a+b+2c=1 (1)

-a-b+c=1 (2)

-5a-4b+c=2 (3)

2a+0b+6c=4 (4)

Adding (1) and (2), we get a+3c=2. Putting c=0, then a=2

Putting a=2,c=0 in (1), we get b= -3

∴ (1,1,2,4)= 2(2,-1,-5,2)-3(1,-1,-4,0)+0(2,1,1,6)

⇒ 1 (1,1,2,4)- 2(2,-1,-5,2)+3(1,-1,-4,0)-0(2,1,1,6)=(0,0,0,0)

∴ The given vectors are linearly dependent in R^4

Example: Show that the set of vectors {(1,2,0),(0,3,1),(-1,0,1)} in V_3(R) is linearly independent.

Solution: Let a,b,c be the real numbers such that

a(1,2,0)+b(0,3,1)+c(-1,0,1)=(0,0,0)

(a-c,2a+3b,b+c)=(0,0,0)

⇒a-c=0,2a+3b=0,b+c=0

These equations will have a non-zero solution if the coefficient matrix is less than 3, the number of unknowns a,b,c. If the rank is 3, then a=0, b=0, c=0 will be the only solution.

The coefficient matrix A= $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

|A|=1(3-0)-2(0+1)=1≠0 and Rank A=3

Hence the zero solution a=0,b=0,c=0 is the only solution and the given system is linearly independent

Example: Find whether the vectors (-1,2,1), (3,0,-1), (-5,4,3) in V_3(R) are linearly independent or not.

Solution: Let a,b,c be scalars such that

a(-1,2,1)+b(3,0,-1)+c(-5,4,3)=(0,0,0)

⇒(-a+3b-5c,2a+0b+4c,a-b+3c)=(0,0,0)

⇒ -a+3b-5c=0,2a+0b+4c=0,a-b+3c=0

The coefficient matrix is A== $\begin{bmatrix} -1 & 3 & -5 \\ 2 & 0 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

|A|= -1(0+4)-2(9-5)+1(12-0)=0

∴ Rank<3 and the given system of equations will possess a non-zero solution.

Hence the given vectors are linearly dependent in R^4

Example: If α,β,γ are linearly independent vectors of V(R), show that α+β, β+γ, γ+α are also linearly independent

Solution: Let a,b,c∈R

a(α+β)+b(β+γ)+c(γ+α)=ō ⇒ (a+c)α+(a+b)β+(b+c)γ=ō

α,β,γ are linearly independent⇒a+0b+c=0, a+b+0c=0, 0a+b+c=0

The coefficient matrix A= $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Rank of A = 3 which is equal to the number of unknowns

⇒a=0, b=0, c=0 is the only solution of the given equations

∴α+β,β+γ,γ+α are also linearly independent

Example: Is the Vector (2,-5,3) in the subspace of R^3 spanned by the vectors (1,-3,2), (2,-4,-1),(1,-5,7)?

Solution: Let α=(2,-5,3), $\alpha_1 = (1,-3,2)$, α_2=(2,-4,-1), α_3=(1,-5,7)

Let α= $a_1\ \alpha_1 + a_2\ \alpha_2 + \ a_3\ \alpha_3$, where a_1, a_2, a_3 ∈ R

(2,-5,3)= $a_1\ (1,-3,2) + a_2\ (2,-4,-1) + \ a_3$ (1,-5,7)

⇒ (2,-5,3)= $(a_1 + 2a_2 \ + \ a_3$, -3 $a_1 - 4a_2 - 5\ a_3$,2 $a_1 - a_2 \ + 7\ a_3)$

⇒$a_1 + 2a_2 \ + \ a_3 = 2,$ (1)

-3 $a_1 - 4a_2 - 5\ a_3 = -5$ (2)

2 $a_1 - a_2 \ + 7\ a_3 = 3$ (3)

Multiplying (1) by 3 and adding to (2), we get $2a_2 - 2\ a_3$=1⇒$a_2 - a_3$=1/2 (4)

Multiplying (1) by 2 and subtracting from (3) ,we get $-5a_2 + 5\ a_3 = -1$ ⇒$a_2 - a_3$=1/5 (5)

From (4) and (5), the above equations are inconsistent

∴ α cannot be expressed as a linear combination of the vectors $\alpha_1, \alpha_2, \alpha_3$

Hence the vector (2,-5,3) is not in the subspace of R^3 spanned by the vectors (1,-3,2), (2,-4,-1),(1,-5,7)

Theorem: Every superset of a linearly dependent set of vectors is Linearly dependent.

Proof: Let S={$\alpha_1, \alpha_2, \dots, \alpha_n$} be a linearly dependent set of vectors

Then there exists scalars $a_1, a_2, \dots, a_n$ϵF, not all zero such that

$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$=ō ...(1)

Let S'={$\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n$} be a superset of S.

Then from (1) $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n + 0\,\beta_1 + 0\beta_2 + \cdots + 0\beta_n$=ō

Here all the scalars are not zero, we have $S'$is linearly dependent

Hence any superset of a linearly dependent set is linearly dependent

Theorem: Every non-empty subset of a linearly independent set of vectors is linearly independent.

Proof: Let S={$\alpha_1, \alpha_2, \dots, \alpha_m$} be a linearly dependent set of vectors

Consider the subset {$\alpha_1, \alpha_2, \dots, \alpha_k$} where 1≤k≤m.

Now $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k$=ō

⇒$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k + 0\,\alpha_1 + 0\alpha_{k+1} + \cdots + 0\alpha_m$=ō
⇒a_1=0,a_2=0,…a_k=0 (Since S is L.I)

Hence the subset {$\alpha_1, \alpha_2, \dots, \alpha_k$} is Linearly independent

Theorem: Let V(F) be a vector space and S={$\alpha_1, \alpha_2, \dots, \alpha_n$} is a finite subset of non zero vectors of V(F). Then S is linearly independent iff some vector α_kϵS, 2≤k≤n can be expressed as a linear combination of its preceding vectors

Proof: Suppose S={$\alpha_1, \alpha_2, \dots, \alpha_n$} is linearly dependent.

Then there exists $a_1, a_2, \dots, a_n$ϵF, not all zero such that

$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$=ō

Let k be the greatest suffix of a for which α_k≠ō

Then $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k + 0\alpha_{k+1} + \cdots + 0\alpha_n$=ō

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k = \bar{o}$

Suppose k=1 then $a_1\alpha_1 = \bar{o}$

But $a_1 = 0 \Rightarrow \alpha_1 = \bar{o}$ which contradicts that each element of S is a non- zero vector.

Hence k>1, ie.,$2 \leq k \leq n$

Also $a_k\alpha_k = -a_1\alpha_1 - a_2\alpha_2 - \cdots - a_{k-1}\alpha_{k-1}$

$\Rightarrow a_k^{-1}(a_k\alpha_k) = a_k^{-1}(-a_1\alpha_1 - a_2\alpha_2 - \cdots - a_{k-1}\alpha_{k-1})$

$\alpha_k = (-a_k^{-1}a_1)\alpha_1 + (-a_k^{-1}a_2)\alpha_2 + \cdots + (-a_k^{-1}a_{k-1})\alpha_{k-1}$

= Linear combination of preceding vectors

Conversely suppose that some $\alpha_p \in S$ can be expressible as a linear combination of preceding vectors

$\therefore \alpha_p = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{p-1}\alpha_{p-1}$

$\Rightarrow b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{p-1}\alpha_{p-1} + (-1)\alpha_p = \bar{o}$

$\Rightarrow \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ is Linearly dependent

Hence the superset $S = \{\alpha_1, \alpha_2, \ldots, \alpha_p, \ldots, \alpha_n\}$ is Linearly dependent

- **Basis of vector space**
- **Finite dimensional vector spaces**
- **Basis extension**
- **Coordinates**
- **Dimension of a vector space**
- **Dimension of a subspace**

- **Quotient space and dimension of quotient space**

Basis of a vector space: A subset S of a vector space V(F) is said to be a basis of V(F), if (i) S consists of linearly independent vectors (ii)L(S)=V

Example: A system S consisting of n vectors

e_1=(1,0,0,…,0),e_2=(0,1,0,…,0),…e_n=(0,0,…,0,1) is a basis of V_n over the field F.

Solution: Suppose S={$e_1, e_{2,} \ldots, e_n$}

Let $a_1, a_2, \ldots, a_n$∈F then $a_1e_1 + a_2e_2 + \cdots + a_ne_n$=ō

⇒$a_1(1,0,0, \ldots ,0) + a_2(0,1,0, \ldots ,0) + \cdots + a_n(0,0, \ldots ,0,1)$ =ō

⇒$(a_1, a_2 \ldots a_n)$ =(0,0,…,0)⇒$a_1 = 0, a_2 = 0, \ldots, a_n = 0$

⇒the given vectors are linearly independent

Let α=$(a_1, a_2 \ldots a_n)$∈V_n(F)

α=$(a_1, a_2 \dots a_n) = a_1(1,0,0,\dots,0) + a_2(0,1,0,\dots,0) + \cdots + a_n(0,0,\dots,0,1)$=$a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$=linear combination of elements of the set S⇒α∈L(S)

∴ V_n(F)⊆L(S). We have L(S)⊆ V_n(F)

∴V_n=L(S) and hence S is a basis of V_n(F)

Note1: The basis S={$e_1, e_2, \dots, e_n$}is called standard basis of V_n(F)

Note2: The standard basis of V_2 (F) is {(1,0),(0,1)}

Note3: The standard basis of V_3 (F) is {(1,0,0),(0,1,0),(0,0,1)}

Example: Show that the infinite set S={1,x,$x^2, x^3, \dots, x^n$ ...} is a basis of the vector space F[x] of all polynomials over the field F

Solution:Let S'= {$x^{m_1}, x^{m_2}, \dots\ , x^{m_n}$ } be any finite subset of S having n vectors where $m_1, m_2 \dots m_n$ are some non-negative integers.

Let $a_1, a_2, \dots, a_n$∈F be scalars such that

$a_1 x^{m_1} + a_2 x^{m_2} + \cdots + a_n\ x^{m_n}$=0(zero polynomial)

⇒$a_1 = 0, a_2 = 0, \dots, a_n = 0$

Thus every finite subset of S is linearly independent and hence S is linearly independent.

Let f(x)=$a_0 + a_1 x + a_2 x^2 + \cdots + a_t x^t$ be a polynomial of degree t

Then f(x)=$a_0 1 + a_1 x + a_2 x^2 + \cdots + a_t x^t$

Hence S is a basis of F[x]

Example: Show that the vectors (1,2,1), (2,1,0), (1,-1,2) form a basis of R^3

Solution: Since the set {(1,0,0), (0,1,0), (0,0,1)} forms a basis of R^3, dimR^3=3

Let S = {(1,2,1), (2,1,0), (1,-1,2)}

Consider a(1,2,1)+b(2,1,0)+c(1,-1,2)=(0,0,0)

⇒(a+2b+c, 2a+b-c, a+2c)=(0,0,0)

a+2b+c=0 (1)

2a+b-c=0 (2)

a+2c=0 (3)

Multiplying (2) by 2, we get 4a+2b-2c=0 (4)

Subtracting (4) from (1) we get -3a+3c=0⇒ -a+c=0 (5)

Adding (3) and (5), 3c=0⇒c=0

Put c=0 in (3) we get a=0 and put c=0,a=0 in (1), we get b=0

∴ S is linearly independent and hence it forms a basis for R^3

Example: Determine whether or not the following vectors form a basis of R^3:

(1,1,2), (1,2,5), (5,3,4)

Solution: We know that dim R^3=3

We have $a_1(1,1,2) + a_2(1,2,5) + a_3(5,3,4) = (0,0,0)$

⇒$(a_1 + a_2 + 5a_3, a_1 + 2a_2 + 3a_3, 2a_1 + 5a_2 + 4a_3)$ = (0,0,0)

$\therefore a_1 + a_2 + 5a_3$=0 (1)

$a_1 + 2a_2 + 3a_3$=0 (2)

$2a_1 + 5a_2 + 4a_3$=0 (3)

Subtracting (2) from (1), we get - $a_2 + 2a_3$=0

Multiplying (1) by 2, we get $2a_1 + 2a_2 + 10a_3$=0

Subtracting (5) from (3), we get $3\,a_2 - 6a_3$=0 $\Rightarrow a_2 - 2a_3$=0

$\Rightarrow a_2 = 2a_3$

putting $a_2 = 2a_3$ in (1), we get $a_1 = -7a_3$

put a_3=1, we get $a_2 = 2$ *and* $a_1 = -7$

$\therefore$ a_1= -7, $a_2 = 2$ *and* $a_3 = 1$ is a non-zero solution of the above equations.

Hence the given set is linearly dependent and it does not form a basis of R^3

Finite Dimensional Vector Space: The vector space V(F) is said to be finite dimensional or finitely generated if there exists a finite subset S of V such that V= L(S)

Example: The vector space V_n(F) of n-tuples is a finite dimensional vector space.

The vector space F[x] of all polynomials over a field F is not finite dimensional.

Note: A vector space which is not finitely generated is called an infinite dimensional space.

The vector space F[x] of all polynomials over a field F is infinite dimensional

Theorem: There exists a basis for each finite dimensional vector space.

Proof: Let V(F) be a finite dimensional vector space.

Let S={$\alpha_1, \alpha_2, \dots, \alpha_m$} be a finite subset of V such that L(S)=V

Suppose S does not contain ō

If S is linearly independent, then S itself is a basis of V.

If S is linearly dependent, then $\exists \alpha_i$ ∈S which can be expressed as a linear combination of the preceding vectors $\alpha_1, \alpha_2, \dots, \alpha_{i-1}$

If we omit this vector $\alpha_i \in$ S, then the set S' of m-1 vectors $\alpha_1, \alpha_{2,} \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_m$ also generates V ie., V=L(S')

If α∈V, then L(S)=V ⇒α can be written as a linear combination of $\alpha_1, \alpha_{2,} \dots, \dots, \alpha_m$.

Let α=$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1} + a_i\ \alpha_i + a_{i+1}\alpha_{i+1} + \cdots + a_m\alpha_m$

But α_i can be expressed as a linear combination of $\alpha_1, \alpha_{2,} \dots, \alpha_{i-1}$

Let $\alpha_i = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{i-1}\alpha_{i-1}$

∴α=$a_1\alpha_1 + \cdots + a_{i-1}\alpha_{i-1} + a_i\ (b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{i-1}\alpha_{i-1}) + a_{i+1}\alpha_{i+1} + \cdots + a_m\alpha_m$

Thus α is expressed as a linear combination of the vectors $\alpha_1, \alpha_{2,} \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_m$

∴α∈V⇒α can be expressed as a linear combination of the vectors in S'

Thus L(S')=V

If S' is linearly independent, then S'will be a basis of V. If $S'$is linearly dependent, then proceeding as above we shall get a new set of n-2 vectors which generates V.

Continuing this process, we shall after finite number of steps, obtain a linearly independent subset of S which generates V and hence a basis of V.

Theorem: Let V(F) be a finite dimensional vector space and S={$\alpha_1, \alpha_2, \dots, \alpha_m$} be a linearly independent subset of V. Then either S itself a basis of V or S can be extended to form a basis of V.

Proof:S={$\alpha_1, \alpha_2, \ldots, \alpha_m$} is a linearly independent subset of V

Since V(F) is finite dimensional, it has a finite basis say B

Let B={$\beta_1, \beta_2, \ldots, \beta_n$ }

Consider the set S_1={$\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_n$}

Then L(S_1)=V

Each α can be expressed as a linear combination of β's since B is a basis of V⇒$S_1$is linearly dependent.

Hence some vector in S_1can be expressed as a linear combination of its preceding vectors.

This vector cannot be any of α's, since S is linearly independent. So this vector must be some β_i

Consider S_2={$\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \ldots, \beta_{i-1}, \beta_{i+1} \ldots, \beta_n$}=$S_1$-{$\beta_i$}

Then L(S_2) $=$ L(S_1)=V

If S_2 is linearly independent, then S_2 forms a basis of V and it is the extended set.

If S_2 is linearly dependent, then continue this procedure till we get $S_k \subseteq S$ such that S_k is linearly independent.

∴L(S_k) $=$ L(S)=V

Hence $S_k$will be extended set of S forming a basis of V

Definition: Let S={$\alpha_1, \alpha_2, \ldots, \alpha_n$} be the basis of a vector space over V

Let β= $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$∈V, where $a_1, a_2, \ldots, a_n$ ∈ F then the scalars {$a_1, a_2, \ldots, a_n$ (F). } are called the coordinates.

Example: Show that the set {(I,0,0),(1,1,0),(1,1,1)} is a basis of C^3(C). Hence find the coordinates of the vector (3+4i,6i,3+7i) in C^3(C)

Solution: Let S={(I,0,0),(1,1,0),(1,1,1)}

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} R_2 - R_1, R_3 - R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ Rank A=3 and the given set of vectors is linearly independent.

Let z=(a,b,c)∈ C^3

(a,b,c)= p(1,0,0)+q(1,1,0)+r(1,1,1)=(p+q+r, q+r , r)

⇒a=p+q+r, b=q+r, c=r⇒ r=c, q=b-c, p=a-b

∴ z=(a-b)(1,0,0)+(b-c)(1,1,0)+c(1,1,0)∈L(S)

∴ S is a basis of C^3 If (a,b,c)=(3+4i,6i,3+7i), then p= a-b=3+4i-6i=3-2i, q=b-c=6i-3-7i=-3-i and r=c=3+7i ∴ 3-2i, -3-i, 3+7i are the coordinates of the given vector.

Dimension of a vector space:

The number of elements in any basis of a finite dimensional vector space V(F) is called the dimension of the vector space V(F) and is denoted by dimV

Example: Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V which has 4 elements.

Sol: α= $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, β= $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, γ=$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and δ=$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in V

aα+bβ+cγ+dδ=0⇒ a$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ + b $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ + c$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ +d$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$=$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

⇒$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ⇒a=0,b=0,c=0,d=0

∴ S={α,β,γ,δ} is linearly independent

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is any vector in V, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$= aα+bβ+cγ+dδ

∴ L(S)=V and hence S is a basis of V

dim V = 4

Theorem: If V(F) is a finite dimensional vector space, then any two bases of V have the same number of elements

Proof: Let S_m and S_n be the two bases of V(F) where S_m ={$\alpha_1, \alpha_2, \dots, \alpha_m$}, S_n={$\beta_1, \beta_2, \dots, \beta_n$}

∴S_m and S_n are linearly independent subsets of V

(i) Consider S_m as the basis of V and S_n as linearly independent

⇒L(S_m)=V and n(S_m)=m

∴S_n can be extended to be a basis of V⇒n≤m

(ii) Consider S_n as the basis of V and S_m as linearly independent

⇒L(S_n)=V and n(S_n)=n

∴S_m can be extended to be a basis of V⇒m≤n

But both S_m and S_n are bases of V.

∴n≤m and m≤n⇒ m=n

Hence any two bases of V have the same number of elements.

Ex: For the vector space V_3 , the set S_1={(1,0,0),(0,1,0),(0,0,1)} and S_2={(1,0,0),(1,1,0),(1,1,1)} are clearly bases and contain the same number of elements

Theorem: Each set of (n+1) or more vectors of a finite dimensional vector space V(F) of dimension n is linearly dependent

Proof: Let V(F) be a finite dimensional vector space of dimension n. Let S be a linearly independent subset of V containing n+1 or more vectors. Then S will form a part of a basis of V. Thus we shall get a basis of V containing more than n vectors. But every basis of V will contain exactly n vectors. Hence our assumption is wrong. ∴ If S contains n+1 or more vectors, then S must be linearly dependent.

Theorem: Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, \ldots, \beta_{m.}$ Then any linearly independent set of vectors in V is finite and contains no more than m vectors.

Proof: Let S ={ $\beta_1, \beta_2, \ldots, \beta_m$ } Since L(S)=V, V has a finite basis and dim V ≤ m. ∴ Every subset S' of V which contains more than m vectors is linearly dependent.

Hence the theorem is proved.

Theorem: If a set S of n vectors of a finite dimensional vector space V(F) of dimension n generates V(F), then S is a basis of V

Proof: Let V(F) be a finite dimensional vector space of dimension n. Let S={$\alpha_1, \alpha_2, \ldots, \alpha_n$} be a subset of V such that L(S)=V. If S is linearly independent, then S will form a basis of V.
If S is not linearly independent, then there will exist a proper subset of S which will form a basis of V. Thus we shall get a basis of V containing less than n elements. But every basis of V must contain exactly n elements. ∴ S cannot be linearly dependent and hence S must be a basis of V

Theorem: If V(F) is a finite dimensional vector space of dimension n, then any set of linearly independent vectors in V forms a basis of V.

Proof: Let {$\alpha_1, \alpha_2, \ldots, \alpha_n$} be a linearly independent subset of a finite dimensional vector space V(F) of dimension n. If S is not a basis of V, then it can be extended to form a basis of V. Thus we shall get a basis of V containing more than n vectors. But every basis of V must contain exactly n vectors. ∴Our assumption is wrong and S must be a basis of V

Dimension of a subspace:

Theorem: Let V(F) be a finite dimensional vector space of dimension n and W be the subspace of V. Then W is a finite dimensional vector space with dim W≤n.

Proof: dim V = n⇒ each (n+1) or more vectors of V form a linearly dependent set.

W is a subspace of V(F)⇒each set of (n+1) vectors in W is a subset of V and hence linearly dependent.

Thus any linearly independent set of vectors in W can contain at the most n vectors.

Let S = $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be the largest linearly independent subset of W, where m≤n

Now we shall prove that S is the basis of W.

For any β∈W, consider $S_1 = \{\alpha_1, \alpha_2, \dots, \alpha_m, \beta\}$

Since S is the largest set of linearly independent vectors, S_1 is linearly dependent.

$\therefore \exists a_1, a_2, \dots, a_m, b \in$ F not all zero such that
$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m + b\beta = 0$

Let b=0, then $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m = 0 \Rightarrow a_1 = 0, a_2 = 0, \dots, a_m = 0 \Rightarrow S_1$ is linearly independent which is a contradiction.

$\therefore b \neq 0$. Then $\exists\ b^{-1} \in F \ni bb^{-1} = 1$

$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m + b\beta = 0 \Rightarrow b\beta = -a_1\alpha_1 - a_2\alpha_2 - \cdots - a_m\alpha_m$

$\Rightarrow \beta = (-b^{-1}a_1)\alpha_1 + (-b^{-1}a_2)\alpha_2 + \cdots + (-b^{-1}a_m)\alpha_m$

⇒β= a linear combination of elements of S⇒β∈L(S)

Also S is linearly independent and hence S is the basis of W

∴ W is a finite dimensional vector space with dim W≤n.

Theorem: *If W_1, W_2 are two subspaces of a finite dimensional vector* space V(F) then dim($W_1 + W_2$)=dimW_1+dimW_2-dim($W_1 \cap W_2$)

Proof: let dim($W_1 \cap W_2$)=k and S=$\{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k\}$ be a basis of ($W_1 \cap W_2$)

Then S⊆ W_1 and S⊆ W_2

Since S is linearly independent and S⊆ W_1,

S can be extended to form a basis of W_1.

Let $\{\gamma_1, \gamma_2, \ldots, \gamma_k, \alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a basis of W_1

Then dimW_1=k+m

Similarly let $\{\gamma_1, \gamma_2, \ldots, \gamma_k, \beta_1, \beta_2, \ldots, \beta_t\}$ be a basis of W_2

Then dimW_2=k+t

dimW_1+dimW_2-dim($W_1 \cap W_2$)=k+m+t

Let S_1= $\{\gamma_1, \gamma_2, \ldots, \gamma_k, \alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_t\}$ be a basis of $W_1 + W_2$

Let $c_1\gamma_1 + c_2\gamma_2 + \cdots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t$ =ō

⇒$b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t$= -($c_1\gamma_1 + c_2\gamma_2 + \cdots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m$) $\in W_1 \cap W_2$

$b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t$=$d_1\gamma_1 + d_2\gamma_2 + \cdots + d_k\gamma_k$

⇒$b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t - d_1\gamma_1 - d_2\gamma_2 - \cdots - d_k\gamma_k = \bar{o}$

But $\beta_1, \beta_2, \ldots, \beta_t, \gamma_1, \gamma_2, \ldots, \gamma_k$ are linearly independent vectors.

Therefore $b_1 = 0, b_2 = 0, \ldots, b_t = 0$

$c_1\gamma_1 + c_2\gamma_2 + \cdots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m$=ō⇒$c_1 = 0, c_2 = 0, \ldots, c_k = 0, a_1 = 0, a_2 = 0, \ldots, a_m = 0$

Since $\gamma_1, \gamma_2, \ldots, \gamma_k, \alpha_1, \alpha_2, \ldots, \alpha_m$ are linearly independent, $c_1 = 0, c_2 = 0, \ldots, c_k = 0, a_1 = 0, a_2 = 0, \ldots, a_m = 0$,$b_1 = 0, b_2 = 0, \ldots, b_t = 0$

Therefore S_1 is linearly independent.

Now we show that L(S_1)=$W_1 + W_2$

Since $W_1 + W_2$ is a subspace of V and each element of $S_1 \in W_1 + W_2$, L(S_1) ⊆ $W_1 + W_2$

Let α∈$W_1 + W_2$. Then α=some element of W_1+some element of W_2

= a linear combination of elements of basis of W_1+ a linear combination of elements of basis of W_2

=a linear combination of elements of S_1

∴ α∈ L(S_1) and $W_1 + W_2$⊆ L(S_1)

∴ L(S_1) = $W_1 + W_2$

∴ S_1 is a basis of $W_1 + W_2$ and dim($W_1 + W_2$)= k+m+t

Hence the theorem.

Example: Let $W_1 \, and \, W_2$ be two subspaces of R^4 given by $W_1 = \{(a,b,c,d): b - 2c + d = 0\}$, W_2={(a,b,c,d):a=d, b=2c}. Find the basis and dimension of (i) W_1 (ii) W_2(iii) $W_1 \cap W_2 \, and$ hence find dim($W_1 + W_2$)

Solution: Given $W_1 = \{(a,b,c,d): b - 2c + d = 0\}$

Let (a,b,c,d)∈W_1 then (a,b,c,d)=(a,2c-d,c,d)=a(1,0,0,0)+c(0,2,1,0)+d(0,-1,0,1)

∴ (a,b,c,d)=linear combination of linearly independent set

{(1,0,0,0),(0,2,1,0),(0,-1,0,1)} which form a basis of W_1.

∴ dimW_1=3

(ii) Given W_2={(a,b,c,d):a=d, b=2c}

Let (a,b,c,d)∈W_2 then (a,b,c,d)=(d,2c,c,d)=d(1,0,0,1)+c(0,2,1,0)

∴ (a,b,c,d)=linear combination of linearly independent set

{(1,0,0,1),(0,2,1,0)} which form a basis of W_2.

∴ dimW_2=2

(iii) $W_1 \cap W_2$ ={(a,b,c,d):b-2c+d=0, a=d, b=2c}

Now b-2c+d=0, a=d, b=2c ⇒ b=2c, a=0, d=0

∴ (a,b,c,d)=(0,2c,c,0)=c(0,2,1,0)

∴ Basis of $W_1 \cap W_2$ = (0,2,1,0)⇒ dim($W_1 \cap W_2$)= 1

dim($W_1 + W_2$)=dim$W_1 + dimW_2$- dim($W_1 \cap W_2$)=3+2-1=4

Let W be any subspace of a vector space V(F). Let α∈V. Then the set W+α = {γ+α: γ∈W} is called a right coset of W in V generated by α.

Similarly the set α+W = {α+γ: γ∈W} is called a left coset of W in V generated by α.

Let V/W denote the set of all cosets of W in V i.e., V/W = {W+α: α∈V}

Quotient space: If W is any subspace of a vector space V(F), then the set V/W of all cosets W+α where α∈V, is a vector space over F for addition and scalar multiplication compositions defined as follows:

(W+α)+(W+β) = W+(α+β), ∀α,β∈V and a(W+α) = W+aα, a∈ F, α∈V. The vector space V/W is called the Quotient space of V

Theorem: If W is a subspace of a finite dimensional vector space V(F), then dim V/W = dim V – dim W.

Proof: Let m be the dimension of the subspace W of the vector space V(F).

Let S = $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a basis of W.

Since S is a linearly independent subset of V, it can be extended to form a basis of V.

Let S' = $\{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_l\}$ be a basis of V

Then dim V = m+l

∴ dim V – dim W = (m+l) - m=l

Now we shall prove that dim V/W=l

Suppose $S_1 = \{W + \beta_1, W + \beta_2, \dots, W + \beta_l\}$

Now we prove that S_1 is a basis of V/W

Let $a_1(W + \beta_1) + a_2(W + \beta_2) + \cdots + a_l(W + \beta_l)$=W

⇒$(W + a_1 \beta_1) + (W + a_2\beta_2) + \cdots + (W + a_l\beta_l)$=W

⇒ W+$(a_1\beta_1 + a_2\beta_2 + \cdots + a_l\beta_l)$=W+$\boldsymbol{O}$

⇒$a_1\beta_1 + a_2\beta_2 + \cdots + a_l\beta_l$ ∈ W

⇒$a_1\beta_1 + a_2\beta_2 + \cdots + a_l\beta_l = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_m\alpha_m$

⇒$a_1\beta_1 + a_2\beta_2 + \cdots + a_l\beta_l - b_1\alpha_1 - b_2\alpha_2 - \cdots - b_m\alpha_m = \boldsymbol{O}$

⇒ $a_1=0, a_2 = 0, \dots\ a_l = 0$ (∵$\beta_1, \beta_2, \dots, \beta_l, \alpha_1, \alpha_2, \dots, \alpha_m$ are L I)

∴ S_1 is linearly independent.

Now we show that L(S_1) = V/W.

Let W+α ∈ V/W

α=$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m + d_1\beta_1 + d_2\beta_2 + \cdots + d_l\beta_l$

=$\gamma + d_1\beta_1 + d_2\beta_2 + \cdots + d_l\beta_l, where\ \gamma = c_1\alpha_1 + c\alpha_2 + \cdots + c_m\alpha_m$ ∈W

W+α=W+$(\gamma + d_1\beta_1 + d_2\beta_2 + \cdots + d_l\beta_l)$
=(W+γ) $+ d_1\beta_1 + d_2\beta_2 + \cdots + d_l\beta_l$ =W +$(d_1\beta_1 + d_2\beta_2 + \cdots + d_l\beta_l)$ =$(W + d_1\beta_1) + (W + d_2\beta_2) + \cdots + (W + d_l\beta_l)$ =$d_1(W + \beta_1) + d_2(W + \beta_2) + \cdots + d_l(W + \beta_l)$

Thus any element W+α of V/W can be expressed as a linear combination of S_1.

∴ V/W =L(S_1)

$\therefore S_1$ is a basis of V/W and $\dim V/W = l$

Hence the theorem.

- Linear Transformations
- Linear operators
- Properties of Linear Transformations
- Sum and product of Linear Transformations
- Algebra of Linear operators
- Range and null space of Linear Transformation

- Rank and Nullity of Linear Transformation
- Rank – Nullity theorem

LINEAR TRANSFORMATION

Definition: Let U(F) and V(F) be two vector spaces over the same field F. A linear transformation from U into V is a function T from U into V such that $T(a\alpha+b\beta)=aT(\alpha)+bT(\beta)$ for all $\alpha,\beta\in U$ and $a,b\in F$

Zero Transformation: Let U(F) and V(F) be two vector spaces. The function T from U into V defined by $T(\alpha)=\bar{o}$ for all $\alpha\in U$ is a linear transformation from U into V. It is called zero transformation

Identity operator:: Let V(F) be a vector space. The function I from V into V defined by $I(\alpha)=\alpha$ for all $\alpha\in V$ is a linear transformation from V into V. The transformation I is called identity operator on V

Negative of a linear Transformation: Let U(F) and V(F) be two vector spaces. Let T be a linear transformation from U into V. The correspondence -T defined by $(-T)(\alpha)= -[T(\alpha)]$ for all $\alpha\in U$ is a linear transformation from U into V. The linear transformation –T is called the negative of the linear transformation T.

Properties of linear transformations:

Theorem: **Let T be a linear transformation from a vector space U(F) into V(F). Then**

(i) $T(\bar{o})=\bar{o}$, where $\bar{o}$ on the left hand side is zero of U and $\bar{o}$ on the right hand side is zero vector of V

ii) $T(-\alpha)=-T(\alpha)$, for all $\alpha\in U$

(iii) $T(\alpha-\beta)=T(\alpha)-T(\beta)$, for all $\alpha,\beta\in U$

(iv)T$(\boldsymbol{a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) = a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_nT(\alpha_n)}$

Where $\boldsymbol{\alpha_1, \alpha_2, \ldots, \alpha_n}$∈U, $\boldsymbol{a_1, a_2, \ldots, a_n}$∈F

Proof: Let α∈U then T(α)∈V

T(α)+ ō=T(α)=T(α+ ō)=T(α)+T(ō)⇒ō=T(ō)

(ii) T[α+(-α)]=T(α)+T(-α)

But T[α+(-α)]=T(ō)=ō∈V

Therefore T(α)+T(-α)=ō and T(-α)=-T(α)

(iii) T(α-β)= T[α+(-β)]=T(α)+T(-β)= T(α)+[-T(β)]= T(α)-T(β)

(iv) We prove this by using mathematical induction.
We know that T($a_1\alpha_1$)=$a_1T(\alpha_1)$ Suppose
T($a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{n-1}\alpha_{n-1}$) = $a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_{n-1}T(\alpha_{n-1})$

Then T($a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$)

=T[($a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{n-1}\alpha_{n-1}$)+$a_n\alpha_n$]

=T($a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{n-1}\alpha_{n-1}$)+ $T(\, a_n\alpha_n)$

=[$a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_{n-1}T(\alpha_{n-1})$] + $a_nT(\alpha_n)$

=$a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_{n-1}T(\alpha_{n-1})$ + $a_nT(\alpha_n)$

Example: **The function T: $\boldsymbol{V_3(R)} \to \boldsymbol{V_2(R)}$ defined as T(a,b,c)= (a,b) ∀ a,b,c ∈ R is a linear transformation from $\boldsymbol{V_3(R)} \to \boldsymbol{V_2(R)}$.**

Solution: Let $\alpha = (a_1, b_1, c_1), \beta = (a_2, b_2, c_2) \in V_3(R)$

T(aα+bβ)=T$[a(a_1, b_1, c_1) + b(a_2, b_2, c_2)]$

=T$[aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2]$

=$(aa_1 + ba_2, ab_1 + bb_2)$

=$(aa_1, ab_1) + (ba_2, bb_2)$

=a$(a_1, b_1) + b(a_2, b_2)$

=aT$(a_1, b_1, c_1) + bT(a_2,\ b_2, c_2)$

=aT$(\ \alpha) + bT(\beta)$

Therefore T is a linear transformation

Example: Describe explicitly the linear transformation T:R^2→R^2 such that T(2,3)=(4,5) and T(1,0)=(0,0)

Solution: Let S = {(2,3),(1,0)}

a(2,3)+b(1,0)=ō⇒(2a+b,3a) = (0,0)⇒2a+b=0, 3a=0⇒ a=0, b=0

∴ S is linearly independent

Let (x,y)∈ R^2

(x,y) = a(2,3)+b(1,0)=(2a+b,3a)⇒ 2a+b=x, 3a=y ⇒ a=$\frac{y}{3}$, b=$\frac{3x-2y}{3}$

∴ L(S) = R^2

T(x,y)= $T\left[\frac{y}{3}(2,3) + \frac{3x-2y}{3}(1,0)\right]=\frac{y}{3}T(2,3) + \frac{3x-2y}{3}T(1,0)$ $=\frac{y}{3}(4,5) + \frac{3x-2y}{3}(0,0)= \left(\frac{4y}{3},\frac{5y}{3}\right)$

Example: Find T(x,y,z) where T: $\boldsymbol{R}^3 \rightarrow \boldsymbol{R}$ is defined by T(1,1,1) =3, T(0,1,-2) = 1, T(0,0,1)= -2

Solution: Let S= {(1,1,1),(0,1,-2),(0,0,1)}

Let a(1,1,1)+b(0,1,-2)+c(0,0,1)=ō⇒(a,a+b,a-2b+c)=(0,0,0)⇒a=0,a+b=0, a-2b+c=0⇒a=0,b=0,c=0

∴ S is linearly independent

Let (x,y,z) ∈R^3

(x,y,z)=a(1,1,1)+b(0,1,-2)+c(0,0,1)=(a,a+b,a-2b+c) ⇒a=x, a+b=y, a-2b+c=z⇒a=x, b=y-x, c=z+2y-3x

∴ L(S)= R^3

T(x,y,z)=T[x(1,1,1)+(y-x)(0,1,-2)+(z+2y-3x)(0,0,1)] =xT(1,1,1)+(y-x)T(0,1,-2)+(z+2y-3x)T(0,0,1) =x(3)+(y-x)(1)+(z+2y-3x)(-2)=8x-3y-2z

Sum of linear transformations:

Definition: Let T_1 *and* T_2 be two linear transformations from U(F) into V(F). Then their sum $T_1 + T_2$ is defined by $(T_1 + T_2)(\alpha) = T_1(\alpha) + T_2(\alpha), \forall\alpha \in U$

Theorem: **Let U(F) and V(F) be two vector spaces. Let T_1 *and* T_2 be two linear transformations from U into V. Then the mapping $T_1 + T_2$ defined by**

$(T_1 + T_2)(\alpha) = T_1(\alpha) + T_2(\alpha), \forall\alpha \in U$ is a linear transformation.

Proof: $(T_1 + T_2)(\alpha) = T_1(\alpha) + T_2(\alpha), \forall\alpha \in U$

$T_1(\alpha) \in V$ *and* $T_2(\alpha) \in V \Rightarrow T_1(\alpha) + T_2(\alpha) \in V$

Let a,b∈F and α ,β∈U

Then $(T_1 + T_2)(a\alpha + b\beta) = T_1(a\alpha + b\beta) + T_2(a\alpha + b\beta)$=a $T_1(\alpha) + b\,T_1(\beta)$+ a $T_2(\alpha) + b\,T_2(\beta)$=a$[T_1(\alpha) + T_2(\alpha)]$+b$[T_1(\beta) + T_2(\beta)]$

=a$(T_1 + T_2)(\alpha)$+b$(T_1 + T_2)(\beta)$

∴ $T_1 + T_2$ is a linear transformation from U into V

Scalar multiplication of a Linear Transformation: Let T:U(F)→V(F) be a linear transformation and a∈F. Then the function aT defined by (aT)(α)=aT(α) ∀α∈U is a linear transformation.

Example: **Let T: $V_3(R) \to V_2(R)$ and H: $V_3(R) \to V_2(R)$ be the two linear transformations defined by T(x,y,z)=(x-y,y+z) and H(x,y,z)=(2x,y-z)**

Find (i) H+T (ii) aH

Solution: (i) (H+T)(x,y,z) = H(x,y,z) + T(x,y,z) = (2x, y-z) + (x-y, y+z)= (3x-y,2y)

(ii) (aH)(x,y,z) = aH(x,y,z)= a(2x, y-z) = (2ax, ay-az)

Product of Linear Transformations:

Theorem: **Let U(F), V(F) and W(F) are three vector spaces and T:V→W and H:U→V are two linear transformations . Then the composite function TH defined by (TH)(α)=T[H(α)] = T[H(α)] ∀α ∈U is a linear transformation from U into W.**

Proof: $\alpha \in U \Rightarrow H(\alpha) \in V$

$H(\alpha) \in V \Rightarrow T[H(\alpha)] \in W \Rightarrow (TH)(\alpha) \in W$

∴ TH is a mapping from U into W

Let a,b∈F, α,β∈U.

Then $(TH)[a\alpha+b\beta] = T[H(a\alpha+b\beta)] = T[aH(\alpha)+b\,H(\beta)] = a(TH)(\alpha)+b(TH)(\beta)$

∴ TH is a linear transformation from U into W:

Example: Let T:$R^3 \rightarrow R^2$ and H:$R^3 \rightarrow R^2$ be defined by T(x,y,z) = (3x, y+z) and H(x,y,z)= (2x-z, y). Compute (i) T+H (ii) 4T-5H (iii) TH (iv) HT

Solution: (T+H)(x,y,z) = T(x,y,z) + H(x,y,z) = (3x, y+z)+(2x-z,y) =(5x-z, 2y+z)

(ii) (4T-5H)(x,y,z)=4T(x,y,z)-5H(x,y,z)=4(3x, y+z)-5(2x-z,y) =(2x+5z, -y+4z)

(iii) TH and HT are not defined because R(T) is not equal to domain of H and vice versa.

Algebra of Linear operators:

Let A,B,C be linear operators on a vector space V(F). Let 0 be the zero operator and I be the identity operator on V. Then (i) A0=0A=0 (ii) AI=IA=A (iii) A(B+C)= AB+AC (iv) (A+B)C=AC+BC (v) A(BC)=(AB)C

Range of a linear transformation: Let U(F) and V(F) be two vector spaces and T be a linear transformation from U into V. Then the range of T written as R(T) is the set of all vectors β in V such that β=T(α), for some α in U.

Range (T)={T(α)∈V:α∈U}

Theorem: **If U(F) and V(F) are two vector spaces and T is a linear transformation from U into V , then range of T is a subspace of V.**

Proof: $\bar{o} \in U \Rightarrow T(\bar{o}) = \hat{\bar{o}} \in R(T)$

∴ R(T) is a non-empty subset of V

Let $\beta_1, \beta_2 \in$ R(T). Then there exists $\alpha_1, \alpha_2 \in$ U such that T($\alpha_1) = \beta_1$, T($\alpha_2) = \beta_2$

Let a,b∈F . a $\beta_1 + b\beta_2 =$ aT(α_1) +bT(α_2) = T (a $\alpha_1 + b\alpha_2$)

Since U is a vector space , a $\alpha_1 + b\alpha_2$ ∈U

T (a $\alpha_1 + b\alpha_2$)= a $\beta_1 + b\beta_2$∈R(T)

∴ R(T) is a subspace of V

Null space of a linear transformation: Let U(F) and V(F) be two vector spaces and T be a linear transformation from U into V. Then the null space of T written as N(T) is the set of all α in U such that T(α)= $\hat{\bar{o}}$ (zero vector of V)
N(T)={α∈U: T(α)= $\hat{\bar{o}}$ }

Theorem: If U(F) and V(F) are two vector spaces and T is a linear transformation from U into V then the kernel of T or null space of T is a subspace of U.

Proof : Let N(T) = {α∈U:T(α)= $\hat{\bar{o}}$∈V}

Since T(ō)= $\hat{\bar{o}}$∈V,therefore at least $\hat{\bar{o}}$∈N(T)

Thus N(T) is a non-empty subset of U.

Let α_1, α_2∈N(T) Then T(α_1)= $\hat{\bar{o}}$ and T(α_2)= $\hat{\bar{o}}$

Let a, b∈F. Then a α_1+ b α_2 ∈U and T(a α_1+ b α_2)=a T(α_1)+b T(α_2)

=a $\hat{\bar{o}}$+b $\hat{\bar{o}}$= $\hat{\bar{o}}$+ $\hat{\bar{o}}$= $\hat{\bar{o}}$∈V

Therefore a α_1+ b α_2 ∈ N(T)

Thus a,b∈ F and α_1, α_2∈N(T)⇒ a α_1+ b α_2 ∈N(T)

Therefore N(T) is a sub space of U.

Rank and nullity of a linear transformation: Let T be a linear transformation from a vector space U(F) into V(F) with U as finite dimensional. The rank of T denoted by ρ(T) is the dimension of the range of T ie., ρ(T) =dim R(T)

The nullity of T denoted by ν(T) is the dimension of the null space of T ie., ν(T) =dimN(T)

Theorem: **Let U and V be vector spaces over the field F and T be a linear transformation from U into V. Suppose U is finite dimensional then**

ρ(T) + ν(T)=dimU

Proof: Let N be the null space of T. Then N is a subspace of U. Since U is finite dimensional , N is finite dimensional. Let dim N=k and let $\{\alpha_1, \alpha_2 \ldots, \alpha_k\}$ be a basis of N

Since $\{\alpha_1, \alpha_2 \ldots, \alpha_k\}$ is a linearly independent subset of U, we can extend it to form a basis of U. Let dim U=n and $\{\alpha_1, \alpha_2 \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_n\}$be a basis of U

$T(\alpha_1), T(\alpha_2) \ldots, T(\alpha_k), T(\alpha_{k+1}), \ldots, T(\alpha_n) \in R(T)$

To Prove That $\{T(\alpha_{k+1}), T(\alpha_{k+2}), \ldots, T(\alpha_n)\}$ is a basis of R(T)

(i) First we shall prove that $T(\alpha_{k+1}), T(\alpha_{k+2}), \ldots, T(\alpha_n)$ span R(T)

Let $\beta \in R(T)$. Then $\exists \alpha \in U$ such that $T(\alpha)=\beta$. $\alpha \in U \Rightarrow \exists$ $a_1, a_2, \ldots, a_n \in F$ such that $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$

$\Rightarrow T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$

$\Rightarrow \beta = a_1T(\alpha_1) + \cdots + a_kT(\alpha_k) + a_{k+1}T(\alpha_{k+1}) + \cdots + a_nT(\alpha_n)$

$\Rightarrow \beta = a_{k+1}T(\alpha_{k+1}) + a_{k+2}T(\alpha_{k+2}) + \cdots + a_nT(\alpha_n)$ ∴ $T(\alpha_{k+1}), T(\alpha_{k+2}), \ldots, T(\alpha_n)$ span R(T)

(ii) Now we prove that $T(\alpha_{k+1}), T(\alpha_{k+2}), \ldots, T(\alpha_n)$ are linearly independent

Let $c_{k+1}, c_{k+2}, \ldots, c_n \in F$ such that $c_{k+1}T(\alpha_{k+1}) + \cdots + c_nT(\alpha_n) = \bar{o}$

$\Rightarrow T(c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n) = \bar{o}$

$\Rightarrow c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n \in N(T)$
$\Rightarrow c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n = b_1\alpha_1 + \cdots + b_k\alpha_k$

$\Rightarrow b_1\alpha_1 + \cdots + b_k\alpha_k - c_{k+1}\alpha_{k+1} - \cdots - c_n\alpha_n = \bar{o}$

$\Rightarrow b_1 = \cdots = b_k = c_{k+1} = \cdots = c_n = 0$

$\Rightarrow T(\alpha_{k+1}), \ldots, T(\alpha_n)$ are linearly independent and form a basis of R(T)

Rank T=dim R(T)=n-k

Hence rank (T)+nullity(T)=(n-k)+k=n=dim U

Example: **Show that the mapping T:$V_2(R) \rightarrow V_3(R)$ defined as**

T(a,b)= (a+b,a-b,b) is a linear transformation from $V_2(R) \to V_3(R)$. Find the range,rank, null space and nullity of T

Solution: Let $\alpha = (a_1, b_1), \beta = (a_2, b_2) \in V_2(R)$

T(α)=$T(a_1, b_1) = T(a_1 + b_1, a_1 - b_1, b_1)$ and

T(β)=$T(a_2, b_2) = T(a_2 + b_2, a_2 - b_2, b_2)$

Let a,b∈R. Then aα+bβ∈ $V_2(R)$

T(aα+bβ)= T[a$(a_1, b_1) + b(a_2, b_2)$]

=$T(aa_1 + ba_2, ab_1 + bb_2)$

=(a$a_1 + ba_2 + ab_1 + bb_2, aa_1 + ba_2 - ab_1 - bb_2, ab_1 + bb_2)$

=(a$(a_1 + b_1) + b(a_2 + b_2), a(a_1 - b_1) + b(a_2 - b_2), ab_1 + bb_2)$

=a$(a_1 + b_1, a_1 - b_1, b_1) + b(a_2 + b_2, a_2 - b_2, b_2)$

=aT(α)+bT(β)

Hence T is a linear transformation from $V_2(R) \to V_3(R)$

Now {(1,0),(0,1)} is a basis for $V_2(R)$

We have T(1,0)=(1+0,1-0,0)=(1,1,0) and T(0,1)=(0+1,0-1,1)=(1,-1,1)

The vectors T (1,0), T(0,1) Span the range of T.

Thus the range of T is the sub space of $V_3(R)$ spanned by the vectors (1,1,0) , (1,-1,1).

Now the vectors (1,1,0) , (1,-1,1)∈ $V_3(R)$ are L.I because if x,y ∈ R ,then

x(1,1,0)+y(1,-1,1) = (0,0,0)

⇒(x+y,x-y,y)=(0,0,0) ⇒ x+y=0, x-y=0, y=0 ⇒ x=0,y=0

Therefore the vectors (1,1,0) , (1,-1,1) form a basis for range of T

Hence rank T = dim of range of T = 2

Nullity of T = dim of $V_2(R)$ - rank T = 2 -2 = 0

Therefore null space of T must be the zero sub space of $V_2(R)$.

Otherwise , (a,b) ∈ null space of T

⇒ T(a,b)=(0,0,0)

$\Rightarrow$(a+b,a-b,b)=(0,0,0) $\Rightarrow$ a+b=0, a-b=0, b=0 $\Rightarrow$ a=0,b=0

Therefore (0,0) is the only element of $V_2(R)$ which belongs to null space of T.

Therefore null space of T is the zero sub space of $V_2(R)$.

Example: If T: $V_4(R) \rightarrow V_3(R)$ is a linear transformation defined by

T(a, b, c, d) = (a-b+c+d , a+2c-d , a+b+3c-3d) for a , b, c , d $\in$ R , verify that $\rho(T) + \vartheta(T) = \dim V_4(R)$.

Solution : Let S = {(1 , 0 , 0 , 0) (0 , 1 , 0 , 0) (0 , 0 , 1 , 0) (0 , 0 , 0 , 1)} be the basis set of $V_4(R)$.

$\therefore$ The transformation T on B will be T(1 , 0 , 0 , 0) = (1 , 1 ,1) ,

T(0 , 1 , 0 , 0) = (-1 , 0 , 1) ,

T(0 , 0 , 1 , 0) = (1 , 2 , 3) , T(0 , 0 , 0 , 1) = (1 , -1 , -3).

Let S_1 = { (1 , 1 ,1) ,(-1 , 0 , 1) , (1 , 2 , 3) , (1 , -1 , -3) }.

$\therefore S_1 \subseteq R(T)$

Now we verify whether S_1 is Linearly independent or not,. If not, we find the least Linearly independent set by forming the matrix, $S_1 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix}$

Applying $R_2 + R_1$, $R_3 - R_1$, $R_4 - R_1$

$$S_1 \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}$$

Again applying $R_4 + 2R_3$, $R_3 - R_2$

$$S_1 \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The non-zero rows of vectors $\{(1, 1, 1), (0, 1, 2)\}$

constitute the linearly independent set forming the basis of R(T)

$\Rightarrow \dim R(T) = 2$

Basis for null space of T

Let $\alpha = (a, b, c, d) \in V_4(R)$

$\alpha \in N(T) \Rightarrow T(\alpha) = \hat{o}$

$\Rightarrow T(a, b, c, d) = \hat{o}$ where $\hat{o} = (0, 0, 0) \in V_3(R)$

$\Rightarrow (a-b+c+d, a+2c-d, a+b+3c-3d) = (0, 0, 0)$

$\Rightarrow a-b+c+d = 0$; $a+2c-d = 0$; $a+b+3c-3d = 0$

We have to solve these for a , b , c ,d .

Co-efficient matrix = $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

Applying $R_2 - R_1$, $R_3 - R_1$.

$$= \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix}$$

Again applying $R_3 - 2R_2$, the echelon form is

$$= \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore the equivalent systems of equations are

$a-b+c+d = 0$, $b+c-2d = 0$

$\Rightarrow$ b = 2d - c , a = d - 2c

The number of free variables is 2 namely c , d and the values of a , b depend on these .

And hence nullity of T = dim N(T) = 2.

Choosing c = 1 , d = 0 , we get a = -2 , b = -1

Therefore (a , b , c , d) = (-2 , -1 , 1 , 0)

Choosing c = 0 , d = 1 , we get a = 1 , b = 2

Therefore (a , b , c , d) = (1 , 2 , 0 , 1)

Therefore {(-2 , -1 , 1 , 0) , (1 , 2 , 0 , 1)} constitute a basis of N(T)

$\therefore$ dim R(T) + dim N(T) = 2 +2 = 4 = dim V_4(R)

Example: Find the null space, range, rank and nullity of the transformation T: $R^2 \rightarrow R^3$ defined by T (x , y) = (x+y , x-y , y) .

Solution : Given that the transformation T: $R^2 \rightarrow R^3$ defined by

T (x , y) = (x+y , x-y , y) .

To find the null space, range, rank and nullity of the given transformation.

Null Space and Nullity of T :

Let α = (x,y) ∈ R^2 then α ∈ N(T) $\Rightarrow$ T(α) = $\hat{o}$

i.e., T(x , y) = (0 , 0 , 0)

$\Rightarrow$ (x+y , x-y , y) = (0 , 0 , 0)

$\Rightarrow$ x+y = 0 , x-y = 0 , y= 0

$\Rightarrow$ x = 0 , y= 0

$\therefore$ α = (0 , 0) = $\hat{o}$ ∈ R^2

Thus the null space of T consists of only zero vector of R^2

$\therefore$ nullity of T = dim N(T) = 0

Range and Rank of T :

Range Space of T = {β∈ R^2 : T(α) = β for α∈ R^2 }

$\therefore$ The range space consists of all vectors of the type (x+y , x-y , y)

for all(x,y) ∈ R^2 .

By rank nullity theorem , dim R(T) + dim N(T) = dim R^2

⇒ dim R(T) + 0 = 2

⇒ dim R(T) = rank of T = 2

Example : Verify Rank - nullity theorem for the linear transformation

T: $R^3 \rightarrow R^3$ defined by T (x , y , z) = (x-y , 2y+z , x+y+z) .

Solution : Given that T: $R^3 \rightarrow R^3$ defined by T (x , y , z) = (x-y , 2y+z , x+y+z) is a linear transformation .

We know that dimension of R^3 = 3 → (1)

Let α = (x,y,z) ∈ R^3

if α ∈ N(T) then T(α) = $\hat{0}$

⇒ T (x , y , z) = $\hat{0}$

⇒ (x-y , 2y+z , x+y+z) = (0 , 0 , 0)

Comparing the components , x-y = 0 ; 2y+z = 0 ; x+y+z = 0

Taking y = k we get x = k and z = -2k

$\therefore$ (x , y , z) = (k , k , -2k) = k(1 , 1 , -2)

Thus every element in N(T) is generated by the vector (1 , 1 , -2)

Thus dim N(T) = 1 → (2)

Again T (x , y , z) = (x-y , 2y+z , x+y+z)

From this $T(1 , 0 , 0) = (1 , 0 ,1)$, T(0 , 1 , 0) = (-1 , 2 , 1) , and

T(0 , 0 , 1) = (0 , 1 , 1)

Let S = {$(1 , 0 ,1)$,(-1 , 2 , 1) ,(0 , 1 , 1)} and let A = $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

R_2+R_1 , gives ~$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

$R_2/2$ gives ~$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

R_3- R_2 gives ~$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Thus the set {(1 , 0 , 0) (0 , 1 , 1)} consists the basis of R(T) i.e., the range of T

Thus, dim R(T) = 2 → (3)

Substituting (1) , (2) , (3) in rank – nullity theorem , rank + nullity = dimension

$\Rightarrow 1 + 2 = 3$

This verifies the theorem.

UNIT - 4

CHARACTERISTIC VECTOR AND CHARACTERISTIC VALUE OF A LINEAR OPERATOR:

DEFINITION**:** Let T be a linear operator on a finite dimensional vector space V(F). A non-zero vector $\alpha \in V$ is called a characteristic vector of T if there exists a scalar c such that $T(\alpha) = c\alpha$. The scalar c is called characteristic value of T corresponding to a characteristic vector α.

Each non-zero vector is called a characteristic vector of T corresponding to a characteristic value c .

CHARACTERISTIC VECTORS AND CHARACTERISTIC VALUES OF A MATRIX:

DEFINITION: Any non- zero vector X is said to be a characteristic vector of a square matrix A if there exists a scalar λ such that $AX = \lambda X$.

Here A can be a $n \times n$ matrix and X can be a $n \times 1$ matrix.

Then λ is said to be a characteristic value of the matrix A corresponding to a characteristic vector X . Also X is said to be characteristic vector corresponding to the characteristic value λ of the matrix A.

If X is a Characteristic vector of a matrix A, X cannot corresponded to more than one characteristic value of A .

Let the characteristic vector X of A correspond to two distinct characteristic values λ_1, λ_2 then $AX = \lambda_1 X$ and $AX = \lambda_2 X$.

Therefore $\lambda_1 X = \lambda_2 X \Rightarrow (\lambda_1 - \lambda_2) X = \bar{o} \Rightarrow \lambda_1 - \lambda_2 = 0 \quad \Rightarrow \lambda_1 = \lambda_2$

Similarly, if α is a characteristic value of T then α cannot corresponded to more than one characteristic value of T.

CHARACTERISTIC POLYNOMIAL, CHARACTERISTIC EQUATION OF A SQUARE MATRIX:

DEFINITION: Let $A = [a_{ij}]_{n\times n}$ and λ any indeterminate scalar . The matrix A- λI is called the characteristic matrix of A , where I is the unit matrix of order n.

Also| A- λI | =

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$$

is a polynomial in λ of degree n , is called the characteristic polynomial of A .

It is denoted by f(λ). The equation | A- λI |=0 is called the characteristic equation of A **EXAMPLE:** The characteristic polynomial of the matrix $A = \begin{vmatrix} 1 & 0 & 5 \\ 0 & 2 & 6 \\ 3 & 1 & 4 \end{vmatrix}$ is det (A- λI)

i.e., $\begin{vmatrix} 1-\lambda & 0 & 5 \\ 0 & 2-\lambda & 6 \\ 3 & 1 & 4-\lambda \end{vmatrix} = \lambda^3 + 7\,\lambda^2 + 7\,\lambda - 28$

Note: A scalar λ is a characteristic root of a square matrix A if and only if| A- λI |= 0.

Theorem: The characteristic vectors corresponding to distinct characteristic roots of a matrix are linearly independent.

Proof: Let A be a square matrix.

Let X_1 , X_2 ,….,X_m be characteristic vectors of A corresponding to respective distinct characteristic roots λ_1 , λ_2 ,….,λ_m .

Then $AX_i = \lambda_i X_i$ for i= 1,2,…m → (1)

Now we prove that the set of vectors { $X_1, X_2,\ldots, X_m$} is linearly independent. Since $x_1 \neq \bar{o}$, the set { x_1} is L.I

If { X_1 , X_2 ,….,X_m} is linearly dependent , then we can choose r (r<m) such that { X_1 , X_2 ,….,X_r} is linearly independent and { X_1 , X_2 ,….,X_r , X_{r+1} } is linearly dependent

Hence we can choose scalars k_1 , k_2 ,….,k_r , k_{r+1} not all zeros such that $k_1 X_1 + k_2 X_2 + \ldots + k_r X_r + k_{r+1} X_{r+1} = \bar{o}$ → (2)

$\Rightarrow A(k_1 X_1 + k_2 X_2 + \ldots + k_r X_r + k_{r+1} X_{r+1}) = A(\bar{o})$

$\Rightarrow k_1 (AX_1) + k_2 (AX_2) + \ldots + k_r (AX_r) + k_{r+1}(A X_{r+1}) = \bar{o}$

$\Rightarrow k_1 (\lambda_1 X_1) + k_2 (\lambda_2 X_2) + \ldots + k_r (\lambda_r X_r) + k_{r+1}(\lambda_{r+1} X_{r+1}) = \bar{o}$ → (3)

(3) - λ_{r+1} (2)$\Rightarrow k_1 (\lambda_1 - \lambda_{r+1}) X_1 + \ldots + k_r (\lambda_r - \lambda_{r+1}) X_r = \bar{o}$ → (4)

Since { X_1 , X_2 ,….,X_r} is linearly independent and λ_1 , λ_2 ,….,λ_{r+1} are distinct, we have $k_1 = 0$, …., $k_r = 0$.

Putting $k_1 = 0$, ….,$k_r = 0$ in (2),we obtain $k_{r+1} X_{r+1} = \bar{o}$ But $X_{r+1} \neq \bar{o}$. So , $k_{r+1} = 0$

Thus (2) $\Rightarrow k_1 = 0$, …., $k_r = 0$, $k_{r+1} = 0$

But this contradicts our assumption that the scalars k_1 , k_2 ,….,k_r , k_{r+1} are all not zeros .

Hence our assumption that {X_1 , X_2 ,….,X_m} is linearly dependent is wrong .

∴ {X_1 , X_2 ,….,X_m} , which corresponding to distinct characteristic roots of a matrix A are linearly independent .

Note: Distinct characteristic vectors of T corresponding to distinct characteristic values of T are linearly independent.

CHARACTERISTIC POLYNOMIAL OF A LINEAR OPERATOR:

DEFINITION: Let T be a linear operator on an n-dimensional vector space V with ordered basis β . We define the characteristic polynomial f(λ) of T to be the characteristic polynomial of $A = [T]_\beta$ i.e., f(λ) = det (T-λI) = det (A-λI)

The equation det(T-λI) =0 is called the characteristic equation of T

Example:Prove that the square matrices A and A^I have the same characteristic values.

Solution: If λ is any scalar, then $(A-\lambda I)^I = A^I - \lambda I^I = A^I - \lambda I$

$\Rightarrow |(A-\lambda I)^I| = |A^I - \lambda I|$

$\Rightarrow |A-\lambda I| = |A^I - \lambda I|$

$\Rightarrow |A-\lambda I| = 0 \Leftrightarrow |A^I - \lambda I| = 0$

i.e., λ is a characteristic value of A ⇔ λ is a characteristic value of A^I.

Example: Show that 0 is a characteristic root of a matrix if and only if the matrix is singular.

Solution: 0 is a characteristic value of A

$\Leftrightarrow \lambda = 0$ satisfies the equation $|A-\lambda I| = 0$

$\Leftrightarrow |A - 0I| = 0$

$\Leftrightarrow |A| = 0$

⇔ A is singular.

NOTE:

λ is a characteristic root of a non -singular matrix. $\lambda \neq 0$.

At least one characteristic root of every singular matrix is zero

EXAMPLE: T is a linear operator on a finite dimensional vector space V(F) . Show that T is not invertible iff 0 is a characteristic value of T **.**

Solution: Let T be not invertible ie., T is singular Therefore, there exists a non- zero vector α in V such that

$T\alpha = 0 = 0\alpha$.

Therefore 0 is a characteristic value of T Conversely suppose 0 is a characteristic value of T.

Then there exists a non- zero vector α in V such that $T\alpha = 0\alpha$.

$\Rightarrow T\alpha = 0 \Rightarrow$ T is singular $\Rightarrow$ T is not invertible .

EXAMPLE: If $\lambda_1 , \lambda_2 ,\ldots,\lambda_n$ are the characteristic values of a n-rowed square matrix A and k is a scalar, show that $k\lambda_1 , k\lambda_2 ,\ldots, k\lambda_n$ are the characteristic values of kA

Solution: Let $k \neq 0$.

Now $|kA- \lambda kI | = |k(A- \lambda I) | = k| A- \lambda I |$

$\Rightarrow |kA- (\lambda k)I | = 0 \Leftrightarrow | A- \lambda I | = 0$

i.e., $k\lambda$ is a characteristic values of kA $\Leftrightarrow$ λ is a characteristic value of A.

Thus $k\lambda_1 , k\lambda_2 ,\ldots, k\lambda_n$ are the characteristic values of kA if $\lambda_1 , \lambda_2 ,\ldots,\lambda_n$ are the characteristic values of A.

Example: Find the eigen roots and the corresponding eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

Solution: The characteristic equation of A is | A- λI | =0

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

⇒ (λ+2)(λ-5) = 0

Hence the eigen roots of A are -2 , 5.

Case 1: Let λ = -2.

Eigen vectors X corresponding to the eigen root -2 are given by (A-(-2)I)X =0

i.e., $\begin{bmatrix} 1+2 & 4 \\ 3 & 2+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - R_1$ $\begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

⇒ $3x_1 + 4x_2 = 0$

Let $x_2 = k$, then $x_1 = -4k/3$

∴ Eigen vectors corresponding to the eigen root -2 are given by

k $\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$ where k is a non-zero parameter.

Clearly, the subspace generated by $\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$

is a one dimensional characteristic space of R^2

Case 2: Let $\lambda = 5$.

Eigen vectors X corresponding to the eigen root 5 are given by

(A-5I)X =0

i.e., $\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \rightarrow R_2 + \left(\frac{3}{4}\right) R_1 \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -4 x_1 + 4 x_2 = 0$

Let $x_2 = k$ then $x_1 = k$

$\therefore$ Eigen vectors corresponding to the eigen root 5 are given by k$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where k is a non-zero parameter. Clearly the subspace generated by$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a one dimensional characteristic space of R^2

Example: Find the characteristic roots and the corresponding characteristic vectors

of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Solution: The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)[21-10\lambda+\lambda^2-16]+6[-18+6\lambda+8]+2[24-14+2\lambda]= 0$$

$\Rightarrow -\lambda^3+10\lambda^2-5\lambda+8\lambda^2-80\lambda+40-60+36\lambda+20+4\lambda = 0$

$\Rightarrow -\lambda^3+18\lambda^2-45\lambda = 0$

$\Rightarrow \lambda(\lambda-3)(\lambda-15)=0$

$\Rightarrow \lambda = 0 , 3, 15$

∴ The characteristic roots of A are 0 , 3, 15

Case 1: Let $\lambda = 0$.

Characteristic vectors corresponding to the characteristic root 0 are given by

$(A-0I)X = 0 \Rightarrow$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_3 \leftrightarrow R_1 \begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_3 \to 3\,R_3 + 4\,R_2 \quad , R_2 \to R_2 + 3\,R_1 \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 + 2\,R_2$

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 2x_1$-$4x_2$+$3x_3 = 0$, -$5x_2$+$5x_3 = 0$

Let $x_3 =$ k therefore x_2= k and 2 $x_1 =$ k i.e., $x_1 =$ k/2

∴Characteristic vectors corresponding to the characteristic root 0 are given by

$k\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$ where k is a non zero parameter

Similarly , by considering characteristic equations (A-3I)X = 0 , (A-15I)X = 0

We get characteristic vectors $k_1 \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$, $k_2\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ for non-zero parameters k_1 , k_2 respectively corresponding to the characteristic roots 3, 15.

MATRIC POLYNOMIAL

DEFINITION: An expression of the form F(x) = $A_0+A_1x+A_2x^2+\ldots+A_mx^m$, $A_m \neq 0$,where $A_0,A_1,A_2,\ldots A_m$ are matrices each of order n×n over a field F, is called a matric polynomial of degree m.

The matrices themselves are matric polynomials of zero degree.

EQUALITY OF MATRIC POLYNOMIALS

DEFINITION : Two matric polynomials are equal if and only if the coefficients of like powers of x are the same.

ADDITION AND MULTIPICATION OF POLYNOMIALS

Let $G(x) = A_0+A_1x+A_2x^2+\ldots+A_m x^m$ and $H(x) = B_0+B_1x+B_2x^2+\ldots+B_Kx^k$ We define : if m>k then $G(x)+H(x)=(A_0+B_0)+(A_1+B_1)x+\ldots+(A_k+ B_k) x^k + A_{k+1} x^{k+1}+\ldots.+ A_mx^m$ similarly we have G(x)+H(x) where m=k and m<k.

Also $G(x)H(x)=A_0B_0+(A_0B_1+A_1B_0)x +(A_0B_2+A_1B_1+A_2B_0)x^2+\ldots.+A_kB_mx^{k+m}$

The degree of the product of two matric polynomials is less than or equal to the sum of their degrees

CAYLEY – HAMILTON THEOREM (MATRICES)

THEOREM: Every square matrix satisfies its characteristic equation **.**
Proof: Let $A = [a_{ij}]_{n\times n}$

The characteristic equation of A is det $(A-\lambda I) = f(\lambda)$

$$=\begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix}$$

$= (-1)^n [\lambda_n+a_1 \lambda_{n-1}+a_2 \lambda_{n-2}+\ldots +a_n]$ where a_i's ϵ F

Let adj $(A-\lambda I) = B_0 \lambda^{n-1}+B_1\lambda^{n-2} + \ldots.+B_{n-2}\lambda^1+B_{n-1}$ where $B_0, B_1 ,\ldots. B_{n-1}$ are n-rowed square matrices

Now (A-λI) adj (A-λI) = det (A-λI) I

⇒ (A-λI)($B_0 \lambda^{n-1}$+$B_1 \lambda^{n-2}$+ ..... +$B_{n-2} \lambda$+B_{n-1}) = $(-1)^n$ [λ^n+a_1 λ^{n-1}+$a_2 \lambda^{n-2}$

Comparing coefficients of like powers of λ , we obtain

- $B_0 = (-1)^n$ I ,

A B_0- $B_1 = (-1)^n$ a_1 I ,

A B_1- $B_2 = (-1)^n$ a_2 I ,

....

.... A

$B_{n-1} = (-1)^n$ a_n I .

Premultiplying the above equations successively by A^n, A^{n-1}, ...,I and adding ,

we obtain

$0 = (-1)^n A^n + (-1)^n a_1 A^{n-1} + (-1)^n a_2 A^{n-2} + ... + (-1)^n a_n I$

⇒ $(-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + ... + a_n I] = 0$

⇒ $A^n + a_1 A^{n-1} + a_2 A^{n-2} + ... + a_n I = 0$

⇒ A satisfies its characteristic equation.

A satisfies its characteristic equation

⇒ $(-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ ... + a_n I] = 0$

⇒ $A^n + a_1 A^{n-1} + a_2 A^{n-2} + ... + a_n I = 0$

⇒ $A^{-1} [A^n + a_1 A^{n-1} + a_2 A^{n-2} + ... + a_n I] = 0$ ⇒ $a_n A^{-1}$ = - A^{n-1} - $a_1 A^{n-2}$ - a_{n-1} I

⇒ $A^{-1} = (-1/a_n) [A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I]$

CAYLEY – HAMILTON THEOREM (LINEAR OPERATOR)

THEOREM: T is a linear operator on a vector space V(F) of dimension n.

If f(x) is the characteristic polynomial of T , then f(T) = 0 (zero operator). i.e., T satisfies its characteristic equation .

Example: Verify cayley –Hamilton Theorem when T is a linear operator defined by T(a,b)=(a+2b,-2a+b).

Solution: Let $\beta = \{e_1, e_2\}$

$T(e_1)=T(1,0)=(1+2(0),-2(1)+0)=(1,-2)$ and $T(e_2)= T(0,1)=(2,1)$

Thus $A=[T]_\beta = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

Let A be an n×n matrix and let f(t) be the characteristic polynomial of A. Then f(A)=0, the n×n zero matrix.

The characteristic polynomial of T is f(T) = det (A-λI $)=\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix}=\lambda^2$-2λ+5

Now f(T) = T^2 -2T+5

Given T(a,b) = (a+2b,-2a+b)
Therefore T^2 (a,b) = T(a+2b,-2a+b)

= (a+2b+2(-2a+b),-2(a+2b)-2a+b)

= (a+2b-4a+2b,-2a-4b-2a+b)

= (-3a+4b,-4a-3b)

2T (a,b) = (2a+4b,-4a+2b) , 5I = 5 (a,b) = (5a,5b)

T^2 -2T+5I = (-3a+4b-2a-4b+5a,-4a-3b+4a-2b+5b) = (0,0) = T_0

Thus T satisfies its characteristic equation.

f(A) = A^2 -2A+5I $=\begin{bmatrix}-3 & 4\\-4 & -3\end{bmatrix}+\begin{bmatrix}-2 & -4\\4 & -2\end{bmatrix}+\begin{bmatrix}-3 & 4\\-4 & -3\end{bmatrix}+$ $\begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$=0

Example: Using Cayley-Hamilton theorem , find the inverse of the matrix $A=\begin{bmatrix}1 & 2 & 3\\2 & -1 & 4\\3 & 1 & -1\end{bmatrix}$

Solution:

Given $A=\begin{bmatrix}1 & 2 & 3\\2 & -1 & 4\\3 & 1 & -1\end{bmatrix}$

The characteristic equation of A is |A- λI | = 0

.e., $\begin{vmatrix}1-\lambda & 2 & 3\\2 & -1-\lambda & 4\\3 & 1 & -1-\lambda\end{vmatrix}$=0

⇒ (1-λ)[(1+ λ^2)-4]-2[(1+λ)-12]+3[2+3(1+λ)] = 0

⇒ $\lambda^3+\lambda^2$ -18λ-40 = 0

Since every square matrix satisfies its characteristic equation,

we have A^3+A^2 -18A-40I = 0

Multiplying with A^{-1} on both sides A^2 +A-18I = 40 A^{-1}

⇒ A^{-1} = 1/40 [A^2 +A-18I]

we have $A^2 = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$

therefore $A^{-1} = \frac{1}{40}\left\{\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}\right\}$

$A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$

UNIT-5

INNER PRODUCT SPACES

DEFINITION: Let V(F) be a vector space where F is a field of real numbers or the

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field of complex numbers. The vector space V(F)is said to be an inner product space if there is defined for any two vectors α,β∈V an element ‹α,β› ∈ F such that

(1). ‹α,β› = $\overline{‹β, α›}$

(2). ‹α,α› > 0 (zero element in F) for α≠$\overline{0}$

(3). ‹aα+bβ,γ› = a‹α,γ› + b‹β,γ› for any α,β,γ ∈ V and a,b ∈ F.

A function f:VxV→F satisfying the above properties is called an inner product.

If f is the inner product function then f‹α,β› = ‹α,β› or (α,β) for all α,β∈V

From the definition it is clear that a vector space V over F endowed with a specific inner product is an inner product space.

If F = R the field of real numbers then V(F) is called **Euclidean space or Real inner product space.**

If F = C the field of complex numbers then V(F) is called **Unitary space or Complex inner product space.**

An inner product space having only zero vector is called zero space or nullspace.

If V(F) is an inner product space then V(F) is a vector space. A sub space W(F) of the vector space V(F) is also inner product space with the same inner product as in V(F).

PROBLEMS:

If α = (a_1,a_2,a_3),β = (b_1,b_2,b_3) are the elements of a vector space R^3 ,then

‹α,β› = $a_1b_1 + a_2b_2 + a_3b_3$ defines an inner product on R^3 .

Solution : Let α = (a_1,a_2,a_3),β = (b_1,b_2,b_3) and γ = (c_1,c_2,c_3) ∈ R^3 .

Then $a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3 \in R$.

(1). $\langle\alpha,\beta\rangle = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = \langle\beta,\alpha\rangle = \overline{\langle\beta,\alpha\rangle}$.

(2). $\langle\alpha,\alpha\rangle = a_1a_1 + a_2a_2 + a_3a_3 = a_1^2 + a_2^2 + a_3^2$

If $\alpha = (a_1,a_2,a_3) \neq (0,0,0)$ then at least one of a_1,a_2,a_3 is not zero.

So, $\langle\alpha,\alpha\rangle = a_1^2 + a_2^2 + a_3^2 > 0$.

(3). For $a,b \in F$ and $\alpha,\beta,\gamma \in R^3$ we have

$a\alpha + b\beta = a(a_1,a_2,a_3) + b(b_1,b_2,b_3) = (aa_1 + bb_1 , aa_2 + bb_2 , aa_3 + bb_3)$

$\langle a\alpha + b\beta,\gamma\rangle = (aa_1 + bb_1)c_1 + (aa_2 + bb_2)c_2 + (aa_3 + bb_3)c_3$

$= (aa_1c_1 + aa_2c_2 + aa_3c_3) + (bb_1c_1 + bb_2c_2 + bb_3c_3)$

$= a(a_1c_1 + a_2c_2 + a_3c_3) + b(b_1c_1 + b_2c_2 + b_3c_3)$

$= a\,\langle\alpha,\gamma\rangle\ b\,\langle\beta,\gamma\rangle$

Therefore the product $\langle\alpha,\beta\rangle = a_1b_1 + a_2b_2 + a_3b_3$ is an inner product on the vector space R^3 .

Hence R^3 is an inner product space with the above inner product and $R^3(R)$ is real inner product space.

NOTE : The inner product of α and β namely , $\langle\alpha,\beta\rangle = a_1b_1 + a_2b_2 + a_3b_3$ is Called the dot product of α and β and denoted by $\alpha.\beta$. This is Called the standard inner product in R^3.

If $\alpha = (a_1,a_2,\dots, a_n),\beta = (b_1,b_2,\dots, b_n)$ are the elements of a vector space $V_n(C)$ where C is the field of complex numbers, then $\langle\alpha,\beta\rangle = a_1\bar{b}_1 + a_2\bar{b}_2 + \dots + a_n\bar{b}_n = \sum_{i=1}^{n} a_i\bar{b}_i$ defines an inner product on $V_n(C)$.

Solution : Let a,b ∈ C and $\alpha,\beta,\gamma \in V_n$ so that $\alpha = (a_1,a_2,\ldots, a_n)$, $\beta = (b_1,b_2,\ldots, b_n)$, $\gamma = (c_1,c_2,\ldots, c_n)$ where a's, b's and c's are complex numbers.

(1) $\overline{\langle\beta, \alpha\rangle} = \overline{b_1\bar{a}_1 + b_2\bar{a}_2 + \ldots + b_n\bar{a}_n} = \bar{b}_1 a_1 + \bar{b}_2 a_2 + \ldots + \bar{b}_n a_n$

$= a_1\bar{b}_1 + a_2\bar{b}_2 + \ldots + a_n\bar{b}_n = \langle\alpha,\beta\rangle$.

(2) $\langle\alpha,\alpha\rangle = a_1\bar{a}_1 + a_2\bar{a}_2 + \ldots + a_n\bar{a}_n = |a_1|^2 + |a_2|^2 + \ldots + |a_n|^2$

If $\alpha \neq 0$ then at least one of $a_1, a_2,\ldots, a_n$ is non zero complex number .

So $\langle\alpha,\alpha\rangle = |a_1|^2 + |a_2|^2 + \ldots + |a_n|^2 > 0$.

(3) $a\alpha+b\beta = a(a_1,a_2,\ldots, a_n)+b(b_1,b_2,\ldots, b_n)$

$= (aa_1+bb_1, aa_2+bb_2,\ldots, aa_n+bb_n)$

$\langle a\alpha+b\beta,\gamma\rangle = (aa_1+bb_1)\bar{c}_1 + (aa_2+bb_2)\bar{c}_2,\ldots,(aa_n+bb_n)\bar{c}_n$

$= (aa_1\bar{c}_1 + aa_2\bar{c}_2+\ldots+ aa_n\bar{c}_n) + (bb_1\bar{c}_1 + bb_2\bar{c}_2+\ldots+ bb_n\bar{c}_n)$

$= a(a_1\bar{c}_1 + a_2\bar{c}_2+\ldots+ a_n\bar{c}_n) + b(b_1\bar{c}_1 + b_2\bar{c}_2+\ldots+ b_n\bar{c}_n)$

$= a\langle\alpha,\gamma\rangle+b\langle\beta,\gamma\rangle$

Therefore the product $\langle\alpha,\beta\rangle = a_1\bar{b}_1 + a_2\bar{b}_2 + \ldots + a_n\bar{b}_n$ is an inner product on $V_n(C)$.

Therefore $V_n(C)$ or $C^n(C)$ is the unitary space.

Let V(C) be the vector space of all continuous complex valued functions on the closed interval [0,1]. For f,g ∈ V if $\langle f,g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt$ **, then V is an inner product space.**

Solution :

NORM OR LENGTH OF A VECTOR

DEFINITION : Let V be an inner product space over the field F. The narm (length) of $\alpha \in V$ denoted by $\| \alpha \|$ is defined as the positive square root of $\langle\alpha,\alpha\rangle$.

Norm or length of $\alpha \in V = \| \alpha \| = \sqrt{(\alpha, \alpha)} \Rightarrow \| \alpha \|^2 = \langle\alpha,\alpha\rangle$.

NOTE : 1. For $\alpha \in V$, $\langle\alpha,\alpha\rangle$ is non- negative real number and hence the norm of α is always non- negative real number.

2. $\alpha = \overline{0} \Leftrightarrow \| \alpha \| = 0$

EXAMPLE : 1. In the inner product space $V_2(R) = R^2(R)$;If $\alpha = (a,b) \in V_2$

then $\| \alpha \| = \|(a,b)\| = \sqrt{a^2 + b^2} = \sqrt{\langle\alpha, \alpha\rangle}$.

2. In the inner product space $V_3(R) = R^3(R)$;If $\alpha = (a,b,c) \in V_3$

then $\| \alpha \| = \|(a,b,c)\| = \sqrt{a^2 + b^2 + c^2} = \sqrt{\langle\alpha, \alpha\rangle}$.

3. In the inner product space $V_n(R) = R^n(R)$;If $\alpha = (a_1,a_2,\ldots,a_n) \in$

$V_n$then $\| \alpha \| = \|(a_1,a_2,\ldots,a_n)\| = \sqrt{|a_1|^2+|a_2|^2+\ldots+|a_n|^2} = \sqrt{\langle\alpha, \alpha\rangle}$.

THEOREM : In an inner product space V(F) (1) $\| \alpha \| > 0$ if $\alpha \neq \overline{0}$ and

(2) $\|a \alpha \| = |a|\| \alpha \|$ where 0, $a \in F$ and 0, $\overline{\alpha} \in V$.

Solution : (1) If $\alpha \neq \overline{0}$ then $\langle\alpha, \alpha\rangle > 0$.

$\| \alpha \| = \sqrt{\langle\alpha, \alpha\rangle} > 0$. for any $\alpha \in V$, $\| \alpha \| \geq 0$.

(2) By the definition of norm , $\| a \alpha \|^2 = \langle a\alpha, a\alpha\rangle$

$= a\langle\alpha, a\alpha\rangle$

$= a \overline{a}\langle\alpha, \alpha\rangle$

$= |a|^2 \, \| \alpha \|^2$

$= (|a| \, \| \alpha \|)^2$

Therefore $\| a\alpha \| = |a| \, \| \alpha \|$ where $0, a \in F$ and $\overline{0}, \alpha \in V$.

NOTE : If $\alpha \in V$ and $\alpha \neq \overline{0}$ by the above theorem $\| \alpha \| > 0$. Since $\| \alpha \| (> 0) \in F$ and F is a field , there exists $\frac{1}{\| \alpha \|} \in F$ such that $\| \alpha \| \frac{1}{\| \alpha \|} = 1$. Now for $\frac{1}{\| \alpha \|} \in F$ and $\alpha \in V$ we have $\frac{1}{\| \alpha \|} \alpha \in V$, such that $\langle \frac{1}{\| \alpha \|} \alpha , \frac{1}{\| \alpha \|} \alpha \rangle = \frac{1}{\| \alpha \|} \left(\frac{1}{\| \alpha \|} \right) \langle \alpha, \alpha \rangle$

$= \left(\frac{1}{\| \alpha \|} \right) \left(\frac{1}{\| \alpha \|} \right) \| \alpha \|^2 = 1$

Hence $\alpha \in V$ and $\alpha \neq \overline{0}$, $\frac{1}{\| \alpha \|} \alpha \in V$ is a vector of length 1.

DEFINITION : Let V(F) be an inner product space. $\alpha \in V$ is called a unit vector if $\| \alpha \| = 1$. If $\alpha \in V$ then $\frac{1}{\| \alpha \|} \alpha \in V$ is unit vector.

Example : (1) In the inner product space R^2 , i = (1,0) , j = (0,1) are unit vectors.

(2) In the inner product space R^3 with standard inner product

i = (1,0,0) , j = (0,1,0) and k = (0,0,1) are vectors of length 1.

THEOREM : Cauchy-Schwarz's inequality

In an inner product space V(F) , $|\langle \alpha,\beta \rangle| \leq \| \alpha \| \, \| \beta \|$ for all $\alpha , \beta \in V$.

Proof : Case (1). Let $\alpha = \overline{0}$. Then $\langle \alpha,\beta \rangle = \langle 0,\overline{\beta} \rangle = 0$ and $\| \alpha \|^2 = \langle \alpha, \alpha \rangle = \langle 0, \overline{0} \rangle = 0$.

Therefore $|\langle \alpha,\beta \rangle| = 0$ and $\| \alpha \| \, \| \beta \| = 0$.

Therefore $|\langle\alpha,\beta\rangle| = \|\alpha\| \, \|\beta\|$.

Case (2). Let $\alpha \neq \overline{0}$. Then $\|\alpha\| > 0$ so that $\frac{1}{\|\alpha\|} > 0$.

Take $\gamma \in V$ so that $\gamma = \beta - \dfrac{\langle \beta,\alpha \rangle}{\| \alpha \|^2}\alpha$.

$$\text{Now } \langle \gamma, \gamma \rangle = \left\langle \beta - \frac{\langle \beta,\alpha \rangle}{\| \alpha \|^2}\alpha \, , \, \beta - \frac{\langle \beta,\alpha \rangle}{\| \alpha \|^2}\alpha \right\rangle$$

$$= \langle \beta,\beta \rangle - \frac{\overline{\langle \beta,\alpha \rangle}}{\| \alpha \|^2} \langle \beta,\alpha \rangle - \frac{\langle \beta,\alpha \rangle}{\| \alpha \|^2} \langle \alpha,\beta \rangle + \frac{\langle \beta,\alpha \rangle}{\| \alpha \|^2} \frac{\overline{\langle \beta,\alpha \rangle}}{\| \alpha \|^2} \langle \alpha,\alpha \rangle$$

$$= \|\beta\|^2 - \frac{\overline{\langle \beta,\alpha \rangle} \langle \beta,\alpha \rangle}{\| \alpha \|^2} - \frac{\langle \beta,\alpha \rangle \langle \alpha,\beta \rangle}{\| \alpha \|^2} + \frac{\langle \beta,\alpha \rangle \overline{\langle \beta,\alpha \rangle}}{\| \alpha \|^2}$$

$$= \|\beta\|^2 - \frac{\langle \beta,\alpha \rangle \langle \alpha,\beta \rangle}{\| \alpha \|^2} = \|\beta\|^2 - \frac{\overline{\langle \alpha,\beta \rangle} \langle \alpha,\beta \rangle}{\| \alpha \|^2}$$

But by the definition of the norm ; $\langle \gamma, \gamma \rangle \geq 0$

Therefore $\|\beta\|^2 - \dfrac{\overline{\langle \alpha,\beta \rangle} \langle \alpha,\beta \rangle}{\| \alpha \|^2} \geq 0$

$\Rightarrow \quad \|\beta\|^2 \geq \dfrac{|\langle \alpha,\beta \rangle|^2}{\| \alpha \|^2}$

$\Rightarrow \quad \| \beta \|^2 \; \| \alpha \|^2 \geq |\langle \alpha,\beta \rangle|^2$

$\Rightarrow \quad (\| \beta \| \; \| \alpha \|)^2 \geq |\langle \alpha,\beta \rangle|^2$

Therefore $\| \beta \| \| \alpha \| \geq |\langle \alpha,\beta \rangle|$ as $\| \beta \| \| \alpha \|$ and $|\langle \alpha,\beta \rangle|$ are non-negative.

Hence $|\langle \alpha,\beta \rangle| \leq \| \alpha \| \| \beta \|$.

NOTE : For $\gamma \in V$, $\langle \gamma, \gamma \rangle = 0 \Rightarrow \gamma = \bar{0} \Rightarrow \beta - \dfrac{\langle \beta,\alpha \rangle}{\| \alpha \|^2}\alpha = \bar{0} \Rightarrow \beta = \dfrac{\langle \beta,\alpha \rangle}{\| \alpha \|^2}\alpha$

$\Rightarrow$ β vector = scalar multiple of the vector α

$\Rightarrow$ α , β are linearly dependent.

Hence α , β are linearly dependent vectors of V ⇔ $|\langle\alpha,\beta\rangle| = \|\alpha\| \; \|\beta\|$.

THEOREM : (Triangle inequality)

In an inner product space V(F) , $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ for all $\alpha , \beta \in V$.

Proof : By the definition of norm , $\|\alpha+\beta\|^2 = \langle \alpha+\beta , \alpha+\beta \rangle$

$$= \langle\alpha,\alpha\rangle + \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle + \langle\beta,\beta\rangle$$

$$= \|\alpha\|^2 + \langle\alpha,\beta\rangle + \overline{\langle\alpha,\beta\rangle} + \|\beta\|^2$$

$$= \|\alpha\|^2 + 2\,\mathrm{Re}\,\langle\alpha,\beta\rangle + \|\beta\|^2$$

$$\leq \|\alpha\|^2 + 2\,|\langle\alpha,\beta\rangle| + \|\beta\|^2$$

$$\leq \|\alpha\|^2 + 2\,\|\alpha\|\,\|\beta\| + \|\beta\|^2$$

$$\leq (\|\alpha\| + \|\beta\|)^2$$

Therefore $\|\alpha+\beta\|^2 \leq (\|\alpha\| + \|\beta\|)$

As both $\|\alpha+\beta\|$ and $\|\alpha\| + \|\beta\|$ are non-negative we have $\|\alpha+\beta\| \leq \|\alpha\| + \|\beta\|$.

THEOREM : (Parallelogram law)

If α , β are two vectors in an inner product space V(F) then

$$\|\alpha - \beta\|^2 + \|\alpha + \beta\|^2 = 2 (\|\alpha\|^2 + \|\beta\|^2)$$

Proof : $\|\alpha-\beta\|^2 = \langle\alpha-\beta , \alpha-\beta\rangle$

$$= \langle\alpha,\alpha\rangle - \langle\alpha,\beta\rangle - \langle\beta,\alpha\rangle + \langle\beta,\beta\rangle$$

$$= \|\alpha\|^2 - \langle\alpha,\beta\rangle - \langle\beta,\alpha\rangle + \|\beta\|^2$$

$\|\alpha+\beta\|^2 = \langle\alpha+\beta , \alpha+\beta\rangle = \langle\alpha,\alpha\rangle + \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle + \langle\beta,\beta\rangle = \|\alpha\|^2 + \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle + \|\beta\|^2$

Therefore $\|\alpha-\beta\|^2 + \|\alpha+\beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2 = 2 (\|\alpha\|^2 + \|\beta\|^2)$

NORMED VECTOR SPACE AND DISTANCE

DEFINITION : Let V(F) be an inner product space in which norm of a vector α ϵ V is defined as $\|\alpha\| = \sqrt{\langle\alpha, \alpha\rangle}$. The inner product space with this definition of norm is called a normed vector space if the following conditions are true :

_ _

(i) $\| \alpha \| \geq 0$ and $\| \alpha \| = 0 \Leftrightarrow \alpha = 0$

(ii) $\| a\alpha \| = |a| \, \| \alpha \|$ and

(iii) $\|\alpha+\beta\| \leq \| \alpha \| + \| \beta \|$ for all $\alpha , \beta \in V$, $a \in F$

As the above three conditions are true in every inner product space , every inner product space is a normed vector space.

DEFINITION : Let α , β be two vectors in an inner product space V(F). The distance between the vectors α , β denoted by d‹α,β› is defined as ‖α-β‖.

NOTE : (1) If $\alpha , \beta \in V$ then $d\langle\alpha,\beta\rangle = \|\alpha-\beta\|$

$$\Rightarrow \quad d\,[\langle\alpha,\beta\rangle]^2 = \|\alpha-\beta\|^2 = \langle \alpha-\beta , \alpha-\beta \rangle$$

(2) d‹α,β› is a non-negative real number.

THEOREM : If α,β,γ ϵ V(F) an inner product space then (1) d (α,β) ≥ 0 and d (α,β) = 0 ⇔ α = β (2) d (α,β) = d (β,α) and (3) d (α,β) + d (β,γ) ≥ d (α,γ).

Proof : (1) By the definition, $d\langle\alpha,\beta\rangle = \|\alpha-\beta\| \geq 0$ since norm of a vector is a non-negative real number.

$d\langle\alpha,\beta\rangle = 0 \Leftrightarrow \|\alpha-\beta\| = 0 \Leftrightarrow \|\alpha-\beta\|^2 = 0 \Leftrightarrow \langle \alpha-\beta , \alpha-\beta \rangle = 0 \Leftrightarrow \alpha-\beta = 0$ i.e., $\alpha = \beta$

(2) $d\langle\alpha,\beta\rangle = \|\alpha-\beta\| = \| (-1)(\beta- \alpha) \| = |-1| \, \| \beta- \alpha \| = 1 \, \| \beta- \alpha \| = d \langle\beta, \alpha\rangle$.

(3) $d \langle\alpha,\beta\rangle + d \langle\beta,\gamma\rangle = \|\alpha-\beta\| + \| \beta-\gamma \| \geq \|\alpha-\beta + \beta-\gamma \|$ By triangle inequality

$\geq \|\alpha-\gamma\| = d \langle\alpha,\gamma\rangle$.

NOTE : (1) In an inner product space V(F) the distance function d : V→F ,defined as $d\langle\alpha,\beta\rangle = \|\alpha-\beta\|$ for all $\alpha , \beta \in V$ is satisfying the properties (1),(2),(3) of the metric space.

(2) For $\alpha,\beta,\gamma \in V$; $d \langle\alpha+\gamma , \beta+\gamma\rangle = \| \alpha+\gamma-\beta-\gamma \| = \|\alpha-\gamma\| = d \langle\alpha,\gamma\rangle$.

PROBLEMS

1. **If α = (2,1,1+i) is a vector in C^3 with standard inner product find ‖α‖ andthe unit vector of α.**

   Solution : $\| \alpha \|^2 = \langle \alpha,\alpha \rangle = (2)(\bar{2}) + 1(\bar{1}) + (1+i)(\overline{1+i})$
   $= (2)(2) + 1(1) + (1+i)(1-i)$
   $= 4 + 1 + 2 = 7.$

   Unit vector of α = $\frac{1}{\| \alpha \|} \alpha = \frac{1}{\sqrt{7}}(2,1,1+i).$

2. **If α = (4,1,8),β = (1,0,-1) are two vectors in R^3 find the angle between α and β.**

   Solution : $\| \alpha \| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$ and

   $\| \beta \| = \sqrt{1^2 + 0^2 + (-3)^2} = \sqrt{10}$

   $\langle \alpha,\beta \rangle = (4)(1)+(1)(0)+(8)(-3) = -20.$

If θ = angle between α and β then $\cos\theta = \frac{|\langle \alpha,\beta \rangle|}{\| \alpha \| \; \| \beta \|} = \frac{|-20|}{9(\sqrt{10})} = \frac{20}{9(\sqrt{10})} = \frac{2}{9}\sqrt{10}.$

3. **If α,β are two vectors in an inner product space , then α,β are linearly dependent if and only if $|\langle \alpha,\beta \rangle| = \| \alpha \| \; \| \beta \|$.**

   Solution : Let α,β be linearly dependent.

   Then either $\alpha = \bar{0}$ or $\beta = \bar{0}$ or $\alpha = a\beta$ where 'a' is a scalar.

   When $\alpha = \bar{0}$: $\langle \alpha,\beta \rangle = \langle \bar{0},\beta \rangle = 0$ and $\| \alpha \| = 0.$

   When $\beta = \bar{0}$: $\langle \alpha,\beta \rangle = \langle \alpha, \bar{0} \rangle = \overline{\langle \bar{0},\alpha \rangle} = 0$ and $\| \beta \| = 0.$

   When $\alpha = a\beta$: $\langle \alpha,\beta \rangle = \langle a\beta,\beta \rangle = a \langle \beta,\beta \rangle = a \| \beta \|^2$ and $\| \alpha \| = \| a\beta \| = |a| \; \|\beta\|$

Therefore $|\langle\alpha,\beta\rangle| = |a|\ \|\beta\|^2 = (|a|\ \|\beta\|)\ (\|\beta\|) = \|\alpha\|\ \|\beta\|$.

Conversely , let $|\langle\alpha,\beta\rangle| = \|\alpha\|\ \|\beta\|$.

When $\alpha = \overline{0}$ the vectors α,β be linearly dependent.

When $\alpha \neq \overline{0}$; we have $\| \alpha \| > 0$.

Consider the vector $\gamma = \beta - \dfrac{\langle\beta,\alpha\rangle}{\| \alpha \|^2}\alpha$.

$$\text{Now } \langle\gamma, \gamma\rangle = \left\langle \beta - \frac{\langle\beta,\alpha\rangle}{\| \alpha \|^2}\alpha , \beta - \frac{\langle\beta,\alpha\rangle}{\| \alpha \|^2}\alpha \right\rangle$$

$$= \langle\beta,\beta\rangle - \frac{\overline{\langle \beta,\alpha \rangle}}{\| \alpha \|^2} \langle\beta,\alpha\rangle - \frac{\langle\beta,\alpha\rangle}{\| \alpha \|^2}\langle\alpha,\beta\rangle + \frac{\langle\beta,\alpha\rangle}{\| \alpha \|^2} \frac{\overline{\langle \beta,\alpha \rangle}}{\| \alpha \|^2}\langle\alpha,\alpha\rangle$$

$$= \|\beta\|^2 - \frac{\langle\beta,\alpha\rangle \overline{\langle\beta,\alpha\rangle}}{\| \alpha \|^2} - \frac{\langle\beta,\alpha\rangle \langle\alpha,\beta\rangle}{\| \alpha \|^2} + \frac{\langle\beta,\alpha\rangle \overline{\langle\beta,\alpha\rangle}}{\| \alpha \|^2}$$

$$= \|\beta\|^2 - \frac{\langle\beta,\alpha\rangle \langle\alpha,\beta\rangle}{\| \alpha \|^2} = \|\beta\|^2 - \frac{\overline{\langle\alpha,\beta\rangle} \langle\alpha,\beta\rangle}{\| \alpha \|^2}$$

$$= \|\beta\|^2 - \frac{|\langle\alpha,\beta\rangle|^2}{\| \alpha \|^2} = \|\beta\|^2 - \frac{\| \alpha \|^2 \|\beta\|^2}{\| \alpha \|^2} = 0.$$

Therefore $\langle\gamma, \gamma\rangle = 0 \Rightarrow \gamma = \overline{0}$

⇨ $\beta - \dfrac{\langle\beta,\alpha\rangle}{\| \alpha \|^2}\alpha = \underline{0}$

⇨ $\beta = \dfrac{\langle\beta,\alpha\rangle}{\| \alpha \|^2}\alpha = \underline{0}$

⇨ $\beta = a\alpha$ where $a = \dfrac{\langle\beta,\alpha\rangle}{\| \alpha \|^2}$ is a scalar.

Therefore α,β be linearly dependent.

4. Two vectors α,β in an unitary space V(C) are such that $\langle\alpha,\beta\rangle = 0$ iff $\|a\alpha+b\beta\|^2 = |a|^2\,\|\alpha\|^2 + |b|^2\|\beta\|^2$ for all a,b ∈ C.

Solution : Let $\langle\alpha,\beta\rangle = 0$. Then $\overline{\langle\alpha,\beta\rangle} = 0$.

$$\|a\alpha+b\beta\|^2 = \langle a\alpha+b\beta , a\alpha+b\beta\rangle = a\,\langle\alpha, a\alpha+b\beta\rangle + b\,\langle\beta, a\alpha+b\beta\rangle$$

$$= a[\bar{a}\langle\alpha,\alpha\rangle + \bar{b}\langle\alpha,\beta\rangle] + b[\bar{a}\langle\beta,\alpha\rangle + \bar{b}\langle\beta,\beta\rangle]$$

$$= a\bar{a}\langle\alpha,\alpha\rangle + a\bar{b}\langle\alpha,\beta\rangle + b\bar{a}\langle\beta,\alpha\rangle + b\bar{b}\langle\beta,\beta\rangle$$

$= |a^2| \| \alpha \|^2 + a\bar{b}\langle\alpha,\beta\rangle + b\bar{a}\overline{\langle\alpha,\beta\rangle} + |b^2| \| \beta \|^2 \rightarrow (1)$

$= |a^2| \| \alpha \|^2 + 0 + 0 + |b^2| \| \beta \|^2$

$= |a^2| \| \alpha \|^2 + |b^2| \| \beta \|^2$

Conversely, Let $\|a\alpha+b\beta\|^2 = |a|^2 \| \alpha \|^2 + |b|^2 \| \beta \|^2$ for all $a,b \in C$.

Using (1) we have

$|a^2| \| \alpha \|^2 + a\bar{b}\langle\alpha,\beta\rangle + b\bar{a}\overline{\langle\alpha,\beta\rangle} + |b^2| \| \beta \|^2 = |a^2| \| \alpha \|^2 + |b^2| \| \beta \|^2$

⇨ $a\bar{b}\langle\alpha,\beta\rangle + b\bar{a}\overline{\langle\alpha,\beta\rangle} = 0 \rightarrow (2)$

Take $a = 1$, $b = 1$ so that $\bar{a} = 1$, $\bar{b} = 1$

Then (2) : $(1)(1)\langle\alpha,\beta\rangle + (1)(1)\overline{\langle\alpha,\beta\rangle} = 0$

⇨ $\langle\alpha,\beta\rangle + \overline{\langle\alpha,\beta\rangle} = 0$

⇨ $2\,\mathrm{Re}\,\langle\alpha,\beta\rangle = 0$

⇨ $\mathrm{Re}\,\langle\alpha,\beta\rangle = 0$

Take $a = i$, $b = 1$ so that $\bar{a} = -i$, $\bar{b} = 1$

Then (2) : $i\langle\alpha,\beta\rangle - i\overline{\langle\alpha,\beta\rangle} = 0$

⇨ $i[\langle\alpha,\beta\rangle - \overline{\langle\alpha,\beta\rangle}] = 0$

⇨ $\langle\alpha,\beta\rangle - \overline{\langle\alpha,\beta\rangle} = 0$

⇨ $2\,\mathrm{Im}\,\langle\alpha,\beta\rangle = 0$

Thus we have $\mathrm{Re}\,\langle\alpha,\beta\rangle = 0$ and $\mathrm{Im}\,\langle\alpha,\beta\rangle = 0$.

Hence $\langle\alpha,\beta\rangle = 0$.

5. If u,v are two vectors in a complex inner product space with standardinner product then prove that

$4\langle u,v\rangle = \| u+v \|^2 - \| u-v \|^2 + i\| u+iv \|^2 - i\| u-iv \|^2$.

Solution : $\|u+ v\|^2 = \langle u+v , u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$

$= \| u \|^2 + \langle u,v\rangle + \langle v, u\rangle + \| v \|^2 \rightarrow (1)$

$\|u-v\|^2 = \langle u-v , u-v\rangle = \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle$

$= \| u \|^2 - \langle u,v\rangle - \langle v, u\rangle + \| v \|^2 \rightarrow (2)$

$\|u+iv\|^2 = \langle u+iv , u+iv\rangle = \langle u,u\rangle + \bar{\imath}\langle u,v\rangle + i\langle v,u\rangle + i\,\bar{\imath}\langle v,v\rangle$

$= \| u \|^2 - i\langle u,v\rangle + i\langle v, u\rangle + \| v$

$\|^2 i\ \|u+iv\|^2 = i\| u \|^2 + \langle u,v\rangle - \langle v, u\rangle + i\| v \|^2 \rightarrow$

(3)

$\|u-iv\|^2 = \langle u-iv , u-iv\rangle = \langle u,u\rangle - \bar{\imath}\langle u,v\rangle - i\langle v,u\rangle + i\bar{\imath}\langle v,v\rangle$

$= \| u \|^2 + i\langle u,v\rangle - i\langle v, u\rangle + \| v \|^2$

$i\ \|u-iv\|^2 = i\| u \|^2 - \langle u,v\rangle + \langle v, u\rangle + i\| v \|^2 \rightarrow (4)$

From (1),(2),(3) and (4) : $\| u+v \|^2 - \| u-v \|^2 + i \| u+iv \|^2 - i \| u-iv \|^2$

$= \{2\langle u,v\rangle + 2\langle v,u\rangle\} + \{2\langle u,v\rangle - 2\langle v,u\rangle\}$

$= 4 \langle u,v\rangle$