DNR COLLEGE (A), BHIMAVARAM

P.G DEPARTMENT MATHEMATICS

305 – LATTICE THEORY

II MSc, SEMESTER -III

UNIT - 1

PREPARED BY:

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M.Sc Mathematics

SYLLABUS (M 305- LATTICE THEORY)

UNIT – 1:

(Additional input: Relations) Partially ordered Sets – Diagrams – Special subsets of a poset – length – lower and upper bounds – the minimum and maximum condition – the Jordan Dedekind chain conditions – Dimension functions.

(Sections 1 – 9 of chapter 1)

UNIT – 2:

Algebras – lattices – the lattice theoretic duality principle – semi lattices – lattices as posets – Diagrams of lattices – sub lattices – ideals – bound elements of lattices – atoms and dual atoms – complements, relative complements – semi complements – irreducible and prime elements of a lattice – the homomorphism of a lattice – axioms system of lattices.

(Sections 10 – 21 of chapter 2)

UNIT – 3:

Complete lattices – complete sub lattices of a complete lattice – conditionally complete lattices – lattices – compact elements, compactly generated lattices – sub algebra lattice of an algebra – closure operations – Galois connections, Dedekind cuts – partially ordered sets as topological spaces.

(Sections 22 – 29 0f chapter 3)

UNIT – 4:

Distributive lattices – infinitely distributive and completely distributive lattices – modular lattice – characterization of modular and distributive lattices by their sub lattices – distributive sub lattices of modular lattices – the isomorphism theorem of modular lattices – covering conditions – meet representation in modular and distributive lattices – some special sub classes of the class of modular lattices – preliminary theorems – modular lattices of locally finite length – the valuation of a lattice – metric and quasi metric lattices – complemented modular lattices.

(sections 30 – 40 of chapter 4)

➢ DEFINITIONS :

1. REFLEXIVE RELATION: ***Let ϕ be a* relation *on a set M. we say that ϕ is reflexive relation if all elements x of M satisfies $x\phi x$.***

EXAMPLE:* *let Z be the set of all integers defined a relation ϕ on Z as

$x\phi y \Leftrightarrow x=y \ \forall \ x, y \in Z$.

This relation is reflexive because $x \in Z$, $x = x \Leftrightarrow x \phi x$.

2. SYMMETRIC RELATION:* *Let ϕ be a* relation *on a set M. we say that ϕ is symmetric relation if $x\phi y$ then $y\phi x$.

EXAMPLE:* *let Z be the set of all integers defined a relation ϕ on Z as

$x\phi y \Leftrightarrow x=y \ \forall \ x, y \in Z$.

This is a symmetric relation.

3. ANTISYMMETRIC RELATION:* *Let ϕ be a* relation *on a set M. we say that ϕ is an antisymmetric relation if $x\phi y$ & $y\phi x$ then $x = y$.

EXAMPLE:* *let Z be the set of all integers defined a relation ϕ on Z as

$x\phi y \Leftrightarrow x \leq y \ \forall \ x, y \in Z$.

This is an ant symmetric relation.

4. TRANSITIVE RELATION:* *Let ϕ be a* relation *on a set M. we say that ϕ is transitive relation if $\forall$ $x, y, z \in M$, $x\phi y$, $y\phi z \Rightarrow x\phi z$.

EXAMPLE:* *let Z^+ be the set of positive integers defined a relation ϕ on Z as $x\phi y \Leftrightarrow x/y \ \forall$ pairs $x, y \in Z^+$.

This is a transitive relation on Z^+.

5. **EQUIVALENCE RELATION**: ***Let*** θ ***be a*** **relation** ***on a set M,*** θ ***is turned an equivalence relation. If the following conditions are satisfied***

I. θ ***is reflexive***
II. θ ***is symmetric***
III. θ ***Is transitive.***

Usually an equivalence relation is denoted by $x \cong y\ (\theta)$

Or $x \cong y \; \forall\ x, y \in M$.

EXAMPLE: let Z be the set of all integers defined a relation θ ***on Z as***

$$x\theta\, y \Leftrightarrow x=y \;\forall\ x, y \in Z.$$

6. PARTIAL ORDERING: let Π ***be a relation defined on a set M.*** Π ***is called a partial ordering relation on M if the following conditions are satisfied.***

I. Π ***is reflexive***
II. Π ***is ant symmetric***
III. Π ***Is transitive.***

Usually a partial ordering is denoted by $x \leq y\ (\Pi)$ ***are simply*** $x \leq y$ $\forall\ x, y \in M$.

- ***PARTIALLY ORDERED SET: Let P be a set. P is called a partially ordered set if there is defined on ordering*** Π ***on P. It is written as*** (P, Π).

EXAMPLE: let Z be the set of integers then $(Z, \leq)$ ***is a poset.***

Define a relation $\leq$ ***on P (M) as*** $A \leq B \Leftrightarrow A \subseteq B,\ A, B \in P(M)$

We can easily see that $\subseteq$ ***is an ordering relation P (M).***

Hence (P (M),$\subseteq$) is a poset.

- ***QUASI ORDERING RELATION:*** *A relation θ defined on a set M. It is turned as quasi ordering relation .If the following conditions are satisfied.*

  1. *θ is reflexive*
  2. *θ is transitive*

- ***COMPARABLE:*** *let P be a poset and a, b € P. If a$\leq$ b or b$\leq$ a then a , b are said to be comparable otherwise a , b are said to be incomparable.*
- ***TOTALLY ORDERED (OR) CHAIN:*** *let (P,$\leq$) be a partly ordered set. If for all x, y € P either x$\leq$ y (or) y $\leq$ x then P is called completely ordered (or) totally ordered (or) chain.*

  EXAMPLE: *let V be the set of real numbers.*

  Define a relation $\leq$ on V as for all x, y € V, x $\leq$ y$\Leftrightarrow$ y-x is not negative.

- ***NOTE:*** *Let P be a poset with respective to the relation Π then the relation D(Π) defined by a$\leq$ b(D(Π))$\Leftrightarrow$ a$\geq$ b(Π) $\forall$ a, b € P is also a partial ordering relation on P. This relation D (Π) is called DUAL RELATION of Π.*

SOLUTION:

1.RELEXIVE: *Let a € P*

$$a \geq a\ (\Pi)$$

$$\Rightarrow a \leq a\ (D\ (\Pi\ ))$$

$\therefore$ D (Π) is reflexive.

2. ANTISYMMETRIC: ***let a, b € P***

$$a \leq b \; (D(\Pi)) \text{ and } b \leq a \; (D(\Pi))$$

$$\Rightarrow a \geq b \; (\Pi) \text{ and } b \geq a \; (\Pi)$$

$$\Rightarrow a = b$$

$\therefore$ D (Π) is anti symmetric.

3. TRANSITIVE: ***let a, b, C € P***

$$a \leq b \; (D(\Pi)) \text{ and } b \leq c \; (D(\Pi))$$

$$\Rightarrow a \geq b \; (\Pi) \text{ and } b \geq c \; (\Pi)$$

$$\Rightarrow a \geq c \; (\Pi)$$

$$\Rightarrow a \leq c \; (D(\Pi))$$

$\therefore$ D (Π) is transitive.

Hence, dual relation D (Π) is an ordering relation on P.

- ***ORDERING HOMOMORPHISM:*** ***Let (P_1 Π_1), (P_2, Π_2) be two partly ordered sets. A single value function $\phi : P_1 \rightarrow P_2$ is called an ordering homomorphism if $x \leq y \; (\Pi_1) \Leftrightarrow \phi(x) \leq \phi(y) \; (\Pi_2) \; \forall \; x, y$ € P_1.***

  In this case those posets are called order homomorphic.

  If ϕ is one-to-one mapping from P_1 onto P_2 then ϕ is called order isomorphism.

  In this case posets P_1 and P_2 are called order isomorphic.

$\rightarrow$ ***DIAGRAMS:***

- ***COVERS:*** ***Let P be a poset and a, b € P. If a < b holds and there is no element x such that a < x < b then we say that a is covered by b (or) b covers a.***

  This representation symbolically as a -<b (or) b >- a.

If we use the symbol a =< b this means that a is covered by b (or) a is equal to b.

- ***MAXIMAL ELEMENT:*** *Let P be a poset. An element a of P is called a maximal element if P has no element x for which a<x.*

  An element a of P is called a maximal element of P if a is not covered by any element of P (or) there is no element in P that covers a.

- ***MINIMAL ELEMENT:*** *Let P be a poset. An element a of P is called minimal element if P has no element x for which x<a.*

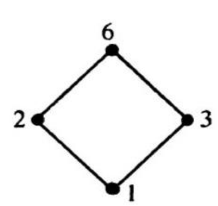
  An element a of P is called a minimal element of P if a covers no element of P.

- ***NOTE:*** *Every finite poset can be represented by a diagram. This diagram is called "HASSE DIAGRAM".*

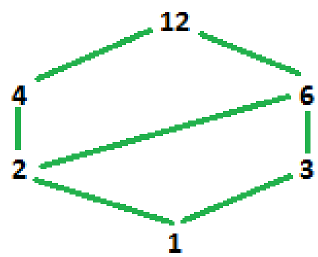
- ***DRAW THE DIAGRAMS OF THE FOLLOWING OSETS:***

  1) ({1,2,4,8},/)

  2) ({1,2,3,6},/)

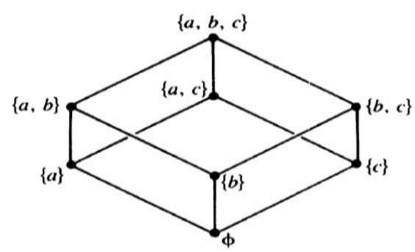


3) ({1,2,3,4,6,12},/)



4) (P(a, b ,c),$\subseteq$)

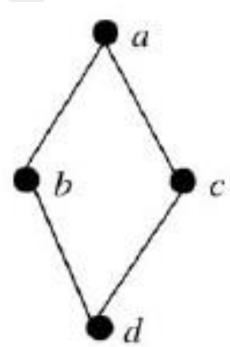
P(a , b , c) = {ϕ , {a} ,{b} , {c} , {a ,b} , {b , c} , {c , a} , {a , b , c}}



- ***Given an example of a partly ordered set with unique maximal element and unique minimal element.***

  Solution: consider the posets

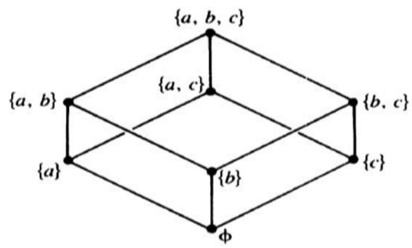
  i)



Here the maximal element a is unique and

The minimal element d is also unique.

ii)



Here the maximal element {a, b, c} is unique and
The minimal element ϕ is also unique.

- ***THEOREM:*** ***Every non-empty finite partly ordered set can be represented by a diagram.***

PROOF: ***let P be a non-empty partly ordered set containing n elements.***
We use mathematical induction on n to prove this result
STEP: 1) let n = 1
Let P = {a}
This can be represented by a diagram.
å because a ≤ a .
STEP: 2) assume that the statement is true for a poset with n-1 elements.
Now we prove that a poset with n elements can be also be represented a diagram.
Let P be a poset with n elements.
Since every finite poset has atleast a maximal element, we have a maximal element m.
By deleting the maximal element m from P.

We get a subset R with n-1 elements.

Clearly R is a poset with respective to the ordering defined on P.

By assumption, the poset R is represented by a diagram.

Since R is a poset we have that R has a maximal element.

Now put a circle above that maximal element and join element x € R such that $x -< m$.

The resulting diagram is the diagram of the poset P

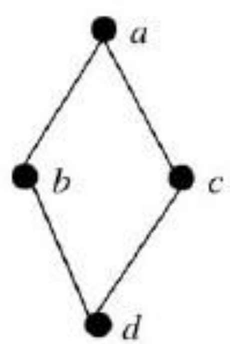
$\therefore$ *by mathematical induction the result is true for n elements.*

Hence, *every non-empty finite partly ordered set can be represented by a diagram.*

- ***DEFINITION:*** *Let P be a poset and a be an arbitrary element in P. we write (a] = {x € P / $x \leq a$},*

  [a) = {x € P / $x \geq a$}.

EXAMPLE: *Let us consider the poset*



Here (a] = {x € P / $x \leq a$}

= {a, b, c, d}

[a) = {x € P / $x \geq a$}

= {a}

- ***DEFINITION:*** ***Let P be a poset and a, b € P such that $a \leq b$ then the interval [a, b] bounded by the elements a and b is a set.***

  $$[a, b] = \{x \in P \,/\, a \leq x \leq b\}$$

- ***CONVEX SET:*** ***Let P be a poset. A subset R of P is said to be CONVEX if for any pair of elements a, b (a < b) R contains all elements x of P for which $a \leq x \leq b$***

  i.e., every element of the interval [a, b] is a member of R.

- ***COMPLETELY UNORDERED SET:*** ***Let P be a poset. A subset R of P is called a COMPLETELY UNORDERED SET if any two elements of R are incomparable with respective to the relation $\prod_P$***

- ***SUBCHAIN:*** ***Let P be a poset. A subset C of P is called a sub chain of P if C is a chain.***

  i.e., any two elements in C are comparable.

  - ***NOTE:*** ***A sub chain of a poset is not convex.***

- ***MAXIMAL CHAIN:*** ***Let $\bar{C}$ be a sub chain in the poset C. $\bar{C}$ is called a maximal chain in P if $\bar{C}$ is maximal element in the poset C that means there is no sub chain c € C such that $\bar{C} \supset C$***

- ***CHAIN AXIOM:*** ***For any sub chain C of a partly ordered set P there exists atleast one maximal chain $\bar{C} \supset C$.***

- ***STATE AND PROVE KURATOWSKE - ZORNS LEMMA :***

- ***STATEMENT:*** ***If every sub chain of a nonempty partly ordered set P has an upper bound then P contains maximal element .***

  PROOF: ***let P be a partly ordered set.***

Assume that every sub chain of P has an upper bound
Let C be any sub chain of the poset P then by chain axiom there exists atleast one maximal chain $\overline{C}$ *such that* $\overline{C} \supset C$.
Then by assumption, the maximal chain $\overline{C}$ *also has an upper bound m.*
Further, this upper bound m is itself the maximal element of P.
If not, then there is an element s € P such that s>m.
Since m is an upper bound of $\overline{C}$
We have m>t for t€ $\overline{C}$
Now we have s>m>t.
Now consider, $\overline{C}$ ⊠ *{s}*
This is a sub chain containing $\overline{C}$.
This is a contradiction to the fact that $\overline{C}$ *is a maximal chain*
Therefore m is the maximal element of P.
Hence, if every sub chain of a non- empty partly ordered set P has an upper bound then P contains maximal element.

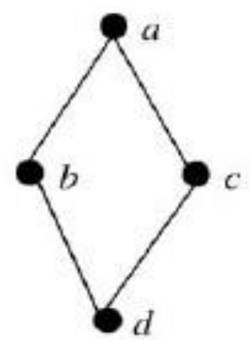
→ ***LENGTH :***

DEFINITION:

- ***LENGTH:*** *By the length of a chain consisting of r elements being of the form* $x_0 < x_1 <- - - < x_{r-1}$. *we mean the non - negative integers r-1.*
If a chain consisting of an infinite number of elements then the length of the chain is infinite.
Symbolized by ∞ .
By the length of a partly ordered set P. we mean that the least upper bound of the set of all lengths of the chains of

the poset P.

✓ ***PROBLEM:** Find the length of the poset*



***SOLUTION:** The length of chains with one element = 0*

The length of chains with two elements = 1

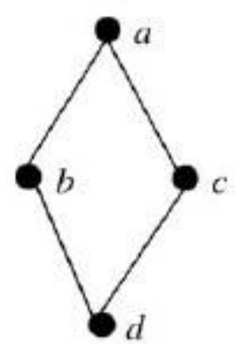
The length of chains with three elements = 2

∴ {0, 1, 2} is the set of lengths of all sub chains of the given poset.

LUB of {0, 1, 2} = 2.

- ***LOCALLY FINITE LENGTH:** Let P be a poset. P will be said to be of locally finite length if every one of its intervals is of finite length.*

- ***EXAMPLE:** Let us consider the poset*



This poset is of locally finite length.

- ***NOTE:* Every finite partly ordered set is of finite length.**

→ *Lower and upper bounds:*

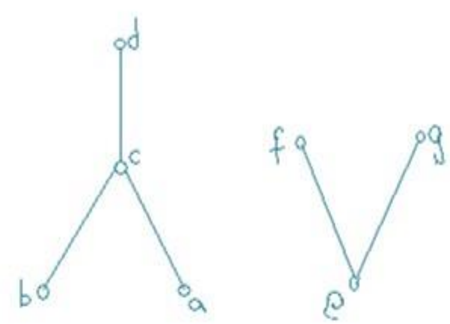
Let R be a non –empty subset of a partly ordered set P. An element a is called the upper bound of R. if $x \leq a \ \forall x \in R$.
An element b is called the lower bound of R if $x \geq b \ \forall x \in R$.
R is called a bounded above subset of P if R has a upper bound.
R is called a bounded below subset of P if R has a lower bound.
R is called a bounded subset of P if R is both bounded Above & bounded below subset of p.
Clearly, the set of lower bounds of a in the poset P is (a] , similarly the set of upper bounds of a in the poset P is [a).
Let R be a non – empty subset of the poset P. An element a in P is called the least element of R if a is a lower bound of R and a is contained in R.
An element b is called the greatest element of R if b is an upper bound of R and b is contained in R.

- ***NOTE:* The least & greatest element of R are called bounds of R. Further, other elements of R are called inner elements.**

❀ ***DEFINITION:* Let R be a non – empty subset of a poset P. Let U be the set of all upper bounds of R.**
Let L be the set of all lower bounds of R. The least element of U is called the least upper bound of R or supremum of R. It is denoted by $SUP_p R$.

The greatest element of L is called the greatest lower bound of R or infimum of R. It is denoted by INF_pR.

✓ ***PROBLEM:*** ***Find the infimum& supremum of the subsets {a, b}, {c, d} in the following poset***



SOLUTION: ***Let R = {a, b}***

The set of upper bounds of a is [a) = {a, c, d}

The set of upper bounds of b is [b) = {b, c, d}

The set of upper bounds of R = {a, b} = {c, d}

The least upper bound of R is c.

$\Rightarrow SUP_pR = c.$

The set of lower bounds of a = {a}

The set of upper bounds of b = {b}

The set of upper bounds of R = ϕ

$\therefore$ ***R = {a, b} has no infimum.***

The set of upper bounds of c is [c) ={ c , d}

The set of upper bounds of d is [d) = {d}

The set of upper bounds of R = {c , d}

The least upper bound of R is d.

$\Rightarrow SUP_pR = d.$

The set of lower bounds of c = {a, b, c}

The set of lower bounds of d = {a, b, c, d}

The set of lower bounds of R = {a, b, c}

$\Rightarrow INF_pR = c.$

- ***ISOMORPHISM INVARIANT PROERTY OF SUPREMUM AND INFIMUM:***
- ***STATEMENT:*** ***Let ϕ be an ordered isomorphism of the partly ordered set P_1 onto the partly ordered set P_2. If a subset R_1 of P_1 has an infimum in P_1. The set $P_2 = \{\phi(x)/x€R_1\}$ will have an infimum in P_2. $INF_{P2}R_2 = \phi(INF_{P1}R_1)$. The corresponding statement for suprema also holds.***

PROOF: ***Given that***

P_1 and P_2 are two partly ordered sets & ϕ: $P_1 \to P_2$ is an ordered isomorphism. Also R_1 be a non-empty subset of P_1 and R_1 has infimum in P_1 say $INF_{P1}R_1 = U$.

To prove that the set $P_2 = \{\phi(x)/x€R_1\}$ will have an infimum in P_2 . $INF_{P2}R2 = \phi(INF_{P1}R_1)$.

Let y be any arbitrary element of R_2 then $\phi^{-1}(y)$ € R_1

Since INF R_1 = U

We have $U \leq \phi^{-1}(y)$

Since ϕ is an ordered homomorphism.

So $\phi(u) \leq \phi(\phi^{-1}(y))$

$\Rightarrow \phi(u) \leq y$.

Because y is arbitrary, we get $\phi(u)$ is a lower bound of R_2.

Next, we prove that $\phi(u)$ = INF R_2.

Let t be any lower bound of R then $\phi^{-1}(t)$ € R_1

Further, we have $\phi^{-1}(t) < U$

$\Rightarrow \phi^{-1}(\phi(t)) \leq \phi(u)$.

$\Rightarrow t \leq \phi(u)$.

$\phi(u)$ = INF R_2

Hence, INF $R_2 = \phi(INF_{P1}R_1)$

On similar lines we can prove that the statement for suprema also.

$\rightarrow$ ***THE MINIMUM AND MAXIMUM CONDITIONS:***

- ***DEFINITION: Let C_0 be any arbitrary elements of a partly ordered set P. Let us form a sub chain of P in the following way. Let the greatest element of the sub chain be c_o .***
  Otherwise there exists an element c_1€ p such that $c_0 < c_1$.
  Let c_1 be the greatest element of a sub chain otherwise there exist an element c_2 € p such that $c_0 < c_1 < c_2$. If each of the chains so formed coming at any c_0 is finite then P is said to satisfy the minimum condition.

In the above discussion if symbol < is replaced by >then P is said to satisfy the maximum condition.

Example: *- let us consider the poset (z^+,$\leq$)*

We have the chain $1 \leq 2 \leq 3 \leq$ -----

Clearly, this poset satisfies minimum chain condition. But no maximum chain condition.

Example: *- let us consider the poset (Z^-,$\geq$) we have the chain*

$-1 \geq -2 \geq -3 \geq$ --------In this poset satisfies maximum chain condition but no minimum chain condition.

- ***THEOREM:***- *If a partly ordered set P satisfies the minimum condition (maximum) then any x € p there exists atleast one minimal (maximal)element m of P such that $x \geq m$ ($x \leq m$).*

PROOF: *let P be a partly ordered set satisfying the minimum condition.*

Let x€ p.

If x is itself a minimal element of P. Then x=m.

If x is not minimal then there exist an element X_1€ p such that $x>x_1$.

If x_1 is minimal then the proof will be completed by taking $m=x_1$.

If x_1 is not minimal then there exist an element x_2€ p such that $x>x_1>X_2$.

Continuing this argument in this way,

Since P satisfies minimum condition, there exists an element x_r in P such that $x>x_1>x_2>$------- $> x_r$.

Take $x_r = m$.

$\therefore$ For every element x€ p there exist a minimal element m of P such that $x \geq m$.

We repeat the above proof to where the poset Satisfies maximum condition by interchanging the symbols > and <.

- ***COROLLARY:-** Every sub chain of a po subset satisfying the maximum(minimum)condition has a greatest (least)element.*

PROOF:-Let every sub chain of a po subset satisfying the maximum condition.

BY known theorem, a chain can have no more than one maximal element and further in any sub chain the maximal Element is same as the greatest element.

Hence, every sub chain of a po subset satisfying the maximum condition has a greatest element.

Let every sub chain of a po subset satisfying the minimum condition.

BY known theorem, a chain can have no more than one minimal element and further in any sub chain the minimal Element is same as the least element.

Hence, every sub chain of a po subset satisfying the minimal condition has a least element.

- ***THEOREM:-A partly ordered set can be satisfy both the maximum and minimum conditions if and only if every one of its sub chains is finite.***

PROOF: - Let P be a poset.

First we prove the converse part.

Assume that every one of its sub chains is finite.

Immediately, we have poset satisfies the both maximum and minimum conditions.

Next, we prove the necessary part.

Assume that the poset satisfies both maximum and minimum conditions.
By the known corollary of a theorem,
Every Sub chain of p has a least element.
Let c_0 be a sub chain of P.
Let us construct a sequence of sub chains in the following way.
$C_{t+1} \supset C_t$, where C_t is a sub chain obtained from C_{t+1} by deleting the least element of C_{t+1} then we get a sub chain

$C_0 \supset C_1 \supset C_2 \supset \text{--------} \supset C_i \supset C_{j+1} \supset \text{------}$

This is an infinite sub chain.
Further we can form a sub chain.

$C_0 > C_1 > C_2 > \text{-------} C_j > C_{j+1} > \text{------}$

Which is also an infinite chain. But, it is a contradiction.
Because P satisfies both minimum and maximum conditions.
$\therefore$ Every sub chain of P is finite.

→ JORDAN DEDEKIND CHAIN CONDITIONS AND DIMENSION FUNCTION:

- **DEFINITION**: Let P be a poset. a and b are two elements of P such that a < b. Let c be a sub chain of P having a as the least element and b as the greatest element, then we say that the chain c is situated between the elements a and b (or) c connects the elements a and b.
If the chain c is maximal chain then we say that the maximal chain is connecting the elements a and b. we can easily observe

that the number of maximal chains between the two elements a and b is > 1 and of different lengths.

- ***<u>JORDAN – DEDEKIND CHAIN CONDITION:</u>***

  If for every of elements a, b ($a \leq b$) of a poset P, if it is true that all maximal chains connecting the elements a and b are of the same length then the set is said to satisfy the JORDAN – DEDEKIND CHAIN CONDITION.

- ***<u>DEFINITION</u>: By the height (or) dimension h(a) of the element a of the poset P. A bounded below is meant the length of the interval [0 a].***

- ***<u>NOTE:</u> h(x) = 0 if and only if x = 0.***

THE END

DNRCOLLEGE(A),BHIMAVARAM

P.GDEPARTMENTMATHEMATICS

305–LATTICETHEORY

IIMSc,SEMESTER-III

UNIT-2

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LATTICESINGENERAL

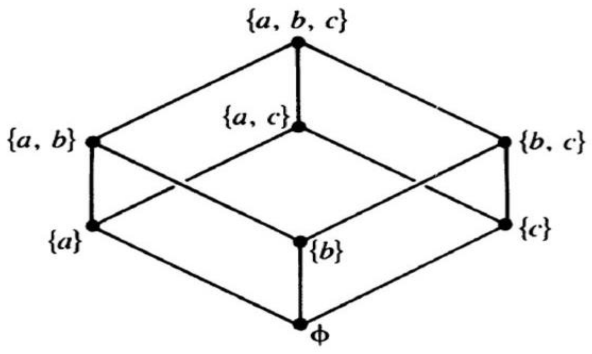
- **DEFINITION:LATTICE:**

  A set L is called a "LATTICE" if there are defined in L two operations MEET ($\cap$) and JOIN ($\cup$) which assign to each pair a, b of elements of L uniquely an element $a\cap b$ as well as an element $a\cup b$ satisfying the following lattice axioms.

  1) **Foranyelementsa, b, c€L,$(a\cap b)\cap c=a\cap(b\cap c)$**
  2) **Foranyelementsa, b, c€L,$(a\cup b)\cup c=a\cup(b\cup c)$**
  3) **Foranyelements a, b€L,$a\cap b=b\cap a$**
  4) **Foranyelements a, b€L,$a\cup b=b\cup a$**
  5) **Foranyelementsa,b€L,$a\cap(a\cup b)=a$**
  6) **Foranyelementsa,b€L,$a\cup(a\cap b)=a$**

- **EXAMPLE:LetMbetheset{a,b,c}**

  ConsiderthepowersetP(M)whichisalatticeundertheset intersection ($\cap$) and set the union ($\cup$)



- **PROBLEM:usingabsorptionlawsinlattices.Provetheidempotent laws $a\cup a = a\cap a = a\ \forall a \in L$.**

  SOLUTION:LetLbelatticeanda,b€L Now

  by absorption laws we have

  I. **$a\cap(a\cup b)=a$**
  II. **$a\cup(a\cap b)=a$**

  CLAIM: **$a\cup a=a, a\cap a=a\ \forall a \in L$.**

Now $a\cup a=a\cup(a\cap(a\cup b))$

$=a$

$\Rightarrow a\cup a=a$

Next, $a\cap a=a\cap(a\cup(a\cap b))$

$=a$

$\Rightarrow a\cap a=a$

Henceidempotentlawscanbe obtainedfromabsorptionlawsin Lattice.

- **COROLLARY: For any elements a, b of a lattice $a\cap b = a\cup b$ if and only if $a = b$.**

PROOF: **LetLbealatticeanda,b€L**

First,weprovetheconversepart. Assume that $a = b$

CLAIM: $a\cap b= a\cup b$

Now $a \cap b=a\cap a$
$=a$
$=a\cup a$
$=a\cup b$
$\therefore a \cap b = a \cup b$ Therefore,
$a = b \Rightarrow a \cap b = a \cup b$ Next,
Assume that $a \cap b = a \cup b$

CLAIM: $a=b$

Now, $a=a\cup(a\cap b)$
$=a\cup(a\cup b)$
$=(a\cup a)\cup b$
$\therefore a=a\cup b$ --------→(1)
$b=b\cup(b\cap a)$
$=b\cup(b\cup a)$
$=(b\cup b)\cup a$
$=b\cup a$
$\therefore b=a \cup b$------- →(2)

From(1)&(2)

$$a = b$$

$\therefore a \cap b = a \cup b \Rightarrow a = b$

$\therefore a \cap b = a \cup b \Leftrightarrow a = b$

→THELATTICETHEORETICALDUALITYPRINCIPLE:

By a lattice theoretical proposition. We mean a statement A whetheritistrueorfalseandvariablesoninterchangingthesymbols $\cap$and$\cup$. We again obtain a lattice theoretical statement.This statementiscalledthedualofthestatementA.itisdenotedbyD(A). This process is called dualization again dualizating D (A) i.e., D (D (A)) we get the original lattice theoretical statement.

Hence,D(D(A))=A.

LetAbealatticetheoreticalpropositionondualizationofD(A)=A then A is called self – dual statement.

THEOREM: Every lattice has the following property:

Foranyelementsa,boflattice,$a \cap b = b$ifandonlyif $b \cup a = a$.

PROOF: Let L be a lattice and for any element a,b€L.

Assume $a \cap b = b$.

Claim:$b \cup a = a$.

$$
\begin{aligned}
b \cup a &= (a \cap b) \cup a \\
&= (a) \cup (a \cap b) \\
&= a
\end{aligned}
$$

$\therefore a \cap b = b \Rightarrow b \cup a = a$

Next,Assume$b \cup a = a$.

Claim:$a \cap b = b$

$$
\begin{aligned}
a \cap b &= (b \cup a) \cap b \\
&= b \cap (b \cup a) \\
&= b
\end{aligned}
$$

$\therefore b \cup a = a \Rightarrow a \cap b = b$

Hence$a \cap b = b \Leftrightarrow b \cup a = a$.

- **DEFINATION:DUALOFALATTICE:**

Let(L∩∪)bealattice.Nowdefinethetwooperations∧and∨ asa∧b=a∩banda∨b=a∪bfora,b€Lunderthesetwo operations

∧and ∨. We get a newlattice D(L). This new lattice is called the DUAL OF THE LATTICE L.

IfD(L)isisomorphictoL. i.e.,D(L)≈Lthenwesaythatthelattice L is SELF-DUAL.

→SEMI-LATTICES:

- **DEFINATION:SUBLATTICE:** **letusrecallthedefinitionofasublattice.Let(L∩∪)be a lattice. S is a nonempty subset of L. If S is formed a lattice under theoperations ∩and ∪defined on L then S is called sublattice of L.**
- **SEMI LATTICE: A set H is called a SEMI LATTICE if there is defined in H an operation which assigns to each pair of elements a, b an element a ob satisfying the following axioms:**
    1. **(aob)oc=ao(boc)∀a,b,c€H**
    2. **a ob=boa∀a,b€ H**
    3. **a oa =a∀a €H**

Letusconsiderthelattice(L∩∪).

Clearly, (L∩)isasemilatticeandfurther(L∪)isalsoasemilattice.

The semi lattice (L ∩) is called a MEET SEMI LATTICE and the semi lattice (L ∪) is called JOIN SEMI LATTICE.

Hence,everylattice(L∩∪)isbothmeetsemilatticeandjoinsemilattice.

→ LATTICESASPOSETS:

❁**THEOREM:Withrespecttotheordering≤definedby a≤b⇔a∩b=afora,b€LonalatticeL.Everyfinitesubset $\{a_1,a_2,---,a_n\}$ofLhasinfimumandsupremum namely $inf_L\{a_1,a_2, ---, a_n\}=\bigcap^n_{j=1} aj$,$sup_L\{a_1, a_2,---, a_n\}=\bigcup^n_{j=1} aj$.**

PROOF: **Giventhat**

The ordering≤isdefined onLbya≤b $\Leftrightarrow a\cap b=a$fora,b€L Let $\{a_1, a_2, ---, a_n\}$ be a finite subset of L.

Toproveourtheorem,weprovethefollowing.

$\bigcap_{j=1}^{n} aj \le a_1,a_2,---,a_n \le \bigcup_{j=1}^{n} aj \quad \rightarrow(1)$

$u \le a_1,a_2,---,a_n \Rightarrow u \le \bigcap_{j=1}^{n} aj \quad \rightarrow(2)$

$u \ge a_1,a_2,---,a_n \Rightarrow u \ge \bigcup_{j=1}^{n} aj \quad \rightarrow(3)$

where

$\bigcap_{j=1}^{n} aj=\inf_L\{a_1,a_2,---,a_n\}$

$\bigcup_{j=1}^{n} aj=\sup_L\{a_1,a_2,---,a_n\}$

Now, $\bigcap_{j=1}^{n} aj \cap a_k=(a_1\cap a_2\cap---a_k\cap---\cap a_n)\cap a_k$

$=(a_1\cap a_2\cap---a_k\cap---\cap a_{n-1})\cap(a_n\cap a_k)$

$=(a_1\cap a_2\cap---a_k\cap---\cap a_{n-1})\cap(a_k\cap a_n)$

Bycommutativeandassociativeaxioms,

continuing this way, we get

$=a_1\cap a_2\cap---\cap a_k\cap a_k\cap---\cap a_n$

$=a_1\cap a_2\cap---\cap a_k\cap---\cap a_n$

$=\bigcap_{j=1}^{n} aj$

$\Rightarrow \bigcap_{j=1}^{n} aj\cap ak=\bigcap_{j=1}^{n} aj$

$\Rightarrow \bigcap_{j=1}^{n} aj \le a_k$ fork=1,2,---,n

$\Rightarrow \bigcap_{j=1}^{n} aj \le a_1,a_2,---,a_n$

Letubeany otherlowerboundof$\{a_1,a_2,---,a_n\}$ $u \le a_1, a_2, ---, a_n$

Now, $u\cap\bigcap_{j=1}^{n} aj=u\cap(a_1\cap a_2\cap---\cap a_k\cap---\cap a_n)$

$=(u\cap a_1)\cap(a_2\cap---\cap a_k\cap---\cap a_n)$

$=u\cap(a_2\cap---\cap a_k\cap---\cap a_n)$

Continuingthisprocess,weget

$=u$

$\Rightarrow u\cap\bigcap_{j=1}^{n} aj=u$

$\Rightarrow u\le\bigcap_{j=1}^{n} aj$

$\therefore \bigcap_{j=1}^{n} aj = glb\{a_1,a_2,---,a_n\}$

i.e., $\bigcap^{n}{}_{j=1} aj=\inf_L\{a_1,a_2,---, a_n\}$

Bydualizingtheaboveproof,wecanget

$a_1, a_2, ---, a_n \leq \cup^n{}_{j=1}\, aj$ and $u \geq a_1, a_2, ---, a_n$

$$\Rightarrow u \geq \cup_{j=1}^{n} aj$$

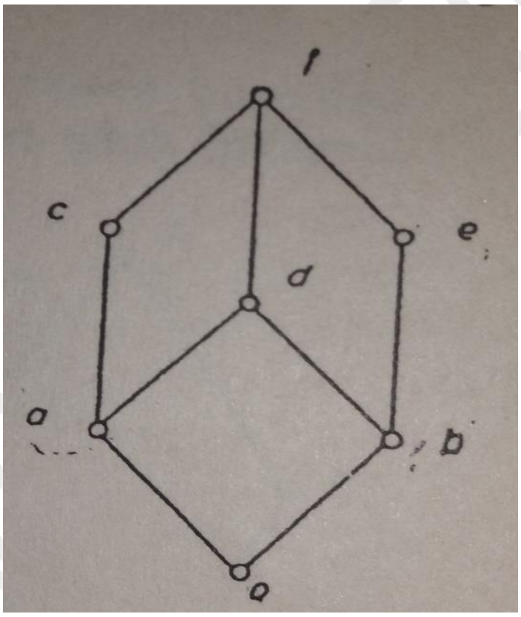
Hence,thiscompletestheproofofthetheorem.

→DIAGRAMSOFLATTICES:

We know that Every finite partly ordered set can be represented by a diagram. Further, we have prove that the two lattices are same diagram if and only if there are order isomorphic.

Clearly,anyfinitelatticeisuniquelydeterminedbyitsdiagram up to isomorphic.

- **EXAMPLE:** **Verifythefollowingdiagramislattice(or)not**



SOLUION: **Toshowthatthediagramrepresentsalattice(or) not We verify the following fact for all a, b € L**

$a \cup b = lub\{a,b\}$€L

$a \cap b = glb\{a,b\}$€L

Foranyelementsxof thelattice,$o \cap x = o, o \cup x = x$

$i \cap x = x, i \cup x = i$,and

∩	a	b	c	d	e
a	a	o	a	a	o
b	a	b	o	b	b
c	a	o	c	a	o
d	a	b	a	d	b
e	o	b	o	b	e

∪	a	b	c	d	e
a	a	d	c	d	i
b	b	b	i	d	e
c	c	i	c	i	i
d	d	d	i	d	i
e	i	e	i	i	e

Fromthetables, itisclearthateverypairhaslub, glbinL. Hence, the given diagram represents a lattice.

→SUBLATTICES&IDEALS

- **DEFINITION:SUBLATTICE:** **Let (L ∩∪) be a lattice. A non empty subset R of L is called a SUBLATTICE of L if**
  - **I. Foralla,b€R,a∩b€R**
  - **II. Foralla,b€R,a∪b€R.**

EXAMPLE:LetNbeasetofpositiveintegers

a∩b=glb{a,b}anda∪b=lub{a,b}thenNisa lattice.

Letα€Nbethesetofallmultiplesofαisasublatticeof N

Next,letβ€NbethesetofalldivisorsofβisasublatticeofN.

THEOREM: Let Lbe a lattice thenany subchain of L is a sublattice of L.

PROOF: LetLbealatticeandCbeanysubchainofL

CLAIM: CisasublatticeofL

Leta,b€C

thenwe have,either$a\leq b$(or)$b\leq a$ suppose

$a \leq b$ then $a \cap b = a$ € C

$\Rightarrow a\cap b$€C

$a \cup b=b$€ C

$\Rightarrow a\cup b$€C

Next, $b\leq a$thena$\cap b=b$€C

$\Rightarrow a\cap b$€C

$a\cup b=a$€C

$\Rightarrow a\cup b$€C

$\therefore$ThesubchainCisasublattice.

Hence,everysubchainofLisasublatticeofL.

- **DEFINITION:IDEAL:**AsubsetIofalatticeLiscalledanIDEAL of L if I satisfies the following conditions.
  i. a,b€I$\Rightarrow a\cup b$€I
  ii. foranyelementx€L,a€I $\Rightarrow a\cap x$€I

RESULT: Every ideal of a lattice L is a sublattice.

PROOF: LetLbealatticeandletIbeanideal of L

CLAIM:I is a sublattice of L

I. Leta,b€I

$\Rightarrow a \cup b$€I

II Leta,b€I

$\Rightarrow$a€Iandb€L

then $a \cap b$ € I

$\therefore$IisasublatticeofL.

→BOUND ELEMENTS OF A LATTICE, ATOMS & DUAL ATOMS

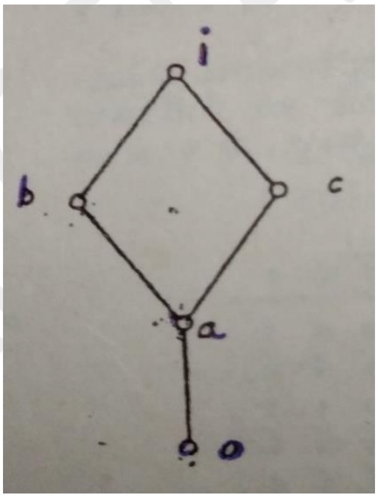
- **DEFINITION: BOUNDELEMENTSOFALATTICE:**

IfalatticeLhasanelementosuchthat $o \le x \; \forall x €L$ iscalled the LEAST ELEMENT of L.

If alatticeLhasanelementisuchthat $i \ge x \; \forall x €L$ iscalled the GREATEST ELEMENT of L.

Theelementsoandiarecalledthe BOUNDELEMENTSofL.

- **EXAMPLE:** **considerthelattice**



Inthislatticetheboundelements are

o, i

- **ATOM:**Anelementpof alatticeboundedbelowiscalledanatom if p >−o
- **ATOMICLATTICE:**LetLbealatticeifforeachelement $x(\neq 0)$ of L there can be found all atoms p such that $x > p$ then the lattice L is called atomic lattice.
- **DUALATOM:**Anelementm ofalatticeLisboundedaboveis called a dual atom if m −<i

  i.e.,thereisnoelementinmandi.Liscalledduallyatomic lattice for any element $x(\neq i)$there is a dual atom m suchthat $x \le m$.

→COMPLEMENTS,RELATIVELYCOMPLEMENTSAND SEMI COMPLEMENTS

LetLbeaboundedlatticeandube anyelementofL.

Byacomplementofuismeantany xofLsatisfyingtheequation u $\cap$x = o and u $\cup$x = i

- **DEFINITION: In an element u of a bounded lattice L has atleast one complement then u is called a complemented element of l.**

  IfallelementsofLarecomplementedthenLiscalledacomplemented lattice.

  Ifuhasexactlyonecomplementthenuiscalledauniquelycomplemented element of L.

  IfallelementsareuniquelycomplementedthenLiscalled uniquely complemented lattice.

THEOREM: Every lattice has at most one minimal and one minimal element these elements are at the same time the least and greatest element of that lattice.

PROOF:LetLbealattice.

Letx,mbeanyarbitraryelementsofL. We have m $\cap$x $\leq$ m

Ifmistheminimal,thenm$\cap$x<misnotpossible. This is possible only of m $\cap$x = m $\Rightarrow$m $\leq$ x

Wherexisarbitraryelement.

$\therefore$Thestatementisprovedforminimalelement.

Clearly,wecanprovethestatementistrueformaximalelement.

- **DEFINITION: RELATIVELYCOMPLEMENTINLATTICE:AlatticeL issaidtoberelativelycomplementedifforanytripletof elements,a, b, u(a $\leq$ u$\leq$ b) there is atleast one complement of "u'' in [a, b].**
- **DEFINATION: SECTIONCOMPLEMENTEDLATTICE:LetLbealattice boundedbelow.Liscalledsectioncomplementedifeachintervalof the form [o, a] for a € L is a complemented sublattice of L**
- **DEFINITION:Let L be a bounded below lattice by a semi complement ofanelement uofL.wemeaneveryelement xofLsuch that u $\cap$x = 0**

  bytheabovedefinitionitisclearthatuisthesemicomplementof x in L.
- **DEFINITION:WEAKLYCOMPLEMENT: LetLbeaboundedbelowlattice.Liscalledweaklycomplementedif for a, b (a < b) in L. a has a semi complement that is not a complement of b.**

THEOREM: Every weakly complemented lattice is semi complemented.

PROOF: LetLbea lattice
which is weakly complemented and a be any inner element of L
Then"a"cannotbeamaximalelementthenthereexistanelement b € L such that $a < b$
since,Lisweaklycomplemented.
ahasasemicomplement ofxwhichis notasemicomplement ofb. i.e., $a \cap x = 0$ & $b \cap x \neq 0$
from this discussion it is clear that x is a proper semi complement of a.
$\therefore$ ThelatticeLissemicomplemented.

Hence,everyweaklycomplementedlatticeissemicomplemented.

THEOREM: Everysectioncomplementedlatticeboundedbelowis weakly complemented.

PROOF: LetLbeasectioncomplementedlatticeboundedbelow.

CLAIM: Lisweaklycomplemented Let
a, b € L, $a \leq b$
Clearly, $0 < a < b \Rightarrow a$ € [0, b]
Since,Lissectioncomplemented.
ahas acomplementsayxin[0, b]thena $\cap x=0$, a $\cup x=b$ This says that a is semi complemented.
Nowwe needtoshowthatb$\cap x \neq 0$ b $\cap x$
$= (a \cup x) \cap x = x$
Nowweprovethat $x \neq 0$
Ifpossible,supposethatx=0 then b
$= a \cup x = a \cup 0 = a$
$\Rightarrow b=a$
Whichisacontradictiontoa<b
$\therefore x \neq 0$thatmeansb$\cap x \neq 0$
Hence,Lisweaklycomplementedlattice.

THEOREM: Every uniquely complemented lattice is weakly complemented.

PROOF: LetLbeauniquelycomplementedlattice

CLAIM: Lisweaklycomplemented Let
a, b € L, $a < b$
Since,aisuniquelycomplemented.

Wehavethat$\exists$anelementa$'$inL $\ni a\cap a'=0$&$a\cup a'=i$ Now we need to prove that $b\cap a'\neq 0$

Wehavea<borb>a

$\Rightarrow b\cup a'\geq a\cup a'$

$\Rightarrow b\cup a'\geq i$

$\Rightarrow b\cup a'=i$

$\therefore b\cap a'=0, b\cup a'=i$

$\therefore$biscomplementofa$'$

Ifb$\cap$a$'$=0thusbisacomplementof a$'$

Fromtheabovediscussionaiscomplementofa$'$

$\therefore$Theelementa$'$hastwocomplementsinL

Thisisacontradictiontothedefinitionofuniquelycomplemented lattice.

Hence, $b\cap a'\neq 0$

Hence,Lisweaklycomplemented.

→IRREDUCIBLE&PRIMEELEMENTSOFALATTICE

- **DEFINITION:**
  MEETREDUCIBLE&MEETIRREDUCIBLE:

AnelementaofalatticeLissaidtobemeetreducibleifthereexists an element $a_1, a_2 \in$ L such that $a = a_1\cap a_2 (a_1, a_2 > a)$

If some a has no decomposition $a = a_1\cap a_2(a_1, a_2 > a)$ then it is said to be meet irreducible.

- **DEFINITION:**
  MEETREDUCIBLE&MEETIRREDUCIBLE:

Duallyspeakingwegetthefollowing.

An element a of a lattice L is said to be join reducible if there exists an element $a_1, a_2 \in$ L such that $a = a_1\cup a_2 (a_1, a_2 < a)$

If some a has no decomposition $a = a_1\cup a_2(a_1, a_2 < a)$ then it is said to be join irreducible.

THEOREM: In a lattice satisfying maximum condition, every one of its elements can be represented as the meet of a finite number of meet irreducible elements.

PROOF:LetLbealatticesatisfyingmaximumconditionthen

immediately we have L has a greatest element i

CLAIM:Everyoneofitselementscanberepresentsasthemeetofa

finitenumberofmeetirreducibleelements.

LetH=thesetofelementsinLwhichcannotberepresentedasthe meet of finite number of meet irreducible elements.

NowweshowthatH= ϕ

Ifa€Hthenacanbewrittenas a=a $\cap$i(or)a=a $\cap$a

Therefore, a cannot be meet irreducible.

Hence,Hcontainsnomeetirreducibleelements. If possible, suppose H $\neq\phi$

Since,LsatisfiesmaximumconditionwehaveHisalsosatisfiesmaximum condition then H has atleast one maximal element say m.

Inviewoftheaboveargumentmismeetreduciblethen$\exists$two elements $m_1, m_2 \in L \ni m = m_1 \cap m_2$ $(m_1, m_2 > m)$ $\dashrightarrow(1)$

Since,mismaximalinH m $< m_1$; $m < m_2$

we havem$_1$,m$_2$arenottheelementsofH. By the def of H,

m$_1$,m$_2$arecanberepresentedasthemeetofafinite numberofmeet irreducible elements then $m_1= q_1\cap q_2 \cap\text{---} \cap q_s$

$m_2=r_1\cap r_2\cap\text{---}\cap r_t$ where $q_j > m_1$, j = 1, 2, --- , s

$r_i>m_2$, i=1,2,---, t

from(1),

$m=(q_1\cap q_2\cap\text{---}\cap q_s)\cap(r_1\cap r_2\cap\text{---}\cap r_t)$

wegetmisrepresentedasthemeetofafinitenumberofmeetirreducible elements $q_1, q_2, \text{---}, q_s, r_1, r_2, \text{---}, r_t$

whichis a contradiction

$\therefore$H= ϕ

Hence,everyoneofitselementscanberepresentsasthemeetof a finite number of meet irreducible elements.

- **DEFINITION:**

  MEET PRIME: An element "a" of a lattice L is called meet prime if $a_1\cap a_2 \leq a \Rightarrow a_1\leq a$ (or) $a_2\leq a$ (or) both hold.**

- **DEFINITION:**

  JOINPRIME: Anelement"a"ofalatticeLiscalledjoinprimeifa $\leq a_1\cup a_2\Rightarrow a\leq a_1$(or)a$\leq a_2$(or)both hold.

THEOREM: In a complemented lattice every join (meet) prime element except the least (greatest) one is an atom (dual atom) of the lattice.

PROOF: Let L be a complemented lattice and p be a join prime element (is not equal to zero) $\neq 0$ in L.

CLAIM: pisanatominL

Supposethat, qbeanelement of lsuchthat$0 \leq q \leq p$ Since, q € L

Itiscomplementedlattice.

$\Rightarrow$qhasacomplementq'inL

Then $q \cap q' = 0$, $q \cup q' = i$

Then $p \leq i$

$\Rightarrow p \leq q \cup q'$

Sincepisajoinprimep $\leq q$(or)$p \leq q'$butp$\nleq q$

$\Rightarrow p \leq q'$

Now$q = p \cap q$

$\leq q' \cap q'$

$= 0$

$\therefore q = 0$

$\therefore$ pisanatominL

Hence, Inacomplementedlatticeeveryjoinprimeelementexceptthe least one is an atom of the lattice.

Duallyspeaking,similarlywecanproveinacomplementedlattice every meet prime element except the greatest one is an atomof the lattice.

→HOMOMORPHISMOFALATTICE

- **ORDERHOMOMORPHISM(OR)ORDERPRESERVING:** Let us recall the definition of order homomorphism. Let L_1, L_2 be two latticesand$\phi: L_1 \rightarrow L_2$beasinglevalued mapping. Wesaythat ϕisan order homomorphism or order preserving
  Ifforeverypairof elements, a, b€L_1 $a \leq b \Rightarrow \phi(a) \leq \phi(b)$
- **HOMOMORPHISM:** LetL_1&L_2 betwolattices and $\phi: L_1 \rightarrow L_2$be a single valued mapping. We say that ϕ is a homomorphism.
  Ifforeverypairofelements,a,b€L_1
  i. $\phi(a \cap b) = \phi(a) \cap \phi(b)$
  ii. $\phi(a \cup b) = \phi(a) \cup \phi(b)$

- **JOINHOMOMORPHISM:** **LetL$_1$&L$_2$ betwolatticesand ϕ: L$_1$$\to$L$_2$ be a single valued mapping. We say that ϕ is a join homomorphism.**
  Ifforeverypairofelements,a,b€L$_1$
  i. $\phi(a\cup b)=\phi(a)\cup\phi(b)$
- **MEETHOMORPHISM:** **LetL$_1$&L$_2$betwolatticesand ϕ:L$_1$$\to$L$_2$ beasinglevaluedmapping.Wesaythat ϕisameet homomorphism.**
  Ifforeverypairofelements,a,b€L$_1$
  i) $\phi(a\cap b)=\phi(a)\cap\phi(b)$

from the above definitions it is clear that ϕ is a homomorphism of $L_1^{\cup}$into $L_2^{\cup}$and $L_1^{\cap}$into $L_2^{\cap}$

- **PROBLEM:** **provethateveryhomomorphicimageofalattice bounded below is likewise bounded below.**

SOLUTION: **LetLbealatticeboundedbelowthen0€L Let ϕ(L) be a homomorphic image of L**

CLAIM: **ϕ(L) is bounded below.Since,Lhasaboundedbelowthen0€ L We have $0 \le x\ \forall x$ € L**

$\Rightarrow 0\cap x=0$

$\Rightarrow \phi(0\cap x)=\phi(0)$

$\Rightarrow \phi(0)\cap\phi(x)=\phi(0)$

$\Rightarrow \phi(0)\le \phi(x)\forall x$€L

$\therefore$ **ϕ(0)istheleastimageinϕ(L)**

Hence,thehomomorphicimageϕ(L)isboundedbelow.

- **DEFINITION:**
  KERNELOFAHOMOMORPHISM:

LetL$_1$,L$_2$betwolatticesandletϕ:L$_1$$\to$L$_2$beahomomorphism.If L$_2$ has the least element 0_2 then the set $\{x$ € $L_1/\ \phi(x) = 0_2\}$ is called the kernel of a homomorphism ϕ and it is denoted by K_ϕ

- **THEOREM:** **Ifahomomorphismofalatticehasakernel,thiskernel is an ideal of the lattice.**

PROOF: **Letϕ:L$_1$$\to$L$_2$ beahomomorphismand0_2 €L$_2$ We know that $K_\phi = \{x$ € $L\ /\ \phi(x) = 0_2\}$**

CLAIM: **$K_\phi$isanideal**

Leta,b€K_ϕ

Then$\phi(a)=0_2$ and$\phi(b)=0_2$

i) $\phi(a\cup b)=\phi(a)\cup\phi(b)$

$=0_2\cup 0_2=0_2$

$\Rightarrow\phi(a\cup b)=0_2$

$\therefore a\cup b \in K_\phi$

ii) Let $a\in L_1$ and $b\in K_\phi$

$\Rightarrow\phi(b)=0_2$

$\phi(a\cap b)=\phi(a)\cap\phi(b)$

$=\phi(a)\cap 0_2$

$=0_2$

$\Rightarrow\phi(a\cap b)=0_2$

$\Rightarrow a\cap b\in K_\phi$

$\therefore K_\phi$ is an ideal.

DNR COLLEGE (A), BHIMAVARAM

P.G DEPARTMENT MATHEMATICS

305 – LATTICE THEORY

II MSc, SEMESTER -III

UNIT - 3

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COMPLETE LATTICES

- **NOTATION:** For any non – empty subset $R = \{a_\gamma\}_{\gamma \in \Gamma}$ of a lattice L. we understand by the meet of elements $a_\gamma, \gamma \in \Gamma$ the elements $\inf_L R$ by their join the elements $\sup_L R$.

  If $\inf_L R$, $\sup_L R$ exists we symbolize them by $\cap_{\gamma \in \Gamma} a\gamma$, $\cup_{\gamma \in \Gamma} a\gamma$ or simply $\cap R$, $\cup R$.

REMARK:

1. If inf R, sup R do not exist, then it is said that R has no infimum and supremum.
2. If $\{a_\gamma\}_{\gamma \in \Gamma}$, $\{a_\gamma\}_{\gamma \in \Gamma}$ be subsets of a lattice and both infima and suprema exist then if $a_\gamma \leq b_\gamma$ $(\gamma \in \Gamma) \Rightarrow \cap_{\gamma \in \Gamma} a\gamma \leq \cap_{\gamma \in \Gamma} b\gamma$,
   $\cup_{\gamma \in \Gamma} a\gamma \leq \cup_{\gamma \in \Gamma} b\gamma$

   and $a_\gamma \leq b_\delta$ for each pair $\gamma, \delta \Rightarrow \cup_{\gamma \in \Gamma} a\gamma \leq \cap_{\delta \in \Delta} b\delta$

- **DEFINITION: MEET COMPLETE LATTICE:**
  If for any non – empty subset R of a lattice L the meet $\cap R$ exists then L is said to be a lattice complete with respect to meet.
- **DEFINITION: JOIN COMPLETE LATTICE:**
  If for any non – empty subset R of a lattice L the join $\cup R$ exists then L is said to be a lattice complete with respect to join.
- **DEFINITION: COMPLETE LATTICE:**
  If a lattice L is complete with respect to both operations then L is said to be a complete lattice.
  If a complete lattice is a chain then it is called a complete chain.

- **NOTE:** 1) every finite lattice is complete.
  2) every complete lattice is bounded.

- **DEFINITION: DUAL COMPLETE LATTICE:**
  We know the definition of a complete lattice the dual to every true position in complete lattice is also true.
- **DEFINITION: COMPLETE HOMOMORPHISM:**
  Let L_1 and L_2 be two complete lattices and $\phi: L_1 \rightarrow L_2$ be a single valued mapping. ϕ is called a complete homomorphism if for any arbitrary subset $\{a_\gamma\}$ $\gamma \in \Gamma$ of L_1.

  $\phi(\cap_{\gamma \in \Gamma} a\gamma) = \cap_{\gamma \in \Gamma} \phi(a\gamma)$,

  $\phi(\cup_{\gamma \in \Gamma} a\gamma) = \cup_{\gamma \in \Gamma} \phi(a\gamma)$

- **DEFINITION: COMPLETE MEET HOMOMORPHISM:**
  Let L_1 and L_2 be two complete lattices and $\phi: L_1 \to L_2$ be a single valued mapping. ϕ is called a complete meet homomorphism if for any arbitrary subset $\{a_\gamma\}$ $\gamma \in \Gamma$ of L_1.
  $\phi(\cap_{\gamma \in \Gamma} a\gamma) = \cap_{\gamma \in \Gamma} \phi(a\gamma)$
- **DEFINITION: COMPLETE JOIN HOMOMORPHISM:**
  Let L_1 and L_2 be two complete lattices and $\phi: L_1 \to L_2$ be a single valued mapping. ϕ is called a complete join homomorphism if for any arbitrary subset $\{a_\gamma\}$ $\gamma \in \Gamma$ of L_1.
  $\phi(\cup_{\gamma \in \Gamma} a\gamma) = \cup_{\gamma \in \Gamma} \phi(a\gamma)$

  In other words, a single valued mapping $\phi : L_1 \to L_2$ is a complete homomorphism if ϕ is both complete meet homomorphism and ϕ is complete join homomorphism.
- **DEFINITION: COMPLETE ISOMORPHISM:**
  a complete homomorphism of lattices is a complete isomorphism if the mapping ϕ is one – one and onto.

➢ **THEOREM: If P is a partly ordered set bounded above each of whose non – empty (non - void) subsets R has an infimum then each non – void subset of P will have a supremum too and by definitions $\cap R$ = inf R, $\cup R$ = sup R then P becomes a complete lattice.**

PROOF: Let P be a partly ordered set bounded above and each of whose non – empty subset R has an infimum.

CLAIM: R has supremum too

Let R be any non – void subset of P

and let U = the set of upper bounds of R.

Since P is bounded above, we have R has an upper bound and hence $U \neq \phi$.

Clearly, U is also a non – void subset of P then by data U has an infimum.

Now we show that inf U = sup R

By definition of U, $r \leq \inf U \;\; \forall \; r \in R$

Hence inf U is also an upper bound of R.

On the other hand, let u be any upper bound of R then u $\in$ U and inf U $<$ u.

This is clear that inf U = lub R.

$\therefore$ sup R exists.

Hence the first part of the theorem.

Next, put $\bigcap R$ = inf R, $\bigcup R$ = sup R

In view of the theorem,

In a poset if a $\cap$ b = inf {a, b}, a $\cup$ b = sup {a, b} then P is lattice.

$\therefore$ We get P is a lattice.

By data, every non – void subset R has infimum and we have just proved that R has supremum.

- **THEOREM: If a lattice satisfying both minimum and maximum condition (in particular if it is finite length) is complete.**

PROOF: Let L be a lattice satisfying both maximum & minimum conditions

CLAIM: L is complete.

Since L satisfies both maximum & minimum conditions.

We have that L has greatest & least elements.

$\therefore$ L is bounded lattice.

Let R be a non – void subset of L and k be the set of all lower bounds of R.

i.e., k = the set of all lower bounds of R.

Clearly, k $\neq \phi$ because 0$\in$ L $\Rightarrow$ 0 $\in$ k

Since k $\subseteq$ L we have k also satisfies both maximum and minimum conditions.

Then k has a maximal element say m then m is a lower bound of R.

Now we shall show that m = glb R.

Let s be any element of k.

CLAIM: **$s \leq m$**

We have $\forall$ r $\in$ r, r $\geq$ s and r $\geq$ m

Then r $\geq s \cup m \Rightarrow s \cup m \in$ k.

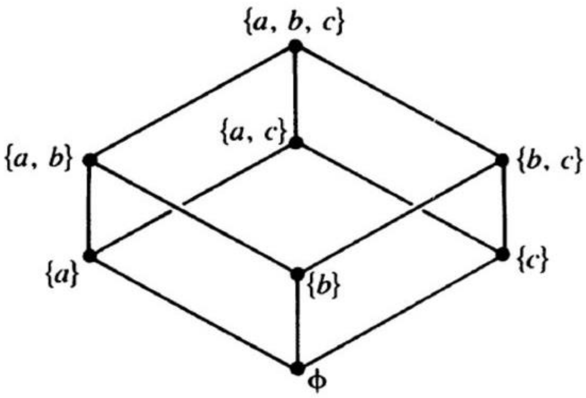
$\therefore$ Neither s > m nor s $\|$m is possible s $\leq$ m.

i.e., m = glb R.

$\therefore \cap R$ exists, by known theorem, $\cup R$ exists too.

Hence, L is complete lattice.

- **EXAMPLE OF COMPLETE LATTICE:**



Let L be a subset lattice. L is always a complete lattice for M = {a, b, c}

P(M) = {{a}, {b}, {c}, {ab}, {bc}, {ac}, {abc}, {ϕ}}

(P(M), $\subseteq$) is a lattice.

Which is always a complete.

Clearly for any subset R of P(M), $\cap R$, $\cup R$ exists.

Next, let M be an infinite set then P(M) is also infinite for any

subset R_δ , $\delta \in \Lambda$.

We have $\bigcap_{\delta \in \Lambda} R\delta$, $\bigcup_{\delta \in \Lambda} R\delta$ exists.

$\therefore$ (P(M), $\subseteq$) is a complete lattice.

- **DEFINITION: FIX ELEMENT:**
  Let L be an arbitrary lattice and let σ be some mapping of L into itself. The elements a € L such that σ (a) = a are called FIX ELEMENT of σ.

➢ **FIX ELEMENT THEOREM:**

STATEMENT: Every order preserving mapping of a complete lattice into itself has a fix element.

PROOF: Let L be a complete lattice and σ be an order preserving mapping.

CLAIM: σ has a fix element.

Let S be the set of all x € L such that $x \le \sigma(x)$
$S \neq \phi$ because $o \le \sigma(o)$.
Clearly, $\phi \neq S \subseteq L$
Since L is complete lattice S has supremum
i.e., sup S exists.
Let $u = \sup_L S$ then $u \ge s$ for s € S
Since σ is an order preserving mapping.
$\sigma(u) \ge \sigma(s) \ge s$
$\Rightarrow \sigma(u) \ge s$
$\Rightarrow \sigma(u)$ is an upper bound of S.
Since $\sup_L S = u$, $\sigma(u) \ge u$ --→ (1)
Next, since σ is an order preserving mapping
We have $\sigma(\sigma(u)) \ge \sigma(u)$
$\Rightarrow \sigma(u)$ € S
Also $\sigma(u) \le u$ --→ (2)
From (1) & (2)
We get $\sigma(u) = u$
$\therefore$ u is a fix element of σ
Hence every order preserving mapping of a complete lattice into itself has a fix element.

- **DEFINITION: CONDITIONALLY COMPLETE:**

  A lattice is said to be conditionally complete if each of its non – empty bounded subsets has infimum and supremum.

PROBLEM: **show that every complete lattice is conditionally complete.**

SOLUTION: **let L be a complete lattice.**

Then every non – void subset R of L has both infimum and supremum.

i.e., $\inf_L R$, $\sup_L R$ exists.

In particular every non – void bounded subset H of L has both infimum and supremum.

i.e., $\inf_L H$, $\sup_L H$ exists.

Hence L is conditionally complete lattice.

- **THEOREM:** **If we affix bound element to a conditionally complete lattice then we obtain a complete lattice.**

PROOF: **Let π be an ordering relation of the set P under which every non – empty subset of P bounded below has infimum.**

now we affix the bounded elements $\bar{o}$ and $\bar{i}$ to P we obtain the set $\bar{P}$

Define on the set $\bar{P}$ an ordering relation $\bar{\pi}$ as follows.

For any pair of elements x, y € P

$x \leq y(\pi) \Leftrightarrow x \leq y(\bar{\pi})$

let x € P, $\bar{o} < x < \bar{i}(\bar{\pi})$ then $\bar{P}$ is a complete lattice.

Let $\bar{X}$ be a non – empty subset of $\bar{P}$

If $\bar{o}$ € $\bar{X}$ (or) X is a subset of P without any lower bounds in P then $\inf_{\bar{P}} \bar{X} = \bar{o}$

If $\bar{X} = \{i\}$ then $\inf_{\bar{P}} \bar{X} = i$

In every case the subset X = $\bar{X} - \{\bar{i}\}$ is a non – empty subset bonded below of P has infimum

Therefore, $\inf_P X$ exists.

On the other hand, we have I > x ∀ x € X

The affixing of $\bar{i}$ to X does not affect the value of infimum.

Therefore, every non – empty subset $\bar{X}$ of $\bar{P}$ has $\inf_{\bar{P}} \bar{X}$

Further, $\sup_{\bar{P}} \bar{X}$ also exists.

∴ $\bar{P}$ is a complete lattice.

- **DEFINITION: COMPACT:**

  By a covering of an element c of the lattice L is meant any subset $\{c_\gamma\}_{\gamma \in \Gamma}$ of L such that $c \leq \bigcup_{\gamma \in \Gamma} c\gamma$ then the element c is called compact.

If out of every covering $\{c_\gamma\}$ we can select a finite covering $\{c_{\gamma 1}, c_{\gamma 2}, ---, c_{\gamma n}\}$ so that $c \le \bigcup_{j=1}^{n} c_{rj}$

- **DEFINITION:**
  COMPACTLY GENERATED LATTICE:
  A complete lattice is called a compactly generated if every of it can be represented as join of (finite or infinite) number of compact elements.

➢ **THEOREM:** Every element of a lattice satisfying maximum condition is compact.

PROOF: Let $\{r_\lambda\}$ be a subset of lattice satisfying the maximum condition such that $\bigcup_\lambda r\lambda$ exists.

now we select a finite subset $\{r_{\lambda 1}, r_{\lambda 2}, ---, r_{\lambda n}\}$ such that

$\bigcup_\lambda r\lambda = \bigcup_{j=1}^{n} r\lambda j$

Let us choose for $r_{\lambda j}$ any element of R.

For which $r_{\lambda j} \nleq r_{\lambda 1} \cup r_{\lambda 2} \cup --- \cup r_{\lambda j-1}$

Then we have

$r_{\lambda 1} < r_{\lambda 1} \cup r_{\lambda 2} < r_{\lambda 1} \cup r_{\lambda 2} \cup r_{\lambda 3} < --- < \bigcup_{j=1}^{m-1} r\lambda j < \bigcup_{j=1}^{m} r\lambda j < ---$

since the lattice L is satisfying the maximum condition, we get there exists an integer n such that $r_\lambda < r_{\lambda 1} \cup --- \cup r_{\lambda n}$

then in view of this we get

$r_\lambda \le r_{\lambda 1} \cup --- \cup r_{\lambda n}$

$\Rightarrow \bigcup_\lambda r\lambda \le \bigcup_{j=1}^{n} r\lambda j \dashrightarrow (1)$

The inequality $\bigcup_{j=1}^{n} r\lambda j \le \bigcup_\lambda r\lambda \dashrightarrow (2)$ can be obtained

from (1) & (2) we get

$$\bigcup_\lambda r\lambda = \bigcup_{j=1}^{n} r\lambda j$$

- **DEFINITION:** By a closure operation on a partly ordered set p we mean an order preserving mapping $\phi : p \to p$ such that
  i) $\phi(x) \ge x \ \forall \ x \in p$
  ii) $\phi^2(x) = \phi(x) \ \forall \ x \in p$

➢ **THEOREM:** let ϕ be a closure operation of a complete lattice L the set $\mathbb{Z}_\phi$ of the complete lattice with respect to the ordering L that is $\inf_{\mathbb{Z}_\phi} R = \inf_L R$, $\sup_{\mathbb{Z}_\phi} R = \phi(\sup_L R)$

PROOF: let ϕ be a closure operation of a complete lattice L.

$\mathbb{Z}_\phi$ = the set of ϕ closed elements of L

CLAIM: $\mathbb{Z}_\phi$ is complete lattice i.e., $\inf_{\mathbb{Z}_\phi} R = \inf_L R$

Further, the formula sup $_{\mathbb{Z}_\phi}$ R = ϕ (sup $_L$ R) exists.
Let R be an arbitrary subset of $\mathbb{Z}_\phi$.
Since L is complete lattice inf $_L$ R exists &
by known theorem inf $_L$ R is also ϕ closed.
$\therefore$ inf $_L$ R € $\mathbb{Z}_\phi$
Hence inf $_L$ R = inf $_{\mathbb{Z}_\phi}$ R
By known theorem, sup $_L$ R = sup $_{\mathbb{Z}_\phi}$ R
Thus $\mathbb{Z}_\phi$ is complete sublattice of L.
Next, we prove the formula
sup $_{\mathbb{Z}_\phi}$ R = ϕ (sup $_L$ R)
let us denote u = ϕ(sup $_L$ R) $\geq$ sup $_L$ R
$\Rightarrow$ u $\geq$ sup $_L$ R
U is an upper bound of R.
By a known remark, u is also ϕ closed.
$\Rightarrow$ u € $\mathbb{Z}_p$
Therefore, u is an upper bound of R in $\mathbb{Z}_\phi$
Let y be any upper bound of R in $\mathbb{Z}_\phi$
We have y $\geq$ sup $_L$ R
Since ϕ is order preserving mapping.
We get ϕ(y) $\geq$ ϕ(sup $_L$ R) = u
$\Rightarrow$ ϕ(y) $\geq$ u.
Next, y € $\mathbb{Z}_\phi$ then ϕ(y) = y
Then y $\geq$ u or u $\leq$ y
Therefore u = sup $_{\mathbb{Z}_\phi}$ R
Hence sup $_{\mathbb{Z}_\phi}$ R = ϕ (sup $_L$ R).

- **COROLLARY:** **Every lattice is isomorphic to some sublattice of a complete lattice.**

PROOF: **let L be any lattice.**
We know that every lattice is a partly ordered set.
In view of the known theorem,
The set D(L) is a partly ordered set by set inclusion and forms a complete lattice.
Furthermore, by the known theorem
The mapping ϕ : x $\rightarrow$ (x], x € L is an isomorphism of L onto a subset D(L).
Hence every lattice is isomorphic to some sublattice of a complete lattice.

DNRCOLLEGE(A),BHIMAVARAM

P.GDEPARTMENTOFMATHEMATICS

305–LATTICETHEORY

IIMSc,SEMESTER-III

UNIT-4

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DISTRIBUTIVELATTICE

- **DEFINITION:** **Let (L $\cap\cup$) be a lattice. We say that L is distributive lattice if the following conditions hold.**

L_{10} : $a\cap(b\cup c)=(a\cap b)\cup(a\cap c)$

L_{11}: $a\cup(b\cap c)=(a\cup b)\cap(a\cup c)$

NOTE: **L_{10},L_{11}arecalleddistributivelawsalsoL_{10}iscalledthe distributive identity of meet. L_{11} iscalledthe distributive identity of join.**

REMARK: **In order to prove the distributivity of a lattice it is suffices to prove that for any triplet a, b, c of the lattice.**

$$a\cap(b\cup c)\leq(a\cap b)\cup(a\cap c)$$

$$a\cup(b\cap c)\geq(a\cup b)\cap(a\cup c)$$

- **THEOREM:** **Wheneveralatticesatisfies $a\cap(b\cup c) = (a\cap b)\cup(a\cap c)$ it also satisfies $a\cup(b\cap c)=(a\cup b)\cap(a\cup c)$andconversely.**

PROOF: **let L be a lattice satisfying**

CLAIM: **$a\cup(b\cap c) = (a\cup b)\cap(a\cup c)$**

Now $(a\cup b)\cap(a\cup c)=((a\cup b)\cap a)\cup((a\cup b)\cap c)$

$$=a\cup((a\cap c)\cup(b\cap c))$$

$$=(a\cup(a\cap c))\cup(b\cap c)$$

$$=a\cup(b\cap c)$$

$$\therefore a\cup(b\cap c)=(a\cup b)\cap(a\cup c)$$

Hence $a\cap(b\cup c)=(a\cap b)\cup(a\cap c)\Rightarrow a\cup(b\cap c)=(a\cup b)\cap(a\cup c)$ **Conversely L satisfies** $a\cup(b\cap c) = (a\cup b)\cap(a\cup c)$

CLAIM: **$a\cap(b\cup c)=(a\cap b)\cup(a\cap c)$**

Now $(a\cap b)\cup(a\cap c)=(((a\cap b)\cup a)\cap((a\cap b)\cup c))$

$=a\cap((a\cap b)\cup c)$

$=a\cap((a\cup c)\cap(b\cup c))$

$=(a\cap(a\cup c))\cap(b\cup c)$

$=a\cap(b\cup c)$

$\Rightarrow a\cap(b\cup c)=(a\cap b)\cup(a\cap c)$

Hence $a\cup(b\cap c)=(a\cup b)\cap(a\cup c)\Rightarrow a\cap(b\cup c)=(a\cap b)\cup(a\cap c)$

- **THEOREM:** **Thedual,everysublatticeandeveryhomomorphicimage of a distributive lattice is likewise a distributive lattice.**

PROOF: **LetusrecallthedefinitionofduallatticeD(L)**

LetLbealattice define $a\vee b=a\cap b, a\wedge b=a\cup b\ \forall a,b\in L$.

Inviewofthisdefinitionof $\vee$ **and** $\wedge$ $(D(L)\wedge\vee)$ **isalsoalatticecalled dual lattice.**

wehave $a\cup(b\cap c)=(a\cup b)\cap(a\cup c)\ \forall a,b,c\in L$.

$\Rightarrow a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$

$\therefore$ **D(L)isadistributivelattice.**

Let $S\neq\phi\subseteq L$ **and** $(S\cap\cup)$ **beasublatticeofthedistributive lattice** $(L\ \cap\cup)$.

CLAIM: $(S\cap\cup)$ **isdistributivesublattice ofL Let**

$a, b, c\in S$

$\Rightarrow a\cap(b\cup c)=(a\cap b)\cup(a\cap c)$ **and** $a\cup(b\cap c)=(a\cup b)\cap(a\cup c)$

$\therefore (S,\cap,\cup)$ **isadistributivesublattice ofL.**

ThehomomorphicimageofaDistributiveLatticeisf(L) Let $f(a), f(b), f(c)\in f(L)$ **where** $a, b, c\in L$.

Now $f(a)\cap\{f(b)\cup f(c)\}=f(a)\cap\{f(b\cup c)\}$

$=f(a\cap(b\cup c))$

$=f\{(a\cap b)\cup(a\cap c)\}$

$=f(a\cap b)\cup f(a\cap c)$

$\therefore f(a)\cap\{f(b)\cup f(c)\}=f(a\cap b)\cup f(a\cap c)$

Next $f(a)\cup\{f(b)\cap f(c)\}=f(a)\cup\{f(b\cap c)\}$

$=f\{a\cup(b\cap c)\}$

$=f((a\cup b)\cap(a\cup c))$

$=f(a\cup b)\cap f(a\cup c)$

$\therefore f(a)\cup\{f(b)\cap f(c)\}=f(a\cup b)\cap f(a\cup c)$

$\therefore$ Thehomomorphicimageofadistributivelatticeisalsoadistributive lattice

Hencethedual,everysublattice andeveryhomomorphicimageof a distributive lattice is likewise a distributive lattice.

- **THEOREM:Showthateverychainisdistributivelattice.**

PROOF:LetCbeachaina,b,c€ C. since C is a chain.

wehavethatfollowingcases:

i) $a\leq b\leq c$
ii) $a\leq c\leq b$
iii) $c\leq a\leq b$
iv) $b\leq a\leq c$
v) $c\leq b\leq a$
vi) $b\leq c\leq a$.

Letusconsidercasei) $a\leq b\leq c$

Nowweverifythedistributivelaw $a\cap(b\cup c)=(a\cap b)\cup(a\cap c)$ $b\leq c\Rightarrow b\cup c=c$,

$a\cap(b\cup c)=a\cap c=a$ **because** $a\leq c$.

Next, $a\cap b=a$ **because** $a\leq b$

$a\cap c=a$ **because** $a\leq c$.

Now$(a\cap b)\cup(a\cap c)=a\cup a=a$

$\therefore a\cap(b\cup c)=(a\cap b)\cup(a\cap c)$

HencethechainCisadistributivelattice.

- **DEFINATION:** **Alatticeissaidtobeinfinitelymeetdistributiveifit isjoincompleteand $a\cap(\bigcup_{\beta\epsilon B} b\beta)=\bigcup_{\beta\epsilon B}(a\cap b\beta)$ holds.**

Alatticeissaidtobeinfinitelyjoindistributiveifitisbothinfinitelymeet distributive and infinitely join distributive.

✓ **LEMMA:Everycompleteringofsetsiscompletelydistributive.**

PROOF: **let $\mathcal{M}=\{M_\gamma\}_{\gamma\epsilon\Gamma}$ be a complete ring of sets.** **CLAIM:** **$\mathcal{M}$ is completely distributive.**

Let$A=\{1,2,---,r\},B_1=\{1,2,---,n_1\},---,B_r=\{1,2,---,n_r\}$and

γbeachoicefunction.

Let$X_{\alpha\beta}$where$\alpha\epsilon A$, $\beta\epsilon\bigcup_{\alpha\epsilon A}B\alpha$

Nowwehavetoprove

i) $\bigcap_{\alpha\epsilon A}\bigcup_{\beta\epsilon B\alpha}X\alpha\beta=\bigcup_{\gamma\epsilon\Gamma}\bigcap_{\alpha\epsilon A}X\alpha\gamma(\alpha)$

ii) $\bigcup_{\alpha\epsilon A}\bigcap_{\beta\epsilon B\alpha}X\alpha\beta=\bigcap_{\gamma\epsilon\Gamma}\bigcup_{\alpha\epsilon A}X\alpha\gamma(\alpha)$

let$x\epsilon\bigcap_{\alpha\epsilon A}\bigcup_{\beta\epsilon B\alpha}X\alpha\beta\Leftrightarrow x\epsilon\bigcup_{\beta\epsilon B\alpha}X\alpha\beta \quad \forall\alpha\epsilon A$

$\Leftrightarrow$**forevery** $\alpha\epsilon A$, $x\epsilon X\alpha\beta$**forsome**$\beta\epsilon B_\alpha$

$\Leftrightarrow$**forevery** $\alpha\epsilon A$, $x\epsilon X\alpha\gamma(\gamma)$**where**$\gamma(\alpha)=\beta\epsilon B_\alpha$

$\Leftrightarrow\exists a_\gamma\ni x\epsilon\bigcap_{\alpha\epsilon A}X\alpha\gamma(\alpha)$

$\Leftrightarrow x\epsilon\bigcup_{\gamma\epsilon\Gamma}\bigcap_{\alpha\epsilon A}X\alpha\gamma(\alpha)$

Hence$\bigcap_{\alpha\epsilon A}\bigcup_{\beta\epsilon B\alpha}X\alpha\beta=\bigcup_{\gamma\epsilon\Gamma}\bigcap_{\alpha\epsilon A}X\alpha\gamma(\alpha)$

Thus,thecompleteringofsets$M$iscompletelymeetdistributive.

ii)let$x\epsilon\bigcup_{\alpha\epsilon A}\bigcap_{\beta\epsilon B\alpha}X\alpha\beta$

$\Leftrightarrow\exists$**an**$\alpha\epsilon A,\ni x\epsilon\bigcap_{\beta\epsilon B\alpha}X\alpha\beta$

$\Leftrightarrow \exists$ an $\alpha \epsilon A, \ni x \epsilon X_{\alpha\beta}$ for every $\beta \epsilon B_\alpha$

$\Leftrightarrow \exists$ an $\alpha \epsilon A, \ni$ for every choice function $\gamma \epsilon \Gamma \ni x \epsilon X_{\alpha\gamma(\alpha)}$

$\Leftrightarrow$ for every choice function $\gamma \epsilon \Gamma, x \epsilon \bigcup_{\alpha \epsilon A} X_{\alpha\gamma(\alpha)}$

$\Leftrightarrow x \epsilon \bigcap_{\gamma \epsilon \Gamma} \bigcup_{\alpha \epsilon A} X_{\alpha\gamma(\alpha)}$

$$\therefore \bigcup_{\alpha \epsilon A} \bigcap_{\beta \epsilon B_\alpha} X_{\alpha\beta} = \bigcap_{\gamma \epsilon \Gamma} \bigcup_{\alpha \epsilon A} X_{\alpha\gamma(\alpha)}$$

- **MODULAR LATTICE:** let L be a lattice. We say that L is a modularlatticeifanytriplet ofelements a,b,cwitha≤cinLsatisfies the equation $a \cup (b \cap c) = (a \cup b) \cap c$

REMARK: Theidentity $a \cup (b \cap c) = (a \cup b) \cap c \ \forall a, b, c \in L$ satisfying $a \leq c$ is modular identity.

REMARK: Itisobviousthateverydistributivelatticeismodular.

PROOF: LetLbeadistributivelattice.

CLAIM: Lismodular.

Leta, b,c€Lsatisfyinga≤c

Now $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$

$= (a \cup b) \cap c$

Because $a \leq c \Rightarrow a \cup c = c$

$\therefore$ Lsatisfiesmodularlattice.

Hence,everydistributivelatticeismodular.

➢ **THEOREM:** Thedualeverysublatticeandeveryhomomorphic image of a modular lattice is modular.

PROOF: Letusrecallthedefinitionofdual latticeD (L)

LetLbealattice. Definea $\vee b = a \cap b$, $a \wedge b = a \cup b \ \forall a, b \in L$ In view of this definition of $\vee$ and $\wedge$.

$(D(L) \vee \wedge)$isalsoalatticecalledduallattice.

Wehave $a \cup (b \cap c) = (a \cup b) \cap c \ \forall a, b, c \in L$ with $a \leq c$

$\therefore$ **D(L)isamodularlattice.**

LetS$\neq\phi\subseteq$Land(S$\cap\cup$)beasublatticeofthemodularlatticeL.

CLAIM:(S$\cap\cup$)ismodular. Leta,

b,c€Ssatisfyinga$\leq$c

$\Rightarrow$a$\cup$(b$\cap$c)=(a $\cup$b)$\cap$c$\forall$a,b,c€S

Becausea, b, c€Lwitha$\leq$cthena$\cup$(b$\cap$c)=(a$\cup$b)$\cap$c

$\therefore$ **Sisamodularlattice.**

Hence,Sisamodularsublatticeofthemodularlattice L.

Letf(L)bethehomomorphicimageofamodularlatticeLand the homomorphism f.

CLAIM:f(L)ismodular.

f(a),f(b), f(c)€f(L)wherea,b, c€Lwithf(a)$\leq$f(c)

$$\begin{aligned} f(a)\cup(f(b)\cap f(c)) &= f(a)\cup(f(b\cap c)) \\ &= f(a\cup(b\cap c)) \\ &= f((a\cup b)\cap c) \\ &= f(a\cup b)\cap f(c) \\ &= (f(a)\cup f(b))\cap f(c) \end{aligned}$$

$$\therefore f(a)\cup(f(b)\cap f(c))=(f(a)\cup f(b))\cap f(c)$$

❖ **EXAMPLE:Everychainisamodularlatticebecauseeverychainisa distributive lattice and moreover every distributive lattice is modular.**

OBESERVATION:foranyelementsa,b,cofalatticethefollowing inequality holds.

$$(a\cap b)\cup(b\cap c)\cup(c\cap a)\leq(a\cup b)\cap(b\cup c)\cap(c\cup a)$$

Ifforanytripletofelements,a,b,cofalattice

$(a\cap b)\cup(b\cap c)\cup(c\cap a)=(a\cup b)\cap(b\cup c)\cap(c\cup a)$holds.

Thecommonvalue ofthetwosidesofthis,equalityiscalledmedian.

Itiswrittenas med(a,b,c).

✓ **LEMMA:** **Alatticeismodulariffeverytripletofelements a, b, c ($a \leq c$) has median.**

PROOF: **LetLbealattice.**

FirstassumethatLisamodularlattice.

CLAIM: **Foranytripleta,b,c($a\leq c$)ofelements ofLhasmedian. i.e., $(a \cap b) \cup (b \cap c) \cup (c \cap a) = (a \cup b) \cap (b \cup c) \cap (c \cup a)$**

Leta,b, c($a\leq c$)beanytripletofL.

Since,Lismodular we havea $\cup(b\cap c)=(a\cup b)\cap c$ Using absorption laws respectively.

Now, $a\cup(b\cap c)=a\cup(a\cap b)\cup(b\cap c)$

$$=(a\cap c)\cup(a\cap b)\cup(b\cap c)$$

$\therefore a\cup(b\cap c)=(a\cap b)\cup(b\cap c)\cup(c\cap a)$ **Next,** $(a \cup b) \cap c = (a \cup b) \cap (b \cup c) \cap c$

$$= (a\cup b)\cap(b\cup c)\cap(c\cup a)$$

$\therefore (a\cup b)\cap c=(a \cup b)\cap(b\cup c)\cap(c \cup a)$

Hence, $(a \cap b)\cup(b\cap c)\cup(c \cap a)=(a\cup b)\cap(b\cup c)\cap(c \cup a)$ **i.e., a, b, c has median.**

Ontheother hand,

assumethatanytripleta,b,c($a\leq c$)ofLhasamedian.

i.e., $(a \cap b)\cup(b\cap c)\cup(c \cap a)=(a\cup b)\cap(b\cup c)\cap(c \cup a)$

CLAIM: **Lismodular. Let**

a, b, c ($a \leq c$)

Just wehaveprovedthat $(a\cap b)\cup(b\cap c)\cup(c\cap a)=a\cup(b\cap c)$

$$(a\cup b)\cap(b\cup c)\cap(c\cup a)=(a\cup b)\cap c$$

$$\therefore a\cup(b\cap c)=(a\cup b)\cap c$$

Hence,Lismodularlattice.

✓ **LEMMA:** **Alatticeisdistributiveiffeveryoneofits triplets ofelements has a median.**

PROOF: **LetLbea lattice.**

Firstassumethat,Lisdistributive.

CLAIM: **everyoneofitstripletsofelementshasamedian.**

Leta,b,canytripletofelements ofL. Now,

$$(a\cap b)\cup(b\cap c)\cup(c\cap a)=(a\cap b)\cup\{(b\cup(c\cap a))\cap(c\cup(c\cap a))\}$$

$$=(a\cap b)\cup\{(b\cup c)\cap(b\cup a)\cap c\}$$

$$=(a\cap b)\cup\{(b\cup c)\cap c\cap(b\cup a)\}$$

$$=(a\cap b)\cup(c\cap(b\cup a))$$

$$=((a\cap b)\cup c)\cap((a\cap b)\cup(b\cup a))$$

$$=(a\cup c)\cap(b\cup c)\cap(b\cup a)$$

$$=(a\cup b)\cap(b\cup c)\cap(c\cup a)$$

$\therefore (a\cap b)\cup(b\cap c)\cup(c\cap a)=(a\cup b)\cap(b\cup c)\cap(c\cup a)$

Hence,everytripleta,b,cofelementsofLhasa median.

Ontheotherhand,assumethatanytripleta,b,celementsofLhasamedian. Then by known lemma, immediately we have L is modular.

Thenwehave $a\cap(b\cup c)=(a\cap b)\cup c$

Now $a\cap(b\cup c)=a\cap(a\cup b)\cap(b\cup c)$

$$=a\cap(a\cup c)\cap(a\cup b)\cap(b\cup c)$$

$$=a\cap(b\cup c)\cap(a\cup c)\cap(a\cup b)$$

$$=a\cap((b\cap c)\cup(c\cap a)\cup(a\cap b))$$

$\therefore a\cap(b\cup c)=a\cap((b\cap c)\cup(c\cap a)\cup(a\cap b))\text{---}\rightarrow(1)$

If $a\geq c$ **then** $a\cap(b\cup c)=(a\cap b)\cup c$

SinceLismodulara≥$(c\cap a)\cup(a\cap b)$

RHSof(1)becomes

$a\cap((b\cap c)\cup(c\cap a)\cup(a\cap b))=(a\cap(b\cap c))\cup((c\cap a)\cup(a\cap b))$ since $a\cap(b\cap c)$ = glb {a, b, c} and

$(c\cap a)\cup(a\cap b)$=lub{$c\cap a$, $a\cap b$}

Then$a\cap(b\cap c)\leq(a\cap c)\cup(a\cap b)$

∴RHS of (1) = $(c\cap a)\cup(a\cap b)$ Hence,

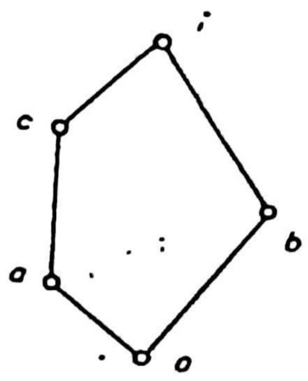
$a\cap(b\cup c)=(a\cap b)\cup(a\cap c)$ By a known theorem,

∴Lisadistributivelattice.

➢ THEOEREM:

DEDEKINDMODULARITYCRITERION:

Alatticeismodulariffifnosublattice ofitisisomorphic withthelattice shown in the following figure

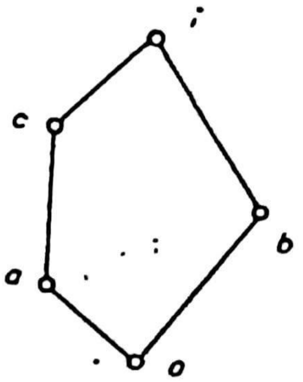


PARAPHRASED: **Alatticeismodulariffnointerval[ab]ofthelattice includes one element having two different comparable relative complements in [a b]**

PROOF: **letLbealattice.**

FirstassumethatLismodular.

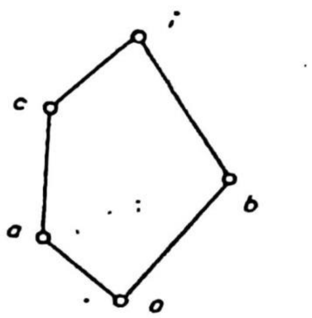
CLAIM:nosublattice ofLisisomorphicwiththelattice.



We prove this part indirectly.

Assume that L has a sub lattice.

Whichisisomorphicwiththelattice.



Letusconsidertheelementsa,b, c(a≤c)inthissublattice. Now a

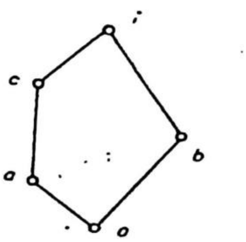
$\cup$(b $\cap$c) = a $\cup$o = a

Next, (a $\cup$b) $\cap$c = i $\cap$c = c

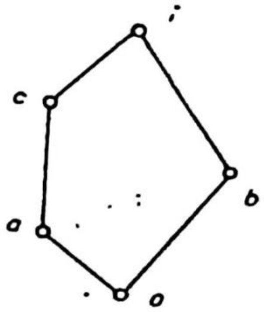
Clearly,a$\cup$(b$\cap$c)$\neq$(a$\cup$b)$\cap$c

$\therefore$ThelatticeLisnotmodular.

Hence,Lhasnosublatticeisomorphicwith

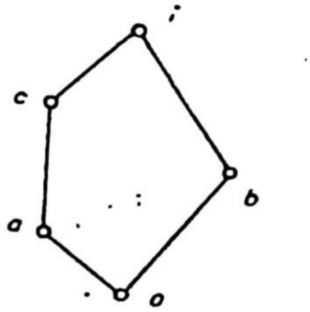


Onthe other hand,weprovethatiftheLhas nosublatticeisomorphic with



ThenLismodular.

Toprovethesestatements,weprovethecontrapositiveifLis nonmodular lattice then L has a sub lattice isomorphic with



FirstassumethatLisnon--modular.

ByusingthedefinitionofmodularlatticedefinitelywecanfindinL.

Atriplet ofelements x,y,zsuchthat $x \leq z$ **and** x

$\cup (y \cap z) = (x \cup y) \cap z$ **- - -→ (1)**

LetusconsiderasubsetRconsistingofthefollowingelements.

$u = y \cap z$

$v = x \cup y$

$a = x \cup (y \cap z)$ **---→(2)**

$b = y$

$c = (x \cup y) \cap z$

$u \leq a < c \leq v, u \leq b \leq v$

Clearly,u,varetheboundsofthesubsetR.

$\therefore u \leq a \cap b \leq c \cap b = ((x \cup y) \cap z) \cap y$

$= y \cap z$

$= u$

Hence, $a \cap b = c \cap b = u$ **---→(3) Next,**

$v \geq a \cup b,\ v \geq c \cup b$

$\Rightarrow v \geq c \cup b \geq a \cup b = (x \cup (y \cap z)) \cup y$

$= x \cup y$

$= v$

Hence, $c \cup b = a \cup b = v$ **---→(4) From**

(3) & (4)

WehaveRissublatticewiththeboundsu& v.

ToprovethesublatticeRisisomorphictothesublattice

Itissufficienttoshowthat $u \neq b,\ v \neq b,\ u \neq a, v \neq c,\ a \neq b,\ c \neq b$ **If** $u = b$

$\Rightarrow a \cap b = b$

$\Rightarrow a \cup b = a$

$\Rightarrow v = a$

Thisisacontradictionto

$\therefore u \neq b$

If $v = c \Rightarrow c \cup b = c$

$\Rightarrow c \cap b = b$

$\Rightarrow u = b$

Thisisacontradictionto $u \neq b$

$\therefore v \neq c$ **Next,**

if $a = b$

$\Rightarrow a \cap b = b \cap b$

$\Rightarrow \quad u = b$

Thisisacontradictiontou≠b

∴a≠ b

Likewise,dualizingtheaboveconsideration. We

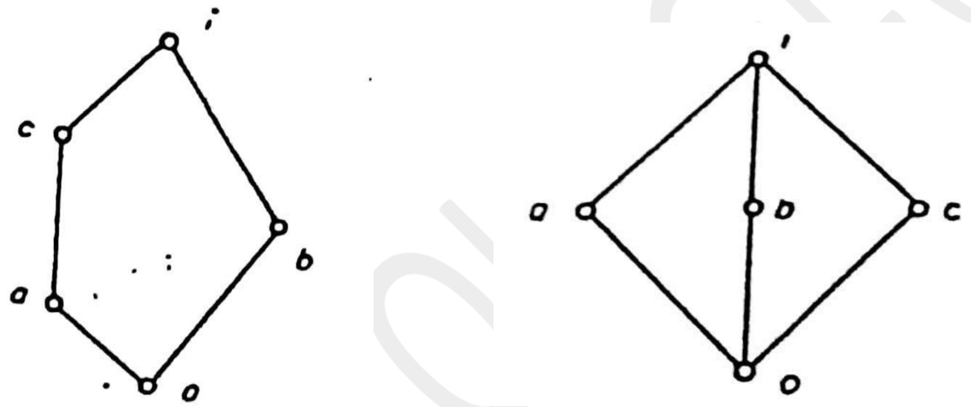
can see that v ≠ b, v ≠ a, c ≠ b

Hence,theproofofthetheorem.

THEOEREM:

BIRKHOFFSDISTRIBUTIVITYCRITERION:

Alatticeisdistributiveiffithasnosublatticeisomorphic witheither one of the lattices shown as the figures.



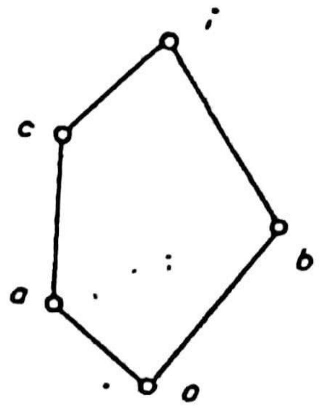
PARAPHRASED:A lattice is distributive iff no interval [a b] of the latticeincludesanelementhavingtwodifferentrelativecomplementsin

[ab].

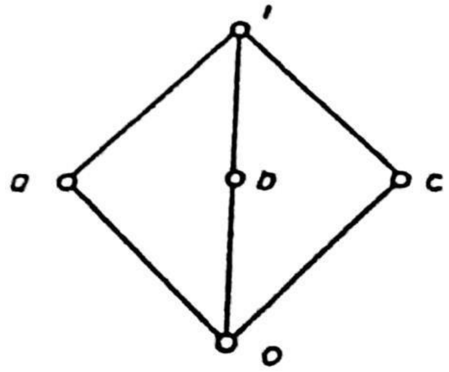
PROOF:LetLbea lattice.

First,assumethatLisdistributive.

CLAIM:Lhasnosublatticeisomorphicwitheither oneofthelattices

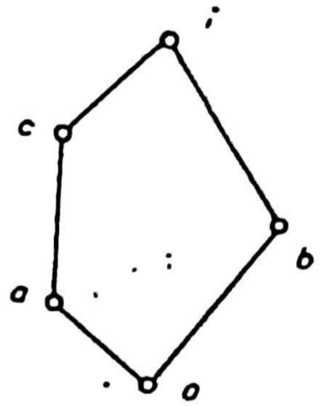


and

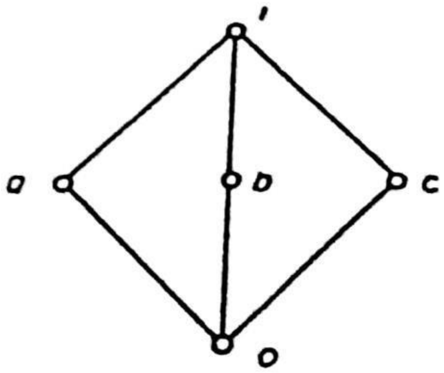


Weprovethispartindirectly.

SupposethatLhasasublatticeisomorphicwitheitheroneofthelattice



and



Takea,b,c

$a\cup(b\cap c)=a\cup o=a$

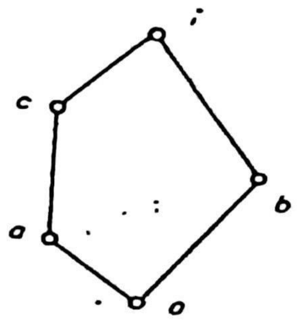
Next, $(a\cup b)\cap(a\cup c)=i\cap c=c$

Clearly,$(a\cup b)\cap(a\cup c)\neq a\cup(b\cap c)$

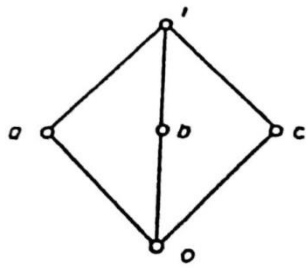
$\therefore$Thelatticeisnotdistributive.

Thisisa contradiction.

Hence,Lhasnosublatticeisomorphicwiththelattice



Next,takea,b,cinthelattice



Now, a $\cup$(b $\cap$c) = a $\cup$o = a

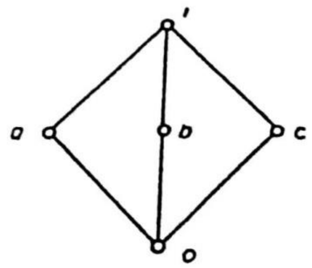
(a$\cup$b)$\cap$(a$\cup$c)=i$\cap$i=i

Clearly,a $\cup$(b$\cap$c)$\neq$(a $\cup$b)$\cap$(a$\cup$c)

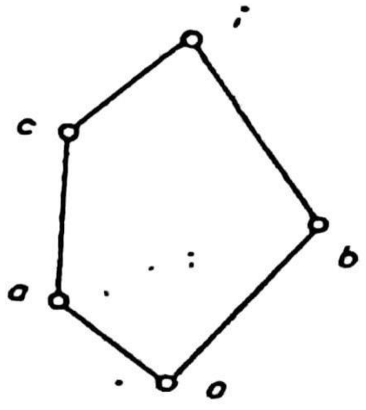
$\therefore$ThelatticeLisnot distributive.

Thisisacontradiction.

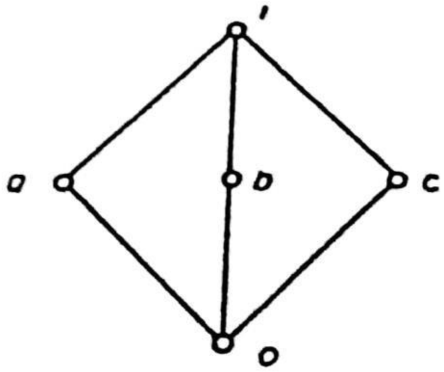
Hence,Lhasnosublatticeisomorphicwiththelattice



onthe other hand,assumethatL hasnosublatticeisomorphicwitheither are of the lattice

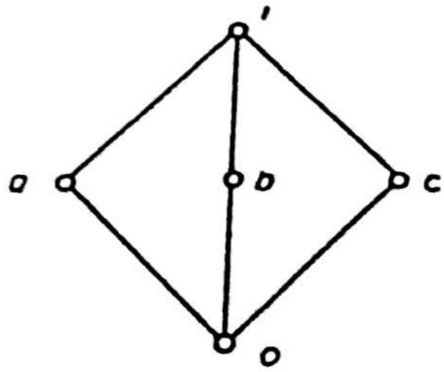


and



CLAIM: **Lisdistributive.**

Ifamodularlatticeis non–distributive.Ithasasublatticeisomorphic with the figure.



Since,thelatticeLismodularbutnon–distributivelatticeL hasatriplet of elements p, q, r such that

$(p\cap q)\cup(q\cap r)\cup(r\cap p)<(p\cup q)\cap(q\cup r)\cap(r\cup p)$---→(1)

Letusconsiderthefollowingelements u

$= (p \cap q) \cup(q \cap r) \cup(r \cap p)$

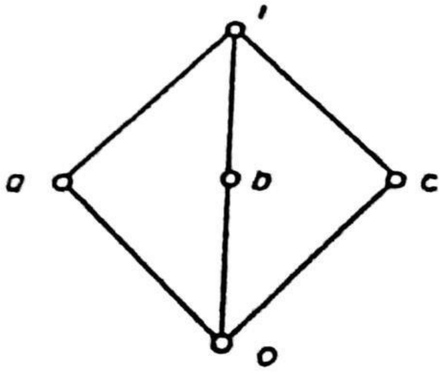
$v=(p\cup q)\cap(q\cup r)\cap(r \cup p)$

$a=u\cup(p\cap v)=(u\cup p)\cap v$

$b=u\cup(q\cap v)=(u\cup q)\cap v$

$c=u\cup(r\cap v)=(u\cup r)\cap v$

Now, weshowthatthesefiveelementsu, v,a,b,cformasublattice isomorphic with



$$\left.\begin{array}{l} \textbf{First,weshowthat}\ a\cap b=b\cap c=c\cap a=u \\ \qquad\qquad a\cup b=b\cup c=c\cup a=v \end{array}\right\} \dashrightarrow (2)$$

Now,

$a\cup b=(u\cup(p\cap v))\cup(u\cup(q\cap v))$

$\qquad =u\cup(p\cap v)\cup(q\cap v)$

$=u\cup(p\cap(p\cup q)\cap(q\cup r)\cap(r\cup p))\cup(q\cap(p\cup q)\cap(q\cup r)\cap(r\cup p))$

$=u\cup(p\cap(q\cup r)\cap(r\cup p))\cup(q\cap(q\cup r)\cap(r\cup p))$

$=u\cup(p\cap(q\cup r)\cup(q\cap(r\cup p))$

Now,wehave $p\cap(q\cup r)\leq p\leq r\cup p$

$\qquad a\cup b=u\cup((p\cap(q\cup r))\cup q)\cap(r\cup p)$

$\qquad\qquad =u\cup(q\cup(p\cap(q\cup r)))\cap(r\cup p)$

$\qquad\qquad =u\cup((q\cup p)\cap(q\cup r))\cap(r\cup p)$

$\qquad\qquad =u\cup v=v$

$\therefore a\cup b=v$

Onsimilarlines, wecanshowthat $b\cup c=v$,

$c\cup a=v$,

$a\cap b=u$,

$b\cap c=u$,

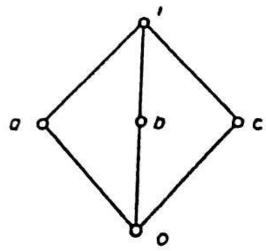
$c\cap a=u$

Hence, a ∩b=b∩c=c ∩a=u

a∪b=b∪c=c∪a =v

Thus, we havetoprovethatthe elements u,v,a,b,cin(2)formasub lattice of L.

Toshowthatthissublatticeisisomorphicwiththefigure



Forthatweproveallelementsu,v,a,b,cin(2)aredifferent.

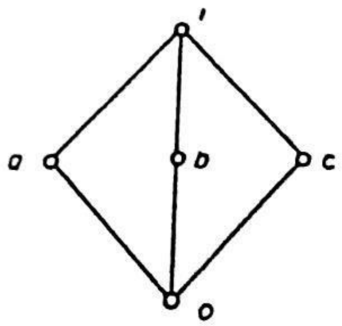
If possible, assumethat u=a ⇒a∩b=a⇒a∪b=b⇒v=b Next, v = v ∩v = b ∩c ⇒b ∩c = u

Thisisacontradictionbecause u<v

On similar lines we canshowthat u ≠b,u≠c bydualconsideration

Wecanshowthatv≠a, v≠b, v≠candfurthera,b, cisalldifferent.

Hence,thesublatticeformedbytheelements u,v,a,b,cisisomorphic with the figure



- **THEOREM:Inamodularlatticethesublatticegeneratedbyx,y,zof the lattice is distributive iff x ∩(y ∪z) = (x ∩y) ∪(x ∩z)**

PROOF:LetLbeamodularlattice.

Firstassumethatthesublatticegeneratedbyx,y, zofLisdistributive

Thenimmediately we have x ∩(y ∪z) = (x ∩y) ∪(x ∩z)

Next,weprovethesufficientpart

Assumethatx,y, zofLsatisfiesthecondition $x \cap(y \cup z) = (x \cap y) \cup(x \cap z)$

CLAIM: **Thesublatticegeneratedbyx,y, zisdistributive. To prove this by a known theorem, we havetoshowthat**

$x\cap(y\cup z)=(x\cap y)\cup(x\cap z)\Rightarrow y\cap(z\cup x)=(y\cap z)\cup(y\cap x)$ and $x \cap(y \cup z) = (x \cap y) \cup(x \cap z) \Rightarrow z\cap(x \cup y) = (z\cap x) \cup(z\cap y)$ Assume that $x \cap(y \cup z) = (x \cap y) \cup(x \cap z)$

Now $y\cap(z\cup x)=(y\cap(z\cup y))\cap(z\cup x)$

$=y\cap((z\cup x)\cap(z\cup y))$

$=y \cap(z\cup(x\cap(z\cup y)))$

$=y\cap(z\cup(x\cap z)\cup(x\cap y))$

$=y\cap(\{z\cup(x\cap z)\}\cup(x\cap y))$

$=y\cap(z\cup(x\cap y))$

$y\cap(z\cup x)=(y\cap z)\cup(x\cap y)$

$\therefore x\cap(y \cup z) = (x\cap y) \cup(x\cap z)) \Rightarrow y \cap(z\cup x) = (y \cap z) \cup(y \cap x)$

Next,usingcyclic order,onsamelinesofthoughwecan showthat $x \cap(y \cup z) = (x \cap y) \cup(x \cap z) \Rightarrow z \cap(x \cup y) = (z\cap x) \cup(z\cap y)$ Hence, the sub lattice generated by x, y, z is distributive.

➢ THEOREM:

(ISOMORPHISMTHEOREMOFMODULARLATTICES)

STATEMENT: **Transposedintervalsofamodularlatticesareisomorphic.**

SUPPLEMENT: **If $H=[u\cap v,v]$ and $k=[u, u\cup v]$ aretwointervalsof a modular lattice L then ϕ: $x \rightarrow\phi(x) = u \cup x$, x€ H is anisomorphism of the sublatticeHontothesublatticeKwhereasψ:$y\rightarrow\psi(y)=v\cap y$,y€Kisan**

isomorphismofthesublatticeKontothesublatticeHandofthesetwo mappings each is the inverse of the other.

PROOF:By data H = [u $\cap$v, v] and k = [u, u $\cup$v] are two intervals of a modular lattice L.

Clearly,H&KaretransposedintervalsofLandwehaveH,Karesub lattices of L.

Firstobservethat$\phi(H)\subseteq K$

Letx€Hthen$u\cap v \le x \le v$

$\Rightarrow u\cup(u\cap v)\le u\cup x\le u\cup v$

$\Rightarrow u\le\phi(x)\le u\cup v$

$\Rightarrow\phi(x)$€$[u, u\cup v]$

$\Rightarrow\phi(x)$€K

$\therefore \phi(H)\subseteq K$

Duallyspeaking, weshallprovethat$\psi(K)\subseteq H$

Clearly, $\psi\phi$: H $\rightarrow$H and $\phi\psi$: K $\rightarrow$K

Letx€H, y€K then$u\cap v\le x\le v, u\le y\le u\cup v$ Now $\psi\phi(x)$

$= \psi(\phi(x))$

$=\psi(u\cup x)$

$=v\cap(u\cup x)$

$=(v\cap u)\cup x$

$=x$

$=I_H(x)\forall x$€H---$\rightarrow$(1)

$\therefore \psi\phi=I_H$

Next,$\phi\psi(y)=\phi(\psi(y))$

$=u\cup(v\cap y)$

$=(u\cup v)\cap y$

$=y$

$=I_K(y)\forall y€K$ --- $\rightarrow$(2)

$\therefore \phi\psi=I_K$

$\psi\phi(H)=H$, $\phi\psi(K)=K$.

Now $H=\psi\phi(H)=\psi(\phi(H))\subseteq H$ **From**

(1) & (2)

Weget ϕ **and** ψ **areonetoonemapping, If**

$\phi(x_1) = \phi(x_2)\ \forall x_1, x_2 € H$

Now, $x_1=\psi\phi(x_1)$

$=\psi(\phi(x_1))$

$=\psi(\phi(x_2))$

$=\psi\phi(x_2)$

$=x_2$

$\therefore x_1=x_2$

Onsimilarlinesweprovethat ψ **isalsoone – onemapping.**

Further, from (1) & (2)

We get ψ **and** ϕ **are inverse mappings of each other.**

Finally,itremainstoprovethat ϕ **and** ψ **areisomorphism.**

Furtheritissufficienttoprovethat ϕ,ψ **areorderisomorphic. We prove** ϕ **&** ψ **are order homomorphism.**

Firstassume $\phi(x_1)\leq\phi(x_2)\forall x_1,x_2€H$ **Now,**

$x_1 = \psi\phi(x_1)$

$=\psi(\phi(x_1))$

$\leq\psi(\phi(x_2))$

$=\psi\phi(x_2)$

$=x_2$

$\therefore \phi(x_1)\leq\phi(x_2)\Rightarrow x_1\leq x_2$

Similarly,$\psi(y_1)\leq\psi(y_2)\Rightarrow y_1\leq y_2$ On the other hand, $x_1 \leq x_2$

$$\Rightarrow \quad \psi\phi(x_1)\leq\psi\phi(x_2)$$

$$\Rightarrow\psi(\phi(x_1))\leq\psi(\phi(x_2))$$

$$\Rightarrow \quad \phi(x_1)\leq\phi(x_2)$$

Similarly, $\quad y_1\leq y_2$

$$\Rightarrow \quad \psi(y_1)\leq\psi(y_2)$$

Hence,ϕ&ψarebothhomomorphism.Inviewofabovediscussionthe mappingϕ:H$\rightarrow$Kisanisomorphismandψ:K$\rightarrow$Hisalsoanisomorphism.

- **COROLLARY:In a modular lattice bounded below the elementsof finite height form an ideal.**

  PROOF:LetLbeamodularlatticeboundedbelow.

  CLAIM:Theelementsoffiniteheightformanideal.

  Leta,bbeanytwoelementsofLthen[a$\cap$ba],[ba$\cup$b]aretransposed intervals of L.

  Byisomorphismtheorem, [a$\cap$ba]$\approx$[ba$\cup$b].

  Ifaisanelementoffiniteheightthena$\cap$b$\leq$aisalsoanelementoffinite height.

  Leta,bbetwoelementsoffiniteheight. Since

  [a $\cap$ba] $\approx$[ba $\cup$b]

  Wegeta$\cup$bisalsoanelementof finiteheight.

  Hence,theelements offiniteheightformanidealofthemodularlattice bounded below.

- **DEFINITION:LetLbeanarbitrarylattice.Ifforanypairofelements u, v of L.**

  u$\cap$v$-\!<$v$\Rightarrow$u$-\!<$u$\cup$vholdsthenLissaidtosatisfythelowercovering

condition.

$u -\!\!< u\cup v \Rightarrow u\cap v -\!\!< v$ holdsthenLissaidtosatisfytheuppercovering condition.

Ifboth$u\cap v -\!\!< v \Rightarrow u -\!\!< u\cup v$

$u -\!\!< u\cup v \Rightarrow u\cap v -\!\!< v$ holdsforanypairof elementsu, vof L then L is said to satisfy the double covering condition.

- **THEOREM:** Everymodularlatticesatisfiesthedoublecovering condition.

PROOF: Let L bea modular lattice and u, v be any pair of elements of L.

Wealwayshave$u\cap v -\!\!< v$.

Fromtheisomorphismtheorem,wehave thefollowing.

$H=[u\cap v, v], K=[u, u\cup v]$ beanytwointervalsofthe modular latticeLthen$\phi: H\to K$ isanisomorphismand$\psi: K\to H$ isalsoan isomorphism and each is the inverse of the other.

i.e.,$H\approx K$

immediately,wehave$u\cap v -\!\!< v \Rightarrow u -\!\!< u\cup v$ and

$$u -\!\!< u\cup v \Rightarrow u\cap v -\!\!< v$$

Hence,amodularlatticesatisfiesthedoublecoveringcondition.

- **THEOREM:** All irredundantirreduciblemeetrepresentationofany elements of a modular lattice have the same number of components.

PROOF: Let u be any element in modular lattice L.

Let$u=x_1\cap x_2\cap \text{---}\cap x_r \text{---}\to(1)$

$u=y_1\cap y_2\cap \text{---}\cap y_s \text{---}\to(2)$ betwoirreducibleirredundantmeet representations of u.

CLAIM: $r=s$

First,assumethatrisminimal.

Inviewoftheknownlemma,

for the element x_1 in (1) we can find a suitable element y_{k_1} ∍x_1 can be exchanged by y_{k_1}in (1)

Similarly,for$x_2$in (1)wecanfindasuitableelementy_{k_2}into(2)∍$x_2$can beexchangedbyy_{k_2}in(1)

Continuingthisprocessandsincerisminimaland(2)isalsoirredundant we must have s ≤r.

Ontheotherhand,assumingsismaximalandrepeatingthesameargument we get r ≤s.

∴s=r

Hence, all irredundant irreducible meet representation of any elements of a modular lattice have the same number of components.

- **THEOREM:Ifelementxofadistributivelatticehasatmostoneirredundant irreducible meet representation.**

[LEMMA:Ifanelementxofadistributivelatticeismeet –irreducible andx≥$\bigcap_{j=1}^{r} x_j$thenx≥$x_j$forsomej

PROOF:Letxbeameet–irreducibleelementinadistributivelatticeL andx≥$\bigcap_{j=1}^{r} x_j$

CLAIM:x≥x_j forsomej

We have x ≥$x_1 \cap x_2 \cap - - - \cap x_r$

thenx=$x\cup(x_1 \cap x_2 \cap --- \cap x_r)$

$=(x\cup x_1)\cap(x\cup x_2) \cap --- \cap(x\cup x_r)$ Since, x is meet – irreducible.

Wehavethefactthatthereissometerm $x\cup x_j$suchthatx=$x\cup x_j$

⇒x≥$x_j$forsomej

Hence,theproofofthelemma.]

Now, we prove our theorem.

Everyelementofadistributivelatticehasatmostoneirredundantirreducible meet representation.

LetubeanyarbitraryelementofadistributivelatticeL.

Ifpossible,assumethatuhastwoirredundantirreduciblemeet representation given below.

$u=x_1\cap x_2\cap---\cap x_r---\to(1)$

$u=y_1\cap y_2\cap---\cap y_s---\to(2)$ **from**

(1)

$u\leq x_j, x_j=1,2,---r$

wehave $x_j=y_1\cap y_2\cap---\cap y_s, j=1,2,---s$

i.e., $x\geq\bigcap_{i=1} y_i$

inviewoftheabove lemma, we have $x_j\geq y_k$ **forsome k. on similar lines we get some** $x_l \ni y_k\geq x_l$

Hence, $x_j\geq y_k\geq x_l$

Since,(1)&(2)areirredundantmeetrepresentation. This is compatible only if j = l i.e., $x_j = x_l$

Hence,toeveryjthereexistsexactlyonesuchk.

Thus,therepresentationsof(1)&(2)areidenticaltowrittentheorderof the components.

- **THEOREM:If every element of a lattice has a unique irredundant irreducible meet representation then the lattice satisfies the lower covering conditions.**

PROOF:Let us suppose that L is a lattice in which every element has irredundant irreducible meet representation.

Weprovethistheorembyindirectmethod.

AssumethatLdoesnotsatisfylowercoveringcondition. That

means for u, v € L $u \cap v \prec v \not\Rightarrow u \prec u \cup v$

Then∃inLthreeelements a, b,csuchthat$a\cap b \prec b$--- →(1)

and$a<c<a\cup b$--- →(2)

theninviewof(2)∃elementsq_1,q_2 €Lsuchthat $q_1 \geq a$

and q_1 c - - →(3)

$q_2 \geq a$and$q_2 a \cup b$--- →(4)

from(3),$b \cap a \leq b \cap q_1 \leq b$--- →(5)

from(4), we have$b \cap a \leq b \cap q_2 \leq b$--- →(6) In view of

(1)

Wehaveeither$b \cap a = b \cap q_1$(or)$b \cap q_1 = b$ If $b \cap q_1$

$= b$ then we get $q_1 \geq a \cup b > c$.

$\Rightarrow q_1 > c$

whichisacontradiction.

$\therefore b \cap a = b \cap q_1$

Similarly,weget$b \cap a = b \cap q_2$

Furthermore $q_1 \neq b$, $q_2 \neq b$.

Sinceb€L

byassumption,

$b = b_1 \cap b_2 \cap --- \cap b_r$beanirredundantirreduciblemeetrepresentation.

Wehave$b \cap a$hastwoirredundantandirreduciblemeetrepresentations.

thisisacontradiction.

∴ThelatticeLsatisfieslowercoveringcondition.