2022 Mid MP 3m **Defenition :-**

A Discrete Dynamic system is a function which is composed with itself over and over again.

Ex: Let us consider the function $f(x) = -x^3$

$$(f \circ f)(x) = f(f(x)) = f(-x^3) = -(-x^3)^3 = x^9$$

$$\therefore f^2(x) = x^9$$

'f' is composed with 'f' itself

Next, $(f \circ f \circ f)(x) = f^2(f(x))$

$$= f^2(-x^3) = (-x^3)^9 = -x^{27}$$

$$\therefore f^3(x) = -x^{27}$$

$$f^4(x) = (f \circ f \circ f \circ f)(x)$$

$$= f^3 \circ f(x) = f^3(f(x)) = f^3(-x^3) = -(-x^3)^{27} = x^{81}.$$

$$\therefore f^4(x) = x^{81}$$

$$f^n(x) = f^{n-1}(f(x)) = f^{n-1}(-x^3) = (-1)^n x^{3^n}$$

$$x, f(x), f^2(x), f^3(x), \ldots, f^n(x), \ldots$$

Now, the question is what is $\lim_{n \to \infty} f^n(x)$?

Here $x = f(x)$

Note: Here, $f^n(x)$ denotes the composition of $f^{n-1}(x)$ with $f(x)$

But $f^n(x)$ doesn't mean that 'n' is the order (power) of 'f' for n times (or) not the n^{th} derivative (or) not the product of 'f'. ~~of 'f'~~.

Mid 2022 (ii) 3m **Phase portraits :**

Definition:

A phase portrait is a "graphical representation of the dynamics of a system".

Note: The phase portraits consists of a diagram representing possible beginning position in the system and the arrows which ind-
-icate the change in these positions under iterations of the function 'f'.

Problem: Draw the phase portrait of the function $f(x) = x^2$

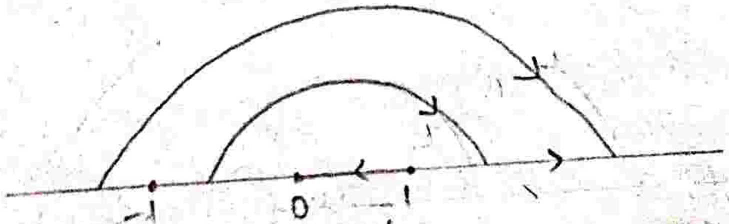
Sol: Given that $f(x) = x^2$

The dynamical system consists of the domain 'f' and the function 'f'

The domain is the set of real numbers which is represented by a line.

If $x=0$, $f^n(0)=0 \Rightarrow$ at $x=0$, the function 'f' is fixed. Thus can be represented by a dot on the real line at '0'.

If $x=1$, $f^n(1)=1 \Rightarrow$ at $x=1$, the function 'f' is fixed. Thus can be represented by a dot on the real line at '1'.



Next, $0<x<1$ then $f^n(x) \to 0$ as $n \to \infty$

Thus can be represented by an arrow from 1 to 0

Next, $x>1$ then $f^n(x) \to \infty$ as $n \to \infty$

Thus can be represented by an arrow from 1 to ∞

Next, $-1<x<0$ then for all values in the interval $(-1,0)$ the function 'f' maps to the values in $(0,1)$

Thus can be represented by an arrow from $(-1,0)$ to $(0,1)$

Finally $x<-1$ then $f^n(x) \to \infty$ as 'n' is very large

Thus can be represented by an arrow from left side of '-1' to right side of '1'

Problem: Draw the phase portrait of the function $f(x) = -x^3$

mid.

Sol: Given that $f(x) = -x^3$

The dynamical system consists of the domain 'f' and the function 'f'.

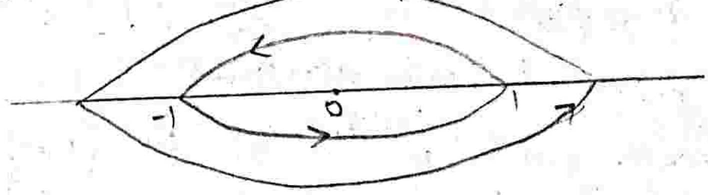
The domain is the set of real numbers which is represented by a line.

If $x=0$ then $f^n(0) = 0$ i.e., '0' is fixed

This can be represented by a dot on the real numbers at $x=0$

If $x=1$, then the values of $f^n(x)$ oscillates between '-1' and '1'

This can be represented by an arrow from '1' to '-1' and another arrow from '-1' to '1'.



If $x>1$, then the values of 'f' in iterations oscillates from side to side of zero.

This can be represented by an arrow from right side of '1' to left side of '-1'.

Periodic points and Stable sets:

In this section we assume that the range of the function is a subset of the domain.

Definition: If 'f' is a function and $f(c) = c$, then 'c' is called a fixed point of 'f'.

Note: A function of real numbers has a fixed point at 'c' iff the point (c, c) is on its graph.

In other words, a function has a fixed point at 'c' iff its graph intersects the straight line $y=x$

Ex: The function $f(x) = x^2$ has two fixed points '0' and '1'.

The function $f(x) = -x^3$ has a fixed point '0'.

Theorem: Let $I = [a,b]$ be a closed interval and $f: I \to I$ be a continuous function. Then 'f' has a fixed point in 'I'.

Proof: By data $I = [a,b]$ be a closed interval and $f: I \to I$ be a continuous function.

Claim: 'f' has a fixed point in 'I'

If $f(a) = a$ (or) $f(b) = b$ then 'f' has fixed points 'a' & 'b' the theorem is immediately proved.

Suppose that $f(a) \neq a$ and $f(b) \neq b$

we have $f(a), f(b)$ are in $[a,b]$ Then $a < f(a) < b$, $a < f(b) < b$

$\Rightarrow a < f(a), f(b) < b$

Let us consider a new function $g(x) = f(x) - x$

Since $x, f(x)$ are continuous functions on $[a,b]$

we have $g(x)$ is also a continuous function

next $g(a) > 0, g(b) < 0$

Clearly, the function $g(x)$ satisfies the Intermediate value theorem.

Then there is a point 'c' between 'a' and 'b' $(a < c < b)$

such that $g(c) = 0$

$\Rightarrow f(c) - c = 0 \Rightarrow f(c) = c$

Therefore, "c" is a fixed point of 'f'

This completes the proof

Problem: Find fixed point of the function $f(x) = 1 - x^2$ in the interval $[0,1]$

Sol. The given function is $f(x) = 1 - x^2$

Clearly, $f(x) = 1 - x^2$ is a continuous function on $[0,1]$

In view of the above theorem, 'f' has a fixed point in $[0,1]$

For that point solve the equation $f(x) = 1 - x^2$, $y = x$

$\Rightarrow 1-x^2 = x \Rightarrow x^2+x-1=0$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1-4(1)(-1)}}{2(1)}$$

$$= \frac{-1+\sqrt{5}}{2}$$

Therefore, the fixed point of 'f' in $[0,1]$ is $\frac{-1+\sqrt{5}}{2}$

Theorem: let 'I' be a closed interval and $f: I \to R$ be a continuous function. If $f(I) \supset I$ then 'f' has a fixed point in 'I'

Proof: By data 'I' be a closed interval and $f: I \to R$ be a continuous function.

Suppose that $f(I) \supset I$

claim: 'f' has a fixed point in 'I'

let $I = [a,b]$. since $f(I) \supset I$ there are two points c,d in 'I' such that $f(c)=a$, $f(d)=b$ if $c=a$ (or) $b=d$, then there is nothing to prove.

Because 'f' has fixed point in $[a,b]$

If $c \neq a$ and $d \neq b$ then $a<c<b$, $a<d<b$.

Let us consider a new function $g(x) = f(x) - x$

Clearly, the function $g(x)$ is continuous on $[c,d]$ and

$g(c) = f(c) - c < 0 \Rightarrow g(c) < 0$

$g(d) = f(d) - d > 0 \Rightarrow g(d) > 0$

Clearly, the function $g(x)$ satisfies the hypothesis of Intermediate value theorem. Then there is a point 'e' between 'c' and 'd' such that $g(e)=0 \Rightarrow f(e) - e = 0$

$\Rightarrow f(e) = e$

Therefore, 'f' has a fixed point 'e' in 'I'

This completes the proof

Definition: The point 'x' is a periodic point of 'f' with Period 'k', If $f^k(x) = x$

In other words, 'x' is a periodic point of 'f' with period 'k'. If 'x' is a fixed point of f^k.

The point 'x' has prime period 'k_0' if $f^{k_0}(x) = x$, and $f^n(x) \neq x$, $0 < n < k_0$

The set of all iterates of the point 'x' is called 'orbit of x' and if 'x' is a periodic point then it and its iterates are called 'a periodic orbit' (or) 'a periodic cycle'.

* Find the orbit and periodic orbit (or) periodic cycle of the function $f(x) = -x^3$

Sol By data the function $f(x) = -x^3$

let $x = 0$, $f(0) = 0$

Therefore, '0' is fixed point and periodic point with prime period '1'.

$x = 1$, $f(1) = -1$, $(f \circ f)(1) = f(f(1)) = f(-1) = 1$ $f(1) = -1$

Therefore, '1' is periodic with prime period '2'

$x = -1$, $f(-1) = 1$, $f^2(-1) = f(f(-1))$
$= -1$

Therefore, '-1' is also periodic with period '2'

The orbit of $1 = \{-1, 1\}$

The Orbit of $-1 = \{1, -1\}$

The set of periodic points $= \{0, 1, -1\}$

The set of periodic points with prime period $2 = \{1, -1\}$

next, the set of periodic points with prime period

* Give an example of a function which has a lot of fixed points and periodic points.

Sol let us consider the function $f(x) = -x$

$x = 0$, $f(0) = 0$ $\Rightarrow$ 0 is a fixed point, also '0' is a periodic

Point with prime period 1

$x=1$, $f(1)=-1$, $f^2(1)=f(f(1))=f(-1)=1$

$x=1, f(1)=-1$
$f^2(1) = f(f(1))$
$= f(-1)$
$= -(-1) = 1$

1 is a periodic point with prime periodic is 2

Clearly, all positive integers are periodic points with prime Period '2'.

Definition: A point 'x' is an eventually fixed point of the function 'f', if there exists N such that $f^{n+1}(x)=f^n(x)$ whenever $n \geq N$

The point "x" is eventually periodic with periodic 'k' if there exist N such that $f^{n+k}(x)=f^n(x)$, whenever $n \geq N$

Ex: Let us consider the function $h(x)=4x(1-x)$

Take $x=0$; $h(0)=0$

Therefore, 0 is a fixed point

$x=1$, $h(1)=4(1)(1-1)=0$

$h^2(1)=h(h(1))=h(0)=0$

Therefore, 1 is eventually fixed

$x=\frac{1}{2}$, $h(\frac{1}{2})=4\cdot\frac{1}{2}(1-\frac{1}{2})=2(\frac{1}{2})=1$

$h^2(\frac{1}{2})=h(h(\frac{1}{2}))=h(1)=0$

$h^3(\frac{1}{2})=h^2(h(\frac{1}{2}))=h^2(0)=0$

Therefore $\frac{1}{2}$ is eventually fixed point of h

Give an example of the function having eventually fixed points and eventually periodic points.

Sol: Let us consider a function $g(x)=|-1+x|$ (or) $g(x)=|x-1|$

Let $x=0$, $g(0)=1$

$g^2(0)=g(g(0))=g(1)=0$

$g^3(0)=g^2(g(0))=g^2(1)=g(g(1))=g(0)=1$

$g^4(0) = g^3(g(0)) = g^3(1) = g(g^2(1)) = g(g(g(1))) = g(g(0))$
$= g(1) = 0$

Now $g^4(0) = 0$

$\Rightarrow g^{2+2}(0) = g^2(0)$ $\boxed{f^{(n+k)} = f^n(x)}$

$\therefore$ The point '0' is eventually periodic.

$g(-3) = |-4| = 4$

$g^2(-3) = g(g(-3)) = g(4) = |4-1| = 3$

$g^3(-3) = g(g^2(-3)) = g(3) = |3-1| = 2$

$g^4(-3) = g(g^3(-3)) = g(2) = |2-1| = 1$

$g^5(-3) = g(g^4(-3)) = g(1) = 0$

$g^6(-3) = g(g^5(-3)) = g(0) = 1$

$\therefore g^6(-3) = g^{4+2}(-3) = g^4(-3) = 1$, $\therefore$ The point '-3' is eventually period

On similar lines we can show that every integer is an eventually periodic point of the function $g(x) = |x-1|$

Take $x = 2$, $g(2) = |2-1| = 1$; $g^2(2) = g(g(2)) = g(1) = 0$

$g^3(2) = g(g^2(2)) = g(0) = 1 = g(2)$

$\therefore$ 2 is eventually periodic

$g^4(2) = g(g^3(2)) = g(1) = 0$

$g^5(2) = g(g^4(2)) = g(0) = 1$

clearly from the above observations the periodic cycle is $\{0,1\}$

2.2 **Definition:** let 'f' be a function and 'p' be a period point of f with prime period k. Then x is forward asymptotic to p if the sequence $x, f(x), f^k(x), f^{2k}(x), \ldots$ converges to p

i.e., $\lim\limits_{n\to\infty} f^{nk}(x) = p$

* The stable set of 'p' denoted by $W^s(p)$, consists of all points which are forward asymptotic to p.
* If the sequence $|x|, |f(x)|, |f^2(x)|, |f^{3}(x)|, \ldots$ grows without bound then 'x' is "forward asymptotic to ∞".

The stable set of "∞" is the set of all points which are forward asymptotic to '∞'. It is denoted by $W^s(\infty)$

Problem: Find the stable set of 0, 1 of the function $f(x) = -x^3$ also find $W^s(-1)$

Sol Given that $f(x) = -x^2$

first we show that 0, 1, -1 are periodic points of f

Take $x = 0$, $f(0) = 0$ $\quad 0 \in W^s(0)$

0 is periodic with prime period 1

$x = 1$, $1, -1, f^2(1) = 1, \ldots \not\to 0$ does not converges to 0

$x = -1$, $-1, 1, -1, 1, \ldots \not\to 0$ $\quad 1, -1 \notin W^s(0)$

clearly, 1 and -1 are not in $W^s(0)$

Next, Take $x = \frac{1}{2}$

Consider the sequence $\frac{1}{2}, \frac{-1}{8}, \frac{1}{512}, \ldots$

This sequence Converges to 0

Therefore, $\frac{1}{2} \in W^s(0)$

It is obvious that the stable set $W^s(0)$ consists of all points between -1 and 1

Hence $W^s(0) = (-1, 1)$

Take $p = 1$, $f(1) = -1$

$$\begin{aligned} f^2(1) &= f(f(1)) \\ &= f(-1) \\ &= 1 \end{aligned}$$

Therefore, $p=1$ is a periodic point of f with prime period 2

Let us find the stable set $W^s(1)$

$x=1$

consider the sequence $x, f^k(x), f^{2k}(x), \ldots$

$1, 1, 1, \ldots \to 1$

Therefore, $1 \in W^s(1)$

Hence, $W^s(1) = \{1\}$

next, $x=-1$, $k=2$

Consider, the sequence $x, f^k(x), f^{2k}(x), \ldots$

$-1, -1, -1, \ldots, -1$

Therefore, $W^s(-1) = \{-1\}$

Next, we find $W^s(\infty)$

Take, $x=2$, $f(x) = -x^3$

Consider, the sequence $|x|, |f(x)|, |f^2(x)|, |f^3(x)|, \ldots$

$\Rightarrow |2|, |8|, |8^3|, |(-8^3)^3|, \ldots$

Clearly, the sequence grows without bounds.

Therefore, $2 \in W^s(\infty)$ and more over all $x \in (1, \infty)$, we have $x \in W^s(\infty)$

likewise, for all $x \in (-\infty, -1)$ we have $x \in W^s(\infty)$

Hence, $W^s(\infty) = (-\infty, -1) \cup (1, \infty)$

Thus, $W^s(0) = (-1, 1)$

$W^s(1) = \{1\}$

$W^s(-1) = \{-1\}$

$W^s(\infty) = (-\infty, -1) \cup (1, \infty)$

① find $W^s(0)$ for the function $g(x) = |x-1|$ //

Sol: Given $g(x) = |x-1|$

First we prove '0' is a periodic point

Now $g(0) = |0-1| = 1$

$g^2(0) = g(g(0)) = g(1) = 0$

$\therefore$ 0 is a periodic point with prime period 2

Take $x = 0$

Consider the sequence,

$$S = \begin{cases} x, f^2(x), f^4(x), f^6(x), \dots \\ 0, 0, 0, 0, \dots \longrightarrow 0 \end{cases}$$

Therefore, $0 \in W^s(0)$

next, $x = 1$

$$x, f^2(x), f^4(x), f^6(x), \dots$$

$$1, 1, 1, 1, \dots \longrightarrow 1$$

Therefore, $1 \notin W^s(0)$

next, $x = 2$

$$x, f^2(x), f^4(x), f^6(x), \dots$$

$$2, 0, 0, \dots \longrightarrow 0$$

$\therefore 2 \in W^s(0)$

On similar lines, we can see that $-2 \in W^s(0)$

Hence all the even integers are in $W^s(0)$

Problem: Find $W^s(1)$ of the function $g(x) = |x-1|$

Sol Given $g(x) = |x-1|$

First we prove '1' is a periodic point

Now $g(1) = |1-1| = 0$

$g^2(1) = g(g(1)) = g(0) = 1$

$\therefore$ 1 is a periodic point with prime period 2

Take $x = 0$

Consider the Sequence,

$$x, f^2(x), f^4(x), f^6(x), \dots$$

$$0, 0, 0, 0, \dots \not\longrightarrow 1$$

$\therefore 0 \notin W^s(1)$

next, $x=1$

$x, f^2(x), f^4(x), f^6(x), \ldots$

$1, 1, 1, 1, \ldots \longrightarrow 1$

$\therefore 1 \in W^s(1)$

next, $x=2$

$x, f^2(x), f^4(x), f^6(x), \ldots$

$2, 0, 0, 0, \ldots \not\rightarrow 1$

$\therefore 2 \notin W^s(1)$

Take $x=3$

$x, f^2(x), f^4(x), f^6(x), \ldots$

$3, 1, 1, \ldots \longrightarrow 1$

$\therefore 3 \in W^s(1)$

Clearly, $1, 3, 5, \ldots, \in W^s(1)$

likewise, $-1, -3, -5, \ldots \in W^s(1)$

Hence, the stable set $W^s(1)$ contains all odd integers.

Theorem: The stable sets of distinct periodic points do not intersect.

In otherwords, if 'p' and 'q' are periodic points and $p \neq q$ then $W^s(p) \cap W^s(q) = \phi$

Proof: Let 'p' and 'q' be two distinct periodic points.

i.e., $p \neq q$, let the prime period of $p = k_1$

Let the prime period of $q = k_2$

Claim: $W^s(p) \cap W^s(q) = \phi$

If possible Assume that $W^s(p) \cap W^s(q) \neq \phi$

then let $x \in W^s(p) \cap W^s(q)$, then $x \in W^s(p)$ and $x \in W^s(q)$

then $x \in W^s(p) \Rightarrow$ the sequence $x, f^{k_1}(x), f^{2k_1}(x), \ldots \longrightarrow p$

then for $\epsilon > 0$, $\exists$ a positive integer N_1, $\ni |f^{nk_1} - p| < \frac{\epsilon}{2}$ $\forall n \geq N_1$

Next, $x \in W^s(q) \Rightarrow \exists$ a positive integer $N_2 \ni |f^{nk_2} - q| < \frac{\epsilon}{2}$ $\forall n \geq N_2$

Take $N = \max\{N_1, N_2\}$

then $|f^{nk_1} - p| < \frac{\epsilon}{2}, \forall n \geq N$, $|f^{nk_2} - q| < \frac{\epsilon}{2}, \forall n \geq N$

consider, $|p-q| = |p - f^{nk_1k_2}(x) + f^{nk_1k_2}(x) - q|$

$$\leq |p - f^{nk_1k_2}| + |f^{nk_1k_2} - q|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore p = q$

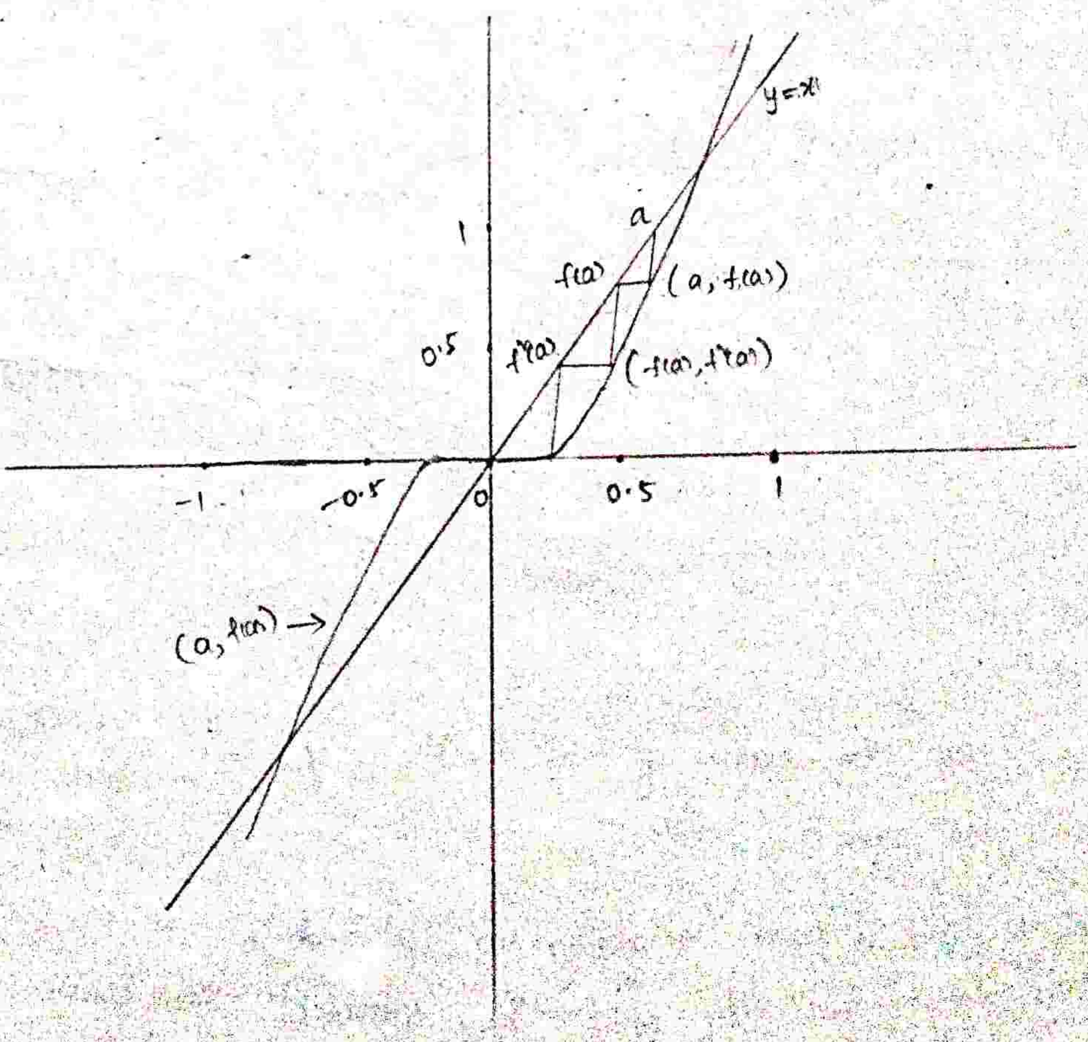
which is a contradiction

$\therefore$ Our assumption is wrong (or) false

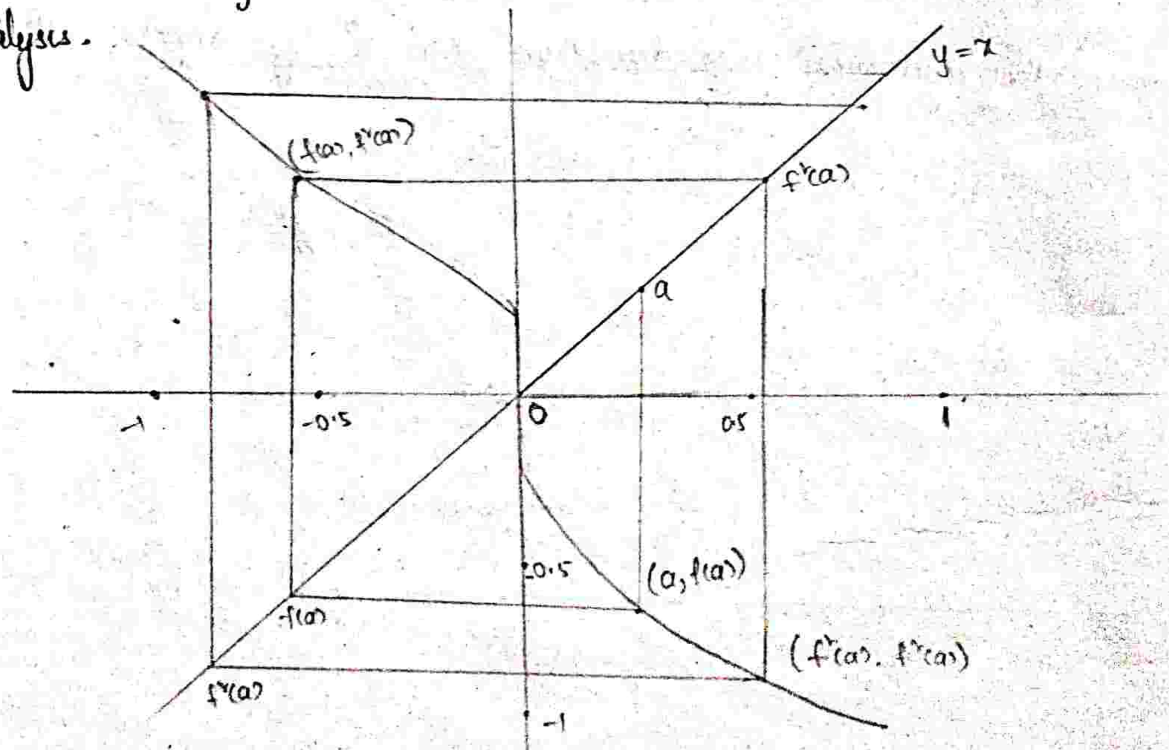
Hence, $W^s(p) \cap W^s(q) = \phi$

Graphical Analysis: To study the dynamical system there is another tool called "Graphical analysis".

Examine the dynamics of a function $f(x) = x^3$ by Graphical analysis



Draw the curve of the function and the line $y=x$ in the cartesian plane.
let us take a point 'a' between '0' and '1'
ie. $a \in (0,1)$
let us study where abouts of the point 'a' on the line $y=x$ with respect to the moment of the point 'a' on x-axis.
move vertically from the point 'a' to the point $(a, f(a))$
ie. meeting the curve of 'f' again move horizontally back of the line $y=x$ and that point of intersection be $f(a)$.
Repeat the same process we get the sequence
$f(a), f^2(a), f^3(a), \ldots, f^n(a) \longrightarrow 0$
Examine the dynamics of the function $f(x) = -x^{1/3}$ by graphical analysis.



Draw the curve of the function $y = -x^{1/3}$ and the line $y=x$ on the cartesian plane.
let us take a point belonging to '0' and '1' ie, $a \in (0,1)$ on the line move vertically until they intersect the curve with the coordinates and then move horizontally back to the line $f(a)$. and again move vertically until we intersect the

Curve with the coordinate $(f(a), f^2(a))$

Again move horizontally back to the line intersecting $f^2(a)$

Continuing in this way, the period cycle with the period '2'

is $\{-1,1\}$

we conclude that 'a' is $W^s(1)$ i.e., $a \in W^s(1)$

and further $W^s(1) = (0,\infty)$, $W^s(-1) = (-\infty, 0)$

$\therefore$ '0' is fixed point.

Theorem: If a continuous function of the real numbers has a periodic point with prime period '3', then it has periodic points of all prime periods.

Proof: Let f be a continuous function of real numbers which has a periodic point say 'a' with prime period '3'.

we have $f(a) = b$, $f^2(a) = f(f(a)) = f(a) = c$,

$$f^3(a) = f(f^2(a)) = f(c) = a$$

Let $\{a,b,c\}$ be the orbit of 'f'

$[f(a) = c,\ f^2(a) = f(f(a)) = f(c) = b,\ f^3(a) = f(f^2(a)) = f(b) = c]$

Without loss of generality we assume $a<b<c$

There are two cases, $f(a) = b$ or $f(a) = c$

Suppose $f(a) = b$, $f^2(a) = f(b) = c$, $f^3(a) = f(c) = a$

Similarly we can prove the other case $f(a) = c$

Let $I_0 = [a,b]$, $I_1 = [b,c]$

By intermediate value theorem, we get

$$f(I_1) \supset I_1$$

By a known theorem, 'f' has a fixed point in I_1

Let $n>1$ be a natural number. Suppose that there is a nested sequence of closed intervals $I_1 = A_0 \supset A_1 \supset A_2 \supset \dots \supset A_n$ with the following properties.

i) $A_0 = I_1$

2) $f(A_k) = A_{k-1}$, for $k = 1, 2, \ldots, n-2$

3) $f^k(A_k) = I_1$, for $k = 1, 2, \ldots, n-2$

4) $f^{n-1}(A_{n-1}) = I_0$

5) $f^n(A_n) = I_1$

Since $I_1 \supset A_n$, (5) and $f^n(A_n) \supset A_n$

by a known theorem,

f^n has a fixed point in "A_n" that means, 'f' has a period Point with prime periodic point 'n' in 'A_n'.

Now our claim is that the periodic point 'p' has the Prime period 'n'

If 'p' is a periodic point of period 'n' in 'A_n' then (1) and (3) imply that $p, f(p), f^2(p), \ldots, f^{n-2}(p) \in I_1 = [b, c]$

Next, from (4) $f^{n-1}(p) \in I_0 = [a, b]$

If $p = c$ then $f(p) = f(c) = a$

Therefore, the first iteration which is not in I_1 is $f^{n-1}(p)$

Then we must have $n = 2$

$[$If $n = 2$, $f^{2-1}(p) = f(p) = f(c) = a \notin I_1]$

This is a contradiction because the prime period of $p = c$ is three.

Therefore, $p \neq c$ Hence $p \in [b, c)$

If $p = b$ then $n = 3$, since $f^2(p) = a$ which is not in I_1

i.e., $f^2(p) = a \notin I_1$ and the only iterate which is not in I_1 is $f^{n-1}(p)$

Suppose that $n \neq 3$ then "p" $\in (b, c)$

Since $f^{n-1}(p) \in I_0 = [a, b]$

$\Rightarrow f^{n-1}(p) \neq p$

'p' is not a periodic point of f with prime period $(n-1)$.

If the prime period of $p < n-1$

Then by ③ and $p \neq b$ and $p \neq c$

we have the orbit of 'p' is contained in (b,c)

This is a contradiction to ④

∴ 'p' has the prime period 'n'

Finally, we prove that for each natural number $n \geq 1$. The nested sequence with the properties ① ② and ③ exists.

let $n > 1$ obviously we can choose $A_0 = I$

Therefore ① is satisfied

Let us use the property. If 'f' is continuous function and J, K are closed intervals such that $f(J) \supset K$ then there exists an interval 'J_0' such that $J_0 \subset I$ and $f(J_0) = K$

Since $A_0 = I_1$ and $f(I_1) \supset I_1$, ie, $f(A_0) \supset A_0$

So there is $A_1 \subset A_0$ such that $f(A_1) = A_0$

Since $A_1 \subset A_0 \Rightarrow f(A_1) \supset A_1$

consequently there is an interval $A_2 \subseteq A_1$ such that $f(A_2) = A_1$

Continuing the same argument. we define A_k for $k = 1, 2, \dots (n-2)$ for $k = 1, 2, \dots, (n-2)$ so that $f(A_k) = A_{k-1}$

In each case $A_k \subset A_{k-1}$

Therefore ② is satisfied

Next, $f(A_k) \supset A_k$ for some k, by this process of defining intervals A_k can continue indefinitely

then $f(A_k) = A_{k-1} \Rightarrow f^2(A_k) = f(f(A_k))$

$$\therefore f^2(A_k) = f(A_{k-1}) = A_{k-2}$$

$$f^3(A_k) = f(f^2(A_k)) = f(A_{k-2}) = A_{k-3}$$

So $f^k(A_k) = A_0 = I_1$, $k = 1, 2, 3, \dots, n-2$

next, note that $f^{n-1}(A_{n-2}) = f(f^{n-2}(A_{n-2}))$

$$= f(I_1) \supset I_0$$

Hence there is $A_{n-1} \subset A_{n-2}$ such that $f^{n-1}(A_{n-1}) = I_0$

$\therefore$ condition ④ is satisfied

Finally, $f^n(A_{n-1}) = f(f^{n-1}(A_{n-1}))$

$= f(I_0) \supset I_1$

Therefore, there is $A_n \subset A_{n-1}$ such that $f^n(A_n) = I$,

i.e., condition ⑤ is satisfied.

This completes the proof of the theorem.

Sarkouskil's Ordering:

mid 3m

Sarkouskil's ordering of the natural numbers is

$3 \succ 5 \succ 7 \cdots \succ 2.3 \succ 2.5 \succ 2.7 \succ \cdots \succ 2^2.3 \succ 2^2.5 \succ 2^2.7 \succ \cdots$

$\succ \cdots 2^n.3 \succ 2^n.5 \succ 2^n.7 \succ \cdots \succ 2^3 \succ 2^2 \succ 2 \succ 1.$

$a \succ b$ indicates 'a' precedes 'b' in the order, when writing the orders, all odd numbers except one are listed in ascending order then two times every odd then four times every odd and it soon. The order is completed by listing the process of '2' in descending order. Every natural number can be find exactly once in sarkouskil's ordering.

Theorem: (Sarkouskil's):

Suppose that $f: R \to R$ is continuous and that 'f' has a Periodic point with prime period 'n' if $n \succ m$ in sarkouskil's ordering then 'f' is also has a periodic point with prime Period 'm'.

Differentiability and its emplications:

Def: Let I be an interval $f: I \to R$ and 'a' be a point in 'I'. The function is differentiable at 'a' if the limit

$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists.

In this case we say 'f' is differential at 'a' and call that limit $f'(a)$ (or) derivative of 'f' at 'a'.

A function is differentiable on I, if it is differentiable at each and every point in the domain "I".

20 **Theorem:** let I be a closed interval. and $f: I \to I$ be a differentiable function satisfying $|f'(x)| < 1$ for all 'x' in 'I' then 'f' has a unique fixed point in 'I'. More over, if x and y are any two points in 'I' and $x \neq y$, then
$|f(x) - f(y)| < |x-y|$.

Proof: By the hypothesis, we have I be a closed interval and $f: I \to I$ be a differentiable function satisfying $|f'(x)| < 1$ for all $x \in I$

First we prove the second part of the statement.
let x and y be two points in 'I' such that $x \neq y$
without loss of generality, we assume that $x < y$
Since f is differentiable on I, we have f is also differentiable over the closed interval $[x, y]$
In view of the mean value theorem, we get
$\exists$ a point $c \in [x, y]$ such that $|f(x) - f(y)| = f'(c)|x-y|$

$$\Rightarrow f'(c) = \frac{|f(x) - f(y)|}{|x-y|}$$

$$\text{cos } f'(c) = \frac{|f(x) - f(y)|}{|x-y|} < 1$$

$$\Rightarrow |f(x) - f(y)| < |x-y| \quad \text{--- (1)}$$

Next, we prove first part of the theorem,
i.e., 'f' has a unique fixed point in I
We know that every differentiable function is continuous.
so, by a known theorem 'f' has a fixed point say 'p' in 'I'. $f(p) = p$
let 'x' be any other point in 'I'
Now $|p - f(x)| = |f(p) - f(x)| < |p - x|$
This inequality is valid if ~~p~~ $p \neq x$
$\therefore$ p is the unique fixed point of 'f' in 'I'.

* We can verify the above theorem by studying some examples.

Ex: 1) $f(x) = \frac{x}{2} + \frac{3}{2}$

Sol. let $f(x) = \frac{x}{2} + \frac{3}{2}$ —— ①

Take the line $y = x$ —— ②

clearly, $f'(x) = \frac{1}{2}$

i.e., 'f' is differentiable function and $|f'(x)| = |\frac{1}{2}| = \frac{1}{2} < 1$

Clearly, 'f' satisfies the hypothesis of the above theorem.

Consequently, 'f' has a unique fixed point

For that fixed point let us solve the equations ① & ②

$$x = \frac{x}{2} + \frac{3}{2} \Rightarrow 2x = x + 3$$
$$\Rightarrow x = 3$$

$\therefore$ the unique fixed point is $x = 3$

By mean value theorem,

there exists a point 'c_1' in $\mathbb{R}$ such that $|f(x) - f(3)| = |f'(c_1)| |x-3|$

$$|f(x) - f(3)| = \frac{1}{2}|x-3|$$

Similarly, by the mean value theorem,

$\exists$ a point c_2 in $\mathbb{R}$ $\ni |f^2(x) - f^2(3)| = |f'(c_2)|^2 |x-3|$

$$\Rightarrow |f^2(x) - f^2(3)| = \frac{1}{2^2}|x-3|$$

Similarly, $\exists$ a point c_3 in $\mathbb{R}$ $\ni |f^3(x) - f^3(3)| = \frac{1}{2^3}|x-3|$

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$$|f^n(x) - f^n(3)| = \frac{1}{2^n}|x-3|$$

Since $f^n(3) = 3$, we get $f^n(x) \longrightarrow 3$ and $x \in W^s(3)$

Thus $W^s(3) = \mathbb{R}$

2) $f(x) = \frac{-x}{2} + \frac{9}{2}$

Let $f(x) = \frac{-x}{2} + \frac{9}{2}$ —①

Take the line $y = x$ —②

Clearly, $f'(x) = \frac{-1}{2}$

ie, 'f' is differentiable function and $|f'(x)| = |\frac{-1}{2}| = \frac{1}{2} < 1$

Clearly, 'f' satisfies the hypothesis of the above theorem

Consequently, 'f' has a unique fixed point

For that fixed point. Let us solve the equation ① and ②

$$\frac{-x}{2} + \frac{9}{2} = x$$

$$\Rightarrow -x + 9 = 2x \Rightarrow 2x + x = 9 \Rightarrow 3x = 9 \Rightarrow x = 3$$

Therefore, the unique fixed point is $x = 3$

By mean value theorem, $\exists$ a point c_1 in $\mathbb{R} \ni |f(x) - f(3)| = |f'(c_1)||x-3|$

$$\Rightarrow |f(x) - f(3)| = \frac{1}{2}|x-3|$$

Similarly, by the mean value theorem,

$\exists$ a point $c_2 \in \mathbb{R} \ni |f^2(x) - f^2(3)| = |f'(c_2)||x-3|$

$$\Rightarrow |f^2(x) - f^2(3)| = \frac{1}{2^2}|x-3|$$

Similarly, $\exists$ a point c_3 in $\mathbb{R} \ni |f^3(x) - f^3(3)| = \frac{1}{2^3}|x-3|$

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$$|f^n(x) - f^n(3)| = \frac{1}{2^n}|x-3|$$

Clearly, $f^n(3) \to 3$, $x \in W^s(3)$

Therefore, $W^s(3) = \mathbb{R}$

Next we examine some examples in which $|f'(x)| > 1$

3) ~~Let~~ $g(x) = 2x - 3$

Let $g(x) = 2x - 3$ —① be the given function,

Take $y = x$ —②

Now $g'(x) = 2 \Rightarrow |g'(x)| = 2 > 1$

So $g(x)$ is a differentiable function

For the fixed point let us solve the equations ① and ②

$$2x-3=x \Rightarrow 2x=x+3 \Rightarrow 2x-x=3 \Rightarrow x=3$$

$\therefore$ "3" is a fixed point

By the mean value theorem,

$\exists$ a point c_1 in $\mathbb{R}$ such that $|g(x)-g(3)| = |g'(c_1)||x-3|$

$$\Rightarrow |g(x)-g(3)| = 2|x-3|$$

Similarly, by the mean value theorem,

$\exists$ a point c_2 in $\mathbb{R}$ such that $|g^2(x)-g^2(3)| = |g'(c_2)||x-3|$

$$\Rightarrow |g^2(x)-g^2(3)| = 2^2|x-3|$$

Similarly, $\exists$ a point c_3 in $\mathbb{R}$ such that $|g^3(x)-g^3(3)| = |g'(c_3)||x-3|$

$$\Rightarrow |g^3(x)-g^3(3)| = 2^3|x-3|$$

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$$|g^n(x)-g^n(3)| = 2^n|x-3|$$

The sequence $g^n(x)$ grows without bounds as 'n' goes to '∞'.

Therefore, $x \in W^s(\infty)$

Hence $W^s(\infty) = (-\infty,3) \cup (3,\infty)$

4) $k(x) = -2x+9$

let $k(x) = -2x+9$ —— ① be a function

Take $y=x$ —— ②

Now $k'(x) = -2$; $|k'(x)| = |-2| = 2 > 1$

so $k(x)$ is a differentiable function

for fixed point solve the equations ① & ②

$$-2x+9 = x$$

$$\Rightarrow 2x+x=9$$

$$\Rightarrow 3x=9$$

$$\Rightarrow x=3$$

$\therefore$ "3" is a fixed point

By the mean value theorem,

$\exists$ a point c_1, c_2, c_3 in $\mathbb{R}$ such that $|k(x)-k(3)| = |k'(c_1)|\,|x-3| = 2|x-3|$

$$|k^2(x)-k^2(3)| = |k'(c_2)|\,|x-3| = 2^2|x-3|$$

$$|k^3(x)-k^3(3)| = |k'(c_3)|\,|x-3| = 2^3|x-3|$$

$$\cdots$$

$$|k^n(x)-k^n(3)| = 2^n|x-3|$$

The sequence $k^n(x)$ grows without bounds as "n" goes to "∞"

$\therefore x \in W^s(\infty)$

Hence $W^s(\infty) = (-\infty, 3) \cup (3, \infty)$

Notes:- A function whose derivative exists and is continuous is called "c^1".

Theorem: Let 'f' be a 'c^1' function and 'p' be a fixed point of 'f', and suppose $|f'(p)|<1$ implies that there is a neighbourhood of 'p' that is contained in the (stable set) $W^s(p)$ and $|f'(p)|>1$ implies that there is a neighbourhood of 'p' all of whose points must leave the neighbourhood under iteration of 'f'.

Proof: By the hypothesis, 'f' be a 'c^1' function and 'p' be a fixed point of 'f'

Since 'f' is a 'c^1' function, we have 'f' is differentiable and its derivative is continuous

Case (i): Suppose that $|f'(p)|<1$

claim: To find a neighbourhood of 'p' contains in $W^s(p)$

Let $\epsilon>0$ and $\delta>0$ such that $|f'(x)| < 1-\epsilon$. $\forall x \in (p-\delta, p+\delta)$

By mean value theorem,

we have to show that $(p-\delta, p+\delta) \subset W^s(p)$

Let $\epsilon = \frac{1}{2}(1-|f'(p)|)$

Since $|f'(p)| < 1$ and $f'(x)$ is continuous $\exists$ a $\delta > 0$ such that

$|f'(x) - f'(p)| < \epsilon$ whenever $|x - p| < \delta$

Equivalently. $|f'(x) - f'(p)| < \epsilon$ whenever $x \in (p-\delta, p+\delta)$

Now, $|f'(x)| = |f'(x) - f'(p) + f'(p)|$

$$\leq |f'(x) - f'(p)| + |f'(p)|$$

$$< \epsilon + |f'(p)|$$

$$< \frac{1}{2}(1 - |f'(p)|) + |f'(p)|$$

$$= 1 - \left(\frac{1}{2} - \frac{f'(p)}{2}\right)$$

$$= 1 - \epsilon$$

Hence, for $\epsilon = \frac{1}{2}(1 - |f'(p)|)$ there exist a $\delta > 0$ such that $|f'(x)| < 1 - \epsilon$ for all $x \in (p-\delta, p+\delta)$

Next, we prove that $(p-\delta, p+\delta) \subset W^s(p)$

Let $x \in (p-\delta, p+\delta)$, $p \neq x$, Now our goal is that the sequence $\{f^n(x)\} \longrightarrow p$

By mean value theorem,

we have that $\exists$ a point 'c' in b/w 'x' and 'p' such that

$$|f(x) - p| = |f(x) - f(p)| = |f'(c)||x - p|$$

Since, $c \in (p-\delta, p+\delta)$, we have $|f(x) - f(p)| < (1-\epsilon)|x - p|$

iterating this process, we get

$$|f^2(x) - p| < (1-\epsilon)^2 |x - p|$$

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$$|f^n(x) - p| < (1-\epsilon)^n |x - p|$$

Since $(1-\epsilon)^n$ converges to '0' as $n \longrightarrow \infty$ because $1 - \epsilon < 1$

then we have $|f^n(x) - p| \longrightarrow 0$ and $f^n(x) \longrightarrow p$

immediately, we have $x \in W^s(p)$

Therefore, the neighbourhood $(p-\delta, p+\delta) \subset W^s(p)$

On similar lines, we prove the theorem when $|f'(p)|>1$

Thus completes the proof of the theorem.

Note: Fixed points where the derivatives are not equal to '1' in absolute values are called "hyperbolic fixed points".

Definition: Let 'p' be a periodic point of the differentiable function 'f' if 'p' is a fixed point of 'f^k' with prime period 'k' then 'p' is a hyperbolic periodic point. if $|(f^k)'(p)| \neq 1$

if $|(f^k)'(p)| = 1$, then 'p' is nonhyperbolic periodic point.

2.2 Theorem: Let 'f' be a 'c^1' function and 'p' be a periodic point of 'f' with prime period 'k' if $|(f^k)'(p)|<1$ then there exist an open interval containing 'p' that is contained in the stable set of 'p'. If $|(f^k)'(p)|>1$ then there exist an open interval containing 'p' such that all points in the interval except 'p' must the interval under the iteration of f^k.

Proof: By the hypothesis, 'f' be a 'c^1' function and 'p' be a periodic point of 'f' with prime period 'k'.

Since 'f' is a 'c^1' function

we have 'f' is differential and its derivative is continuous

Case (i): Suppose that $|(f^k)'(p)|<1$

Claim: To find there exist an open interval continuosly that is contained in $W^s(p)$

Let $\epsilon>0$ and $\delta>0$ such that $|(f^k)'(p)| < 1-\epsilon \ \forall x \in (p-\delta, p+\delta)$

By meanvalue theorem, we have to show that $(p-\delta, p+\delta) \subset W^s(p)$

Let $\epsilon = \frac{1}{2}(1-|(f^k)'(p)|)$

Since $|(f^k)'(p)|<1$ and $(f^k)'(x)$ is continuous $\exists$ a $\delta>0$ such that $|(f^k)'(x)-(f^k)'(p)|<\epsilon$ whenever $|x-p|<\delta$

Equivalently, $|(f^k)'(x) - (f^k)'(p)| < \epsilon$ whenever $x \in (p-\delta, p+\delta)$

Now, $$|(f^k)'(x)| = |(f^k)'(x) - (f^k)'(p) + (f^k)'(p)|$$
$$\leq |(f^k)'(x) - (f^k)'(p)| + |(f^k)'(p)|$$
$$< \epsilon + |(f^k)'(p)|$$
$$< \frac{1}{2}\left(1 - |(f^k)'(p)|\right) + |(f^k)'(p)|$$
$$= 1 - \left(\frac{1}{2} - \frac{(f^k)'(p)}{2}\right)$$
$$= 1 - \epsilon$$

Hence, for $\epsilon = \frac{1}{2}\left(1 - |(f^k)'(p)|\right)$ there exist a $\delta > 0$ such that

$|(f^k)'(x)| < 1 - \epsilon$ for all $x \in (p-\delta, p+\delta)$

Next, we prove that $(p-\delta, p+\delta) \subset W^s(p)$

let $x \in (p-\delta, p+\delta)$, $p \neq x$, Now our goal is that the sequence

$\{(f^k)^n(x)\} \longrightarrow p$

By mean value theorem,

we have that $\exists$ a point 'c' in b/w 'x' and 'p' such that

$$|(f^k)'(x) - p| = |(f^k)'(x) - (f^k)'(p)| = |(f^k)'(c)||x-p|$$

Since, $c \in (p-\delta, p+\delta)$, we have $|(f^k)' - (f^k)'(p)| < (1-\epsilon)|x-p|$

Iterating this process, we get

$$|(f^k)^2(x) - p| < (1-\epsilon)^2 |x-p|$$

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$$|(f^k)^n(x) - p| < (1-\epsilon)^n |x-p|$$

Since $(1-\epsilon)^n$ converges to '0' as $n \to \infty$ because $1-\epsilon < 1$

then we have $|(f^k)^n(x) - p| \longrightarrow 0$ and $(f^k)^n \longrightarrow p$

immediately, we have $x \in W^s(p)$

Therefore, the open interval $(p-\delta, p+\delta) \subset W^s(p)$

On similar lines, we prove the theorem when $|(f^k)'(p)| > 1$

Thus completes the proof of the theorem.

Definition: Let 'p' be a periodic point of f^k with prime period k.

If $|(f^k)'(p)| < 1$ then 'p' is an attracting period point of 'f'.

If $|(f^k)'(p)| > 1$ then 'p' is a repelling periodic point of 'f'.

Unit-2

PARAMETRIZED FAMILIES OF FUNCTIONS AND BIFUR-CATIONS

20 **Definition:** The family of linear functions of the form $f_m(x) = mx$, where "m" is "allowed" to vary the real numbers. The variable "m" is called a parameter and the family represented by "m" is called "parametrized family".

$f_m(x) = mx$ is called "parametrized family".

Ex: 1. Let us consider, the parametrized family $f_m(x) = mx$,

the only fixed point at $m=0$ is "0".

clearly the only fixed point at $m=0$ is "0".

Take $m=1$, then all real numbers are fixed points.

If "$m<-1$" then the hyperbolic fixed point "0" because

$|f_m'(0)| = |m| > 1$

Moreover, it is hyperbolic repelling point of "f"

∴ "0" is the hyperbolic repelling point of "f"

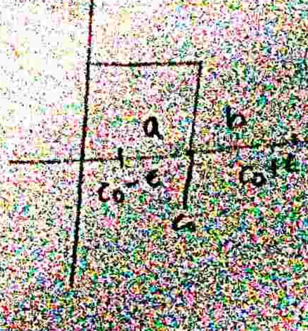
If $m=-1$ then all points except "0" are periodic points with period

Note: The dynamics of a family $f_m(x) = mx$ are unchanged for large intervals of the parameter values. Then at a particular Parameter value the dynamics change suddenly after which they again remain constant for a prolonged interval.

we call these sudden changes in dynamics "Bifurcation".

21, IMP, 22, 20 **Definition:** Let $f_c(x)$ be a parametrized family of functions then there is a bifurcation at "c_0" if there exists $\epsilon > 0$ such that whenever "a" and "b" satisfy $c_0 - \epsilon < a < c_0$ and $c_0 < b < c_0 + \epsilon$, then the dynamics of $f_a(x)$ is different from the dynamics of $f_b(x)$

In otherwords, the dynamics of the function changes when the parameter crosses through the point "c_0".



Ex:2: let us examine the following example

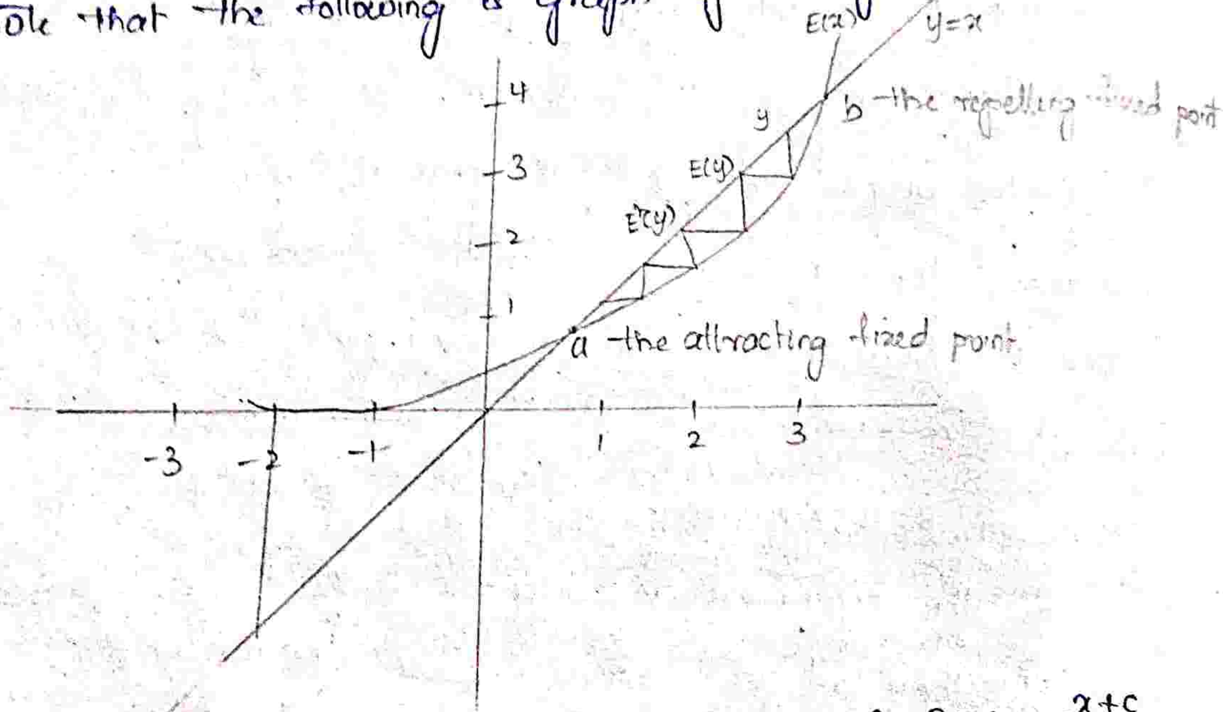
Consider $E_c(x) = e^{x+c}$

we study the dynamics of this function by graphical analysis

we study this exponential function in three cases.

Case 1:- If $c < -1$

we note that the following is graph of the given function



(i) From the graph it is clear that the curve of $E(x) = e^{x+c}$ and line $y=x$ intersect at two points "a" and "b" as shown in the figure.

Therefore, "a" and "b" are fixed points of E

(ii) From the picture $|E'(b)| > 1$, Therefore "b" is the repelling fixed point. also $|E'(a)| < 1$, therefore "a" is the attracting fixed point.

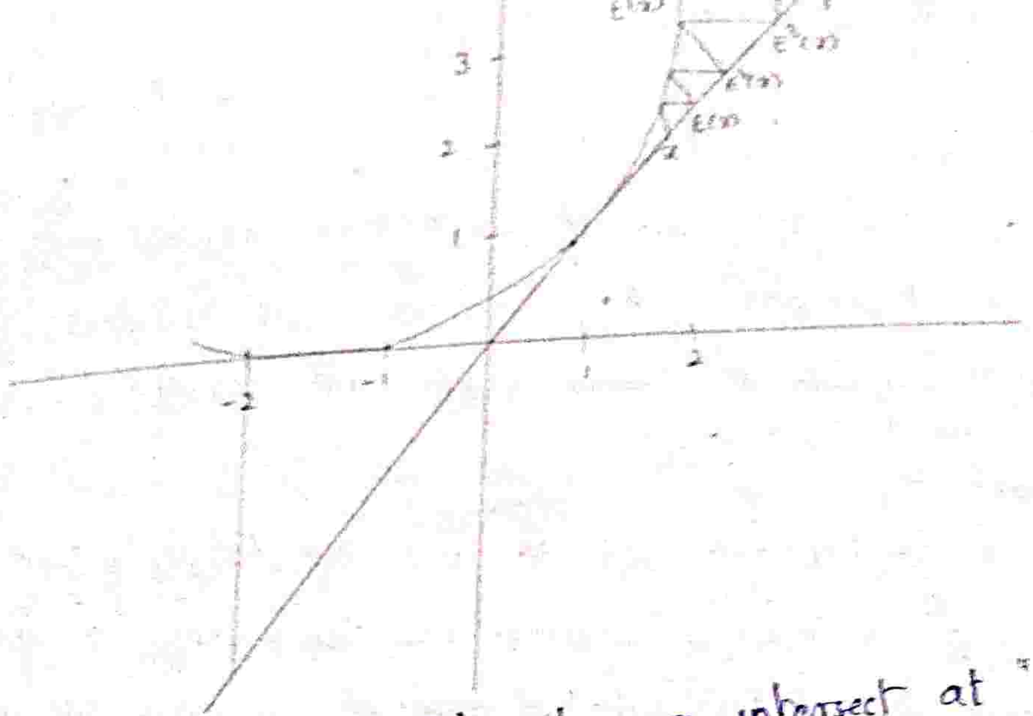
(iii) From the graph it is clear the sequence $y, E(y), E^2(y), \dots \rightarrow a$

Therefore, the stable set of "a" $= W^s(a) = (-\infty, b)$

and $W^s(\infty) = (b, \infty)$

Case 2: If $c = -1$

The graph of this case is as follows



(i) The graph of $E_{-1}(x) = e^{x-1}$ and $y=x$ intersect at "1"
Therefore, "1" is the fixed point. Observe that the fixed points in the previous case coincide at '1'

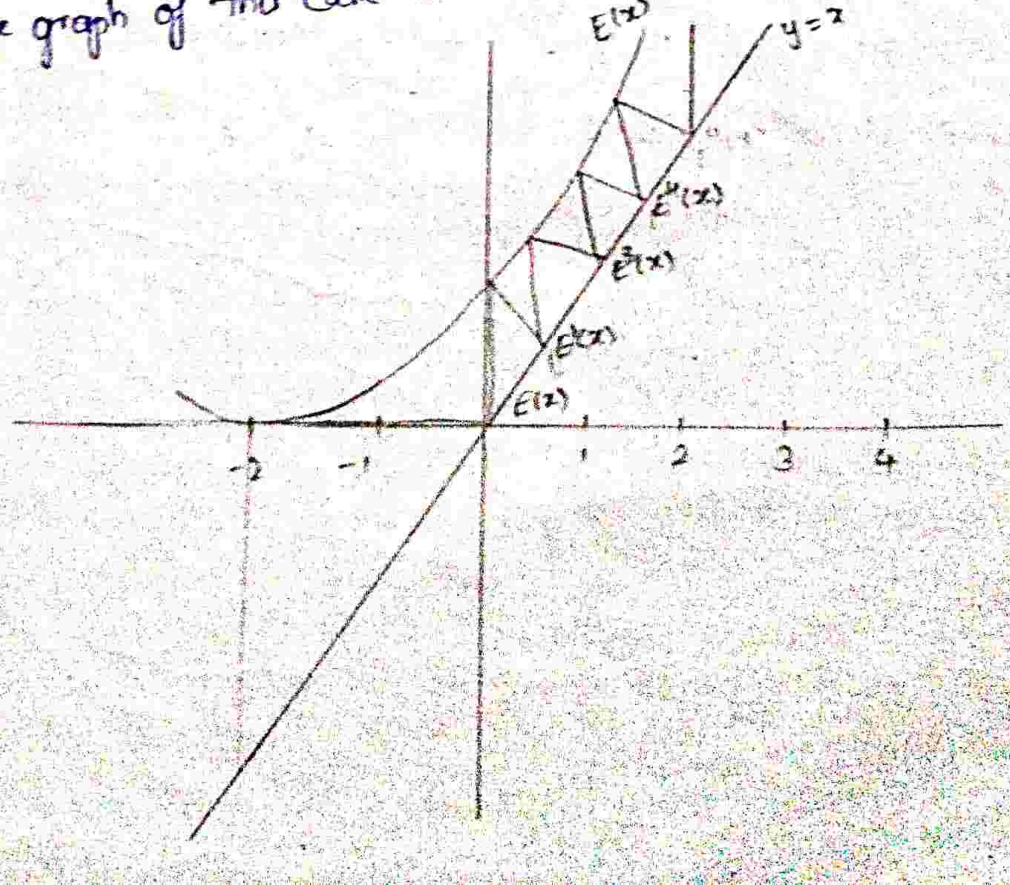
(ii) Since $|E_1'(1)| = 1$, the fixed point is non-hyperbolic

(iii) From the graph it is clear that

$W^s(1) = (-\infty, 1)$ and $W^s(\infty) = (1, \infty)$

Case (iii): If $c > -1$

The graph of this case is as follows



(i) The graph of $E_c(x) = e^{x+c}$ and $y=x$ do not intersect, Therefore where $c>-1$, the function $E_c(x)$ has no fixed points

(ii) Since $E(x)$ is continuous by Sharkovskii's theorem, $E(x)$ has no periodic points

(iii) From the graph it is ~~ealer~~ clear that $W^s(\infty) = \mathbb{R}$

<u>Observation:</u>

From the above example, we have the following observation:-

(i) When $c<-1$, the function $E(x)$ has two fixed points "a" and "b"

(ii) These two fixed points "a" and "b" gradually approach one another until when $c=-1$, they join together and become a single fixed point.

(iii) When $c>-1$ there two fixed points are disappeared.

This type of Bifurcation is called "<u>Saddle-node Bifurcation</u>".

Let us consider the family of functions

$A_k(x) = k \arc\tan(x)$ [$\arc\tan x = \tan^{-1}x$] for parameter values near "1".

Case 1:

If $0<k<1$ then "0" is the attracting fixed point

Because, $A_{\frac{1}{2}}(x) = \frac{1}{2}\tan^{-1}x \Rightarrow A_{\frac{1}{2}}(0) = \frac{1}{2}\tan^{-1}(0) = 0$

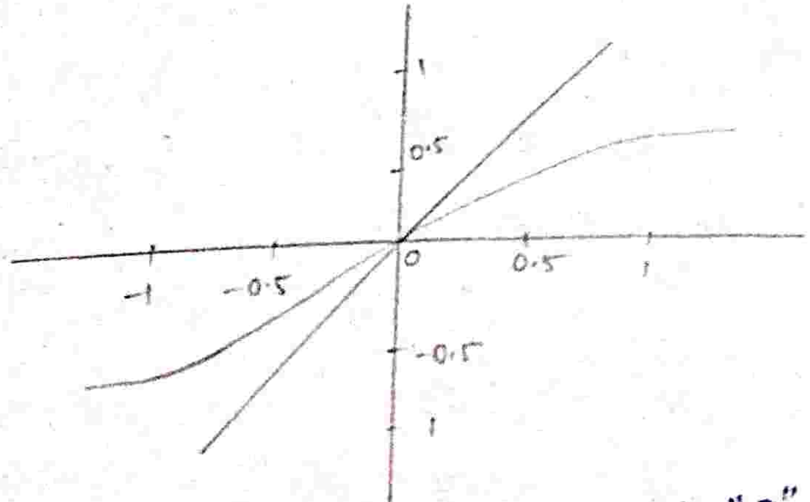
$\therefore$ "0" is the fixed point

Next, $\left|A'_{\frac{1}{2}}(0)\right| = \left|\frac{1}{2}\,\frac{1}{1+0^2}\right|$

$= \frac{1}{2} < 1$

Hence, "0" is the attracting fixed point

The graph of $A_k(x) = k \arc\tan x$ is as follows:



Further, from the graph it is clear that "0" is the attracting fixed point

finally $W^s(0) = \mathbb{R}$

Because $A_{\frac{1}{2}}(x), A^2_{\frac{1}{2}}(x), A^3_{\frac{1}{2}}(x), ----$

Case 2: let $k=1$ then $A_1(x) = \tan^{-1}x$

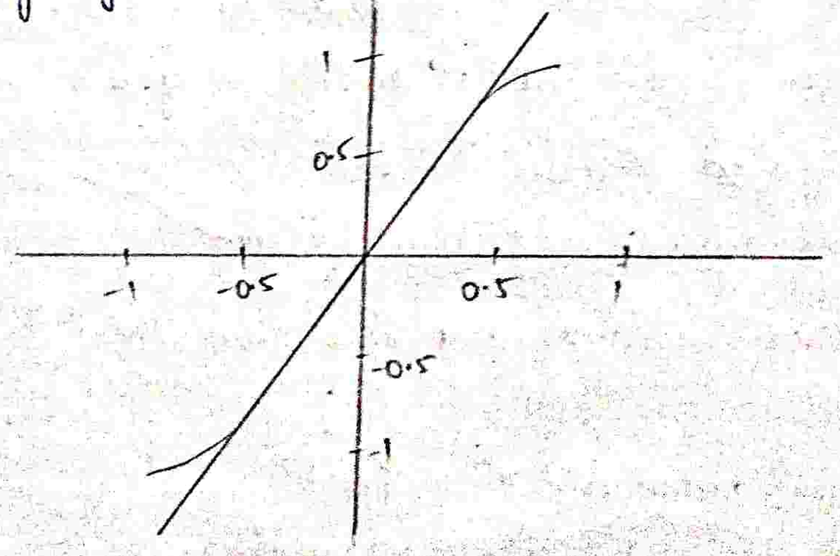
$\Rightarrow A_1(0) = \tan^{-1}0 = 0$ is in A_n

$\therefore$ "0" is the fixed point

Next, $|A_1'(0)| = \left|1 \cdot \frac{1}{1+0^2}\right| = 1$

$\therefore$ "0" is not hyperbolic fixed point

The graph of $A_1(x) = 1.\tan^{-1}x$ is as follows:



Case 3: let $k>1$ then $A_{\frac{3}{2}}(x) = \frac{3}{2}\tan^{-1}x$

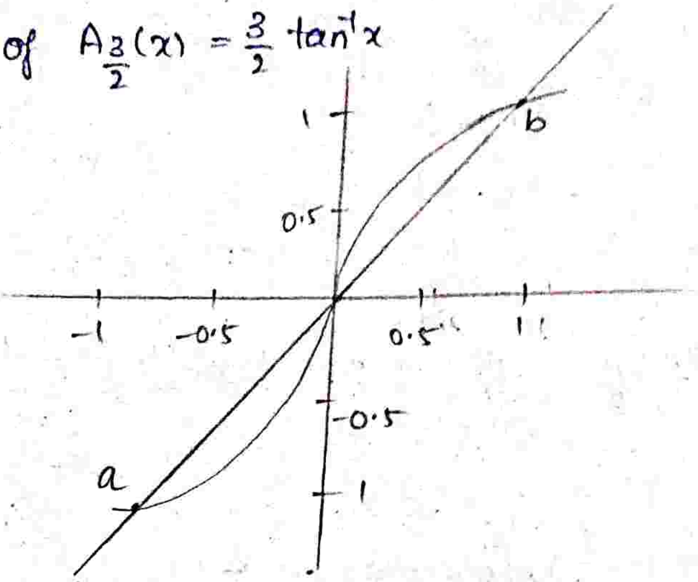
$\Rightarrow A_{\frac{3}{2}}(0) = \frac{3}{2}\tan^{-1}0 = 0$

$\therefore$ "0" is the fixed point

Next, $\left|A'_{\frac{3}{2}}(0)\right| = \left|\frac{3}{2}\frac{1}{1+0^2}\right| = \frac{3}{2} > 1$

$\therefore$ "0" is the repelling fixed point

The graph of $A_{\frac{3}{2}}(x) = \frac{3}{2}\tan^{-1}x$



By graphical analysis, it is clear that the function $A_k(x)$ has two meet fixed points say "a" and "b"

$$W^s(a) = (-\infty, 0)$$
$$W^s(b) = (0, \infty)$$

→ the logistic function (Part-I)

let us consider the family of function

$h_r(x) = rx(1-x)$ where ("r" is the parameter >1) $r>1$

This family of resonable model of population growth

This function is called "logistic function"

we study the dynamics of this function when $r > 4$

Proposition: If $h(x) = rx(1-x)$ and $r>4$ then the following statements are true.

a) The set of real numbers "x" in $[0,1]$ satisfying the condition that $h(x)$ is not in $[0,1]$ is the interval

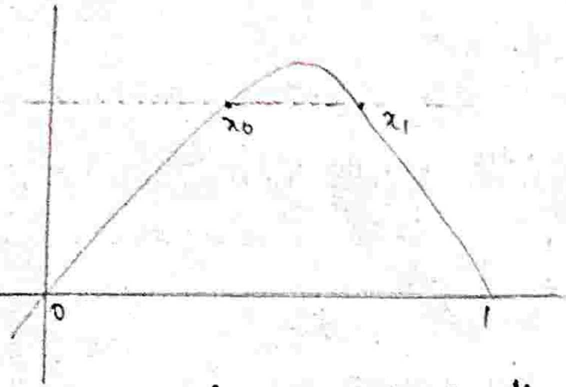
$\left(\frac{1}{2} - \frac{\sqrt{r^2-4r}}{2r}, \frac{1}{2} + \frac{\sqrt{r^2-4r}}{2r}\right)$ and $\Lambda_1 = \left[0, \frac{1}{2} - \frac{\sqrt{r^2-4r}}{2r}\right] \cup \left[\frac{1}{2} + \frac{\sqrt{r^2-4r}}{2r}, 1\right]$ (capital landa)

b) The set "Λ_n" consists of 2^n disjoint closed intervals for all natural numbers "n"

c) If I is one of the closed intervals in Λ_n then $h^n : I \to [0,1]$ is one-to-one and onto

__Proof:__ By data $h(x) = rx(1-x)$ and $r>4$

Let us consider the graph of the function



First observe that if $h(x) \in [0,1]$ then $x \in [0,1]$

Therefore, thus $\Lambda_n \subset [0,1]$

a) From the graph it is clear that, we have to collect the points $x \in [0,1]$ such that $h(x) \notin [0,1]$ i.e., $h(x) > 1$

For that, we solve the equation $h(x) = 1$

That means $rx(1-x) = 1 \Rightarrow -rx^2 + rx - 1 = 0$

$$x = \frac{-r \pm \sqrt{r^2 - 4(r)(-1)}}{2(-r)}$$

$$= \frac{1}{2} \pm \frac{\sqrt{r^2-4r}}{2r}$$

$$\therefore x = \frac{1}{2} - \frac{\sqrt{r^2-4r}}{2r}, \quad x = \frac{1}{2} + \frac{\sqrt{r^2-4r}}{2r}$$

$$\therefore x \in \left[\frac{1}{2} - \frac{\sqrt{r^2-4r}}{2r}, \frac{1}{2} + \frac{\sqrt{r^2-4r}}{2r}\right]$$

and $$\Lambda_1 = \left[0, \frac{1}{2} - \frac{\sqrt{r^2-4r}}{2r}\right] \cup \left[\frac{1}{2} + \frac{\sqrt{r^2-4r}}{2r}, 1\right]$$

Now, we prove (b) and (c) using mathematical induction

If $n=1$, from the above Λ_1 consists of $2'$ disjoint closed subintervals

Next, let "I" be one of the intervals of Λ_1,

then we have to prove that

$h : I \to [0,1]$ is one-one and onto

If $I = \left[0, \frac{1}{2} - \frac{\sqrt{\lambda^2-4\lambda}}{2\lambda}\right]$ then $h(0) = 0 \in [0,1]$,

$$h\left(\frac{1}{2} - \frac{\sqrt{\lambda^2-4\lambda}}{2\lambda}\right) = 1 \in [0,1]$$

Further, "h" is a continuous function

By intermediate value theorem,

immediately we have "h" is onto mapping

analogously, if $I = \left[\frac{1}{2} + \frac{\sqrt{\lambda^2-4\lambda}}{2\lambda}, 1\right]$

we have "h" is an onto mapping

To show that "h" is one-one mapping, we have to verify that "h" is monotone function

Clearly, $h'(x) > 0$ when $x < \frac{1}{2}$

$h'(x) < 0$ when $x > \frac{1}{2}$

Therefore, "h" is one-one function

Assume the statements (b) and (c) for $n = k$

suppose that "Λ_k" consists of "2^k" disjoint closed intervals and if "I" is one of those closed intervals in "Λ_k" then

$h^k : I \to [0,1]$ is one-one and onto and $(h^k)'(x) < 0 \quad \forall x \in I$,

$$(h^k)'(x) > 0 \quad \forall x \in [a,b]$$

we see that "Λ_{k+1}" $\subset \Lambda_k$

Assume $[a,b]$ is one of those intervals of Λ_k, $h^k : [a,b] \to [0,1]$

and $(h^k)'(x) > 0 \quad \forall x \in [a,b]$

Since "h^k" is increasing in $[a,b]$, h^k is continuous and

$h^k[a,b] = [0,1]$

By intermediate value theorem, there exist points x_2 and x_3 such that

1) $a < x_2 < x_3 < b$

2) $h^k([a, x_2]) = \left[0, \frac{1}{2} - \frac{\sqrt{r^2-4r}}{2r}\right]$

3) $h^k([x_2, x_3]) = \left[\frac{1}{2} - \frac{\sqrt{r^2-4r}}{2r}, \frac{1}{2} + \frac{\sqrt{r^2-4r}}{2r}\right]$

4) $h^k([x_3, b]) = \left[\frac{1}{2} + \frac{\sqrt{r^2-4r}}{2r}, 1\right]$

By condition (1) the closed intervals $[a, x_2], [x_2, b]$ are disjoint and by the last three conditions we have

$h^{k+1}([a, x_2]) = [0,1]$, $h^{k+1}(x) > 1 \quad \forall x \in [x_2, x_3)$

and $h^{k+1}([x_3, b]) = [0,1]$

By assumption, $(h^k)'(x) > 0$

$(h^{k+1})'(x) = h'(h^k(x) \cdot (h^k)'(x)) > 0$

On similar lines we can prove that

$(h^{k+1})'(x) < 0 \quad \forall x \in [x_3, b]$

~~Assume~~ that $(h^k)'(x) < 0$, $\forall x \in [a, b]$

with the similar argument "Λ_{k+1}" are contained in two disjoint intervals $[a, x_2], [x_3, b]$

Since $[a,b]$ is arbitrary, it follows that the "Λ_{k+1}" contains "2.2^k" disjoint closed intervals.

Let "J" be one of those "2^{k+1}" disjoint closed intervals in Λ_{k+1}

and $h^{k+1}: J \rightarrow [0,1]$

By conditions (2) & (4), we get

"h^{k+1}" is one-one and onto mapping

By induction process, the statements are true for all values of "n"

Constructing Cantor set:

(5m) To construct cantor middle thirds set we begin with the interval $[0,1]$ and remove the middle open set $(\frac{1}{3}, \frac{2}{3})$

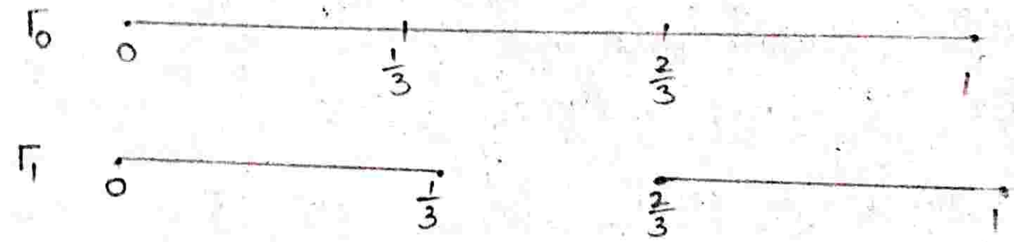
In the 2nd step remove the middle third of each remaining intervals.

That means we remove the intervals $(\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9})$

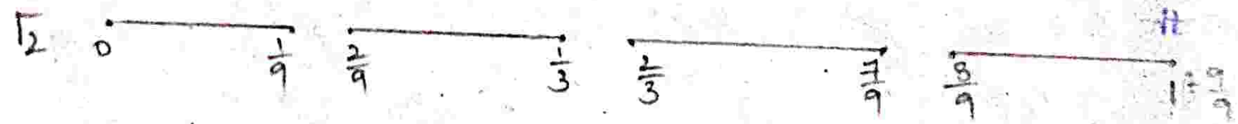
Continuing in this way by removing the middle third of each of the remaining intervals at each step.

The set of points remaining is called "Cantors middle third set"

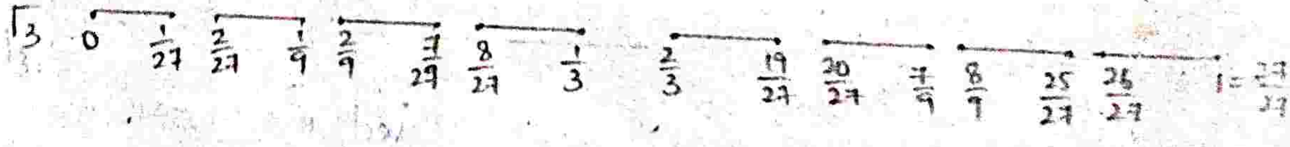
this can be described by the following figure.



Γ_1 contains two intervals and length of each is "$\frac{1}{3}$"



Γ_2 contains 2^2 intervals and length of each is $\frac{1}{3} \times \frac{1}{3}$



Γ_3 contains 2^3 intervals and length of each is $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

Let $0 < \alpha < 1$, just following the above processor.

Let us construct Cantor middle "α" sets

Let $\Gamma_0 = [0,1]$ and Γ_1 be the two closed intervals of equal length left. when the open interval of length "α" is removed from the middle of Γ_0

Define Γ_2 to be the set of "4" closed intervals of equal length obtained by removing an open interval whose length is

$\alpha \times$ (length of the interval of Γ_1)

From the middle of each interval in Γ_1 continuing in this way.

we define Γ_n inductively as the set of 2^n closed intervals of equal length formed by removing an open interval

$\alpha \times$ (length of the interval of Γ_{n-1})

from the centre of the closed intervals in Γ_{n-1}

Therefore, the cantor middle α-set $\Gamma = \bigcap_{n=0}^{\infty} \Gamma_n$

Clearly, the cantor middle α-set $\Gamma \neq \phi$

Lemma: If "Γ_n" is as defined in example 3, then there are "2^n" closed intervals in "Γ_n" and the length of each closed interval is $\left(\frac{1-\alpha}{2}\right)^n$.

Also the combined length of the intervals in 'Γ_n' is $(1-\alpha)^n$ which approaches '0' as 'n' approaches infinitely.

Proof: claim: Γ_n has 2^n closed intervals and the length of each closed interval is $\left(\frac{1-\alpha}{2}\right)^n$.

we prove this assumption by using mathematical induction.

let $n=1$, then Γ_1 has exactly 2^1 closed intervals and the length of each closed interval is $\left(\frac{1-\alpha}{2}\right)$

Therefore, the statement is true for $n=1$

Assume that the statement is true for $n=k$

ie, Γ_k has 2^k closed intervals and the length of each closed interval is $\left(\frac{1-\alpha}{2}\right)^k$

Now, in the closed intervals of Γ_k, each closed interval is made into two closed intervals by deleting the middle open interval.

Therefore, Γ_{k+1} contains 2×2^k closed intervals

ie., 2^{k+1} intervals

Also the length of each closed interval in

$\Gamma_{k+1} = \alpha$ (length of each closed interval Γ_k)

$= \left(\frac{1-\alpha}{2}\right)^{k+1}$

Therefore, by mathematical induction, the statement is true for all "n"

The total length of all closed intervals $= 2^n\left(\frac{1-\alpha}{2}\right)^n$

$= (1-\alpha)^n$

Finally, since $\lim_{n\to\infty} (1-\alpha)^n = 0$,

we have the total length of the intervals approaches to "0" as "n" approaches infinitely.

v.Imp

Proposition: The Cantor middle α-set is a cantor set.

Proof: Let "Γ" be a cantor middle α-set

claim: "Γ" is a cantor set

To show that "Γ" is a cantor set, we need not to show that

(a) "Γ" is closed and bounded

(b) "Γ" contains no intervals

(c) Every point of "Γ" is an accumulation point.

(a) we know that "Γ" $\cong \bigcap_{n=0}^{\infty} \Gamma_n$

Since Γ is the intersection of closed intervals, we get Γ is closed set.

Further, $\Gamma \subset [0,1]$ implies that "Γ" is bounded

Therefore, "Γ" is a closed and bounded set

b) Suppose that Γ contains an open interval (x,y) with the length $|y-x|$

At each step in the construction of $\Gamma_1 (x,y)$ must be contained in one of the remaining closed intervals.

also we have after n-steps the length of one of these intervals is $\left(\frac{1-\alpha}{2}\right)^n$

We can always find an "n_0" such that $\left(\frac{1-\alpha}{2}\right)^{n_0} < |y-x|$ that means the length of each of the closed intervals in Γ_{n_0} is less than the length of (x,y)

Hence, the entire interval (x,y) cannot contain in Γ_{n_0}". That means "Γ" contains "n_0" intervals.

c) Suppose that "x" is a point in "Γ" and

Let $N_\epsilon(x) = (x-\epsilon, x+\epsilon)$ be a neighbourhood of "x"

Now we have to show that $N_\epsilon(x)$ contains a points of "Γ" other than "x"

clearly, "Γ" contains end points of the intervals which were remov

Observe that, at each stage of construction "x" must be in one of the remaining closed intervals.

That is, for each "n" there is an interval in Γ_n which contains "x"

Choose "n" sufficiently large so that $\left(\frac{1-\alpha}{2}\right)^n < \epsilon$

Then there is a closed interval "I" with length $\left(\frac{1-\alpha}{2}\right)^n$ in Γ_n such that "x" $\in I$

Therefore, "x" must be an accumulation point of "Γ"

Hence the Cantor middle α-set is cantor set.

20.21

Theorem: The set $\Lambda = \bigcap_{n=1}^{\infty} \Lambda_n$ is a cantor set

Proof: $\Lambda = \bigcap_{n=1}^{\infty} \Lambda_n$

Claim: Λ is a cantor set

To show that Λ is a cantor set, we need not to show that

a) Λ is closed and bounded

b) Λ contains no intervals

c) every point in Λ is an accumulation point

(a) Since Λ is the intersection of nested closed sets.

By a known theorem, we have this Λ is closed

Since $\Lambda \subset [0,1]$, immediately we have Λ is bounded.

Hence Λ is a closed and bounded set

b) If possible assume that Λ contains an open interval (x,y) with length $|x-y|$ or $|y-x|$, then for each "n", (x,y) is contained in one of the intervals of Λ_n.

By a known lemma, there is an $\epsilon > 0$ such that the length of an interval in Λ_n is less than $\frac{1}{(1+\epsilon)^n}$

we can always an "n_0" such that $|x-y| > \frac{1}{(1+\epsilon)^n}$. Then the interval (x,y) cannot be an interval Λ_{n_0}

Thus is not correct

Hence this Λ contains no intervals.

c) let "x" be any point in Λ

let $\delta > 0$, $N_\delta(x) = (x-\delta, x+\delta)$

we have to show that there is a point in Λ, other than "x" and is contained in $N_\delta(x)$.

Observe, that if "a" is an end point of an interval Λ_n, "a" is a point in Λ.

Since $h^{n+1}(a) = 0$, for each "n", "x" must be contained in one of the intervals of Λ_n.

choose "n" sufficiently large so that $\frac{1}{(1+\epsilon)^n} < \delta$

Then the entire interval of Λ_n must be in $N_\delta(x)$

Therefore, $N_\delta(x)$ contains a point of Λ other than "x"

Hence "x" is an accumulation point

Thus $\Lambda = \bigcap_{n=0}^{\infty} \Lambda_n$ is a cantor set.

Definition: The set "Ω" is a hyperbolic repelling set of the function "f" if Ω is closed and bounded, $f(\Omega) = \Omega$ and there is $N > 0$ such that $|(f^n)'(x)| > 1$ for all $x \in \Omega$ and for all $n \geq N$.

Similarly, the set "Ω" is a hyperbolic attracting set of the function "f" if "Ω" is closed and bounded, $f(\Omega) = \Omega$ and there is $N > 0$ such that $|(f^n)'(x)| < 1$ for all $x \in \Omega$ and forall $n \geq N$.

→ Symbolic dynamics and chaos

In this chapter we discuss the symbolic dynamics

Definition: let Σ_2 be the set of all infinite sequences of 0's and 1's. This set is called the sequence space of "0" and "1" symbol space of "0" and "1". More, precisely $\Sigma_2 = \{(s_0, s_1, s_2, \ldots) / s_i = 0 \text{ or } 1\}$

we will often refer to the elements of Σ_2 as points in Σ_2.

Definition: let "s" $= s_0, s_1, s_2, s_3 \ldots$ and $t = t_0, t_1, t_2, t_3, \ldots$ be the Points in Σ_2 we denote the distance between "s" and "t" as $d[s,t]$ and define it by $d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$, since $|s_i - t_i|$ is either "0" or "1" we have $0 \leq d[s,t] \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 2$

Note:

(i) From the above definition, it is clear that the distance $d[s,t]$ is a number "0" and "2".

(ii) $d[s,t] = 0$ iff $s = t$

(iii) This distance function is an example of a metric

Let us recall the definition of a metric.

let 'X' be a set and let "d" be a function from the set of all Ordered pairs of elements of 'X' into the real numbers.

If the following conditions hold for all x, y, z in X the "d" is a metric on 'X'

a) $d[x,y] \geq 0$ and $d[x,y] = 0$ iff $x = y$

b) $d[x,y] = d[y,x]$

c) $d[x,y] \leq d[x,z] + d[z,y]$

Definition: let 'X' be a set and "d" be a metric on the set

a) A subset "U" of "X" is open if for each "x" in "U" there exist $\epsilon > 0$ such that $d[x,y] < \epsilon \Rightarrow$ "y" is in "U" (Open set)

b) let $\epsilon > 0$ and "x" be in "X" the set $N_\epsilon(x)$.

$N_\epsilon(x) = \{ y \in X / d(x,y) < \epsilon \}$ is called a neighbourhood of "x"

c) let $x_1, x_2, x_3, \ldots$ be a sequence of elements of "X", The sequence converges to "x" if for each $\epsilon > 0$ there exist an integer N such that if $k \geq N$ then $d(x, x_k) < \epsilon$

d) let "S" be a subset of "X", then the point "x" in "X" is an accumulation Point (or limit point) of "S" if every neighbourhood of "x" contains an element of "S" which is distinct from "x".

e) A subset of X is closed if it contains all its accumulation Points.

f) let "A" be a subset of "B" then "A" is dense in "B" if every Point of "B" is an accumulation point of "A", a point of "A" or both. In otherwords, "A" is dense in "B" and "x" is in "B" then every neighbourhood of "x" contains an element of "A".

g) If "Y" is a set and "d_2" is a metric on "Y", then the function $f: X \to Y$ is continuous at the point "x_0" in "X" if for every $\epsilon > 0$ $\exists$ a $\delta > 0 \ni$ if "x" is in "X" and $d[x_0, x] < \delta$ then $d_2[f(x_0), f(x)] < \epsilon$

A function is continuous if it is continuous at each point of its domain.

Proposition: The set "U" is open iff for each $x \in U$ there is a neighbourhood of "x" which is completely contained in "U".

Problem: If "X" is a metric space with metric "d". If "x" is in "X" and "S" is a subset of "X" and $\epsilon > 0$, then neighbourhood $N_\epsilon(x)$ is open.

Proposition: let "X" be a metric with metric "d". If "x" is in "X" and "S" is a subset of "X" then the following statements are equivalent.

a) "x" is an accumulation point of "S"

b) For each $\epsilon > 0$ there exist "y" in "S" such that $0 < |x-y| < \epsilon$

c) If "U" is open set containing x, then $U \cap S$ contains atleast one point other than "x".

d) There is a sequence of points different from "x" and contained in "S" which converges to "x".

Proposition: The complement of an open set is closed. Conversely the complement of a closed set is open.

Proposition: let "X" be a metric space with metric "d" and "A" be a subset of "X". Then the following statements are equivalent.

a) "A" is dense in "X".

b) For all points x in X and all $\epsilon > 0$ there exists "a" in "A" such that $d[x, a] < \epsilon$

c) For each "x" in X there is a sequence of points $a_1, a_2, a_3, \ldots$ contained in "A" that converges to "x".

d) every open subset of "x" contains an element of "A".

Proposition: let "X" and "Y" be a metric spaces and $f: X \to Y$ be a continuous function. If "U" is an open set in "Y" then $f^{-1}(U)$ is an open set in "X".

Lemma: let "s" and "t" be elements of Σ_2. If the first $(n+1)$ digit in "s" and "t" are identical then $d[s,t] \leq \frac{1}{2^n}$. On the otherhand if $d[s,t] \leq \frac{1}{2^n}$ then the first "n" digits in "s" and "t" are identical.

Proof: let $s = s_0, s_1, s_2, \ldots$, $t = t_0, t_1, t_2, \ldots$ be sequences in Σ_2

First assume that the first $(n+1)$ digits in "s" be $s_0, s_1, s_2, \ldots s_n$ and the first $(n+1)$ digits in "t" be $t_0, t_1, t_2, \ldots t_n$

Assume that the first $(n+1)$ digits in "s" and "t" are identical.

claim: $d[s,t] \leq \frac{1}{2^n}$

By assumption, we have $s_i = t_i$, for all $i \leq n$

we know that $d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$

$$= \sum_{i=0}^{n} \frac{0}{2^i} + \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i}$$

$$= \frac{1}{2^{n+1}} \sum_{i=0}^{\infty} \frac{|s_{i+n+1} - t_{i+n+1}|}{2^i}$$

$$\le \frac{1}{2^{n+1}} \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}$$

$\therefore d[s,t] \le \frac{1}{2^n}$

On the other hand, assume that $d[s,t] \le \frac{1}{2^n}$

claim: The first n-digits in "s" and "t" are identical.

If possible suppose that there is $j < n$ such that $s_j \neq t_j$ then

$$d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} \ge \frac{1}{2^j} > \frac{1}{2^n}$$

This is a contradiction because $d[s,t] \le \frac{1}{2^n}$

Hence, the first "n" digits in "s" and "t" are identical.

Problem: Find the distance between the two points "s" and "t",

$s = 0000000 - - -$, $t = 01010101 - - - -$

sol: $d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} = \frac{0}{2^0} + \frac{1}{2^1} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{0}{2^4} + \frac{1}{2^5} + - - -$

$$= \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + - - - -$$

$$= \frac{2}{3}$$

M22 Problem: Let "s" is the set of all elements of Σ_2 which begin with the sequence 011 then show that "s" is closed.

3m

Sol: S = the set of all elements of Σ_2 which begin with the sequence 011

claim: S is closed

For that we need to show that every accumulation point of "s" is in "S"

let "s" be an accumulation points of "S"

we prove $s \in S$, Take $\epsilon = \frac{1}{2^3}$

Since "s" is an accumulation point, we have every neighbourh-
-ood of "s" i.e., $N_{\frac{1}{2^3}}(s)$ contains a point s^* of S such that $s \neq s^*$

By the definition of neighbourhood, we have

$$d[s, s^*] < \frac{1}{2^3}$$

Then by known lemma, the first three digits in "s" and "s^*" are identical.

Therefore, the first three digits in "s" and "s^*" are 011

Therefore, $s \in S$

Hence, "s" is a closed set.

Problem: The set of all sequences in Σ_2 which end with an infinite string of 0's is dense in symbol space. prove it.

Sol: We know that the symbol space is "Σ_2"

let $s = \{ s_0, s_1, s_2, \ldots \mid$ there is a natural number $N \ni s_i = 0 \ \forall i \geq N \}$

claim: "s" is dense in Σ_2

let $t \in \Sigma_2$ to show that "s" is dense in Σ_2

we need to show that every neighbourhood $N_\epsilon(t)$ contains a Point of "s" i.e., $N_\epsilon(t) \cap S = \phi$

let us choose $\epsilon = \frac{1}{2^N}$

let "s" be an element in "S" we need to show that $s \in N_\epsilon(t)$

let "s" be any element in "Σ" such that the first $(n+1)$ digits of "s" are identical with the first $(n+1)$ digits of 't'

then $d[t, s] \leq \frac{1}{2^n}$

$$\Rightarrow s \in N_{\frac{1}{2^n}}(t)$$

i.e., $S \cap N_{\frac{1}{2}}(t) \neq \phi$

Therefore "S" is dense in Σ

Definition: Shift map:

20 M The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is defined by

$\sigma(s_0, s_1, s_2, ---) = s_1 s_2 s_3 ----$

In otherwords, the shift map forgets the first digit of the sequence.

Ex: let $\sigma : \Sigma_2 \to \Sigma_2$ be a map defined by

$\sigma(01110111 0---) = 1110111 0 ----$

mid 20, 21, 22, III M.

Proposition: The shift map is continuous

3m **Proof:** let Σ_2 be the symbol space and "s" be any element of Σ_2

let $\epsilon > 0$, Now we have to show that there is a $\delta > 0$ such that whenever $d[s,t] < \delta$ then $d[\sigma(s), \sigma(t)] < \epsilon$

let us choose "n" such that $\frac{1}{2^n} < \epsilon$,

Take $\delta = \frac{1}{2^{n+2}}$

Now, $d[s,t] < \frac{1}{2^{n+2}} \Rightarrow$ by known lemma, the first "n+2" digits in s and "t" are identical.

Then the first "n+1" digits in $\sigma(s), \sigma(t)$ are identical.

The same lemma, $d[\sigma(s), \sigma(t)] < \frac{1}{2^n} < \epsilon$

Therefore, the shift map "σ" is continuous at the points "s" in Σ and hence the shift map "σ" is continuous on "Σ"

20 mid sm Proposition: The shift function has the property;
The set of all periodic points of the shift map is dense in Σ_2

Proof: "Σ_2" be the symbol space and "σ" be the shift map
let $s = s_0 s_1 s_2 \ldots$ be a periodic point of "σ" with period "k"
Then $\sigma^k(s) = s$ Also $\sigma^n(\sigma^k(s)) = \sigma^n(s)$
Now $\sigma^n(\sigma^k(s)) = s_{n+k}, s_{n+k+1}, s_{n+k+2} \ldots = s_n s_{n+1} s_{n+2} = \sigma^n(s_0 s_1)$
This says that the periodic point of "σ" with period "k",
then "s" is containing $s_0 s_1 s_2 \ldots s_{k-1}$ respectively
that means $s = s_0 s_1 s_2 \ldots s_{k-1}\, s_0 s_1 s_2 \ldots s_{k-1}$
$s_0 s_1 s_2 \ldots s_{k-1} \ldots$

Claim: The set "p" of all periodic points "t" of "σ" in Σ_2 is dense in Σ_2
That is, for $\epsilon > 0$, the neighbourhood $N_\epsilon(t)$ contains a periodic point of "σ"
If $t = t_0 t_1 t_2 \ldots$, we choose "n" such that $\frac{1}{2^n} < \epsilon$
we can find a point say
$s = t_0 t_1 t_2 \ldots t_n t_0 t_1 t_2 \ldots$
Since the first "n+1" digits in "t" and "s" are identical
then $d[s,t] < \frac{1}{2^n} < \epsilon$
$\Rightarrow s \in N_{\frac{1}{2^n}}(t)$

Therefore, the set of all periodic points of "σ" is dense in Σ_2

Definition: The function $f: D \to D$ is topologically transitive if for all open sets "U" and "V" in D
there is "x" in "U" and a natural number "n" such that $f^n(x)$ is in V

V.Imp

Proposition: Let "f" be a function and suppose there is a point whose orbit under iteration of "f" is dense in the domain of "f". Then "f" is topologically transitive.

Proof: Let $f: D \to D$ be a function

and let "U" and "V" be open sets in "D"

Suppose there is a point whose orbit under iteration of "f" is dense in the domain of "f"

Claim: "f" is topologically transitive

For that we must find $x \in U$ and a natural number "n" such that $f^n(x) \in V$

By data there is a point 'D' whose orbit is dense in "D"

let "x_0" be such a point then there exist a "k" such that $f^{(k)}(x_0) \in U$

Now, we show that there is a natural number "n" such that $f^{n+k}(x_0) \in V$

Since $x = f^{(k)}(x_0)$, if $x \in U$, then $f^n(x) = f^n(f^k(x_0))$

$$= f^{n+k}(x_0) \in V$$

Since the orbit of "x_0" is dense in "D",

we know that "V" contains atleast one iterate of "x_0"

Suppose "v" be any point of "V" which is not an iterate of "x_0". Then the number $\epsilon = \min\{|v - f^n(x_0)|\}$.

clearly $\epsilon > 0$, and the neighbourhood $N_{\frac{\epsilon}{2}}(v)$ does not contain any iterate of "x_0".

Therefore, the orbit of "x_0" is not dense in "D" a contradiction.

Therefore, "v" must contain infinitely many iterates of "x_0"

Hence, there exists a natural number "n" such that $f^{n+k}(x_0) \in V$

Thus, "f" is topologically transitive.

Definition. (Sensitive dependence on initial condition).

Let "D" be a metric space with a metric "d". The function $f: D \to D$ exhibits sensitive dependence on initial conditions if there exists a $\delta > 0$ $\ni$ for all $x \in D$ and all $\epsilon > 0$ there is a "y" in "D" and a natural number "n" such that

$d[x, y] < \epsilon$ and $d[f^n(x), f^n(y)] > \delta$

Definition: (Devaney):

Let "D" be a metric space then the function $f: D \to D$ is a ~~chatoic~~ "chaotic" if

a) the periodic points of "f" are dense in "D"

b) "f" is topologically transitive

c) "f" exhibits sensitive dependence on initial condition.

The step function — is called chaotic function

* saddle-node bifurcation
* Pitch-fork bifurcation

If $\lambda > 2+\sqrt{5}$ then show that the set $\Lambda = \bigcap_{n=1}^{\infty} A_n$ is a cantor set

Unit-3

The logistic function part-II, Topological Conjugacy

2M model paper

MP **Definition:** let $f: D \to D$ and $g: E \to E$ be functions.

3M Then "f" is "Topologically Conjugate to g", if there is a homomorphism "τ": $D \to E$ such that $\tau \circ f = g \circ \tau$

In this case τ is called a "topological conjugacy" we represent this relationship by the commutative diagrams.

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "\tau"'] & D \arrow[d, "\tau"] \\ E \arrow[r, "g"'] & E \end{tikzcd} \quad (\text{or}) \quad \begin{tikzcd} x \arrow[r] \arrow[d] & f(x) \arrow[d] \\ \tau(x) \arrow[r] & g(\tau(x)) = \tau(f(x)) \end{tikzcd}$$

Note: Since τ is a homomorphism, we have τ is a one to one and continuous function then τ^{-1} is defined.

Thus, $\tau(y) = g(\tau(x))$ iff $y = \tau^{-1}(g(\tau(x)))$

Put $y = f(x)$ we get $\tau(f(x)) = g(\tau(x))$

$$\Rightarrow \tau^{-1}(g(\tau(x))) = f(x)$$

we represent this by the following diagrams

$$\begin{tikzcd} D \arrow[r, "f"] & D \arrow[d, "\tau"] \\ E \arrow[u, "\tau^{-1}"] \arrow[r, "g"'] & E \end{tikzcd} \quad (\text{or}) \quad \begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d, "\tau"] \\ \tau(x) \arrow[r] & g(\tau(x)) \end{tikzcd}$$

On similar lines if a is an element of E then $\tau(f(\tau^{-1}(a))) = g(a)$

Diagramatically,

$$\begin{tikzcd} D \arrow[r, "f"] & D \arrow[d, "\tau"] \\ E \arrow[u, "\tau^{-1}"] \arrow[r, "g"'] & E \end{tikzcd} \quad (\text{or}) \quad \begin{tikzcd} \tau^{-1}(a) \arrow[r, "f"] & f(\tau^{-1}(a)) \arrow[d, "\tau"] \\ a \arrow[u, "\tau^{-1}"] \arrow[r, "g"'] & \tau(f(\tau^{-1}(a))) = g(a) \end{tikzcd}$$

Note:

1) we know that $\phi: D \to E$ is a homomorphism if and only if ϕ is contains one-to-one an onto and has a continuous inverse.

2) If "D" and "E" are homomorphic then the topologies of "D" and "E" are same.

Imp **Proposition: let "D" and "E" be metric space and $\phi: D \to E$ be a homomorphism then

a) the set "U" in "D" is open iff the set $\phi(D)$ is open in "E"

b) the sequence $x_1, x_2, \ldots$ in D converges to "x" in "D" iff the sequence $\phi(x_1), \phi(x_2), \ldots$ converges to $\phi(x)$ in "E".

c) the set "F" is closed in "D" iff the set $\phi(F)$ is closed in "E"

d) the set "A" is dense in "D" iff the set $\phi(A)$ is dense in "E".

MP Theorem: let "D" and "E" be metric space $f: D \to D$, $g: E \to E$ and 5m $\tau: D \to E$ be a topological conjugacy of "f" and "g" then

MS 20, 21 a) $\tau^{-1}: E \to D$ is a topological conjugacy

MS 21 b) $\tau \circ f^n = g^n \circ \tau$ for all natural numbers

c) p is a periodic point of "f" iff $\tau(p)$ is a periodic point of "g". further the prime periods of "p" and $\tau(p)$ are identical

d) If p is a periodic point of "f" with stable set $W^s(p)$ then stable set of $\tau(p)$ is $\tau(W^s(p))$

20 e) The periodic points of "f" are dense in "D" iff the periodic points of "g" are dense in "E".

MP f) f is topologically transitive on "D" iff "g" is topologically transitive on "E"

20, 21 g) "f" is chaotic on "D" iff "g" is chaotic on "E"

Proof: By the hypothesis, we have "D" and "E" are metric spaces. $f: D \to D$ and $g: E \to E$ are functions. $\tau: D \to E$ is topological conjugacy of "f" and "g". then

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "\tau"'] & D \arrow[d, "\tau"] \\ E \arrow[r, "g"'] & E \end{tikzcd}$$

is commutative ie., $\tau \circ f = g \circ \tau$

(a)

Claim: $\tau^{-1}: E \to D$ is also topological conjugacy for that we prove $\tau^{-1} \circ g = f \circ \tau^{-1}$

Since "τ" is a topological conjugacy, then τ is a homomorphism and τ^{-1} is also continuous

Therefore, $\tau^{-1}: E \to D$ is a homomorphism

Further, the diagram

$$\begin{tikzcd} D \arrow[r, "f"] & D \\ E \arrow[u, "\tau^{-1}"] \arrow[r, "g"'] \arrow[ur] & E \arrow[u, "\tau^{-1}"'] \end{tikzcd}$$

is commutative.

ie., $\tau^{-1} \circ g = f \circ \tau^{-1}$

Hence, τ^{-1} is also topological conjugacy

(b) let "n" be a natural number

then $f^n: D \to D$ and $g^n: E \to E$

Claim: $\tau \circ f^n = g^n \circ \tau$

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "\tau"'] & D \arrow[r, "f"] \arrow[d, "\tau"] & D \arrow[r, dashed] \arrow[d, "\tau"] & D \arrow[r, "f"] \arrow[d, "\tau"'] & D \arrow[d, "\tau"] \\ E \arrow[r, "g"'] & E \arrow[r, "g"'] & E \arrow[r, dashed] & E \arrow[r, "g"'] & E \end{tikzcd}$$

In this diagram each square is commutative also the entire diagram is commutative

ie., $\tau \circ f^n = g^n \circ \tau$

(c) "p" is a periodic point of "f" ~~which~~ with prime period "k"

then $f^k(p) = p$

we prove that $\tau(p)$ is a periodic point of "g"

i.e., $g^k(\tau(p)) = \tau(p)$

Since $f^k(p) = p$, we have the following commutative diagram

$$\begin{tikzcd} p \arrow[r, "f^k"] \arrow[d, "\tau"'] & p \arrow[d, "\tau"] \\ \tau(p) \arrow[r, "g^k"'] & g^k(\tau(p)) \end{tikzcd}$$

$\therefore g^k(\tau(p)) = \tau(p)$

If $0 < n < k$, we say that $g^n(\tau(p)) = \tau(f^n(p)) \neq \tau(p)$

Since τ is one-to-one and "p" has prime period "k" under "f"

Hence, $\tau(p)$ is the periodic point of "g" with prime period "k"

On the other hand τ^{-1} is also topological conjugacy with the same argument we can readily see that if $\tau(p)$ is a period point of "g" with prime period k, then p is a periodic point of "f" with prime period "k".

While proving the above part we have the fact that the Prime period of "p" and "$\tau(p)$" are same ~~(or)~~ identical.

(d) Let "p" be a periodic point of "f" with prime period "k"

then by (c) $\tau(p)$ is the periodic point of "g" with prime period "k"

Claim: $W^s(p) = W^s(\tau(p))$

Let x be an element of $W^s(p)$ then by the definition of the stable set for each $\delta > 0$ there is a natural number "N" such that $d_D(f^{kn}(x), p) < \delta$, $\forall n \geq N$ where "d_D" is a metric on "D".

we have $\tau : D \to E$ is Topological Conjugacy

Then "τ" is a homomorphism from D into E such that topologies are identical.

since τ is topological conjugacy we have $\tau \circ f = g \circ \tau$.

i.e.,
$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "\tau"'] \arrow[dr] & D \arrow[d, "\tau"] \\ E \arrow[r, "g"'] & E \end{tikzcd}$$
is commutative

suppose $\epsilon > 0$, since "τ" is continuous there exists $\delta > 0$ such that $d_D(y, p) < \delta \Rightarrow d_E(\tau(y), \tau(p)) < \epsilon$.

Choose, the positive integer M such that if $n \geq M$ $d_D(f^{nk}(x), p) < \delta$

By continuity,
$$d_E\left(\tau(f^{nk}(x)), \tau(p)\right) = d_E\left(g^{kn}(\tau(x)), \tau(p)\right) < \epsilon \quad \forall n \geq M$$
$$(\tau \circ f = g \circ \tau)$$

Therefore, for each $\epsilon > 0$, there exists a positive integer M such that $d_E(g^{kn}(\tau(x)), \tau(p)) < \epsilon$

Therefore, $W^s(\tau(p))$ is the stable set of "g"

(e) By (c), we have if "p" is a periodic point of "f" with Prime period "k" then $\tau(p)$ is a periodic point of "g" with prime period "k".

Let "A" be the set of all periodic points of "f" then $\tau(A)$ is the set of all periodic points of g.

By a known theorem, from topology, "A" is dense in D iff $\tau(A)$ is dense in E. This Completes the proof of (e)

(f) Suppose "f" is Topologically transitive on D

Our claim is that g is topologically transitive on E

i.e., given an open set $U \subset E$, we have to find an element $x \in U$ such that $g^n(x) \in V \subset E$ where V is an open set and 'n' is some natural numbers.

$$\begin{tikzcd} \tau^{-1}(U) \subset D \arrow[r, "f^n"] \arrow[d, "\tau"'] & \tau^{-1}(V) \subset D \arrow[d, "\tau"] \\ U \subset E \arrow[r, "g^n"'] & V \subset E \end{tikzcd}$$

Since "f" is Topologically Transitive on 'D' we have $y \in \tau^{-1}(U) \subset D$, we get $f^n(y) \in \tau^{-1}(V) \subset D$

Take $x = \tau(y)$,

Now, $g^n(x) = g^n(\tau(y)) = \tau(f^n(y))$

$\therefore g^n(x) = \tau(f^n(y)) \Rightarrow g^n(x) \in V$.

(g) claim: f is chaotic on D iff g is chaotic on E

f is chaotic on $D \Rightarrow$ (i) the periodic points of "f" are dense in "D"

(ii) f is topologically transitive

(iii) "f" exhibits sensitive dependence on initial condition

Immediately by (e)(f) and by known theorem already Proved, we have

$\Longleftrightarrow$ (i) the periodic points of "g" are dense in "E"

(ii) "g" is topologically transitive on E

(iii) g exhibits sensitive dependence on initial conditions

$\therefore$ "g" is chaotic on E.

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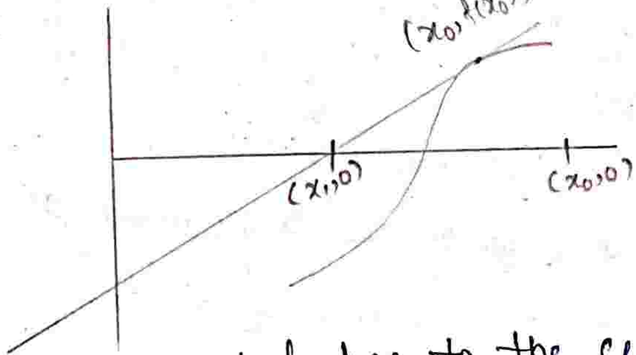
Newton's Method:

we have several methods to solve a quadratic equation, cubic equation and 4^{th} degree equations. But now we are going to study general methods to find a solution to Polynomials of degree "5" (or) higher.

Newton developed a method to solve polynomials of "5" or higher. Later Joseph-Raphson refined that method and hence it is known as "Newton-Raphson" method.

Suppose $f(x)$ is a differentiable function and "x_0" is a reasonable approximation of a solution to the equation $f(x)=0$

Now, we can find a better approximation



Draw a tangental line to the curve of $f(x)$ at the Point $(x_0, f(x_0))$. The, tangent line cut the x-axis at $(x_1, 0)$

Therefore, "x_1" is the next better approximation

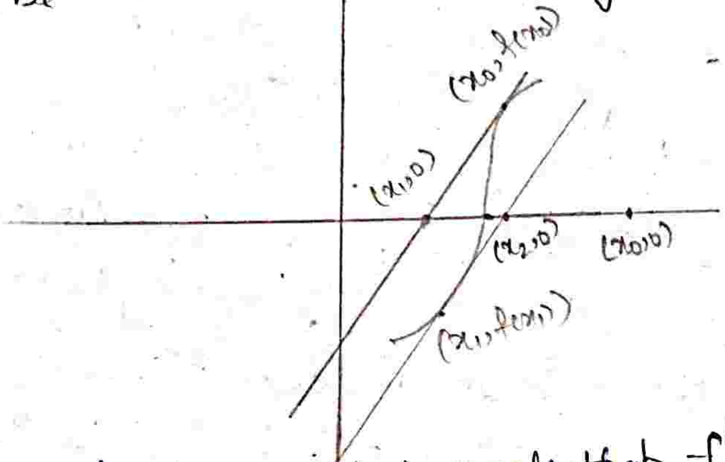
Since f is differentiable the slope of the line is $f'(x_0)$

The graph of the function $f(x)$ is straight on small intervals and if x_0 is close to the root then "x_1" is a better approximation of the polynomial $f(x)$ than "x_0". The method is called "Newtons method"

Next, Draw the tangent line to the curve at the Point $(x_1, f(x_1))$ that tangent line intersects the x-axis. at the point $(x_2, 0)$.

The Next approximation is x_2. As long as the function doesn't come to much between the 1^{st} estimate and the solution.

This can be shown in the following diagram



Suppose the function is $f(x)$ such that $f'(x)$ exists.

Let "x_0" be the initial estimate of the root

To find the next approximation "x_1" in terms of "x_0". we have to solve the equations of the tangent line at $(x_0, f(x_0))$ and the equation of the x-axis.

we know that the slope of the tangent line at $(\overset{x_1}{x_0}, \overset{y_1}{f(x_0)})$ is $f'(x_0)$

Then equation to the tangent line is $y - y_1 = m(x - x_1)$

$$(y - f(x_0)) = f'(x_0)(x - x_0)$$

Put $y=0$, and $x = x_1$

we get $\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$

with the similar argument we can obtain the next estimate x_2 as $\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$

we define $\boxed{N_f(x) = x - \frac{f(x)}{f'(x)}}$

$Nf(x_0) = x_1$, $Nf(x_1) = x_2$

(or) $N_f(N_f(x_0)) = x_2$

(or) $N_f^2(x_0) = x_2$

- - - - - - -

$N_f^n(x_0) = x_n$

Note:- we call $N_f(x)$ is a Newton's function for "f"

Problem: Let $p(x) = x^3 - x$, x^2+x^3 , x^2.

Sol: let $p(x) = x^3 - x$

clearly, the roots are $-1, 0, 1$

we can find them using Newton's method

we know that $N_p(x) = x - \frac{p(x)}{p'(x)}$

$$= x - \frac{x^3 - x}{3x^2 - 1}$$

$$= \frac{3x^3 - x - x^3 + x}{3x^2 - 1} = \frac{2x^3}{3x^2 - 1}$$

If we estimate the root as 0.25

$x_0 = 0.25$

$$x_1 = N_p(x_0) = \frac{2x_0^3}{3x_0^2 - 1} = \frac{2(0.25)^3}{3(0.25)^2 - 1} = -0.0385$$

$$x_2 = N_p(x_1) = \frac{2(-0.0385)^3}{3(-0.0385)^2 - 1} = 0.000114$$

$$x_3 = N_p(x_2) = \frac{2(0.000114)^3}{3(0.000114)^2 - 1} = -0.00000000000296$$

clearly the sequence $x_0, x_1, x_2, x_3, \ldots$ is converging to the root "0"

Let us take the initial estimate be 0.75

$x_0 = 0.75$

$$x_1 = N_p(x_0) = \frac{2(0.75)^3}{3(0.75)^2 - 1} = 1.227$$

$$x_2 = N_p(x_1) = \frac{2(1.227)^3}{3(1.227)^2 - 1} = 1.051$$

$$x_3 = N_p(x_2) = \frac{2(1.051)^3}{3(1.051)^2 - 1} = 1.003$$

Clearly, the sequence $x_0, x_1, x_2, x_3, \ldots$ is converging to the root "1"

Let us take the initial estimate be 0.45

$x_0 = 0.45$

$$x_1 = N_p(x_0) = \frac{2(0.45)^3}{3(0.45)^2-1} = -0.454$$

$$x_2 = N_p(x_1) = \frac{2(-0.454)^3}{3(-0.454)^2-1} = 0.534$$

$$x_3 = N_p(x_2) = \frac{2(0.534)^3}{3(0.534)^2-1} = -7.349$$

$$x_4 = N_p(x_3) = \frac{2(-7.349)^3}{3(-7.349)^2-1} = -5.261$$

$x_5 = -3.550$ $\qquad x_8 = -1.291$

$x_6 = -2.431$ $\qquad x_9 = -1.075$

$x_7 = -1.718$ $\qquad x_{10} = -1.007$

Sit converges to the root -1 but not as quick as the above case. The observation is that this is one disadvantage in Newton's method.

Theorem: Suppose $p(x)$ is a polynomial. If we allow cancellation of common factors in the expression of $N_p(x)$, then $N_p(x)$ is always defined at roots of $p(x)$, a number is a fixed point of $N_p(x)$ iff it is a root of the polynomial and all fixed points of $N_p(x)$ are attracting (i.e., $|f'(p)|<1$)

Proof: Sufficient part:

Let $p(x)$ be a polynomial and suppose "r" is a root of "p"

that is $p(r) = 0$

By a known theorem from algebra, we have

$p(x) = (x-r)^n q(x)$, $q(r) \neq 0$

where "n" is a natural number and "q" is a polynomial

$$\text{Hence } N_p(x) = x - \frac{p(x)}{p'(x)} = x - \frac{(x-r)^n q(x)}{n(x-r)^{n-1} q(x) + (x-r)^n q'(x)}$$

$$= x - \frac{(x-r)q(x)}{(x-r)q'(x) + nq(x)}$$

Therefore, $N_p(x)$ is defined at roots of $p(x)$ because,

$N_p(r)$ is defined $\because q(r) \neq 0$

Now, $N_p(r) = r - \dfrac{0}{0 + nq(r)} = r$, since $q(r) \neq 0$

Therefore, the root r of p is a fixed point of $N_p(x)$

Hence if r is a root then r is a fixed point of $N_p(x)$

Necessary part:

On the otherhand let r be a fixed point of $N_p(x)$

ie, $N_p(r) = r$

claim is that r is a root of p

$$N_p(r) = r \Rightarrow r - \frac{p(r)}{p'(r)} = r$$

$$\Rightarrow \frac{p(r)}{p'(r)} = 0 \text{ (or) } p(r) = 0$$

Therefore, r is the root of the polynomial "p"

Hence a number is a fixed point of $N_p(x)$

$\Longleftrightarrow$ it is a root of "p"

Finally, $$N_p'(x) = 1 - \frac{(p'(x))^2 - p(x)p''(x)}{(p'(x))^2}$$

$$= \frac{p(x)p''(x)}{(p'(x))^2}$$

when r is a root of p & $p'(r) \neq 0$

$N_p(r) = r$ and $|N_p'(r)| = \left|\dfrac{p(r)p''(r)}{(p'(r))^2}\right| = 0 < 1$ $[\because p(r) = 0]$

Therefore, r is attracting fixed point of p

On similar lines, we can show that "r" is an attracting fixed point of p when r is a root of p and $p'(r) = 0$

Proposition:

Let $f(x) = ax^2+bx+c$ and $q(x) = x^2-A$ where $A = (b^2-4ac)$.

then $\tau(x) = 2ax+b$ is a topological conjugacy from $N_f(x)$ to $N_q(x)$

Proof: By the hypothesis, $f(x) = ax^2+bx+c$ and $q(x) = x^2-A$

where $A = b^2-4ac$

$\tau(x) = 2ax+b$

claim: $\tau(x)$ is a topological conjugacy from $N_f(x)$ to $N_q(x)$

It is obvious that a non-constant linear functions are always homomorphism

Consequently $\tau(x)$ is a homomorphism

Finally we show that $\tau \circ N_f = N_q \circ \tau$

we have $N_f(x) = x - \dfrac{ax^2+bx+c}{2ax+b}$ $\left(x - \dfrac{f(x)}{f'(x)} = N_f(x)\right)$

$$N_f(x) = \frac{2ax^2+bx-ax^2-bx-c}{2ax+b}$$

$$\therefore N_f(x) = \frac{ax^2-c}{2ax+b}.$$

$N_q(x) = x - \dfrac{x^2-A}{2x} = x - \dfrac{x^2-b^2+4ac}{2x}$ $\left(x - \dfrac{q(x)}{q'(x)} = N_q(x)\right)$

$$N_q(x) = \frac{2x^2-x^2+b^2-4ac}{2x}$$

$$\therefore N_q(x) = \frac{x^2+b^2-4ac}{2x} = \frac{x^2+A}{2x}$$

Now $\tau \circ N_f(x) = \tau(N_f(x)) = \tau\left(\dfrac{ax^2-c}{2ax+b}\right)$

$$= 2a\left(\frac{ax^2-c}{2ax+b}\right)+b$$

$$= \frac{2a^2x^2-2ca+2abx+b^2}{2ax+b}$$

$$\therefore \tau \circ N_f(x) = \frac{2a^2x^2+2abx+b^2-2ca}{2ax+b} \quad \text{--- (1)}$$

Next, $N_g \circ \tau = N_g(\tau(x)) = N_g(2ax+b)$

$N_g \circ \tau(x)$

$$N_g \circ \tau(x) = \frac{(2ax+b)^2 + A}{2(2ax+b)}$$

$$= \frac{(2ax+b)^2 + b^2 - 4ac}{2(2ax+b)}$$

$$= \frac{4a^2x^2 + b^2 + 4axb + b^2 - 4ac}{4ax+2b}$$

$$= \frac{2(2a^2x^2 + b^2 + 2axb - 2ac)}{2(2ax+b)}$$

$$N_g \circ \tau(x) = \frac{2a^2x^2 + 2axb + b^2 - 2ac}{2ax+b} \quad \text{——} \quad (2)$$

from (1) & (2), we get $\tau \circ N_f = N_g \circ \tau$

Hence $\tau(x)$ is topological conjugacy from $N_f(x)$ to $N_g(x)$

Newton's method for Cubic Equations:

Proposition:

If $f(x) = ax^3 + bx^2 + cx + d$ and $g(x) = x^3 + Ax + B$ where $A = 9ac - 3b^2$ and $B = 27a^2d + 2b^3 - 9abc$ then the function $\tau(x) = 3ax + b$. is a topological conjugacy from $N_f(x)$ to $N_g(x)$

that is $\tau \circ N_f = N_g \circ \tau$

Proof: By the hypothesis, $f(x) = ax^3 + bx^2 + cx + d$, $g(x) = x^3 + Ax + B$

$A = 9ac - 3b^2$, $B = 27a^2d + 2b^3 - 9abc$, $\tau(x) = 3ax + b$

Claim: $\tau(x)$ is a topological Conjugacy from $N_f(x)$ to $N_g(x)$

It is obvious that a non constant linear functions are always homomorphism

Consequently $\tau(x)$ is homomorphism

Finally, we show that $\tau \circ N_f = N_g \circ \tau$

we have, $$N_f(x) = x - \frac{ax^3 + bx^2 + cx + d}{3ax^2 + 2xb + c}$$

$$N_f(x) = \frac{x(3ax^2+2bx+c) - (ax^3+bx^2+cx+d)}{3ax^2+2bx+c}$$

$$N_f(x) = \frac{3ax^3+2bx^2+cx-ax^3-bx^2-cx-d}{3ax^2+2bx+c}$$

$$N_f(x) = \frac{2ax^3+bx^2-d}{3ax^2+2bx+c}$$

$$N_g(x) = x - \frac{x^3+Ax+B}{3x^2+A}$$

$$N_g(x) = x - \frac{x^3+(9ac-3b^2)x+(27a^2d+2b^3-9abc)}{3x^2+9ac-3b^2}$$

$$= x - \frac{x^3+9acx-3b^2x+27a^2d+2b^3-9abc}{3x^2+9ac-3b^2}$$

$$= \frac{3x^3+\cancel{9acx}-\cancel{3b^2x}-x^3-\cancel{9acx}+\cancel{3b^2x}-27a^2d-2b^3+9abc}{3x^2+9ac-3b^2}$$

$$= \frac{2x^3-2b^3-27a^2d+9abc}{3x^2+9ac-3b^2}$$

$$T \circ N_f(x) = T(N_f(x))$$

$$= T\left(\frac{2ax^3+bx^2-d}{3ax^2+2bx+c}\right)$$

$$= 3a\left(\frac{2ax^3+bx^2-d}{3ax^2+2bx+c}\right)+b$$

$$= \frac{6a^2x^3+3abx^2-3ad+3abx^2+2b^2x+bc}{3ax^2+2bx+c}$$

$$\therefore T \circ N_f(x) = \frac{6a^2x^3+6abx^2+2b^2x-3ad+bc}{3ax^2+2bx+c} \quad \text{——} \ (1)$$

$$N_g \circ \Upsilon(x) = N_g(\Upsilon(x))$$

$$= N_g(3ax+b)$$

$$= \frac{2(3ax+b)^3 - 2b^3 - 27a^2d + 9abc}{3(3ax+b)^2 + 9ac - 3b^2}$$

$$= \frac{2(27a^3x^3 + b^3 + 9axb^2 + 27a^2x^2b) - 2b^3 - 27a^2d + 9abc}{3(9a^2x^2 + b^2 + 6abx) + 9ac - 3b^2}$$

$$= \frac{54a^3x^3 + 2b^3 + 18axb^2 + 54a^2x^2b - 2b^3 - 27a^2d + 9abc}{27a^2x^2 + 3b^2 + 18abx + 9ac - 3b^2}$$

$$= \frac{9a(6a^2x^3 + 6abx^2 + 2ab^2x - 3ad + bc)}{9a(3ax^2 + 2bx + c)}$$

$$= \frac{6a^2x^3 + 6abx^2 + 2ab^2x - 3ad + bc}{3ax^2 + 2bx + c} \quad \text{——— (2)}$$

from (1) & (2)

$$\therefore \Upsilon \circ N_f(x) = N_g \circ \Upsilon(x)$$

Unit-4

We know very well that the differential equations can be solved using different techniques in calculus. Most of the differential equation can be solved by integration allow. But here we find a numerical solution to the given differential equation without solving the differential equation.

Problem: let $p' = .5p$ and suppose $p(0) = 1$ without solving the diff. equation. estimate the value of $p(5)$.

3m mid

Sol: The given diff. equation is $p' = .5p$, $p(0) = 1$

suppose that this diff eqn has a solution

Then we know that the slope of the tangent line at the Point $(0,1)$

The slope $= .5(p(0)) = .5(1) = .5$

Next, since the function is differentiable its graph is closed to the tangent line for small intervals around zero.

Now we find the slope of the tangent line at $(1, p(1))$

Next, let us find a point $(1, p(1))$ on the tangent line, that means if the x-coordinate is '1'. what is the "y" coordinate.

y-coordinate = product of the slope and the change in x + current value of y

$= .5(1) + 1$

$p(1) = 1.5$

We find $(2, p(2))$

first we find the slope $p' = .5(p(1))$

$= .5(1.5)$

$= .75$

Now, $p(2) = (.75)(1) + 1.5$

$p(2) = 2.25$

next find $(3, p(3))$

first we find the slope $p' = .5(p(2))$

$= .5(2.25)$

$= 1.125$

Now, $p(3) = (1.125)(1) + (2.25)$

$= 1.125 + 2.25$

$= 3.375$

next find $(4, p(4))$

first we find the slope $p' = .5(p(3))$

$= .5(3.375)$

$= 1.6875$

Now $p(4) = (1.6875)(1) + (3.375)$

$= 5.0625$

next find $(5, p(5))$

first we find the slope $p' = .5(p(4))$

$= .5(5.0625)$

$= 2.53125$

$p(5) = (2.53125)(1) + 5.0625$

$\therefore p(5) = 7.59375$

The next example we generalize this problem :

Suppose that $p' = .5p$, $p(0) = P_0$;

approximate $p(x) = P_n$ by using 'n' steps of the size $h = \frac{1}{n}(x)$

then $P_n = (1 + .5h)^n P_0$ (or) $P_n = g^n(P_0)$ where $g(p) = (1 + .5h)p$

Proof: we know that $P_k = f(x_{k-1}, P_{k-1}) \cdot h + P_{k-1}$

$\Rightarrow P_k = .5P_{k-1}h + P_{k-1}$

Put $g(P) = (1 + .5h)^P$ then

$P_n = (1 + .5h)P_{n-1} = g(P_{n-1})$

$$= g((1+.5h)P_{n-2}) = g^2(P_{n-2})$$

$$\vdots$$

$$= (1+.5h)^n P_0 = g^n(P_0)$$

we know that $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$, since $h = \frac{\dot{x}}{n}$,

$$\lim_{h\to 0} g^n(P_0) = \lim_{h\to 0}(1+.5h)^n P_0 = \lim_{n\to\infty}\left(1+.5\frac{\dot{x}}{n}\right)^n P_0$$

$$= e^{.5\dot{x}} P_0$$

Therefore, by euler method the estimate converges to the solution as $h\to 0$

Dynamics of complex functions:

Problem: let $f(z) = az$, where "a" is a complex number whose modulus is not '1'. Discuss the dynamics of the complex function.

Proof: The given complex function is $f(z) = az$, where 'a' is a complex number is not 1

clearly, 0 is the fixed point of 'f'

Because $f(0) = a.0 = 0$

let $z_0 \neq 0$ be a complex number.

Now we find the orbit of z_0

$$f(z_0) = a z_0$$

$$f^2(z_0) = f(f(z_0)) = f(az_0) = a(az_0) = a^2 z_0$$

— — — — — —

— — — — — —

$$f^n(z_0) = a^n z_0$$

let us take the complex number $a = |a| e^{\theta i}$

then $f^n(z_0) = |a|^n e^{n\theta i} \cdot z_0$ cos) $f^n(z_0) = |a|^n |z_0| e^{(n\theta + \arg(z_0))i}$

Therefore, $|f^n(z_0)| = |a|^n |z_0|$ and $\arg(f^n(z_0)) = \arg z_0 + n\theta$

Next, we find the stable set $W^s(0)$

case(i): Suppose that, $|a| < 1$

Then $|f^n(z_0)| = |a|^n |z_0|$,

since $|a|<1$, as $n \to \infty$ $|a|^n |z_0| \to 0$

Thus $W^s(0) = \mathbb{C}$, where $|a|<1$

case(ii): Suppose that $|a| > 1$

In this case $|f^n(z_0)| = |a|^n |z_0|$ converges to ∞ except '0'

Because $z_0 \neq 0$

Therefore $W^s(0) = \{0\}$

finally, $W^s(\infty) = \mathbb{C} \setminus \{0\}$

The argument of 'a' does not effect the convergence or non-convergence of $f^n(z_0)$ But it effects the orbit.

Therefore, when the argument of 'a' is not zero the orbit of any nonzero point winds around the fixed point.

Theorem: Let "f" be a differentiable complex function and 'p' be a fixed point of 'f' if $|f'(p)| < 1$ then the stable set of p contains a neighbourhood of p. If $|f'(p)| > 1$ then there is a neighbourhood of 'p' all of whose points must leave the neighbourhood under iteration of 'f'.

Proof: By the hypothesis, "f" is a differentiable complex function and 'p' be a fixed point of 'f', $f(p) = p$

case(i): Suppose that $|f'(p)| < 1$

we have $f(p) = p$, $|f'(p)| < 1$

Since $\frac{1}{2}(1 - |f'(p)|) > 0$ then there exist a $\delta > 0$ such that

$0<|z-p|<\delta$ then $\left|\frac{f(z)-f(p)}{z-p}-f'(p)\right|<\frac{1}{2}(1-|f'(p)|)$

Now $\left|\frac{f(z)-f(p)}{z-p}\right| \le \left|\frac{f(z)-f(p)}{z-p}-f'(p)\right|+|f'(p)|$

if $0<|z-p|<\delta$, then $\left|\frac{f(z)-f(p)}{z-p}\right| \le \frac{1}{2}(1-|f'(p)|)+|f'(p)|$ — ①

$\le \frac{1}{2}(1+|f'(p)|)$ — ①

we have $\frac{1}{2}(1+|f'(p)|)<1$, since $|f'(p)|<1$

Put $\lambda=\frac{1}{2}(1+|f'(p)|)$

multiplying ① on both sides by $|z-p|$ and

Substituting 'λ' for $\frac{1}{2}(1+|f'(p)|)$

we get $|f(z)-p|=|f(z)-f(p)| \le \lambda|z-p|$

by mathematical induction,

if $|z-p|<\delta$ and 'n' be a natural number

$|f^n(z)-p| \le \lambda^n|z-p|$

Since $\lambda<1$, we have $\lambda^n\to 0$

i.e., $f^n(z)\longrightarrow p$ as (n increases) $n\to\infty$

Thus, if $|z-p|<\delta$ the neighbourhood of p with radius 'δ'

is in the stable set of p.

On similar lines, we can prove the 2^{nd} part of the theorem.

i.e., when $|f'(p)|>1$

Model paper

MQP, 2022

7(b). Prove that the Orbit of a complex number under iteration of a complex quadratic polynomial is either bounded or the number is in the stable set of infinitely.

24M

Proof: It is suffices to prove that "w" is a complex number it satisfies $|w| > |c|+1$, then w is in the stable set of ∞

Since $q(w) = w^2 + c$

ω - omega

$$|q(w)| = |w^2+c| \geq |w^2| - |c|$$

$$\geq (|c|+1)^2 - |c|$$

$$\therefore |q(w)| \geq |c|^2 + 2|c| + 1 - |c|$$

$$|q(w)| \geq |c|^2 + |c| + 1 \quad \text{--- (1)}$$

Then $|q^2(w)| = |q(q(w))|$

$$= |(q(w))^2 + c|$$

$$\geq |(q(w))|^2 - |c|$$

$$\geq (|c|^2 + |c| + 1)^2 - |c|$$

$$\geq (|c|^2)^2 + |c|^2 + 1^2 + 2(|c|^2 \cdot |c| + |c| \cdot 1 + |c|^2) - |c|$$

$$\geq |c|^4 + |c|^2 + 1 + 2|c|^2|c| + 2|c| + 2|c|^2 - |c|$$

$$\geq |c|^4 + |c|^2 + 1 + 2|c|^3 + 2|c| + 2|c|^2 - |c|$$

$$\geq |c|^4 + 2|c|^3 + 3|c|^2 + |c| + 1$$

$$\geq 3|c|^2 + |c| + 1.$$

By induction, we get

$$\Rightarrow |q^n(w)| \geq (2^n - 1)|c|^2 + |c| + 1$$

If $n \to \infty$ then the right hand side of the equation is tends to ∞

For this, we observe that

"w" is in the stable set of ∞

It follows that the orbit of a complex number under iteration of $q(z) = z^2 + c$ is bounded by $|c|+1$ or the number is in the stable set of ∞.

mid, 22 * Define types of Bifurcations and explain each of type of Bifurcation.

Types of Bifurcations: These are four types of Bifurcation

They are: (i) Saddle-node bifurcation

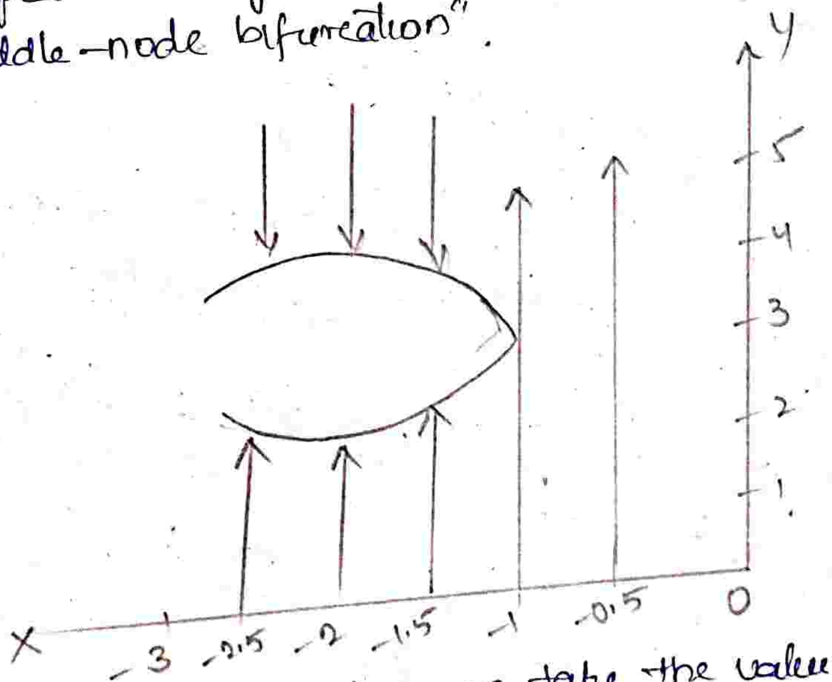
20MP (ii) Pitch fork bifurcation

(iii) Transcritical bifurcation

(iv) Period doubling bifurcation.

MP 20,21 2m (i) Saddle-node bifurcation:

While examining the dynamics of the parameterised family of functions $f_c(x) = e^{x+c}$, we have observe that as the parameter 'c' grows and approaches -1, the two fixed points a and b gradually approach one another until when $c = -1$, they join to become one fixed point. Immediately there after, they disappear all together. The type of bifurcation is called a "saddle-node bifurcation".



In this bifurcation we take the value of the parameter on the x-axis and domain of the function y-axis the solid lines in the diagram are fixed points.

Observe that the bifurcations whole diagrams are similar to this diagram then they are saddle-node bifurcation.

(ii) Pitch-fork bifurcation:

The type of bifurcations seen in $A_k(x) = k$ are various when $k=1$ is called a pitch-fork bifurcation.

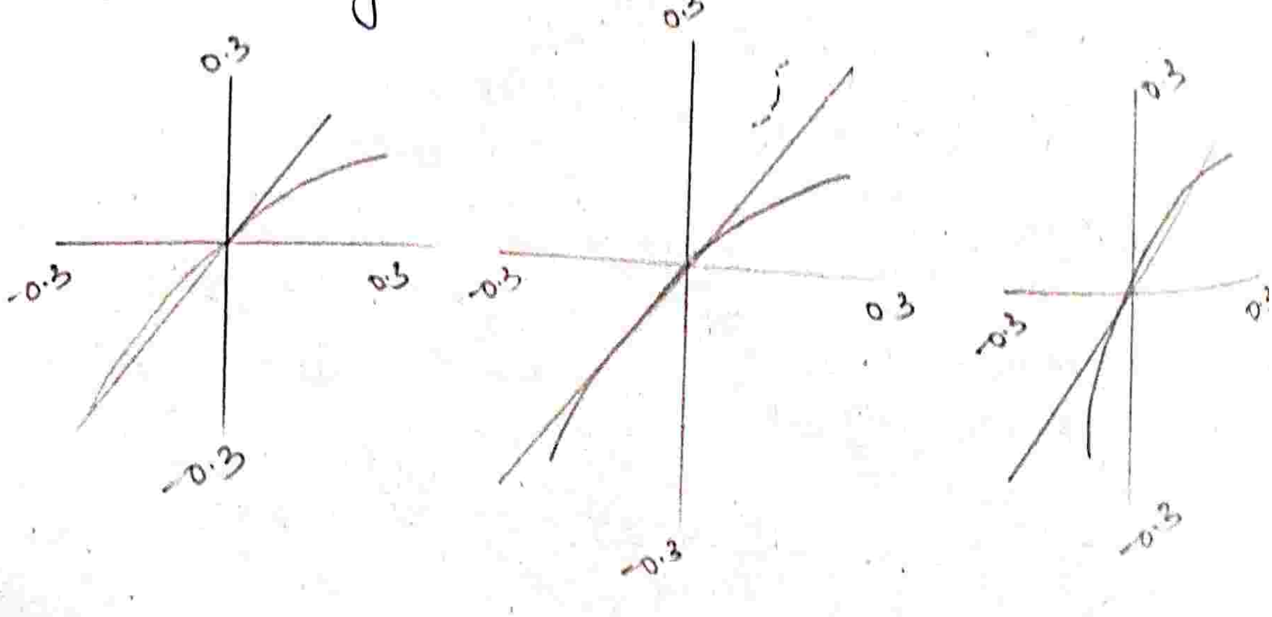
In general, pitch fork bifurcations occurs when either an attracting periodic point splits into a repelling periodic point with an attracting periodic point of the same period as the original point on each side of it.

(iii) Transcritical bifurcations:

The function $h_r(x) = rx(1-x)$ exhibits a third type of bifurcation when $r=1$: a transcritical bifurcation.

* when $0<r<1$, it has two fixed points, one of which is less than one and repelling and another at 0 which is attracting.
* when $r=1$ these two points merge to form one fixed point at '0' which attracts points which are greater than '0' and repels points which are less than '0'.
* Finally, if $1<r<3$, we see that '0' becomes a repelling fixed point and there is an attracting fixed point which is greater than 0.

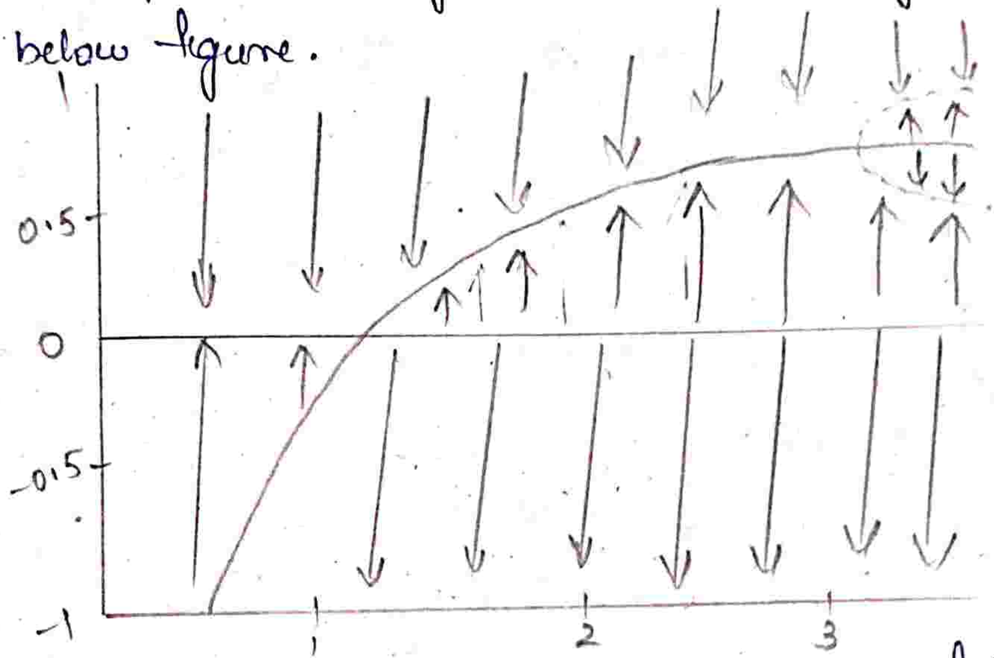
The graphs of these three situations are shown in below figure.



3M **Period Doubling bifurcation:**

A periodic orbit of twice the period of the original periodic point is added then this type of bifurcation is called a period doubling bifurcation.

A bifurcation diagram for the family h_r is shown in below figure.



In closing, we note that there are several ways of drawing bifurcation diagram. The choice of diagram depends on the information we are trying to code.

We also note that of the four types of bifurcations described here, the more important are the saddle-node and the period doubling bifurcations. They arise in many settings and can't be eradicated by small change in the parametrized family.

~~Theorem~~:

Theorem: Let $f: X \to X$ be topologically transitive and suppose the periodic points of f are dense in X. If 'X' contains an infinite number of elements then 'f' exhibits sensitive dependence on initial conditions.

Proof: By data $f: X \to X$ be topologically transitive and suppose that the set of periodic of f is dense in X

Let d be a metric on X

claim:

There is a $\delta_0 > 0 \ni$ for all x in X. There is a periodic point $q \ni d(x, f^n(q)) \geq \delta_0$ $\forall$ natural number n.

choose two periodic points p & q with different orbits.

Take $\delta_0 = \frac{1}{2}\{d[f^n(p), f^m(q)]\}$ m, n are natural number}

Since there are only finitely many points in the orbit of the periodic points P & Q

we have $\delta_0 > 0$

also $\delta_0 \leq d[f^n(p), f^m(q)]$

$\leq d[f^n(p), x] + d[x, f^m(q)]$ $\forall m, n$

Therefore, $d[f^n(p), x] \leq \delta_0$ for some n and

$d[f^m(q), x] \geq \delta_0$ for some m

Also conversely,

claim:

For $\delta > 0$ for all $x \in X$ & $\forall \epsilon > 0$ there is a $y \ni d[x, y] < \epsilon$ and $d[f^n(x), f^m(y)] > \delta$ $\forall m, n$

That means 'f' exhibits sensitive depends on critical conditions.

Take $\delta = \frac{1}{4}\delta_0$ let $x \in X$, & $\epsilon > 0$

without loss of generality we may assume that $\epsilon > \delta$.

since the periodic points of dense in prime period k,

$\ni d[x,p] < \epsilon$,

and also a periodic point $q \ni N_\epsilon(p)$ that every point in its orbit is atleast at a distance $4\delta = \delta_0$

Take $V = \bigcap_{i=0}^{k} f^{-1}(N_\delta(f^i(q)))$ → not opened

$= \{x / d[f^i(x), f^i(q)]\} < \delta$

[since (i) every nbd is an open set
we have $N_\delta(f^i(q))$ is open.

(ii) f is continuous we get $f^{-i}(N_\delta(f^i(p)))$ is also open

(iii) The finite intersection of open set 's' is open
we have V is an open set]

Since V, $N_\epsilon(x)$ are open sets in view of the topologically transitivity of 'f'.

There exist y & $m \ni \frac{y}{U} \in N_\epsilon(x)$ & $f^m(y) \in V$

'J' be an integer satisfying $\frac{m}{k} < j < \frac{m}{k} + r$, (or) $0 < kj - m \leq k$

To prove that assertion we need to prove that

$d[f^{kj}(p), f^{kj}(q)] < \delta$ (or) $d[f^{kj}(x), f^{kj}(y)] > \delta$

Since, $d[x,p] < \epsilon$ or $d[x,y] < \epsilon$

we have 'f' exhibits sensitive dependence on initial conditions.

$f^{kj}(y) = f^{kj-m}(f^m(y))$,

$d[f^{kj}(y), f^{kj-m}(q)] < \delta$ as we choose $y \ni f^m(y) \in V$

Also $d[f^i(q), f^i(q)] < \delta$ whenever $x \in V$ and $i \leq k$ and we choose j so that $k_j - m \leq k$

By triangle inequality, we have

$$d[x, f^{k_j - m}(q)] \leq d(x,p) + d[p, f^{k_j}(y)] + d[f^{k_j}(y), f^{k_j - m}(q)] \quad \text{——} (1)$$

we have $d[x,p] < \epsilon < \delta$ $\quad (\because$ by (2))

$d[f^{k_j}(y), f^{k_j - m}(q)] < \delta$ $\quad (\because$ by (2))

Then (1) becomes

$$d[x, f^{k_j - m}(q)] \leq d[p, f^{k_j}(y)] + 2\delta$$

But from the above consideration

we have $d[x, f^{k_j - m}(q)] > 4\delta = \delta_0$.

we get $4\delta < d[p, f^{k_j}(y)] + 2\delta$ (or)

$$2\delta < d[p, f^{k_j}(y)]$$

Since 'p' is the periodic point with prime period k,

we get $2\delta < d[f^k(p), f^{k_j}(y)]$

By triangle inequality

$$2\delta < d[f^{k_j}(p), f^{k_j}(y)] \leq d[f^{k_j}(p) + f^{k_j}(x)] + d[f^{k_j}(x) + f^{k_j}(y)]$$

Therefore, either $d[f^{k_j}(p), f^{k_j}(x)] > \delta$

(or) $d[f^{k_j}(x), f^{k_j}(y)] > \delta$

② Show that all complex quadratic polynomial and topologically conjugate to a polynomial of the form $q_c(z) = z^2 + c$

MP 21, Nov. III, 3M

Sol We have to already proved the following statement in case of real functions.

There exist a linear function $J(x) = mx + d$ and a parameter $c \ni J$ is a topological conjugacy between

$f(x) = Ax^2 + Bx + c$ & $q_c(x) = x^2 + c$

The same proposition can be extended to the family of complex quadratic polynomials.

Hence all complex quadratic are topologically conjugate to a polynomial of the form.

$\therefore q_c(z) = z^2 + c$

20, 21, 3m

③ If a complex polynomial has an attracting periodic orbit, then there must be a critical point of the polynomial in the stable set of one of the points in the orbit.

Proof: If the derivative of z_0 is 0 then z_0 is called critical point.

then $p'(z_0) = 0$

$z_0 \rightarrow$ critical point

We recall this z_0 as critical point of the polynomial p if $p'(z_0) = 0$

As the only the critical point of $q_c(z) = z^2 + c$ is 0, this implies that q_c can have almost one attracting periodic point and that '0' must be attracted it.

$q_c(z) = z^2 + c$

$q_c'(z) = 2z + 0$

$$q_c'(z) = 2z$$

$$q_c'(0) = 2(0) = 0$$

$$q_c'(0) = 0$$

$\therefore$ 0 is a fixed point

and also 0 is a periodic point with prime period is '1'

$$|q_c'(0)| = 0 < 1$$

'0' is an attracting periodic point.

Examples

MP, 2022 Suppose $f(z) = e^{\theta i} z$ and z_0 is a non zero complex number. Then the orbit of z_0 under iterations of f lies on the circle which is centered at the origin and whose radius is $|z_0|$. If θ is a rational multiple of π, then z_0 is ~~not~~ a periodic point of f

Sol: If the argument of θ is not a rational multiple of π, then z_0 is not a periodic point and its orbit is dense on the circle.

To see this note that $f^n(z_0) = e^{n\theta i} z_0$

Consequently $|f^n(z_0)| = |e^{n\theta i}|\, |z_0| = |z_0|$

for all n and all iterates of z_0 must be on the circle with radius $|z_0|$ and center at 0. Also $\arg(f^n(z_0)) = \arg(z_0) + n\theta$. So z_0 is periodic if and only if there exists a natural number n and an integer k so that

$$\arg(z_0) + n\theta = \arg(z_0) + 2k\pi$$

solving this equation for θ we find $\theta = \frac{2k\pi}{n}$

Hence z_0 is periodic if and only if θ is a rational multiple of π.

Now suppose θ is not a rational multiple of π. we claim that the orbit of z_0 is dense on the circle with radius $|z_0|$ and center at the origin.

To prove this we show that every neighbourhood of every point on the circle with radius $|z_0|$ and center at the origin is $2\pi r$. As θ is not a rational multiple of π. we know that $f^m(z_0) \neq z_0$. for any natural number m. In particular, if m is larger than $2\pi r / \epsilon$, then none of the first m iterates of z_0 are the same and there must be two between which the distance is less than ϵ. (The reader is encouraged to explain why this is so).

If w is any point on the circle of radius $|z_0|$, then the neighbourhood of w with radius ϵ. must contain an iterate of z_0. As ϵ is arbitrary, this implies the orbit of z_0 is dense on the circle with center at the origin and radius $|z_0|$ as desired. $\square$