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PAPER: M 102, REAL ANALYSIS - I

M. Sc. I YEAR, SEMESTER - I

PREPARED BY

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unit - I Basic Topology

* finite, countable and Uncountable sets :

Definition :- Consider two sets A & B. Suppose that with each element x of A there is associated in same manner, an element of B with which we denote by $f(x)$, then f is said to be a function from A to B (or) mapping of A into B. The set A is called the domain of f and the element $f(x)$ are known as the values of f. The set of all values of f is called the range of f. mapped elements

Definition : A and B are two sets and f be a mapping of A into B if $E \subset A$, $f(E)$ is said define to be the set of all elements of $f(x)$, for $x \in E$. We call $f(E)$ is the ~~rano~~ image of E under f. In this notation $f(A)$ is the range of f. It is clear that $f(A) \subset B$. If $f(A) = B$, we say that f maps A onto B

If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ is the inverse image of E under f, if $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$

If for each $y \in B$, $f^{-1}(y)$ consists of the almost one element of A, then f is said to be one-one [1-1 Mapping of A into B].

* If there exists a one-one mapping of A onto B, we say that, A, B can be put in 1-1 correspondance (or) that A & B have some cardinal numbers ore briefly that A & B are equivalen

and we write this relation clearly by specifying that it has the following properties

(i) It is reflexive : $A \sim A$

(ii) It is symmentric : if $A \sim B$ then $B \sim A$

(iii) It is transitive : if $A \sim B$ and $B \sim C$ then $A \sim C$

Any relation with these three properties is known as an equivalence relation.

* Definition : for any +ve integer 'n', let 'J_n' be the set of those elements are integers $1, 2, 3, \ldots n$. let J be the set containing of all positive integers. for any set A, we have say

→ A is finite if $A \sim J_n$, for some 'n' (the empty set is also consider to be finite)

→ A is infinite if A is not finite

→ A is countable if $A \sim J$

→ A is uncountable if A is neither finite nor countable

→ A is atmost countable if A is finite (or) countable.

Countable sets are sometimes called enumerabl or denumerable

Ex ::

1) let A be the set of all integers then A is countable

let A be the set of all integers $f : J \to A$ be a mapping defined by

$$f(A) = \begin{cases} n/2 & ; \text{ if } n \text{ is even} \\ \frac{-n-1}{2} & ; \text{ if } n \text{ is odd} \end{cases}$$

clearly, f is bijection
A is countable.

2) The empty set is finite.

<u>Note</u> :. for two finite sets A and B we have $A \sim B$ if and only A and B contains same no. of elements.

* <u>Sequence</u> : A function f is defined on the set J of all +ve integers if $f(n) = x_n$, for $n \in J$ then the sequence f is denoted by $\{x_n\}$ (or) $x_1, x_2, x_3, \dots\dots$ the values of f i.e the elements of x_n are called the terms of the sequence (or)

If A is a set and if $x_n \in A \ \forall \ n \in J$ then $\{x_n\}$ is said to be a sequence in A (or) a sequence of elements of A

<u>Theorem</u> :. Imp

Statement :. Prove that every infinite subset of a countable set A is countable.

Proof :. Given that A is countable

Suppose that $E \subset A$ and E is infinite

claim : E is countable.

Since A is countable we can arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements.

Consider the sequence $\{n_k\}$ as follows.

Let 'n_1' be the smallest ~~+ve~~ positive integer such that
$$x_{n_1} \in E$$

choose 'n_2' be the another smallest positive integer such that $n_1 < n_2$ and $x_{n_2} \in E$

Having choosen $n_1, n_2, \dots n_{k-1}$

Let n_k be the smallest positive integer such that

$$n_{k-1} < n_k \text{ and } x_{n_k} \in E$$

Now define $f: J \to E$ as $f(k) = x_{n_k}$ where J is set of all positive integers

clearly there is a one-to-one correspondance b/w E & J

$$\Rightarrow E \sim J$$

$$\Rightarrow E \text{ is countable}$$

Hence Every infinite subset of a countab set A is countable.

* Definition :

Let A and Ω be two sets and suppose that with each element $\alpha \in A$ there is associated a subset of Ω which we denoted by E_α

The set whose these elements are the sets E_α will be denoted $\{E_\alpha\}$

The union of the set E_α is defined to be the set S such that $x \in S \Leftrightarrow x \in E_\alpha$

for atleast one $\alpha \in A$ we use the notation

$$S = \cup E_\alpha, \quad \alpha \in A$$

If A consists of the integers $1, 2, \dots n$

We write $S = \bigcup\limits_{m=1}^{n} E_m$

If A is a set of all positive integers

$S = \bigcup\limits_{m=1}^{\infty} E_m$

The intersection of the sets E_α is defined to the set P scale that $x \in P$ if $x \in E_\alpha$ for each $\alpha \in A$, we use the notation $P = AE_\alpha$, $\alpha \in A$

* **Note :.**

1) $A \cup B = B \cup A$, $A \cap B = B \cap A$
2) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$
3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4) $A \subset A \cup B$, $B \subset A \cup B$
5) $A \cap B \subset A$, $A \cap B \subset B$
6) $A \cup \phi = A$, $A \cap \phi = \phi$
7) If $A \subset B$ then $A \cup B = B$, $A \cap B = A$

** Theorem :. / Imp

* Let $\{E_n\}$, $n = 1, 2, 3, \ldots$ be a sequence of countable set and Put $S = \bigcup_{n=1}^{\infty} E_n$. Then S is countable

Proof: Let $\{E_n\}$, $n = 1, 2, 3, \ldots$ be sequence of countable sets and put $S = \bigcup_{n=1}^{\infty} E_n$

claim :. S is countable

Write: $E_m = \{ x_{nk} \mid k = 1, 2, 3, \ldots \}$, $n = 1, 2, 3, \ldots$

[$\because E_n$ is countable]

Consider the infinite array

$$\left.\begin{array}{cccc} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{array}\right\} \quad (1)$$

In which the elements of E_m forms the m th row.

The array contains all elements of S. As indicated by the array. arrows

These elements can be arranged in a sequence.

$$x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, x_{41}, x_{32}, x_{23}, x_{14} \ldots \rightarrow (2)$$

If any two of the sets E_n have elements in common there will appear more than once in (2)

Hence there is a subset T of the set of all positive integers such that $S \sim T$ [there is a bijection]

→ If T is finite then S is finite

If T is the set of all positive integers then

S is countable

Hence S is atmost countable

Since $E_1 \subseteq S$ & E_1 is infinite

$\Rightarrow S$ is infinite

Hence S is countable.

* Note :- If $A \cap B \neq \phi$ then we say that A and B intersect.

Otherwise they are distinct

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Theorem :- (Imp) Let A be a countable set and let B_n be the set of all n-tuples $(a_1, a_2, \ldots a_n)$, where $a_k \in A$ $(k = 1, 2, 3, \ldots n)$ and the element $a_1, a_2, \ldots a_n$ need not be distinct then B_n is countable.

proof : Let A be a countable set

Let $B_n = \{(a_1, a_2, \dots a_n) / a_k \in A ;\ 1 \le k \le n\}$ and the elements $a_1, a_2, \dots a_n$ need not be distinct

claim: B_n is countable

We prove that the theorem by using induction on n

If $n = 1$ then $B_1 = A$

Since A is countable $\Rightarrow B_1$ is countable

$\therefore$ The result is true for $n = 1$.

Assume that the result for $n-1$

i.e, B_{n-1} is countable, $(n = 2, 3, \dots)$

We prove that the result for n

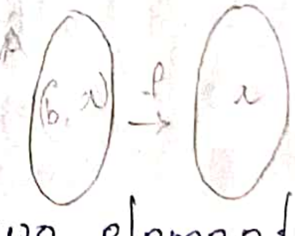
The elements of B_n are of the form (b, a) for $b \in B_{n-1}$ and $a \in A$.

for every fixed $b \in B_{n-1}$, the set of pairs (b, a) is equivalent to A and hence countable.

for, Let us define a mapping

domain codomain

$f : \{(b, a) / a \in A\} \to A$ as $f(b, a) = a$, for every fixed $b \in B_{n-1}$



f is one-one :-

Let $(b_1, a_1), (b_1, a_2)$ are two elements such that $f(b_1, a_1) = f(b_1, a_2)$

$$\Rightarrow a_1 = a_2$$

$$\Rightarrow (b_1, a_1) = (b_1, a_2)$$

$\therefore$ f is one-one

f is onto :

Let $a \in A$ then

$\exists\ (b,a) \in \{(b,a) / a \in A\} \ni f(b,a) = a$

$\therefore$ f is onto

Thus f is bijection.

$\Rightarrow \{(b,a)/a \in A\} \sim A \rightarrow (1)$, for every $b \in B_{n-1}$

Since A is countable.

$\Rightarrow A \sim J \rightarrow (2)$

from (1) & (2) we have $\{(b,a)/a \in A\} \sim J$, for every $b \in B_{n-1}$

But $B_n = \bigcup_{b \in B_{n-1}} \{(b,a) / a \in A\}$

Let $x \in B_n$

then $x = (x_1, x_2, \ldots, x_n)\ \forall\ n$

$x_i \in A,\ 1 \le i \le n$

put $c = (x_1, x_2, \ldots, x_{n-1}) \in B_{n-1}$

then $x = (c, x_n)$

$\Rightarrow x \in \{(b,a)/a \in A\},\ b \in B_{n-1}$

$\Rightarrow x \in \bigcup_{b \in B_{n-1}} \{(b,a)/a \in A\}$.

$B_n \subseteq \bigcup_{b \in B_{n-1}} \{(b,a)/a \in A\}$ — (*)

$(b,a) \in \bigcup_{b \in B_{n-1}} \{(b,a)/a \in A\}$

$(b,a) \in \{(b,a)/a \in A\},\ b \in B_{n-1}$

$b = (b_1, b_2, \ldots b_{n-1})$.

$(b, a) \in B_n$

$(b,a) = (b_1, b_2, b_3, \ldots, b_{n-1}, a) \in B_n$

$\cancel{b \in B_n}$

$\cdot (b, a) \in B_n$

$\bigcup_{b \in B_{n-1}} \{(b,a) / a \in A\} \subseteq B_n$ — (+) (A)

By theorem,

$\bigcup_{b \in B_{n-1}} \{(b,a) / a \in A\}$ is countable

$\Rightarrow B_n$ is countable

Hence by mathematical induction the result is true for all positive integers.

** Imp

Theorem :- Let A is a set of all sequences whose elements are the digits 0 & 1 then the set A is uncountable.

[The elements of A are sequences line 1, 0, 0, 1 0, 1, 1, 1]

$S_n \rightarrow$ 1 0 0 0 1
$S \rightarrow$ 0 1 1 0 0 0

proof :- Let A be the set of all sequences whose elements are the digits 0 & 1.

claim :- A is uncountable

Let E be the countable subset of A and let E consists of the sequences $S_1, S_2, S_3, \ldots$

We construct a sequence s as follows

If the n th digit in S_n is '1', We let the n th digit in s is '0' and vice versa.

Then the sequence s differs from each

number of E atleast one place. proper subset

Hence $s \notin E$, But s is in A $G = \{1,2,3,4\}$

$\therefore$ E is the proper subset of A. $H = \{1,3,4\}$ atleast one

i.e, Every countable subset of A is a proper subset of A

A contains one more than E

Suppose if possible A is countable E proper

Hence, A is the proper subset of A,

which is a contradiction

$\therefore$ A is uncountable.

* Metric Space:

Definition:

Let X be a non-empty set. A mapping $d: X \times X \to \mathbb{R}$ is said to be a distance function (or) metric on X.

If it satisfies the following properties.

(i) $d(x,y) \geq 0$; $d(x,y) = 0$ iff $x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ for every $x,y,z \in X$

If d is a metric on X then (X,d) is called as a metric space.

Exa:

① Let X be a non-empty set and define $d: X \times X \to \mathbb{R}$ by
$$d(x,y) = \begin{cases} 0 \ ; \text{ if } x = y \\ 1 \ ; \text{ if } x \neq y \end{cases}$$

Then d is a metric on X which is called "Trivial (or) Discrete".

② Define $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x,y) = |x-y|$, for any $x,y \in \mathbb{R}$ then d is a metric on $\mathbb{R}$ which is called usual (or) Standard metric on R

(3) Let $n \in Z^+$ and $R^n = \{(x_1, x_2, \dots x_n) / x_i \in R, \forall 1 \leq i \leq n\}$

Define $d : R^n \times R^n \longrightarrow R$ as $d(x, y) = |x - y| = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$

for $x = (x_1, x_2, x_3, \dots x_n) \in R^n$

$y = (y_1, y_2, y_3, \dots y_n) \in R^n$

Then d is a metric on R^n which is called usual (or) Standard metric on R^n.

* For any real numbers a and b with $a < b$. let us define

$(a, b) = \{x \in R / a < x < b\}$

$[a, b] = \{x \in R / a \leq x \leq b\}$

$(a, b] = \{x \in R / a < x \leq b\}$

$[a, b) = \{x \in R / a \leq x < b\}$

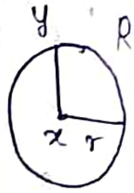
* K-cell : Let k be a +ve integer and $a = (a_1, a_2, \dots a_k)$ and $b = (b_1, b_2, \dots b_k)$ be any points on R^k such that $a_i < b_i \ \forall 1 \leq i \leq k$, the set of all points $x = (x_1, x_2, \dots x_k)$ in R^k whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i, \forall 1 \leq i \leq k$ is called a k-cell. Thus a 1-cell is an interval, a 2-cell is a rectangle.

It is the cartesian product of k-closed intervals on the real line. This means the k-dimensional rectangular solids have each of its edges equal to one of the closed interval

The k-intervals need not be identical.

* If $x \in R^k$ and $r > 0$, the open ball B with centre at x and radius r is defined to be the set of all $y \in R^k$ such that $|x-y| < r$

i.e, $B(x, r) = \{ y \in R^k \mid |x-y| < r \}$



* If $x \in R^k$ and $r > 0$ the closed ball $\bar{B}$ with centre at x and radius r is defined to be the set of all $y \in R^k$ such that $|x-y| \leq r$

i.e, $\bar{B}(x, r) = \{ y \in R^k \mid |x-y| \leq r \}$

* Convex set :- A set $E \subset R^k$ is said to be convex if $\lambda x + (1-\lambda) y \in E$, whenever $x, y \in E$ and $0 < \lambda < 1$

Ex :- Balls are convex.

Let $B(x, r)$ be an open ball with centre at x and $y \in R^k$ is another point and radius 'r'.

Let $y, z \in B(x, r)$ and $0 < \lambda < 1$

We have to prove that $\lambda y + (1-\lambda) z \in B(x; r)$

Since $y, z \in B(x; r)$

$\Rightarrow |x-y| < r$ and $|x-z| < r$

Consider,

$$
\begin{aligned}
|\lambda y + (1-\lambda) z - x| &= |\lambda y + (1-\lambda) z - x + \lambda x - \lambda x| \\
&= |\lambda (y-x) + (1-\lambda) z - (1-\lambda) x| \\
&= |\lambda (y-x) + (1-\lambda)(z-x)| \\
&\leq |\lambda (y-x)| + |(1-\lambda)(z-x)| \\
&< \lambda r + (1-\lambda) r \\
&< r
\end{aligned}
$$

$\therefore |\lambda y + (1-\lambda) z - x| < r$

$\Rightarrow \lambda y + (1-\lambda) z \in B(x, r)$

$\therefore B(x, r)$ is convex.

Definitions:

* Let X be a metric space and $E \subset X$

a) Neighbourhood :- A Neighbourhood of a point $p \in X$ is a set N_r consisting of all points $q \in X$ such that $d(p,q) < r$. The number r is called the radius of $N_r(p)$.

i.e, $N_r(p) = \{q \in X \mid d(p,q) < r\}$

b) Limit point :- A point $p \in X$ is a limit point of the set E if every neighbourhood of p contains a point $q \neq p$ such that $q \in E$

i.e, $N_r(p) \cap E - \{p\} \neq \phi, \ \forall r > 0$.

c) Isolated point : If $p \in E$ and p is not a limit point of E, then p is an isolated point of E

d) Closed set :- E is closed if every limit point of E is a point of E

e) Interior point : A point p is a interior point of E if there is a neighbourhood $N_r(p)$ of p such that $N_r(p)$ is subset or equal to E

i.e, $N_r(p) \subseteq E$

f) Open set :- A set E is open if every point of E is an interior point of E

g) Complement : The complement of E is denoted by E^c is a set of all points $p \in X$ such that $p \notin E$ i.e, $E^c = X - E$

h) **Perfect set**: E is perfect if E is closed if every point of E is a limit point of E.

i) **Bounded set**: E is bounded if there is a real number $m>0$ and $q \in X$ such that $d(p,q) < M \; \forall \; p \in E$ (or) $m > 0 \ni d(p,q) < m \; \forall \; p, q \in E$

j) **Dense set**: E is dense in X if every point of X is a limit point of E (or) a point of E (or) Both.

** Imp

* **Theorem**: Prove that every neighbourhood is a open set.

proof: Consider the neighbourhood $N = N_r(p)$

claim: N is open

It is enough to prove that every point of N is an interior point of N

Let q be any point of N

then $d(p,q) < r$

put $s = r - d(p,q)$

then $s > 0$

We shall prove that $N_s(q) \subseteq N$

Let $t \in N_s(q)$

then $d(q,t) < s$

$< r - d(p,q)$

$\Rightarrow d(q,t) + d(p,q) < r$

$\Rightarrow d(p,q) + d(q,t) < r$

Since d is metric

$d(p,t) \leq d(p,q) + d(q,t)$

$d(p,t) < r$

$\Rightarrow t \in N$

$\therefore N_s(q) \subseteq N_r(p)$

$\therefore N_s(q) \subseteq N$

$\therefore q$ is an interior point of N

Hence N is an open set.

** Imp

Theorem: If p is a limit point of a set E, then every neighbourhood of p contains infinitely many points of E.

proof: Let p be a limit point of a set E.

Consider a Neighbourhood $N_r(p)$

then $N_r(p) \cap E - \{p\} \neq \phi$

$\Rightarrow$ there exists $p_1 (\neq p) \ni p_1 \in N_r(p) \cap E$.

$\Rightarrow p_1 \in N_r(p)$ and $p_1 \in E$

$\Rightarrow 0 < d(p, p_1) < r$

put $r_1 = d(p, p_1)$

then $r_1 > 0$ and $\therefore N_{r_1}(p)$ is a neighbourhood of p.

since p is a limit point of E

$N_{r_1}(p) \cap E - \{p\} \neq \phi$

$\Rightarrow \exists p_2 (\neq p) \ni p_2 \in N_{r_1}(p) \cap E$

$\Rightarrow p_2 \in N_{r_1}(p)$, $p_2 \in E$

$\Rightarrow d(p, p_2) < r_1$ and $p_2 \in E$

$\Rightarrow d(p, p_2) < d(p, p_1) < r$ and $p_2 \in E$

put $r_2 = d(p, p_2)$

and continue the above process with $N_r(p)$ we get points $p_1, p_2, \ldots$ are in E such that $r > d(p, p_1) > d(p, p_2) > \ldots$

which implies that all are distinct points

Hence, if p is a limit point of a set E, then every neighbourhood of p contains infinitely many points of E.

Corollary :-

Every finite set has limit points.

Proof :- Let A be a finite set.

Suppose if possible A has a limit point p (say)

Consider a neighbourhood $N_r(p)$ with $r > 0$

By known theorem, every neighbourhood of p contains infinitely many points of A.

$\Rightarrow N_r(p) \cap A$ is an infinite set

But $N_r(p) \cap A \subseteq A$ and $N_r(p) \cap A$ is an infinite subset of A.

$\Rightarrow A$ is infinite

which is a contradiction

$\therefore A$ has no limit points.

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* Theorem :- Let $\{E_\alpha\}$ be a (finite (or) infinite) collection of sets E_α. Then $(\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha (E_\alpha^c)$

proof : Let $\{E_\alpha\}$ be a (finite (or) infinite) collection of sets E_α

Let $x \in (\bigcup_\alpha E_\alpha)^c$

Then $x \notin \bigcup_\alpha E_\alpha$

$\Rightarrow x \notin E_\alpha$, for each α

$\Rightarrow x \in E_\alpha^c$, for each α

$\Rightarrow x \in \bigcap_\alpha E_\alpha^c$, for all α.

$\therefore \left(\bigcup_\alpha E_\alpha\right)^c \subseteq \bigcap_\alpha (E_\alpha^c) \rightarrow$ 1

let $x \in \bigcap_\alpha E_\alpha^c$

Then $x \in E_\alpha^c$, for each α

$\Rightarrow x \notin E_\alpha$, for each α

$\Rightarrow x \notin \bigcup_\alpha E_\alpha$

$\Rightarrow x \in \left(\bigcup_\alpha E_\alpha\right)^c$

$\therefore \left(\bigcup_\alpha E_\alpha\right)^c \supseteq \bigcap_\alpha (E_\alpha^c) \rightarrow$ 2

from 1 & 2

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c)$$

Note. Every finite set is closed.

** Imp

* Theorem : A set E is open iff its complement is closed

proof : Let (X, d) be a metricspace and $E \subset X$

Suppose E is open

claim : E^c is closed → it contains all its limit points

Let x be a limit point of E^c

If $x \notin E^c$ then $x \in E$

Since E is open and $x \in E$

$\Rightarrow x$ is an interior point of E

$\Rightarrow \exists\, r>0 \ni N_r(x) \subseteq E$

$\Rightarrow N_r(x) \cap E^c = \phi$

which is a contradiction to x is a limit point of E^c.

$\therefore x \in E^c$

$\Rightarrow E^c$ contains all its limit points.

Thus, E^c is closed.

Conversely, suppose E^c is closed.

claim :: E is open

Let $x \in E$.

Then $x \notin E^c$

$\Rightarrow x$ is not a limit point of E^c

$\Rightarrow \exists\, r > 0 \ni N_r(x) \cap E^c - \{x\} = \phi$.

$\Rightarrow N_r(x) \cap E^c = \phi$

$\Rightarrow N_r(x) \subseteq E$

$\Rightarrow x$ is an interior point of E

$\therefore$ Every point of E is an interior point of E and Hence E is open.

corollory :: A set F is closed iff its complement is open.

proof :: We know that $(F^c)^c = F$

Suppose F is closed

$\Rightarrow (F^c)^c$ is closed.

$\Rightarrow F^c$ is open [above theorem]

Theorem :-

(a) for any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open

(b) for any collection $\{F_\alpha\}$ of closed, $\bigcap F_\alpha$ is closed

(c) for any finite collection $G_1, G_2, \dots G_n$ of open sets, $\bigcap_{i=1}^{n} G_i$ is open

(d) for any finite collection $F_1, F_2, \dots F_n$ of closed set, $\bigcup_{i=1}^{n} F_i$ is closed.

proof :-

(a) Let $\{G_\alpha\}$ be a collection of open sets

put $G = \bigcup_\alpha G_\alpha$

claim :- G is open

let $x \in G = \bigcup_\alpha G_\alpha$

Then $x \in G_\alpha$ for some α

Since G_α is open and $x \in G_\alpha$ for some α

x is an interior point of G_α for some α

$\Rightarrow \exists\, r > 0 \ni N_r(x) \subseteq G_\alpha$ for some α

since $G_\alpha \subseteq \bigcup_\alpha G_\alpha = G$

$\Rightarrow N_r(x) \subseteq G_\alpha \subseteq G$

$\exists\, r > 0 \ni N_r(x) \subset G$

(i.e) x is an interior point of G

Therefore, every point of G is an interior point of G and hence G is open.

(b) Let $\{F_\alpha\}$ be a collection of closed sets:

claim: $\bigcap_\alpha F_\alpha$ is closed.

Since F_α is closed for each α

$\Rightarrow F_\alpha^c$ is open for each α

$\Rightarrow \bigcup_\alpha F_\alpha^c$ is open [by (a)]

But $\left(\bigcap_\alpha F_\alpha\right)^c = \bigcup_\alpha F_\alpha^c$ [since theorem]

$\Rightarrow \left(\bigcap_\alpha F_\alpha\right)^c$ is open

$\Rightarrow \bigcap_\alpha F_\alpha$ is closed.

(c) Let $G_1, G_2, \ldots G_n$ be a finite collection of open sets

claim: $\bigcap_{i=1}^{n} G_i$ is open

Let $x \in \bigcap_{i=1}^{n} G_i$

Then $x \in G_i$ for all $i = 1, 2, 3, \ldots n$

Since G_i is open and $x \in G_i$, $\forall\ 1 \leq i \leq n$

$\Rightarrow \exists\ r_i > 0 \ni N_{r_i}(x) \subseteq G_i\ \forall\ 1 \leq i \leq n$

put $r = \min\{r_1, r_2, \ldots, r_n\}$.

Then $r > 0$ and $r \leq r_i\ \forall\ 1 \leq i \leq n$

Let $N_r(x)$ be the neighbourhood of x with the radius 'r'.

for any $y \in N_r(x)$, $d(x, y) < r$

$\Rightarrow d(x, y) < r \leq r_i\ \forall\ 1 \leq i \leq n$

$\Rightarrow d(x, y) < r_i\ \forall\ 1 \leq i \leq n$

$\Rightarrow y \in N_{r_i}(x) \ \forall\ 1 \leq i \leq n$

$\Rightarrow y \in G_i \ \forall\ 1 \leq i \leq n \quad [\because N_{r_i}(x) \subseteq G_i]$

$\Rightarrow y \in \bigcap_{i=1}^{n} G_i$

$\therefore \exists\ r > 0 \ni N_r(x) \subseteq \bigcap_{i=1}^{n} G_i$

$\Rightarrow x$ is an interior point of $\bigcap_{i=1}^{n} G_i$

and hence $\bigcap_{i=1}^{n} G_i$ is open.

(d) Let $F_1, F_2, \ldots F_n$ be the finite collection of closed sets.

claim: $\bigcup_{i=1}^{n} F_i$ is closed

$F_1, F_2, \ldots F_n$ be the finite collection of closed sets.

Then $F_1^c, F_2^c, \ldots F_n^c$ are open.

By (c) $\bigcap_{i=1}^{n} F_i^c$ is open

$\Rightarrow \left(\bigcup_{i=1}^{n} F_i\right)^c$ is open

$\Rightarrow \bigcup_{i=1}^{n} F_i$ is closed.

Definition:

* closure of a point: If X is a metricspace if $E \subset X$ and if E' (derived sets) denotes the set of all limit points of E in X. Then the closure of E is the set $\bar{E} = E \cup E'$

**Theorem: If X is a metric space and $E \subseteq X$, then (a) $\bar{E}$ is closed

(b) $E = \bar{E}$, iff E is closed

(c) $\bar{E} \subset F$; for every closed set $F \subseteq X$ such that $E \subset F$. By (a) and (c), $\bar{E}$ is the smallest closed subset of X that contains E.

proof: Let X be the metric space and $E \subseteq X$

claim: (a) $\bar{E}$ is closed

It is enough to prove that $\bar{E}^c$ is open

Let $x \in \bar{E}^c$

Then $x \in \bar{E}^c$ & $x \notin \bar{E} = E \cup E'$

$\Rightarrow x \notin E$ and $x \notin E'$

$\Rightarrow x$ is not the limit point of E

$\Rightarrow$ Then every $r > 0$ such that $N_r(x) \cap E = \phi$

We have to prove that $N_r(x) \subseteq \bar{E}^c$

Let $y \in N_r(x)$

then $d(x, y) < r$

Since every neighbourhood is open

$\Rightarrow N_r(x)$ is open

$\Rightarrow y$ is an interior point of $N_r(x)$

$\Rightarrow$ there exists $s > 0$, such that $N_s(y) \subseteq N_r(x)$

$\Rightarrow N_{(s)}(y) \cap E \subseteq N_r(x) \cap E = \phi$

$\Rightarrow N_s(y) \cap E = \phi$

$\Rightarrow y \notin E$ and y is not the limit point of E

$\Rightarrow y \notin E$ and $y \notin E'$

$\Rightarrow y \notin E \cup E' = \bar{E} \Rightarrow y \in \bar{E}^c$

Therefore, there exists $r>0$ such that

$N_r(x) \subseteq \bar{E}^c$

$\Rightarrow x$ is an interior point of $\bar{E}^c$ and there $\bar{E}^c$ is open

$\therefore$ Then $\bar{E}$ is closed

(b) Suppose $E = \bar{E}$

by (a) $\bar{E}$ is closed

$\therefore$ E is closed.

Conversely suppose E is closed.

$\Rightarrow$ Then E contains all its limit points

$\Rightarrow E' \subset E$ [$\because$ E is closed]

$\Rightarrow E \cup E' = E$ set of all ele. & limit points

$\Rightarrow \bar{E} = E$

(c) Let F be a closed set on X such that $E \subset F$

claim : $\bar{E} \subset F$

Note that every limit point E is a limit point of F and there $E' \subset F'$

Since F is closed

$\Rightarrow F' \subset F$

$\Rightarrow E' \subset F' \subset F$

$\Rightarrow E' \subset F$

Since $E \subset F$ and $E' \subset F$

$\Rightarrow E \cup E' \subset F$

$\Rightarrow \bar{E} \subset F$

Since $E \cup E' = \bar{E}$

$E \subset E \cup E' = \bar{E}$

by ⓐ $\bar{E}$ is a closed subset of X such that
$E \subset \bar{E}$

let F be a closed subset of X such that $E \subset F$
by ⓒ $\bar{E} \subset F$

Therefore, $\bar{E}$ is the smallest closed subset of $X \ni E \subset \bar{E}$

** Imp.
* **Theorem**:: Let E be a non-empty set of real numbers which is bounded above. Let $y = \sup E$ then $y \in \bar{E}$. Hence $y \in E$ if E is closed.

Proof:: Let $E \subset \mathbb{R}$ and $E \neq \phi$
Suppose E is bounded above and
Let $y = \sup E$
If $y \in E$ then $y \in E \cup E' = \bar{E}$
Assume $y \notin E$
Since, $y = \sup E$, y is an upperbound of E
for every $h > 0$, $y - h$ is not an upperbound of E
[otherwise, $y-h$ is an upperbound of E, for some $h > 0$

$y = \sup E = \text{L.U.B of } E$

$\Rightarrow y < y - h$ for $h > 0$

which is a contradiction]

1) $y \in E \Rightarrow y \in$
2) $y \notin E \Rightarrow y \in$
3) if E is cl
proven
$y \in E$

$\Rightarrow x \in E \Rightarrow y - h < x$

Since $y = \sup E$

$\Rightarrow x < y < y + h$

$\Rightarrow y - h < x < y < y + h$

$\Rightarrow x \in (y-h, y+h) = N_r(y)$

$\Rightarrow x \in N_r(y)$ and $x \in E$ for some $r > 0$

$\Rightarrow N_r(y) \cap E \neq \phi$, for every $r > 0$

$\Rightarrow y$ is a limit point of E

$\Rightarrow y \in E' \subset E \cup E' = \bar{E}$

$\Rightarrow y \in \bar{E}$

Suppose E is closed

$\Rightarrow E = \bar{E}$

Since $y \in \bar{E}$

$\Rightarrow y \in E$

Hence proved.

*** Imp

* Theorem :- Suppose $Y \subset X$. A subset E of Y is open relative to Y iff $E = Y \cap G$, for some open subset G of X.

proof :- Suppose $Y \subset X$

Suppose $E \subset Y$ is open in Y

then each $a \in E$ is an interior point of E with respect to Y.

for each $a \in E$, $\exists\ r_a > 0 \ni N_{r_a}(a)$

$\Rightarrow N_{r_a}(a) \cap Y \subseteq E$, for each $a \in E$

put $G = \bigcup_{a \in E} N_{r_a}(a)$

Since, every neighbourhood $N_{r_a}(a)$ is open in X.

$\Rightarrow G$ is open in X

Also $E \subseteq Y \cap G$

$\Rightarrow E \subseteq Y \cap (\bigcup_{a \in G} N_{r_a}(a))$

$E \subseteq \bigcup_{a \in G} (Y \cap N_{r_a}(a))$ i.e. $a \in E \subset Y$

$a \in Y \ \exists \ r_a > 0 \ni N_{r_a}(a) \subseteq \bigcup$

$a \in N_{r_a}(a) \subseteq G$

$\Rightarrow a \in G$

from $a \in Y$ and $a \in G$

$\Rightarrow a \in Y \cap G$

$E \subseteq Y \cap G$]

$E \subseteq Y \cap G \subseteq E$

$E = Y \cap G$, for some openset G in X

Conversely suppose that

$E = Y \cap G$, for some openset G in X

claim :: E is open in Y

Let $x \in E = Y \cap G$

then $x \in Y$ and $x \in G$

Since G is open and $x \in G$

$\Rightarrow x$ is an interior point of G

$\Rightarrow \exists \ r > 0 \ni N_r(x) \subseteq G$

$\Rightarrow N_r(x) \cap Y \subseteq Y \cap G = E$

$\Rightarrow N_r(x) \cap Y \subseteq E$

$\therefore x$ is an interior point of E w.r.to Y

Hence E is open in Y.

$\{G_\alpha\}$ opencover of E

* Compact Sets ::

Every open cover contains finite subcover. $K \subseteq \bigcup G_{\alpha_i}$

Compact Matric space :: Let E be a subset of a matric space. A family $\{G_\alpha\}_{\alpha \in \Delta}$ of opensets X is called opencover of E. if $E \subseteq \bigcup G_\alpha$ Compact metric space. A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover of K. then there exists finitely many indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots \alpha_n$ in Δ such that $K \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$

K is compact

open cover $E \subseteq \bigcup G_\alpha$

Every " has finite sub

$K \subseteq \bigcup^n G_{\alpha_i}$

Theorem :- Suppose $k \subset Y \subset X$ then k is compact in X iff k is compact in Y. Imp.

proof :- Suppose $k \subset Y \subset X$

Let k be a compact in X

claim :- k be a compact in Y

Let $\{E_\alpha\}_{\alpha \in \Delta}$ be an open cover of k in Y.

then E_α is open in Y, for each α and $k \subseteq \bigcup_{\alpha \in \Delta} E_\alpha$

Since E_α is open in Y, for each $\alpha \in \Delta$

By known theorem, $E_\alpha = Y \cap G_\alpha$, for some open set G_α in X for each $\alpha \in \Delta$

$G_\alpha \in X$

$E_\alpha \in Y$

$$\therefore k \subseteq \bigcup_{\alpha \in \Delta} E_\alpha$$

$$= \bigcup_{\alpha \in \Delta} (Y \cap G_\alpha)$$

$$\subseteq \bigcup_{\alpha \in \Delta} G_\alpha$$

$$k \subseteq \bigcup_{\alpha \in \Delta} G_\alpha$$

$\therefore \{G_\alpha\}_{\alpha \in \Delta}$ is an open cover of k in X

Since k is compact in X

$\exists\ \alpha_1, \alpha_2, \ldots \alpha_n$ in $\Delta \Rightarrow k \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$

$$k \cap Y \subseteq Y \cap \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right)$$

[$\because k \subset Y \subset X$

$k \subset Y$

$k \cap Y = k$

$k \cup Y = Y$]

1) family of openset

2) open cover

3) each open cover has finite subcover

$$k \subseteq \bigcup_{i=1}^{n} (Y \cap G_{\alpha_i})$$

$$k \subseteq \bigcup_{i=1}^{n} E_{\alpha_i}$$

$\therefore k$ has a finite subcover in Y

$\therefore k$ is compact in Y.

Conversely, suppose that k is compact in Y

claim : k is compact in X

Let $\{G_\alpha\}_{\alpha\in\Delta}$ be an open cover in X

Then G_α is open in X, for each α and

$$k \subseteq \bigcup_{\alpha\in\Delta} G_\alpha$$

$$Y \cap k \subseteq Y \cap \left(\bigcup_{\alpha\in\Delta} G_\alpha\right) \quad (\because k \subset Y)$$

$$k \subseteq \bigcup_{\alpha\in\Delta} (Y \cap G_\alpha)$$

$\therefore \{Y \cap G_\alpha\}$ is an open cover of k in Y

Since k is compact in Y.

$\exists\, \alpha_1, \alpha_2, \dots \alpha_n$ in $\Delta \ni k \subseteq \bigcup_{\alpha\in\Delta}^{n} (Y \cap G_{\alpha_i})$

$$\subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$$

$$k \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$$

$\therefore$ k has a finite subcover in X

** $\therefore$ k is compact in X

*** Imp

Theorem : Compact subsets of a metric space are closed.

proof : Let X be a metric space and k be a compact subset of X.

claim : k is closed.

It is enough to prove that k^c is open

Let $x \in k^c$ then $x \notin k$

for each $a \in k$, we have $x \neq a$

then $d(x,a) > 0$

put $r_a = \frac{1}{2} d(x,a)$

then $r_a > 0$ and $N_{r_a}(a) \cap N_{r_a}(x) = \phi$

[otherwise, if $y \in N_{r_a}(a) \cap N_{r_a}(x)$ then

$y \in N_{r_a}(a)$ and $y \in N_{r_a}(x)$

$\Rightarrow d(a,y) < r_a$, $d(x,y) < r_a$

Since d is a metric

$$d(a,x) \leq d(a,y) + d(y,x)$$
$$< r_a + r_a$$
$$= 2r_a$$
$$= 2 \cdot \frac{1}{2} d(x,a)$$
$$= d(x,a)$$

$\therefore d(x,a) < d(x,a)$

which is absurd]

Now $\{N_{r_a}(a) / a \in k\}$ is an open cover of k

Since k is compact

$\exists\ a_1, a_2, \ldots a_n \in k \ni k \subseteq \bigcup_{i=1}^{n} N_{r_{a_i}}(a_i)$

put $r = \min\{r_{a_1}, r_{a_2}, r_{a_3}, \ldots r_{a_n}\}$

then $r > 0$ and $r \leq r_{a_i}$, $N_r(x) \subset N_{r_{a_i}}(x)$,

for each $i = 1, 2, 3, \ldots n$

$$N_r(x) \cap N_{r_{a_i}}(a_i) \subseteq N_{r_{a_i}}(x) \cap N_{r_{a_i}}(a_i)$$
$$= \phi$$

$\therefore N_r(x) \cap N_{r_{a_i}}(a_i) = \phi$, for $i = 1, 2, \ldots n$

Since $k \subseteq \bigcup_{i=1}^{n} N_{r_{a_i}}(a_i)$

$$N_r(x) \cap k \subseteq N_r(x) \cap \left(\bigcup_{i=1}^{n} N_{r_{a_i}}(a_i)\right)$$
$$= \bigcup_{i=1}^{n} \left(N_r(x) \cap N_{r_{a_i}}(a_i)\right)$$
$$= \phi$$

$N_r(x) \cap k = \phi$

$\Rightarrow N_r(x) \subseteq k^c$

$\therefore$ x is an interior point of k^c.

$\therefore$ k^c is open

Hence k is closed

Theorem :: Prove that closed subsets of compact sets are compact.

proof :: Let k be a closed subsets of a compact set.

claim :: k is compact

Let $\{G_\alpha\}_{\alpha \in \Delta}$ be an open cover of k in x.

Then $k \subseteq \bigcup_{\alpha \in \Delta} G_\alpha$

$\Rightarrow k \cup k^c \subseteq \bigcup_{\alpha \in \Delta} G_\alpha \cup k^c$

But $x = k \cup k^c$

$\Rightarrow x \subseteq \bigcup_{\alpha \in \Delta} G_\alpha \cup k^c$.

since k is closed

$\Rightarrow k^c$ is open.

$\therefore \{G_\alpha\}_{\alpha \in \Delta} \cup k^c$ forms an open cover of x.

Since x is compact

$\Rightarrow$ x has a finite subcover

The subcover of x may (or) maynot contains k^c

However, there exists

$\alpha_1, \alpha_2, \ldots \alpha_n$ in $\Delta \ni x \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n} \cup k^c$

$\Rightarrow x = G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n} \cup k^c \; [\because G_\alpha \subseteq x, k^c \subseteq x]$

Now, $k = k \cap X \quad [\because k \subset X]$

$$= k \cap (G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n} \cup k^c)$$

$$= (k \cap G_{\alpha_1}) \cup (k \cap G_{\alpha_2}) \cup \dots \cup (k \cap G_{\alpha_n}) \cup (k \cap k^c)$$

$$= (k \cap G_{\alpha_1}) \cup (k \cap G_{\alpha_2}) \cup \dots \cup (k \cap G_{\alpha_n})$$

$$= k \cap \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right)$$

$$\Rightarrow k \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$$

$\therefore$ k has a finite subcover

$\Rightarrow$ k is compact.

Hence closed subsets of compact sets are compact.

Corollary :- Imp

** If F is closed and k is compact then $F \cap k$ is compact.

proof: Let F be closed and k be a compact subset of metric space X.

Since k is compact $\Rightarrow$ k is closed

$\Rightarrow F \cap k$ is closed

Since $F \cap k \subseteq k$

$\Rightarrow F \cap k$ is a closed subset of a compact set k.

$\Rightarrow F \cap k$ is compact.

Theorem :- If $\{k_\alpha\}$ is a collection of compact subsets of metric space X such that the intersection of every finite subcollection of k_α is non-empty then intersection of k_α is non-empty.

Proof : Let $\{k_\alpha\}$ is a collection of compact subsets of a metric space X.

$\ni$ the intersection of every finite subcollection of $\{k_\alpha\}$ is non-empty.

claim :- $\cap k_\alpha$ is non-empty

fix a number k_{α_0} in the given family for any $\alpha \in \Delta$, put $G_\alpha = k_\alpha^c$.

Since k_α is compact, for each α

$\Rightarrow k_\alpha$ is closed, for each α

$\Rightarrow k_\alpha^c$ is open ; for each α

$\Rightarrow G_\alpha$ is open, for each α

Suppose if possible $\bigcap\limits_{\alpha \in \Delta} k_\alpha = \phi$

then $k_{\alpha_0} \cap \left(\bigcap\limits_{\alpha \in \Delta} k_\alpha\right) = \phi$

$$\Rightarrow k_{\alpha_0} \subseteq \Big(\bigcap_{\substack{\alpha \in \Delta \\ \alpha \neq \alpha_0}} k_\alpha\Big)^c = \bigcup_{\substack{\alpha \in \Delta \\ \alpha \neq \alpha_0}} k_\alpha^c$$

$$= \bigcup_{\substack{\alpha \in \Delta \\ \alpha \neq \alpha_0}} G_\alpha$$

$$\Rightarrow k_{\alpha_0} \subseteq \bigcup_{\substack{\alpha \in \Delta \\ \alpha \neq \alpha_0}} G_\alpha$$

Then $\{G_\alpha\}_{\alpha \in \Delta}$ is an open cover of k_{α_0}.

Since k_{α_0} is compact $\exists\ \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ in Δ

$$K_{\alpha_0} \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$$

$$= \bigcup_{i=1}^{n} K_{\alpha_i}^{c}$$

$$K_{\alpha_0} \subseteq \left(\bigcap_{i=1}^{n} K_{\alpha_i}\right)^{c}$$

$$\Rightarrow K_{\alpha_0} \cap \left(\bigcap_{i=1}^{n} K_{\alpha_i}\right) = \phi$$

which is a contradiction to every finite intersection of K_α is non-empty

$$\therefore \bigcap_{\alpha \in \Delta} K_\alpha \neq \phi$$

Theorem :- If E is an infinite subset of a compact set K, then E has a limit point in K.

proof: Let E be an infinite subset of a compact set K.

claim :- E has a limit point in K.

Suppose if possible, E has no limit point in K.

for each $a \in K$, a is not a limit point of K. then $\exists\, r_a > 0 \ni N_{r_a}(a) \cap E$ has atmost the point 'a' only

i.e, $N_{r_a}(a) \cap E = \{a\} \rightarrow$ ①

Now, $\{N_{r_a}(a) / a \in K\}$ is an open cover of K

Since E is compact, $\exists\, a_1, a_2, \ldots, a_n \ni$

$$K \subseteq \bigcup_{i=1}^{n} N_{r_{a_i}}(a_i)$$

Since $E \subseteq K$, We have $E = E \cap K$

$$E \subseteq E \cap \left(\bigcup_{i=1}^{n} N_{r_{a_i}}(a_i)\right)$$

$$E \subseteq \bigcup_{i=1}^{n} (E \cap N_{r_{a_i}}(a_i))$$

$$E \subset \bigcup_{i=1}^{n} \{a_i\}$$

$$E \subset \{a_1, a_2, \dots \dots a_n\}$$

Hence E is finite.

which is a contradiction to E is infinite

$\therefore$ E has a limit point in k.

Theorem : If $\{I_n\}$ is a sequence of interval in R, such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \dots)$ then $\bigcap_{n=1}^{\infty} I_n$ is not empty.

proof : Let $\{I_n\}$ is a sequence of intervals in R such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \dots \dots)$

claim : $\bigcap_{n=1}^{\infty} I_n$ is not empty i.e) $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

Let $I_n = [a_n, b_n]$, $n = 1, 2, 3, \dots$

Let E be the set of all 'a_n'.

then E is non-empty and bounded above

[$\because$ b_n is an upperbound of E]

Let $a = \text{Sup } E$

Since each b_n is an upperbound of E

$\Rightarrow a \leq b_n \ \forall n$

Also, $a_n \leq a$, $\forall n$.

LUB

UB

Thus, $a_n \leq a \leq b_n$, $\forall n$

$\Rightarrow a \in [a_n, b_n] = I_n \ \forall n$

$\Rightarrow a \in \bigcap I_n$

$\therefore \bigcap I_n \neq \phi$

corollary :

If $\{k_n\}$ is a sequence of non-empty compact sets such that $k_n \supseteq k_{n+1}$ $(n=1,2,3,\dots)$

claim, then then $\bigcap_{n=1}^{\infty} k_n$ is non-empty.

proof : Let $\{k_n\}$ is a sequence of a non-empty compact sets such that $k_n \supseteq k_{n+1}$ $(n=1,2,3,\dots)$

claim : $\bigcap_{n=1}^{\infty} k_n$ is non-empty

since $k_n \supseteq k_{n+1}$ $(n=1,2,3,\dots)$.

then $k_n \cap k_{n+1} = k_{n+1} \neq \phi$ $(n=1,2,3,\dots)$

$\Rightarrow k_n \cap k_{n+1} \neq \phi$ $(n=1,2,3,\dots)$

By known theorem,

$\therefore \bigcap_{n=1}^{\infty} k_n$ is non-empty.

* Theorem : Let k be a positive integer if $\{I_n\}$ is a sequence of k-cell such that $I_n \supset I_{n+1}$ $(n=1,2,3,\dots)$ then $\bigcap_{n=1}^{\infty} I_n$ is not empty.

proof : Let k be a positive integer if $\{I_n\}$ is a sequence of k-cell such that $I_n \supset I_{n+1}$ $(n=1,2,3,\dots)$

claim : $\bigcap_{n=1}^{\infty} I_n \neq \phi$

Let $I_n = \{x=(x_1,x_2,\dots,x_k) \mid a_{n_i} \le x_i \le b_{n_i} \ \forall\ 1<i<k\}$

$I_n = [a_{n_1}, b_{n_1}] \times [a_{n_2}, b_{n_2}] \times \dots \times [a_{n_k}, b_{n_k}]$

$n=1,2,\dots$

Write $I_{n_i} = [a_{n_i}, b_{n_i}]$, $i = 1, 2, \dots k$

Since $I_n \supset I_{n+1}$, $n = 1, 2, 3, \dots$

It follows that $I_{n_i} \supseteq I_{n+1, i}$, $n = 1, 2, 3, \dots$

By known theorem,

$$\bigcap_{n=1}^{\infty} I_{n,i} \neq \phi \quad \forall\ i = 1, 2, \dots k$$

$$\Rightarrow \exists\ x_i \ni x_i \in \bigcap_{n=1}^{\infty} I_{n,i} \quad \forall\ i = 1, 2, 3, \dots k$$

write $x_i = (x_1, x_2, \dots x_k)$

then

$$x = (x_1, x_2, \dots x_k)$$

$$\in \bigcap_{n=1}^{\infty} I_{n,1} \times \bigcap_{n=1}^{\infty} I_{n,2} \times \dots \times \bigcap_{n=1}^{\infty} I_{n,k}$$

$$= \bigcap_{n=1}^{\infty} (I_{n,1} \times I_{n,2} \times \dots \times I_{n,k})$$

$$= \bigcap_{n=1}^{\infty} I_n$$

$$x \in \bigcap_{n=1}^{\infty} I_n$$

Thus, $\bigcap_{n=1}^{\infty} I_n \neq \phi$

***

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*Theorem : Imp

Statement : Prove that every k-cell is compac

proof: Let I be a k-cell consisting of all poin

$x = (x_1, x_2, \dots x_k) \ni a_j \leq x_j \leq b_j \ (1 \leq j \leq k)$

i.e, $I = \{ x = (x_1, x_2, \dots x_k) \mid a_j \leq x_j \leq b_j ,\ 1 \leq j \leq k$

put $\delta = \left(\sum_{j=1}^{k} (b_j - a_j)^2 \right)^{1/2}$

Then for any $x, y \in I$

We have $x = (x_1, x_2, x_3, \ldots, x_k)$

$y = (y_1, y_2, y_3, \ldots, y_k)$

where $a_j \le x_j,\ y_j \le b_j \ \forall\ j = 1, 2, 3, \ldots k$

$$|x-y| = \left(\sum_{j=1}^{k} (x_j - y_j)^2\right)^{1/2}$$

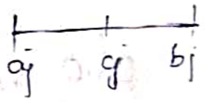
$$\le \left(\sum_{j=1}^{k} (b_j - a_j)^2\right)^{1/2}$$

for any $x, y \in I$, we have $|x-y| \le \delta \rightarrow$ (1)

Suppose I is not compact

Then $\exists$ an open cover $\{G_\alpha\}$ of I which contains no finite subcover

put $c_j = \frac{a_j + b_j}{2}$ $(1 \le j \le k)$



Then by using $[a_j, c_j]$ and $[c_j, b_j]$

We get 2^k number of k-cells, whose union is I.

Among these k-cells atleast one k-cell is not covered by any finite subfamily of $\{G_\alpha\}$ [otherwise, I would be covered by a finite subfamily of G_α]

Let I_1 be such a k-cell

Then $I_1 \subset I$ and

$d(x,y) \le \frac{1}{2}\delta,\ \forall\ x, y \in I_1$

and no subfamily of $\{G_\alpha\}$ cover I_1

Repeat the above process with I_1 in place of I, to get a k-cell, I_2 contains I_1 $\ni$ $I_2 \subset I_1$

$d(x,y) \leq \frac{1}{2^2}\delta, \ \forall \ x,y \in I_2$

and no finite subfamily of $\{G_\alpha\}$ cover I_2

continuing the process, we get k-cells

$I_1, I_2, I_3, \dots$

Satisfying the following:

(a) $I \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots\dots$

(b) $d(x,y) \leq \frac{1}{2^m}\delta, \ \forall \ x,y \in I_m$

(c) No finite subfamily of $\{G_\alpha\}_{\alpha\in\Delta}$ cover I_m, for any I.

By known theorem $\bigcap_{m=1}^{\infty} I_m \neq \phi$ [I_m k-sequence of compact set]

choose an element, $x_0 \in \bigcap_{m=1}^{\infty} I_m$, then $x_0 \in I_m, \ \forall m$

since $I_m \subset I, \ \forall m$.

$\Rightarrow x_0 \in I \subseteq \cup G_\alpha$ [$\because \{G_\alpha\}$ is open cover of I]

$\Rightarrow x_0 \in G_\alpha$, for some $\alpha \in \Delta$

Since G_α is open [open cover]

$\Rightarrow x_0$ is an interior point of G_α

$\Rightarrow \exists \ r > 0 \ni N_r(x_0) \subseteq G_\alpha$, for some $\alpha \in \Delta$.

Now choose sufficiently large m

$\ni \frac{1}{2^m}\delta < r$

Then for any $y \in I_m$, we have

$d(x_0,y) \leq \frac{1}{2^m}\delta < r$

$\Rightarrow d(x_0,y) < r$

$\Rightarrow y \in N_r(x_0) \subseteq G_\alpha$, for some $\alpha \in \Delta$

$\therefore I_m \subseteq G_\alpha$, for some $\alpha \in \Delta$

which is a contradiction to (c)

Thus, there must be a finite subcover $\{G_\alpha\}_{\alpha \in \Delta}$ for I

$\therefore$ I is compact

Hence, every k-cell is compact.

Imp

** Statement and Proven Heine-Borel theorem:

Statement :- Let $E \subset R^k$ then the following statements are equivalent.

(i) E is closed and bounded.

(ii) E is compact

(iii) Every infinite subset of E has a limit point in E

proof: Let $E \subset R^k$

(i) $a \Rightarrow b$:

Suppose E is closed and bounded

claim :- E is compact

Since E is bounded, $\exists M > 0 \ni d(x, y) \leq M \ \forall x, y \in E$

choose $a = (a_1, a_2, \ldots a_k) \in E$

for any $x = (x_1, x_2, \ldots x_k) \in E$

We have $|a_i - x_i| \leq \left(\sum_{i=1}^{k} (a_i - x_i)^2\right)^{1/2}$

$= d(a, x)$

$\leq M$

$\therefore |a_i - x_i| \leq M, \ \forall i = 1, 2, 3, \ldots k$

$\Rightarrow a_i - M \leq x_i \leq a_i + M, \ \forall i = 1, 2, 3, \ldots k$

put $b = (a_1 - M, a_2 - M, \dots \dots a_k - M)$

$c = (a_1 + M, a_2 + M, \dots \dots a_k + M)$

$\therefore E \subseteq \{ x = (x_1, x_2, \dots x_k) \in R^k \mid a_i - M \le x_i \le a_i + M,$

$\forall\, i = 1, 2, \dots k \}$

a k-cell

Since every k-cell is compact and closed subset of a compact set is compact

$\Rightarrow$ E is compact $\quad (\because E$ is closed$)$

Hence (a) $\Rightarrow$ (b).

(ii) (b) $\Rightarrow$ (c) :-

Suppose E is compact.

claim :- Every infinite subset of E has a limit point of E.

Let A be an infinite subset of E

By known theorem,

A has a limit point in E

Hence (b) $\Rightarrow$ (c)

(iii) (c) $\Rightarrow$ (a) :-

Suppose every infinite subset of E has a limit point in E.

claim :- E is closed and bounded

Suppose E is not closed then there exists a limit point x_0 of E $\ni x_0 \notin E$

for each $n \in Z^+$, choose $a_n \in E$ such that $d(x_0, a_n) < 1/n$

This is possible since $N_{1/n}(x_0) \cap E \neq \phi$

Write $S = \{ a_n \mid n \in Z^+ \}$

then $S \subseteq E$ and S is infinite [otherwise, there is an element $a \in E$ which equally to infinitely many n's, so that

$d(x_0, a) = 0$

$\Rightarrow x_0 = a \in E$. [which is a contradiction]].

by (c), S has a limit point in E

Let a_0 be the limit point of S in E

Then $a_0 \neq x_0$ ($a_0 = x_0$, $a_0 \in E \Rightarrow x_0 \in E$)

put $r = \frac{1}{2} d(x_0, a_0) = 0$

Then $N_r(x_0) \cap N_r(a_0) = \phi$

choose $n \ni \frac{1}{n} < r$

then for any $m \geq n$

We have $d(x_0, a_m) < \frac{1}{m} < \frac{1}{n} < r$

$\Rightarrow a_m \in N_r(x_0) \ \forall \ m > n$ $\quad$ [$\because N_r(x_0) \cap N_r(a_0) = \phi$]

$\Rightarrow a_m \notin N_r(a_0) \ \forall \ m \geq n$

$\Rightarrow a_m \in N_r(a_0)$ for $m = 1, 2, \ldots (n-1)$

i.e, $N_r(a_0)$ contains finitely many points of S.

which is a contradiction to S is infinite

$\therefore E$ is closed

Suppose E is not bounded

for any $a_0 \in E$

Then for each $n \in Z^+$, there exists $a_n \in E$

such that $d(a_n, a_0) > n$

$\Rightarrow$ The set $\{a_n \mid n \in Z^+\}$ is an infinite subset of E which has no limits in E.

Which is a contradiction to c

$\therefore$ E is bounded

Hence (c) => (a)

***

Theorem (Weierstran theorem) : Imp

Statement : Every bounded infinite subset of R^k has a limit point in R^k.

proof : (b) => (c) from Heine-Borel theorem

Let E be a bounded infinite subset of R^k.

Then E is contained in some k-cells I (say)

Since every k-cell is compact

=> I is compact

$\therefore$ E is an infinite subset of a compact set I

By Heine Borel theorem,

E has a limit point of I and

$\therefore$ Hence in R^k.

* Connected Sets :-

→ Seperated sets :- Two subsets A and B of a metric space X are said to be seperated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty. i.e, No point of A lies in the closure of B and no point of B lies in the closure of A.

→ Connected sets :- A set $E \subseteq X$ is said to be connected if E is not an union of two non-empty seperated sets.

* Note :-

→ Seperated sets are necessarily disjoint but disjoint sets need not be seperated

*** Theorem :-** A subset E of a real line R^1 is connected iff it has the following property

If $x \in E$, $y \in E$ and $x < y < z \ni z \in E$

proof: Suppose the given property is not satisfied then $\exists$ $x \in E$, $y \in E$ and $x < z < y \ni z \notin E$

We have to prove that E is not connected.

put $A = E \cap (-\infty, z)$ and $B = E \cap (z, \infty)$

Since $x < z$ and $x \in E$

$\Rightarrow x \in (-\infty, z)$ and $x \in E$

$\Rightarrow x \in E \cap (-\infty, z)$ ~~and $x \in E$~~

$\Rightarrow x \in E \cap (-\infty, z) = A$

$\Rightarrow x \in A$

$\therefore A \neq \phi$

Since $z < y$ and $y \in E \Rightarrow y \in (z, \infty)$ and $y \in E$

$\Rightarrow y \in E \cap (z, \infty) = B$

$\Rightarrow y \in B \quad \therefore B \neq \phi$

Since $A = E \cap (-\infty, z) \subset (-\infty, z)$

$B = E \cap (z, \infty) \subset (z, \infty)$

$\therefore A \subset (-\infty, z)$ and $B \subset (z, \infty)$

$\bar{A} \subset \overline{(-\infty, z)}$ and $\bar{B} \subset \overline{(z, \infty)}$

$\bar{A} \subset (-\infty, z]$ and $\bar{B} \subset [z, \infty)$

Also $\bar{A} \cap B \subset (-\infty, z] \cap B$

$= (-\infty, z] \cap E \cap (z, \infty)$

$= E \cap \phi$

$= \phi$

$\therefore \bar{A} \cap B = \phi$

Also $A \cap \bar{B} \subset A \cap [z, \infty)$

$= E \cap (-\infty, z) \cap [z, \infty)$

$= E \cap \phi$

$\therefore A \cap \bar{B} = \phi$

$\therefore$ A and B are non-empty seperated sets

Now $A \cup B = (E \cap (-\infty, z)) \cup (E \cap (z, \infty))$

$= E \cap ((-\infty, z) \cup (z, \infty))$

$= E \cap R$

$A \cup B = E$

$\therefore$ E is not connected.

Conversely suppose that E is not connected then $E = A \cup B$, where A and B are two non-empty seperated sets

i.e, $A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$

Since A and B are non-empty

We can choose $x \in A$ and $y \in B$ and we can

choose without loss of generality $x<y$
put $z = \sup(A \cap [x, y])$ then $x \le z \le y$
By known theorem,

$z \in \overline{A \cap [x,y]} \subset \bar{A}$

$\Rightarrow z \in \bar{A}$

Since $\bar{A} \cap B = \phi$ and $z \in \bar{A}$

$\Rightarrow z \notin B$

We have $x \le z < y$

If $z \notin A$ then $x < z < y$

and hence $z \notin A \cup B = E$

hence $z \notin E$

So, that stated property doesnot hold

If $z \in A$ then $z \notin \bar{B}$

$\Rightarrow z \in \bar{B}$, $z < y$

Hence $\exists\, z_1 \ni z < z_1 < y$ and $z_1 \notin B$

then $x < z_1 < y$ and $z_1 \notin E$

Hence A subset E of a real line R^1 is connected iff it has the following property
if $x \in E$, $y \in E$ and $x < y < z \ni z \in E$

[Unit - II] Numerical Sequence and Series

* Convergent sequence :- A sequence $\{P_n\}$ in a metric space (X, d) is said to be a convergent sequence if there is a point $P \in X$ with the following property for every $\epsilon > 0$ there is an integer N $\ni$ $n \geq N \Rightarrow d(P_n, P) < \epsilon$. In this case We also say that $\{P_n\}$ converges to P (or) P is the limit point of sequence P_n.

→ If $\{P_n\}$ ~~doesn't~~ is not convergent, then we say that the sequence is a divergent sequence.

We know that the range of the sequence P_n is $\{P_n \mid n \geq 1\}$

If this range is a bounded set, then the sequence is said to be a bounded sequence.

Example:

a) Consider $X = \mathbb{R}$ with usual metric, Write $S_n = \frac{1}{n}$ for $n \in N$, then the sequence converges to "0".
Range is infinite. Since $0 \leq \frac{1}{n} \leq 1$, we have the range is bounded. $\frac{1}{n} = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\}$ $0 < \frac{1}{n}$
1, 0.5, 0.3 0

b) Write $S_n = n^2$ for $n \in N$, then the sequence is unbounded, divergent and having Infinite range

c) $S_n = \frac{1 + (-1)^n}{n}$, then the sequence converges, ~~to~~
Here the range is infinite but we have that the range is bounded. $0 \leq S_n \leq 2$

d) Let $i^2 = (-1)$, write $S_n = i^n$ for $n \in N$, the sequence is divergent. the range is $\{1, -1, i, -i\}$ which is finite and bounded. i, i^2, i^3, i^4, i^5

e) Write $S_n = 1$ for each $n \in N$, then the sequence converges to "1", the range is "1". Here the range is finite & bounded.

* Compact Metric space :- Let E be a subset of a metric space. A family $\{G_\alpha\}_{\alpha \in \Delta}$ of opensets X is called opencover of E. If $E \subseteq \bigcup G_\alpha$ Compact metric space. A subset k of a metric space X is said to be compact if every open cover of k contains a finite subcover of k, then there exists finitely many indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots \alpha_n$ in Δ such that $k \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$

* Theorem :- Let $\{P_n\}$ be a sequence in a metric space

(a) $\{P_n\}$ converges to $P \in X$, iff every neighbourhood of P contains all but finitely many of the term of $\{P_n\}$

(b) If $P \in X$, $P' \in X$ and if $\{P_n\}$ converges to P and to p' then $p' = p$

(c) If $\{P_n\}$ converges then $\{P_n\}$ is bounded.

(d) If $E \subset X$ and 'P' is a limit point of E then there is a sequence $\{P_n\}$ in E such $P = \lim_{n \to \infty} P_n$

proof: (a) $P_n \to P$

Consider a neighbourhood $V = \{q \mid d(p,q) < \epsilon\}$ of P. Since $\epsilon > 0$ by definition of a convergent sequence there exists N, such that $n \geq N$ implies $d(P_n, P) < \epsilon$.

Therefore, $n \geq N$ implies $P_n \in V$

Conversely suppose that every neighbourhood of p contains all but ~~infinitely~~ finitely many of the terms of $\{p_n\}$ for $\epsilon > 0$

Consider the neighbourhood $V = \{q \mid d(p,q) < \epsilon\}$

By conversely hypothesis, all but a finite no. of points except $p_{i_1}, p_{i_2}, \ldots, p_{i_m}$ are in V.

Write $n = \max\{i_1, i_2, \ldots, i_m\}$

Then for any $n \geq n+1$ the element $p_n \in V$

Therefore for each $n \geq n+1$, [we have $d(p_n, p) <$ the element $p_n \in V$.

$\therefore$ for each $n \geq N+1$, we have $d(p_n, p) < \epsilon$

Thus, $\{p_n\}$ converges to p.

(b) Suppose $p' \neq p$

then by the definition of metric

$$\epsilon = d(p, p') > 0$$

Since $\frac{\epsilon}{2} > 0$, $p_n \to p'$ $\exists$ an integer N_2 such that

$$d(p_n, p') < \epsilon/2 \quad \forall\, n \geq N_2$$

Since $\frac{\epsilon}{2} > 0$, $p_n \to p$ $\exists$ an integer N_1 such that

$$d(p_n, p) < \epsilon/2 \quad \forall\, n \geq N_1$$

Write $N = \max\{N_1, N_2\}$

Now, for any $n \geq N$

We have $d(p_n, p) < \epsilon/2$, $d(p_n, p') < \epsilon/2$

$$\epsilon = d(p, p') \leq d(p, p_n) + d(p_n, p')$$

$$< \epsilon/2 + \epsilon/2$$

$$= \epsilon$$

Thus it is a contradiction $\therefore$ $\boxed{p = p'}$

(c) Suppose $\{P_n\}$ converges.

claim :: $\{P_n\}$ is bounded.

Recall a set E is bounded if there exi.. positive real number such that and a point $q \in X$, ~~such~~ that $d(x,q) < M$ for all $x \in E$

Suppose that $\{P_n\}$ converges to $P \in X$

since $\epsilon = 1 > 0$, there exists an integer N

$\ni d(P_n, P) < 1 \quad \forall n \geq N$

put $M = \max\{1, d(P_1,P), \ldots, d(P_N, P)\}$

Now for each $n \geq 1$, we have $d(P_n, P) \leq M$

Thus, the set $E = \{P_n \mid n \geq 1\}$ is bounded.

$\therefore$ The $\{P_n\}$ is bounded.

(d) Let $E \subset X$ and P is a limit point of E. We have to prove that there is a sequence $\{P_n\}$ of E such that

$P = \lim_{n\to\infty} P_n$.

Given that P is a limit point of E.

for every neighbourhood of P containing a point $q \in E$ with $q \neq P$.

For each $n \in \mathbb{N}$, consider the neighbourhood of P with radius $\frac{1}{n}$.

Let $P_n \in E$ such that $P_n \neq P$ & $d(P_n, P) < \frac{1}{n}$

Consider the sequence $\{P_n\}$

To show that $P_n \to P$

Let $\epsilon > 0$, $\frac{1}{n} \to 0$ a positive integer $N \ni \epsilon > \frac{1}{N} > 0$

Now for any $n \geq N$, we have that

$d(P_n, P) < \frac{1}{n} \leq \frac{1}{N} < \epsilon \quad \therefore P_n \to P$

$\Rightarrow \lim_{n\to\infty} P_n = P$

* **Theorem** :: Suppose $\{s_n\}$, $\{t_n\}$ are complex sequences and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$ then

a) $\lim_{n\to\infty}(s_n + t_n) = s+t$ Imp

Proof: Let $\epsilon > 0$

Since $\lim_{n\to\infty} s_n = s$ there exists an integer $N_1 \ni$ $d(s_n, s) < \epsilon/2 \ \forall\ n \geq N$.

Since $\lim_{n\to\infty} t_n = t$ there exists an integer $N_2 \ni$ $d(t_n, t) < \epsilon/2 \ \forall\ n \geq N$.

Write $N = \max\{N_1, N_2\}$

then for any $n \geq N$, we have

$$d(s_n + t_n, s+t) = |(s_n + t_n) - (s+t)|$$
$$= |(s_n - s) + (t_n - t)|$$
$$= d(s_n, s) + d(t_n, t)$$
$$< \epsilon/2 + \epsilon/2$$
$$< \epsilon$$

$\therefore d(s_n + t_n, s+t) < \epsilon \ \forall\ n \geq N$.

Hence $\lim_{n\to\infty}(s_n + t_n) = s+t$.

(b) $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty}(c + s_n) = c+s$ for any number

Proof: Let $\epsilon > 0$

Now consider $\frac{\epsilon}{|c|} > 0$

Since $\lim_{n\to\infty} s_n = s$ then there exists an integer $N \ni$

$d(s_n, s) < \frac{\epsilon}{|c|} \ \forall\ n \geq N$.

Now. for any $n \geq N$

Consider, $d(cs_n, cs) = |cs_n - cs|$

$$= |c(s_n - s)|$$
$$= |c||(s_n - s)|$$
$$< |c| \frac{\epsilon}{|c|}$$
$$< \epsilon$$

$\therefore d(cs_n, cs) < \epsilon \quad \forall\, n \geq N.$

Thus, $\lim_{n\to\infty} cs_n = cs$

Let $\epsilon > 0$

Since $\lim_{n\to\infty} s_n = s$, there exists an integer $N \ni$

$d(s_n, s) < \epsilon \quad \forall\, n \geq N.$

Now for any $n \geq N.$

Consider $d(c+s_n, c+s) = |c+s_n - (c+s)|$

$$= |c+s_n - c - s|$$
$$= |s_n - s|$$
$$= d(s_n, s)$$
$$< \epsilon$$

$\therefore d(c+s_n, c+s) < \epsilon$

Thus, $\lim_{n\to\infty} c+s_n = c+s$

(c) $\lim_{n\to\infty} s_n \cdot t_n = st$

Proof: We use the identity

$s_n t_n - st = (s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t)$

Given that $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$

Since all these are complex numbers

$\lim_{n\to\infty} (s_n - s) = 0$, $\lim_{n\to\infty} (t_n - t) = 0$

Since, $\lim_{n\to\infty} (t_n - t) = 0$ and s is constant

By (b), we get $\lim_{n\to\infty} st_n = st$

$$\lim_{n\to\infty} st_n - st = 0$$

$$\lim_{n\to\infty} s(t_n - t) = 0$$

Similarly $\lim_{n\to\infty} t(s_n - s) = 0$

Therefore by (a) we get

$\lim_{n\to\infty} (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s) = 0$

$$\Rightarrow \lim_{n\to\infty} (s_n t_n - st) = 0$$

$$\lim_{n\to\infty} s_n t_n = st$$

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, provided $s_n \neq 0$ $(n = 1, 2, 3, \ldots)$ &

$s \neq 0$

proof: Given $s \neq 0$, $s_n \neq 0$, for all $n \geq 1$

Since $\frac{1}{2}|s| > 0$ there exists an integer N,

$\ni |s_n - s| < \frac{1}{2}|s| \; \forall \; n \geq N$,

Therefore, $|s_n| > \frac{1}{2}|s| \Rightarrow \frac{1}{|s_n|} < \frac{2}{|s|}$

Let $\epsilon > 0$

Now, since $\frac{1}{2}|s^2|\epsilon > 0$, there exists an integer N_2, such that

$|s_n - s| < \frac{1}{2}|s^2|\epsilon \; \forall \; n \geq N_2$

$N = \max\{N_1, N_2\}$

for any $n \geq N$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s \cdot s_n}\right| = |s - s_n| \frac{1}{|s_n \cdot s|}$$

$$= |s - s_n| \cdot \frac{1}{|s_n|} \cdot \frac{1}{|s|}$$

$$< |s - s_n| \cdot \frac{2}{|s|} \cdot \frac{1}{|s|}$$

$$< |s - s_n| \frac{2}{|s^2|}$$

$$< \frac{1}{2}|s^2| \epsilon \frac{2}{|s^2|}$$

$$< \epsilon$$

therefore $\left|\frac{1}{s_n} - \frac{1}{s}\right| < \epsilon$ for all $n \geq N$.

Thus $\lim_{n \to \infty} \frac{1}{s_n} = \frac{1}{s}$

* Theorem :

(a) Suppose $x_n = R^k$ $(n = 1, 2, \dots)$ and $x_n = (\alpha_{1,n}, \dots \alpha_{k,n})$ then $\{x_n\}$ converges to x $(\alpha_1, \alpha_2, \dots \alpha_k)$ iff $\lim_{n \to \infty} \alpha_{j,n} = \alpha_j$ $(1 \leq j \leq k)$.

(b) Suppose $\{x_n\}, \{y_n\}$ are the sequences in R^k, $\{\beta_n\}$ is a sequence of real numbers and $x_n \to x$, $y_n \to y$, $\beta_n \to \beta$ then $\lim_{n \to \infty} (x_n + y_n) = x + y$, $\lim_{n \to \infty} x_n y_n = xy$, $\lim_{n \to \infty} \beta_n x_n = \beta x$

proof : (a) Suppose $\{x_n\}$ converges to x.

Fix j $(1 \leq j \leq k)$

To show that $\lim_{n \to \infty} \alpha_{j,n} = \alpha_j$

Take $\epsilon > 0$

Since $x_n \to x$, $\exists$ an integer $N \ni |x_n - x| < \epsilon \ \forall \ n \geq N$

$\epsilon > |x_n - x| \ \forall \ n \geq N$

$$\epsilon > |x_n - x| = |(\alpha_{1,n}, \ldots \alpha_{k,n}) - (\alpha_1, \ldots \alpha_k)|$$

$$= \left(\sum_{j=1}^{k} (\alpha_{j,n} - \alpha_j)^2 \right)^{1/2}$$

$$> |\alpha_{j,n} - \alpha_j|$$

$$\therefore \epsilon > |\alpha_{j,n} - \alpha_j|$$

$$\therefore |\alpha_{j,n} - \alpha_j| < \epsilon$$

then $\alpha_{j,n} \to \alpha_j$

This is true for all j with $1 \leq j \leq k$

Conversely suppose that $\alpha_{j,n} \to \alpha_j \ \forall \ j \ (1 \leq j \leq k)$

To show that $x_n \to x$

Let $\epsilon > 0$

Consider $\frac{\epsilon}{\sqrt{k}} > 0$

Since $\alpha_{j,n} \to \alpha_j$, $\exists$ an integer $N_j \ni |\alpha_{j,n} - \alpha_j| < \frac{\epsilon}{\sqrt{k}} \ \forall \ n \geq N_j$

Take $N = \max\{N_1, N_2, \ldots N_k\}$

for all $n \geq N$

$$|x_n - x| = |(\alpha_{1,n}, \ldots \alpha_{k,n}) - (\alpha_1, \ldots \alpha_k)|$$

$$= \left(\sum_{j=1}^{k} (\alpha_{j,n} - \alpha_j)^2 \right)^{1/2}$$

$$< \sum_{j=1}^{k} \left(\left(\frac{\epsilon}{\sqrt{k}} \right)^2 \right)^{1/2}$$

$$< \left(\sum_{j=1} \left(\frac{\epsilon^2}{k} \right) \right)^{1/2}$$

$$= \left(k \frac{\epsilon^2}{k} \right)^{1/2} = (\epsilon^2)^{1/2}$$

$$< \epsilon$$

$$|x_n - x| < \epsilon, \forall n \geq N$$

$$\therefore x_n \to x$$

(b) Suppose $x_n = (\alpha_{1,n}, \dots \alpha_{k,n})$

$$y_n = (\beta_{1,n}, \dots \beta_{k,n})$$

$x = (\alpha_1, \alpha_2, \dots \alpha_k)$, $y = (\beta_1, \beta_2, \dots \beta_k)$

Given that $x_n \to x$, $y_n \to y$

Therefore, $\alpha_{j,n} \to \alpha_j$, $\beta_{j,n} \to \beta_j$, $1 \leq j \leq k$

By known theorem,

$$(\alpha_{j,n} + \beta_{j,n}) \to \alpha_j + \beta_j \text{ \& } (\alpha_{j,n} \cdot \beta_{j,n}) \to \alpha_j \cdot \beta_j$$

$$x_n + y_n = (\alpha_{1,n}, \dots \alpha_{k,n}) + (\beta_{1,n}, \dots \beta_{k,n})$$

$$= ((\alpha_{1,n} + \beta_{1,n}), \dots (\alpha_{k,n} + \beta_{k,n})) \to ((\alpha_1 + \beta_1) + \dots \dots (\alpha_k + \beta_k))$$

$$= x + y$$

$$\therefore x_n + y_n = x + y$$

$$x_n y_n = (\alpha_{1,n}, \dots \alpha_{k,n})(\beta_{1,n}, \dots \beta_{k,n})$$

$$= ((\alpha_{1,n}, \beta_{1,n}) \dots (\alpha_{k,n}; \beta_{k,n})) \to ((\alpha_1 \dots \alpha_2) \dots \dots (\beta_1 \dots \beta_k))$$

$$= xy$$

$$\therefore x_n y_n = xy$$

$\therefore \{\beta_n\}$ is a sequence of real numbers such that $\beta_n \to \beta$ & $\alpha_{j,n} \to \alpha_j$, for $1 \leq j \leq k$

By known theorem

$$\beta_n \alpha_{j,n} \longrightarrow \beta \alpha_j \text{ for } 1 \le j \le k$$

$$(\beta_n \alpha_{1,n}, \dots, \beta_n \alpha_{k,n}) \longrightarrow (\beta\alpha_1, \dots \beta\alpha_k)$$

$$\beta_n (\alpha_{1,n} \dots \alpha_{k,n}) \longrightarrow \beta(\alpha_1, \dots \alpha_k)$$

$$\therefore \beta_n x_n \longrightarrow \beta x$$

$$\therefore \lim_{n\to\infty} x_n + y_n = x + y, \lim_{n\to\infty} x_n y_n = xy, \lim_{n\to\infty} \beta_n x_n = \beta x$$

* Subsequence :-

Def: Given sequence $\{P_n\}$ consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < \dots$ then the sequence $\{P_{n_i}\}$ is called subsequence of P_n.

If $\{P_{n_i}\}$ converges, its limit is called a sub-sequential limit of $\{P_n\}$

Examples: Define $x_n = (-1)^n$ for $n \in \mathbb{N}$

Write $n_i = 2i$ for $i \in \mathbb{N}$.

then $n_1 = 2 < n_2 = 4 < \dots$

Consider, $x_{n_i} = (-1)^{n_i} = (-1)^2 = 1 = 1$

So the sequences $\{x_{n_i}\}$ converges to 1

In the average example given here, 1 is not limit of $\{x_n\}$

But it is a subsequential limit of $\{x_n\}$

then for divergent sequences, subsequential limits may exist.

* Theorem :- Let Sequence $\{P_n\}$ be a Sequences then sequences $\{P_n\}$ converges to P iff if every subsequences of $\{P_n\}$ converges to P.

Proof :- Suppose $\{P_n\}$ converges to P.

Let $\{P_{n_i}\}$ be the subsequence of $\{P_n\}$.

To show that $P_{n_i} \longrightarrow P$

Take $\epsilon > 0$

Since $P_n \rightarrow P$, then there exists an integer N

$d(P_n, P) < \epsilon \quad \forall \; n \geq N$.

Now, $n_1 < n_2 < \dots$ & n is a fixed numbers

there exists 'k' such that $n < n_k$

for any, $t \geq k$ We have $n_t \geq n_k \geq n$ and

~~there~~ thus $d(P_{n_t}, P) < \epsilon \; \forall \; n_t \geq n$

$\therefore P_{n_i} \rightarrow P$

Conversely, suppose that every subsequence of $\{P_n\}$ converges to P.

Since $\{P_n\}$ itself is a subsequence of $\{P_n\}$

$\therefore P_n \rightarrow P$.

*****

* Theorem :- Imp

(a) If $\{P_n\}$ is a sequence in a compact metric space X, then some subsequences of $\{P_n\}$ converges to a point of X.

(b) Every bounded sequences in R^k contains a covergent subsequences.

Proof :- We use the known result.

i.e, Every Infinite subset of a compact set K

has a limit point in K.
Write $E = \{P_i \mid i \in N\}$ the range of the sequences

case (i): Suppose E is finite
Then there exists $p \in E$ which occurs infinite tim

$n_1 < n_2 < \dots\dots$

$P_{n_1} = P_{n_2} = \dots\dots = P$

then $\{P_{n_i}\}$ is a subsequence of $\{P_n\}$ &
This subsequences converges to P.

case (ii): Suppose E is infinite
Since X is compact, E has a limit point p in X
[choose n_1 such that $d(P, P_{n_1}) < 1$
choose n_2 such that $d(P, P_{n_2}) < \frac{1}{2}$ & $n_1 < n_2$.
After choosing $n_1, n_2, \dots n_{i-1}$ we choose $n_i > n_{i-1}$

such that $d(P, P_{n_i}) < \frac{1}{i}$

clearly $\{P_{n_i}\}$ is a subsequence of $\{P_n\}$
To show that $P_{n_i} \to P$

Let $\epsilon > 0$
Now, there exists an integer t
such that $\epsilon > \frac{1}{t} > 0$

Now, for any $i \geq t$ We have $\frac{1}{i} \leq \frac{1}{t}$

$d(P_{n_i}, P) < \frac{1}{i} \leq \frac{1}{t} < \epsilon$

$\therefore d(P_{n_i}, P) < \epsilon$

$\therefore P_{n_i} \to P$

$P_n \longrightarrow P$]

(b) We use a known theorem,

Every bounded infinite subset of R^k has a limit point in R^k

Write $E = \{P_i \mid i \in N\}$

If E is finite then as in the case (i) of above a, we get the result.

If E is infinite then by weistress theorem, E has a limit point p.

Now, the proof is as it is in the case (ii) of (a)

Theorem :: The subsequence limits of a sequence $\{P_n\}$ in a metric space X ~~from~~ form a closed subset of X. Imp

proof :: Let E^* be the set of all subsequential limits of $\{P_n\}$.

Let q be a limit point of E^*.

Now we have to show that $q \in E^*$

choose n_i such that $P_{n_i} \neq q$ and put $\delta = d(q, P_{n_i}) > 0$

Now;

$$2^{-i}\delta > 0$$

Since 'q' is a midpoint of E^*, there exist $x \in E^*$

$$d(x, q) < 2^{-i}\delta$$

Since $x \in E^*$ is a subsequential limit there exists $n_2 > n_1$, such that

$$d(x, P_{n_2}) < 2^{-i}\delta$$

Now $d(q, P_{n_2}) \leq d(q, x) + d(x, P_{n_2}) < 2^{-i}\delta + 2^{-i}\delta$

$$= 2^{1-i}\delta$$

In some way, after selecting select P_{n_i} So that $P_{n_1}, \dots, P_{n_{i-1}}$ so that $n_i > n_{i-1}$ & $d(q, P_{n_i}) < 2^{1-i}\delta$

Now, clearly that the subsequence $\{P_{n_i}\}$ conver to q.

$\therefore q \in E^*$

$\therefore E^*$ is closed.

* Cauchy Sequence :-

Def :

* A Sequence $\{P_n\}$ in a metric space 'X' is sai to be a cauchy sequence if for every $\epsilon > 0$, there i an integer 'N' such that $d(P_n, P_m) < \epsilon$. if $n \geq N$ and $m \geq N$.

* Let E be a subset of a metric space 'X' and Let 's' be a set of all real numbers of the form $d(p, q)$ with $p \in E$ and $q \in E$, 'sup of s' is called the diameter of E.

$\text{diam } E = \sup\{d(p,q)\}$

Note :-

(a) $\{P_n\}$ is a sequence in X and $E_n = \{P_m / m \geq n$ then $\{P_n\}$ is a cauchy sequence iff $\lim_{n \to \infty} \text{diam } E_n =$

(b) If $A \subseteq B$ then $\text{diam } A \leq \text{diam } B$.

proof :

(a) Suppose $\{P_n\}$ is a cauchy sequence

Write $S_n = \text{diam } E_n$ for $n \in N$

Now, it is enough to show that $S_n = \text{diam } E_n \to$

Let $\epsilon > 0$

Since $\{P_n\}$ is a cauchy sequence

there is an integer, k, such that

$d(P_n, P_m) < \epsilon \ \forall\ n \geq k,\ m \geq k$ $\quad d(P_n, P_m) < \epsilon$

Now for any $n \geq k$, $m \geq k$ $\quad \sup\{d(P_n, P_m)\} < \epsilon$

$S_n < \epsilon$

Now $S_n = \text{diam}\, E_n = \sup\{d(P_n, P_m) / m \geq k, n \geq k\}$

$S_n - 0 < \epsilon$

$\therefore S_n = \text{diam}\, E_n \to 0$ $\quad \text{diam}\, E_n - 0 < \epsilon$

Conversely suppose $\lim\limits_{n\to\infty} \text{diam}\, E_n = 0$ $\quad \text{diam}\, E_n \to 0$

To show that $\{P_n\}$ is a cauchy sequence.

Let $\epsilon > 0$

Since $\text{diam}\, E_n \to 0$, there exists a positive integer $k \ni \text{diam}\, E_n < \epsilon\ \forall\ n \geq k$

Now $\text{diam}\, E_k < \epsilon$

for any $n \geq k$, $m \geq k$

We have $P_n, P_m \in E_k \Rightarrow d(P_n, P_m) \leq \sup\{d(P_i, P_j) / P_i, P_j \in E_k\}$

$= \text{diam}\, E_k < \epsilon$

$\therefore d(P_n, P_m) < \epsilon\ \forall\ n \geq k,\ m \geq k$

$\therefore \{P_n\}$ is a cauchy sequence.

(b) Suppose $A \subseteq B$

$\Rightarrow \{d(p,q) / p, q \in A\} \subseteq \{d(p,q) / p, q \in B\}$

$\sup\{d(p,q) / p, q \in A\} \subseteq \sup\{d(p,q) / p, q \in B\}$

$\Rightarrow \text{diam}\, A \leq \text{diam}\, B$.

* Theorem :-

(a) If $\overline{E}$ is the closure of a set E in a metric space X, then $\text{diam}\, \overline{E} = \text{diam}\, E$

(b) If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ ($n = 1, 2, 3, \ldots$) & if $\lim\limits_{n\to\infty} \text{diam}\, E_n = 0$ then $\bigcap\limits_{n=1}^{\infty} K_n$ consists of exactly point

Proof : $\operatorname{diam} \bar{E} \leq \operatorname{diam} E$

$\operatorname{diam} E \leq \operatorname{diam} \bar{E}$

Since $E \subseteq \bar{E}$, we have $\operatorname{diam} E \leq \operatorname{diam} \bar{E}$

Let $\epsilon > 0$

let $p, q \in \bar{E}$

Now, by the definition of $\bar{E}$ there exist $p', q' \in$ such that $d(p, p') < \epsilon/2$ and $d(q, q') < \epsilon/2$

Now. $d(p,q) \leq d(p,p') + d(p',q)$

$\leq d(p,p') + d(p',q') + d(q',q)$

$\leq \epsilon/2 + d(p',q') + \epsilon/2$

$= \epsilon + \sup\{d(a,b) \mid a, b \in E\}$

$\leq \epsilon + \operatorname{diam} E$

$d(p,q) \leq \epsilon + \operatorname{diam} E$

$d(p,q) - \operatorname{diam} E \leq \epsilon$.

$\sup\{d(c,d) \mid c, d \in \bar{E}\} - \operatorname{diam} E \leq \epsilon$

$\operatorname{diam} \bar{E} - \operatorname{diam} \bar{E} \leq \epsilon$

This is true for any $\epsilon > 0$

$\operatorname{diam} \bar{E} - \operatorname{diam} E \leq 0$

$\therefore \operatorname{diam} \bar{E} \leq \operatorname{diam} E$

$\therefore \operatorname{diam} \bar{E} = \operatorname{diam} E$

(b) By using a known theorem,

If $\{A_n\}$ is a sequence of non-empty compac sets such that

$A_n \supset A_{n+1}$ for $n \in N$ then $\cap A_n$ is non-empty

By this theorem,

$K = \bigcap_{n=1} k_n$ is non-empty

If 'k' contains two distinct points, say p, q then

$$\text{diam } k \geq d(p,q) > 0$$

Given that $\lim_{n\to\infty} \text{diam } k_n = 0$

$\therefore$ there exists a positive integer $s \ni$ $\text{diam } k_n < \epsilon \ \forall \ n \geq s$

Since $K = \bigcap k_n \subseteq k_n$

We have $\epsilon = \text{diam } k \leq \text{diam } k_n < \epsilon$ which is a contradiction.

Therefore, k do not contain two distinct points.

Since $K \neq 0$, we have that k contains of exactly one point.

[* Theorem :- *]

(a) If any metric space X; every convergent sequences is a cauchy sequence.

(b) If X is a compact metric space and if $\{P_n\}$ is a cauchy sequence in X, then $\{P_n\}$ converges to some point of X.

(c) If R^k, every cauchy sequence converges. (Cauchy criterian)

Proof: (a) Suppose $P_n \to P$

To show that $\{P_n\}$ is a cauchy sequence.

Let $\epsilon > 0$

Since $P_n \to P$, we have that there exists an integer N such that $d(P_n, P) < \epsilon/2 \ \forall \ n \in N$

Now for any $n \geq N$ and $m \geq N$, we have

$$d(P_n, P_m) \leq d(P_n, P) + d(P, P_m)$$
$$< \epsilon/2 + \epsilon/2$$
$$< \epsilon$$

$\therefore \{P_n\}$ is a cauchy sequence.

(b) Let $\{P_n\}$ be cauchy sequence in the compact space 'X'.

Write $E_N = \{P_n / n \geq N\}$

Then by known result, we get that $\lim_{n \to \infty} \dim E_N = 0$

By known theorem, we have $\operatorname{diam} \overline{E_N} = \operatorname{diam} E_n$ and thus $\lim_{n \to \infty} \operatorname{diam} \overline{E_N} = 0$

By definition of E_N, we have $E_N \supset E_{N+1}$

Thus implies $\overline{E_N} \supset \overline{E_{N+1}}$

By known theorem,

each $\overline{E_N}$ is compact.

Now By known theorem, we have that $\cap \overline{E_N}$ contains exactly one point, say P.

Now, we show that $P_n \to P$

$\cap A_n$ is non-empty

By this theorem,

$k = \bigcap_{n=1}^{\infty} k_n$ is non-empty.

If 'k' contains two distinct points, say p, q then

$\operatorname{diam} k \geq d(p, q) > 0$

Given that $\lim_{n \to \infty} \operatorname{diam} k_n = 0$

∴ there exists a +ve positive integers s ∋ diam $k_n < \epsilon$ ∀

since $k = \cap k_n \subseteq k_n$

We have $\epsilon = \text{diam}\, k \leq \text{diam}\, k_n < \epsilon$ a contradiction.

Therefore, k do not contain two distinct points since $k \neq 0$, We have that k contains of exactly one point x]

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* Theorem :- Imp

(a) If any metric space x, every convergent sequence is a cauchy sequence.

(b) If x is a compact metric space and if $\{P_n\}$ is a cauchy sequence in x, then $\{P_n\}$ converges to some points of x.

(c) If R^k, every cauchy sequence converges (Cauchy criterian). ***

Proof :

(a) Suppose $P_n \to P$

To show that $\{P_n\}$ is a cauchy sequence.

let $\epsilon > 0$

Since $P_n \to P$, we have that there exist an integer N such that $d(P_n, P) < \epsilon/2$ for all $n \in N$.

Now for any $n \geq N$ and $m \geq N$, we have

$$d(P_n, P_m) \leq d(P_n, P) + d(P, P_m)$$
$$< \epsilon/2 + \epsilon/2$$
$$< \epsilon$$

∴ $\{P_n\}$ is a cauchy sequence.

(b) Let $\{P_n\}$ be cauchy sequence in the compact space 'X'.

Write $E_N = \{P_n \mid n \geq N\}$

Then by known result, we get that $\lim_{n\to\infty} \operatorname{diam} E_N$

By known theorem, we have $\operatorname{diam} \bar{E}_N = \operatorname{diam} E_N$ and thus $\lim_{n\to\infty} \bar{E}_N = 0$

By definition of E_N, we have $E_N \supset E_{N+1}$

Thus implies $\bar{E}_N \supset \bar{E}_{N+1}$

By known theorem, each $\bar{E}_N$ is compact

Now, By known theorem, we have that $\cap \bar{E}_N$ contains exactly one point, say p.

Now we show that $P_n \to p$

let $\epsilon > 0$

Since $\lim_{n\to\infty} \operatorname{diam} \bar{E}_N = 0$, there exists an integer 'N' such that $\operatorname{diam} \bar{E}_N < \epsilon$ for all $N \geq N_0$

$d(P_n, P) \leq \operatorname{diam} \bar{E}_n < \epsilon$ for all $n \geq N_0$

Thus show that $P_n \to p$.

(c) In R^k, every cauchy sequence converges.

proof: Let $\{x_n\}$ be a cauchy sequence in R^k.

Define $E_n = \{x_n, x_{n+1}, \ldots\}$ for each $n \in N$.

Since $\epsilon = 1$, $\lim_{n\to\infty} \operatorname{diam} E_n = 0$, there exists $k \ni \operatorname{diam} E_k$ $\forall n \geq k$.

Now E_k is bounded.

Since $\{x_1, x_2, \ldots x_k\}$ is bounded because it is a finite set.

Thus, $\{x_n / n \in N\} = F_k \cup \{x_1, \ldots x_k\}$ is a bounded set.

Since $\{x_n / n \in N\}$ is a bounded set in R^k, there exists a k-cell $I \ni \{x_n / n \in N\} \subseteq I'$

By known theorem, every k-cell is compact.

$\{x_n\}$ is a cauchy sequence in a compact set I

$\therefore$ by (b), we have $\{x_n\}$ is convergent.

* Definition:

A metric space in which every cauchy sequence converges is said to be complete.

Example:

Every closed subset E of a complete metric space is complete.

proof: Let $\{P_n\}$ be a cauchy sequence in E

Since $E \subseteq X$, we have $\{P_n\}$ is a cauchy sequence in X.

$\Rightarrow \{P_n\}$ converges to P.

$\Rightarrow P \in E$ (or) P is a limit point of E

$\Rightarrow P \in E$

Thus, $P_n \to P$ & $P \in E$

E is complete.

* Definition:

A sequence $\{S_n\}$ of real numbers is said to be monotonic if

(a) Monotonically increasing if $S_n \leq S_{n+1}$

(b) Monotonically decreasing if $S_n \geq S_{n+1}$, $n = 1, 2, \ldots$

*** Theorem :- Suppose $\{s_n\}$ is monotonic then $\{s_n\}$ converges iff it is bounded. Imp

Proof : By known theorem,
We know that every convergent sequence is bounded.

Suppose $\{s_n\}$ is a monotonic sequence
case - i : let $\{s_n\}$ is monotonically increasing then for each $n \in N$,
We have that $s_n \leq s_{n+1}$

let $E = \{s_n / n \in N\}$

Since $\{s_n\}$ is bounded, there exist an upper bound of E.

Let s be the least upper bound of E

To show that $s_n \to s$

Let $\epsilon > 0$

s is the least upper bound of E

$s - \epsilon$ is not an upperbound of E

there exists an element in E, which is greater than $s - \epsilon$

$s - \epsilon < s_N$ for some $s_N \in E$

$s - \epsilon < s_N \leq s$, for some $s_N \in E$

Since $\{s_n\}$ is monotonically increasing

$s_N \leq s_n \ \forall \ n \geq N$.

Since s is the least upperbound of E

$s_n \leq s \ \forall \ n \geq N$

$s - \epsilon < s_N < s_n < s$ for all $n \geq N$

$|s - s_n| < |s - (s - \epsilon)| = |\epsilon| = \epsilon \ \forall \ n \geq N$

$s - \epsilon < s_n$

$-(s-\epsilon) >$

$-s_n < -(s-$

$\therefore \{S_n\}$ converges to s.

case-ii: Suppose $\{S_n\}$ is monotonically decreasing then for each $n \in N$, we have that $S_n \geq S_{n+1}$

let $E = \{S_n \mid n \in N\}$

Since $\{S_n\}$ is bounded $\exists$ a lowerbound for E

let s be a greatest lowerbound of E

To show that $S_n \to s$

Let $\epsilon > 0$

s is the greatest lowerbound of E

$s+\epsilon$ is not a lowerbound of E

There exists, an element in E which is less than $s+\epsilon$

$s+\epsilon > S_N$ for some $S_N \in E$

$s+\epsilon > S_N > s$ for some $S_N \in E$

Since $\{S_N\}$ is monotonically decreasing

$S_N \geq S_n \quad \forall n \geq N$

Since s is the greatest lowerbound of E

$S_n \geq s \quad \forall n \geq N$

$s < S_n \leq S_N < s+\epsilon$ for all $n \geq N$.

$|S_n - s| < |s+\epsilon - s| < |\epsilon| < \epsilon \quad \forall n \geq N$

$\therefore \{S_n\}$ converges to s.

* Upper and lower limits :-

Let $\{s_n\}$ be a sequence of real numbers with the following property that : for every real 'M' there is an integer N such that $m \geq M$ implies we write $s_n \to +\infty$

Similarly if for every real 'M' there is an integer N. such that $n \geq N$ implies $s_n \leq M$

We write $s_n \to -\infty$

Definition :

Let $\{s_n\}$ be a sequence of real numbers. -

Let E be the set of numbers 'x' [in the exten_real numbers system] such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$.

This is set E containing all sub-sequential limits and possibly the numbers $+\infty, -\infty$

We write $s^* = \sup E$

$s_* = \inf E$

The number s^*, s_* are called the upper and the lower limits of $\{s_n\}$. We use the notation

$\lim\limits_{n\to\infty} \sup s_n = s^*$, $\lim\limits_{n\to\infty} \inf s_n = s_*$

* Theorem :- Let $\{s_n\}$ be a sequence of real numbers. Let E and s^* have the same meaning as in above definition, then s^* has the following two properties.

(a) $s^* \in E$

(b) If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$, moreover s^* is the only number with the properties (a) & (b)

proof:

(a) Case - i : Suppose $s^* = +\infty$

then E is not bounded above thus $\{s_n\}$ is not bounded above.

Therefore, there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \to +\infty$

This show that $s^* = +\infty \in E$

$\therefore s^* \in E$

case - ii : Suppose s^* is real

Then E is bounded above and so $\{s_n\}$ is bounded above.

By known theorem, E is closed

By known theorem, we have $s^* \in E$

[Let E be a non-empty set of real numbers which is bounded above. Let $y = \sup E$, then $y \in \bar{E}$.

Thus $y \in E$ If E is closed]

case - iii :

Suppose $s^* = -\infty$ then $E = \{-\infty\}$

then E is not bounded below.

Thus $\{E_n\}$ is not bounded below.

$\therefore \exists$ a subsequence $\{s_{n_k}\} \ni s_{n_k} \to -\infty$

This shows that $s^* = -\infty \in E$

$\therefore s^* \in E$

(b) Suppose $x > s^*$

In a contrary way, Suppose $s_n \geq x$, for infinite many n choose n_k such that $s_{n_k} \geq x$, for each

$k \in \mathbb{N}$.

We can assume that $n_1 < n_2 < \cdots$

Now. $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$ and $s_{n_k} > x, \forall k$.

$\therefore \{s_{n_k}\}$ is bounded below by x.

<u>case (i)</u>: If $\{s_{n_k}\}$ is bounded above

Then it has a limit y, say

Since $s_{n_k} \geq x$ for all x.

We have $y \geq x$

Given $x > s_*$

$\therefore y > s_*$ a contradiction

Thus, we get a contradiction.

<u>Case-ii</u>: If $\{s_{n_k}\}$ is not bounded below.

Then it has a subsequences which converges to $-\infty$

Then $-\infty \in E \Rightarrow -\infty \leq s_* < x$ a contradiction.

Thus this case do not

Thenus s_* has property (b)

* <u>Uniqueness</u> of s_* :-

Suppose p, q are two numbers

Satisfying (a) & (b)

Suppose $p \neq q$

Assume $p < q$

Consider x such that $p < x < q$

Since $q \in E$, q is subsequential limit of $\{s_n\} \to q$

a contradiction to the above fact $x < q$ thus $p = q$

① a) Let $\{S_n\}$ be a sequences containing all rationals then every real numbers is a subsequential limit and $\lim_{n\to\infty} \sup S_n = +\infty$, $\lim_{n\to\infty} \inf S_n = -\infty$

b) Let $S_n = \frac{(-1)^n}{1+\frac{1}{n}}$ then $\lim_{n\to\infty} \sup S_n = 1$, $\lim_{n\to\infty} \inf S_n = -1$

c) for real value sequence $\{S_n\}$, $\lim_{n\to\infty} S_n = s$, iff $\lim_{n\to\infty} \sup S_n = \lim_{n\to\infty} \inf S_n = s$

proof: a) Let E be the set of all sub-sequential limit of $\{S_n\}$

Given $Q = \{S_n \mid n \in N\}$

To show that ⓐ

Let $x \in R$

for each integer $k \geq 0$, $\exists$ a rational number

$S_{n_k} \ni \_\_ < S_{n_k} < x + \frac{1}{k}$

Now, we may consider this $\{S_{n_k}\}$ as subsequential of $\{S_n\}$

clearly $S_{n_k} \longrightarrow x$

Therefore $x < E$

Thus $E = R$

clearly, every bound sequence $x < S_{n_k} \in E$ in R^k contains $x \in R < \in$ Convergent Sequence

Now it is clear that $\lim_{n\to\infty} \sup S_n = \lim_{n\to\infty} \sup E = +\infty$

if $\lim_{n\to\infty} \inf S_n = \inf E = -\infty$

b) $S_1, S_3, \ldots\ldots$ is a subsequence of $\{S_n\}$ which converges to -1

$S_2, S_4, \ldots\ldots$ is a subsequence of $\{S_n\}$ which converges to $+1$

Therefore $E = \{+1, -1\}$

Thus, $\lim\limits_{n\to\infty} \sup s_n = \sup E = 1$ & $\lim\limits_{n\to\infty} \inf s_n = \inf E$

(c) $\lim\limits_{n\to\infty} s_n = s$ iff $\{s_n\}$ converges to s.

$\Leftrightarrow$ Every subsequential of $\{s_n\}$ converges to s

$\Leftrightarrow E = \{s\}$

$\Leftrightarrow \lim\limits_{n\to\infty} \sup s_n = \sup E = s = \inf E = \lim\limits_{n\to\infty} \inf s_n$

*** Imp

* <u>Theorem</u> :- If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then $\lim\limits_{n\to\infty} \inf s_n \leq \lim\limits_{n\to\infty} \inf t_n$

<u>proof</u>: $\lim\limits_{n\to\infty} \inf s_n = s_*$, $\lim\limits_{n\to\infty} \inf t_n = t_*$

$\lim\limits_{n\to\infty} \sup s_n = s^*$, $\lim\limits_{n\to\infty} \sup t_n = t^*$

In a contrary way, suppose $s_* > t_*$

Let x such that $s_* > x > t_*$

Now, $s_* > x$

There exists N' such that for any $n \geq N'$, we have $s_n > x$

Now take $N^* = \max\{N, N'\}$

for each $n \geq N^*$, we have $t_n \geq s_n > x$

$\Rightarrow t_n > x$ for all $n \geq N^*$

$\Rightarrow$ every subsequential limit for $\{s_n\} \geq x$

x is lower bound for E

$\Rightarrow t_* \geq x$, a contradiction to the fact $s_* > x > t_*$

Therefore: Thus $S_* \le t_*$ → $\liminf_{n\to\infty} S_n \le \liminf_{n\to\infty} t_n$

In a similar way, we can prove $S^* \le t^*$

* Some special sequences :-

problem: If $0 \le x_n \le S_n$ for $n \ge N$ where N is some fixed number and if $S_n \to 0$ then $x_n \to 0$

Sol:. To show that $x_n \to 0$

Let $\epsilon > 0$, where N is some fixed num

Since $S_n \to 0$ $\exists$ an integer $N' \ni |S_n - 0| < \epsilon \; \forall n \ge N'$

Now take $N^* = \max\{N, N'\}$

Then for any $n \ge N^*$, we have

$|x_n - 0| = |x_n| = x_n \le S_n = |S_n| = |S_n - 0| < \epsilon$

which implies $x_n \to 0$

** Theorem: Imp

(a) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$

(b) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$ → Imp.

(d) If $p > 0$ & α is a real then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

(e) If $|x| < 1$ then $\lim_{n\to\infty} x^n = 0$

proof: (a) let $\epsilon > 0$

Consider, $\left(\frac{1}{\epsilon}\right)^{1/p}$ is a positive integer

Let N be a positive integer such that $N > \left(\frac{1}{\epsilon}\right)^{1/p}$

for any $n \ge N$, we have $n \ge N > \left(\frac{1}{\epsilon}\right)^{1/p}$

$\Rightarrow n^p > \left(\frac{1}{\epsilon}\right)^{p/p}$

$\Rightarrow n^p > \frac{1}{\epsilon}$

$\Rightarrow \frac{1}{n^p} < \epsilon$

$\therefore \left|\frac{1}{n^p} - 0\right| = \left|\frac{1}{n^p}\right| \Rightarrow \frac{1}{n^p} < \epsilon \ \forall\ n \geq N.$

Thus $\lim_{n\to\infty} \frac{1}{n^p} = 0$

(b) Case-(i) :: If $P=1$ then it is clear $\Rightarrow \lim_{n\to\infty} \sqrt{1} = 1$

case-ii : Suppose $P>1$

Let $x_n = \sqrt[n]{P} - 1 \Rightarrow x_n + 1 = \sqrt[n]{P}$

$P = (x_n+1)^n \geq 1 + n x_n \quad [\because (1+x_n)^n = 1 + n x_n + \frac{n(n-1)}{2} x_n^2$

$\Rightarrow P \geq 1 + n x_n$

$\Rightarrow P - 1 \geq n x_n$

$\Rightarrow \frac{P-1}{n} \geq x_n$

$\therefore 0 \leq x_n \leq \frac{P-1}{n}$ $\qquad 0 \leq x_n \leq S_n$

Since $\left(\frac{P-1}{n}\right) \to 0$ we have $x_n \to 0$

$\Rightarrow \sqrt[n]{P} - 1 \to 0$

$\sqrt[n]{P} \to 1$

$\lim_{n\to\infty} \sqrt[n]{P} = 1$

case-iii : Suppose $0 < p < 1$

$\Rightarrow \frac{1}{p} > 1$ & thus by case (ii) $\lim_{n\to\infty} \left(\frac{1}{p}\right)^{1/n} = 1$

$\lim_{n\to\infty} \frac{1}{p^{1/n}} = 1$

By known theorem, $\lim\limits_{n \to \infty} \sqrt[n]{p} = 1$

i.e. $\lim\limits_{n \to \infty} \sqrt[n]{p} = 1$

(c) let $x_n = \sqrt[n]{n} - 1$

$x_n + 1 = \sqrt[n]{n} - 1 + 1 \Rightarrow x_n + 1 = \sqrt[n]{n}$

$n = (x_n + 1)^n \geq \frac{n(n-1)}{2} x_n^2 \quad \forall\ n \geq 2$

$\Rightarrow n \geq \frac{n(n-1)}{2} x_n^2 \quad \left[\because (1+x_n) = 1 + n x_n + \frac{n(n-1)}{2} x_n^2 + \ldots\right.$

$\Rightarrow 2n \geq n(n-1) x_n^2 \Rightarrow \frac{2}{n-1} \geq x_n^2$

$\Rightarrow \sqrt{\frac{2}{n-1}} \geq x_n$

$0 \leq x_n \leq S_n$ then as $S_n \to 0$, $x_n \to 0$

$\therefore 0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$

Since $\sqrt{\frac{2}{n-1}} \to 0$, we have $x_n \to 0$

$\Rightarrow (\sqrt[n]{n} - 1) \to 0$

$\sqrt[n]{n} \to 1$

$\lim\limits_{n \to \infty} (n)^{1/n} = 1$

(d) Let k be an integer such that $k > \alpha$ and $k > 0$

Let $n > 2k$

Now, $(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\ldots(n-k+1)}{k!} p^k$

$\Rightarrow (1+p)^n > \frac{n^k}{2^k} \cdot \frac{p^k}{k!}$ $\qquad n(n-1)\ldots(n-k+1)$

$n\,n\left(n - \frac{1}{n}\right)\ldots$

$\Rightarrow \frac{1}{(1+p)^n} < \frac{2^k}{n^k} \cdot \frac{k!}{p^k}$ $\qquad$ Because $n > 2k$

$$\Rightarrow \frac{n^\alpha}{(1+p)^n} < \frac{2^k \cdot k!}{n^k \cdot p^k} \cdot n^\alpha = \frac{2^k \cdot k!\, n^{\alpha-k}}{p^k}$$

$$0 < \frac{n^\alpha}{(1+p)^n} < \left(\frac{2^k \cdot k!}{p^k}\right) n^{\alpha-k}$$

Since $\alpha - k > 0$, by (a) of this theorem

$$\Rightarrow \frac{1}{n^{\alpha-k}} \to 0 \quad (\because s_n \to 0)$$

$$\therefore \left(\frac{2^k \cdot k!}{p^k}\right) n^{\alpha-k} \to 0 \quad (\text{then } x_n \to 0)$$

This shows $\frac{n^\alpha}{(1+p)^n} \to 0$

i.e, $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

(e) If $|x|<1$, then $\lim_{n\to\infty} x^n = 0$

Proof :

case(i) : If $x=0$ then the result is true

case(ii) : Suppose $0<x<1$

Write $p = \frac{1}{x} - 1$ then $1+p = \frac{1}{x}$

$$x = \frac{1}{1+p}$$

By putting $\alpha = 0$ in (d) there $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$, ux

let $x^n = \frac{1}{(1+p)^n} = \frac{n^0}{(1+p)^n} = \frac{n^\alpha}{(1+p)^n} \to 0$

Thus $x^n \to 0$ i.e, $\lim_{n\to\infty} x^n = 0$

Case - iii :: Suppose $-1 < x < 0 \Rightarrow 1 > -x > 0$

$\Rightarrow 0 < -x < 1$ by case - iii, $0 < x < 1$, $\lim_{n \to \infty} x^n = 0$

this implies $\lim_{n \to \infty} (-x)^n = 0$

let $\epsilon > 0$.

Then $\exists$ 'N' $\ni$ $|(-x)^n - 0| < \epsilon \ \forall n \geq N$

Now $|x^n - 0| = |x^n| = |(-x)|^n = |(-x)^n - 0| < \epsilon$

Thus, $\lim_{n \to \infty} x^n = 0$.

* Series :-

Definition: Let $\{a_n\}$ be a sequence. Write $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$. Now $\{S_n\}$ is a sequence for $\{a_n\}$. We also the symbolic expression $a_1 + a_2 + a_3 + \cdots$ (or) $\sum_{n=1}^{\infty} a_n$ is called as infinite series (or) series.

The numbers S_n is called partial sum of the series S_n is called the n^{th} partial sum if S_n converge to s then we say that the series converges. We write $\sum_{n=1}^{\infty} a_n = s$, the number '$s$' is called the sum of series.

Note that s is the limit of the sequence of sums if $\{S_n\}$ diverges then the series is said to be diverges.

For conveince sometimes we consider $\sum_{n=0}^{\infty} a_n$ also, we represent the series by $\sum a_n$, we note that with the notation $a_n = S_n - S_{n-1}$ for $n > 1$

*** Theorem :: $\sum a_n$ converges iff for every $\epsilon > 0$ then the integer N such that $\left|\sum_{k=n}^{m} a_k\right| \leq \epsilon$ if $m \geq n$

In particular by taking $m = n$, $|a_n| \leq \epsilon$ if $n \geq m$

proof :: Suppose $\sum a_n$ converges and $\sum a_n = S$

Consider the partial sums $S_n = \sum_{k=1}^{n} a_k$

Since $s = \sum a_n$, we have $S_n \to S$

Since $\{S_n\}$ converges to S_k, we have that $\{S_n\}$ is also a cauchy sequence

Let $\epsilon > 0$

Now, there exists an integer $N \ni |S_n - S_m| < \epsilon$ for $n, m \geq N$.

$$\left|\sum_{k=n}^{m} a_k\right| = \left|\sum_{k=1}^{m} a_k - \sum_{k=1}^{n-1} a_k\right|$$

$$= |S_m - S_{n-1}| < \epsilon \text{ for } m \geq n \geq N$$

If $m = n$ then

$$|a_n| = \left|\sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k\right| = |S_n - S_{n-1}| < \epsilon \text{ for } n \geq N$$

conversely suppose that for each $\epsilon > 0$ there is an integer $N \ni \left|\sum_{k=n}^{m} a_k\right| \leq \epsilon$ for $m \geq n \geq N$

this means for each $\epsilon > 0$ & $m \geq n \geq N$

We have that $|S_m - S_{n-1}| = \left|\sum_{k=n}^{m} a_k\right| \leq \epsilon$

$\Rightarrow \{S_n\}$ is a cauchy sequence

Since R^k is complete and we have that $\{S_n\}$ is a convergent sequence

$\therefore$ there exist $s \in C$, such that i.e, $S_n \to S$

rms

That means Σa_n convergent.

Theorem: If Σa_n converges then $\lim_{n \to \infty}$ is ...

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proof: Since Σa_n converges.

By known theorem,

We have that for every $\epsilon > 0$ there exists an integer N, such that $n \geq N$, $|a_n| < \epsilon$

for $n \geq N$, we have that $|a_n - 0| = |a_n| < \epsilon$

$\therefore a_n \to 0$ i.e, $\lim_{n \to \infty} a_n = 0$

Theorem:- A series of non-negative terms converges iff its partial sums forms a bounded sequences.

proof: Let Σa_n be a series of non-negative real numbers.

Suppose Σa_n converges

Consider $\{S_n\}$ is the sequence of partial sums

Since $a_n \geq 0$

for each n, we have that $S_{n+1} - a_{n+1} + S_n \geq S$

Therefore, $S_{n+1} \geq S_n$

for each n and so $\{S_n\}$ is monotonically increasing sequence.

By known theorem, we get that $\{S_n\}$ is bounded

Conversely suppose that sequence $\{S_n\}$ of partial sum of Σa_n is bounded.

Since $a_n \geq 0$ for each n as above $S_n \leq S_{n+1}$

$\therefore \{S_n\}$ is monotonic

By known theorem, we get that $\{S_n\}$ is converges.

$\therefore \Sigma a_n$ converges.

**
* Theorem : (a) If $|a_n| \le c_n$ for $n \ge N_0$, where N_0
** is some fixed integer and if Σc_n converges, then Σa_n Converges.

(b) If $a_n \ge d_n \ge 0$ for $n \ge N_0$ & if Σd_n ~~converges~~ diverges then Σa_n diverges.

Proof : a) Let $\epsilon > 0$

Since Σc_n converges $\exists$ an integer $N \ni \left|\sum_{k=n}^{m} c_k\right| < \epsilon$ $\forall\ m \ge n \ge N$

put $N^* = \max\{N_0, N\}$

for any $m \ge N \ge N^*$

Consider, $\left|\sum_{k=n}^{m} a_k\right| \le \sum_{k=n}^{m} |a_k| \le \sum_{k=n}^{m} |c_k| = \left|\sum_{k=n}^{m} c_k\right| \le$

b) If possible, suppose Σa_n converges.

Since $0 \le d_n \le a_n$ for $n \ge N_0$, we have that $|d_n| \le a_n$ for $n \ge N$

Since Σa_n converges by (a), We have that Σd_n converges a contradiction to hypothesis.

$\therefore \Sigma a_n$ diverges.

* Theorem :: Series of Non-negative terms

*** Statement :: If $0 \leq x < 1$ then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ is convergent, if $x \geq 1$ the series diverges.

Proof : If $x = 0$ then clearly Imp

$$\sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n = 1 + 0 = 1 = \frac{1}{1-0} = \frac{1}{1-x}$$

Now suppose $0 < x < 1$

for any positive integer n, consider the partial sum, $S_n = 1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1-x}$ [∵ Geomentric progression $\frac{a - x^n}{1-r}$ differ]

since $0 < x < 1$, we have that $x^n \to 0$ as $n \to \infty$

consider $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1-x} = \frac{1-0}{1-x} = \frac{1}{1-x}$

∴ the given series converges to $\frac{1}{1-x}$ if $0 \leq x < 1$

If $x = 1$, then clearly $\sum_{n=0}^{\infty} x^n = 1 + 1 + 1 + \dots = \infty$.

In this case, it is divergent.

If $x > 1$, then $\sum_{n=0}^{\infty} x^n > 1 + 1 + 1 + \dots = \infty$

In this case also, it is divergent

at $x \geq 1$, $\sum_{n=0}^{\infty} x^n$ divergent

* Theorem :: Suppose $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \dots \geq 0$ then the series $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{k}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_4 + \dots$ — Converges.

proof: Given that $a_i \geq 0$ for each i
Therefore, $S_{n+1} = S_n + a_{n+1} \geq S_n$ and
So the sequences of partial sum of $\{S_n\}$ forms monotonically increasing sequences.
Write, $t_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$
Since each $a_i \geq 0$ we have that $\{t_k\}$ is also monotonically increasing sequence..
first, we prove two relations betweeen $\{S_n\}$ & $\{t_k\}$ for $n \leq 2^k$ we have

$$S_n \leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots a_{2^{k+1}})$$
$$\leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} = t_k$$

On the other hand, if $n > 2^k$ then

$$S_n \geq a_1 + a_2 + (a_3 + a_n) + (a_{2^{k-1}+1} + \cdots + a_{2^k})$$
$$\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} = \frac{1}{2} t_k$$

Therefore, if $n < 2^k$ then $S_n < t_k$ &
if $n > 2^k$ then $S_n > \frac{1}{2} t_k$
Using these two inequalities, first we show that $\{S_n\}$ is bounded iff $\{t_k\}$ is bounded
Suppose $\{S_n\}$ is bounded
Then $\exists$ a real number M such that $S_n \leq M \ \forall n$
Let k be any integer select n such that
Now $\frac{1}{2} t_k \leq S_n \leq M \Rightarrow \frac{1}{2} t_k \leq M \ \forall k$

$$\Rightarrow t_k \leq 2M \ \forall k$$

$\{t_k\}$ is a bounded sequence.

Conversely assume that $\{t^k\}$ is bounded.
there exist a real number M^* such that

$t_k \leq M^* \; \forall k$ and $n \leq 2^k$

Now, $S_n \leq t_k \leq M^* \Rightarrow S_n \leq M^* \; \forall n$

$\therefore \{S_n\}$ is bounded sequence.

$\sum a_n$ is convergent $\Leftrightarrow \{S_n\}$ is bounded

$\Leftrightarrow \{t_k\}$ is bounded

$\Rightarrow \sum t_k$ is convergent.

The series $\sum_{n=1}^{\infty} a_n$

$\sum_{k}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_4 + \dots$ Converges.

*Theorem :: $\sum \frac{1}{n^p}$ converges if $p>1$ & diverges if $p \leq 1$

Proof:

case-(i) : If $p=0$ then $\sum \frac{1}{n^p} = \sum \frac{1}{n^0} = 1+1+\dots$ diverges

case-(ii): If $p \not< 0$ then since $(-p)>0$

We have that $\frac{1}{n^p} = n^{-p} < (n+1)^{-p} = \frac{1}{(n+1)^p}$

Therefore if $p<0$ then the sequence $\{\frac{1}{n^p}\}$ is a monotonic increase.

Now, we show that if $p<0$ then $\{\frac{1}{n^p}\}$ not bounded for this take $M>0$

Now, $M^{-1/p}$ is non-negative $\Rightarrow \exists$ a positive integer k such that

$k > M^{-1/p} \Rightarrow k^{-p} > M \Rightarrow \frac{1}{k^p} > M$

$\Rightarrow \frac{1}{n^p} > \frac{1}{k^p} > M \ \forall \ n \geq k$

Thus $\{\frac{1}{n^p}\}$ is an unbounded sequence

In a contrary, suppose $\sum \frac{1}{n^p}$ is not diverges

$\Rightarrow \sum \frac{1}{n^p}$ converges

$\Rightarrow \frac{1}{n^p} \to 0$

$\Rightarrow \{\frac{1}{n^p}\}$ is a bounded sequence, a contradiction to the fact that $\{\frac{1}{n^p}\}$ is unbounded

Thus, $\sum \frac{1}{n^p}$ is divergent if $p<0$

Case-iii :- Suppose $p \geq 0$

Now, $\frac{1}{1^p} \geq \frac{1}{2^p} \geq \cdots \geq 0$

Therefore, by known theorem, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $\sum_{k=0}^{\infty} 2^k \left(\frac{1}{(2^k)^p}\right)$ converges.

Now consider the series $\sum 2^k \frac{1}{(2^k)^p} = \sum (2^k)(2^k)^{-p}$

$= \sum (2k)^{1-p}$

This is a geometric series

So, by known theorem, $\sum (2^k)^{1-p} = \sum (2^{1-p})^k$ diverge if $x = 2^{1-p} \geq 1$ & converges if $0 \leq 2^{1-p} < 1$ [$\because$ from series of non -ve ter

Therefore, we have the following

$p > 1 \Leftrightarrow (1-p) < 0 \Leftrightarrow 0 < 2^{1-p} < 1$

$\Leftrightarrow \sum (2^{1-p})^k$ converges $\Leftrightarrow \sum_{k=0}^{\infty} 2^k \left(\frac{1}{(2^k)^p}\right)$ converges

$\Leftrightarrow \sum \frac{1}{n^p}$ converges.

$\therefore$ $p>1$ then $\sum \frac{1}{n^p}$ converges.

Now, $p \le 1 \Leftrightarrow (1-p) \geq 0 \Leftrightarrow 2^{1-p} \geq 1$

$\Leftrightarrow \sum 2^k \left(\frac{1}{(2^k)^p}\right) = \sum \left(2^{1-p}\right)^k$ diverges

$\Leftrightarrow \sum \frac{1}{n^p}$ diverges.

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Theorem : If $p>1$, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges

if $p \le 1$ the series diverges.

proof : We know that $\{\log n\}$ is an increasing sequence of positive terms for $n \geq 2$

$\Rightarrow \left\{\frac{1}{n \log n}\right\}$ is a decreasing sequence if $n \geq 2$

Therefore, $\frac{1}{2\log 2} > \frac{1}{3\log 3} > \cdots > 0$

Now by known therorem, we have that

$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges iff $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k (\log_2 k)^p}$ converges

$$\Leftrightarrow \sum_{k=1}^{\infty} 2^k \frac{1}{2^k(\log_2 k)^p} = \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p} = \sum_{k=1}^{\infty} \frac{1}{k^p (\log 2)^p}$$

$$= \frac{1}{p(\log 2)} \sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges}$$

$\therefore$ We have that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges.

$\Leftrightarrow \sum_{k=1}^{\infty} \frac{1}{k^p}$ converges.

Now by known theorem, we have that $p > 1$

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^p}$ converges.

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges.

$\Rightarrow$ By known theorem, we have $p \leq 1$

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ diverges.

* The Number e :-

Definition :

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \text{ for each } n$$

consider the n^{th} partial sum $S_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$

$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$$

$$= 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}\right)$$

$$= 1 + \left(\frac{(1-\frac{1}{2})}{(1-\frac{1}{2})}\right) \cdot \left(\frac{1-(\frac{1}{2})^n}{(1-\frac{1}{2})}\right)$$

$$= 1 + \frac{(1-\frac{1}{2})}{(1-\frac{1}{2})} \cdot \frac{(1-(\frac{1}{2})^n)}{(1-\frac{1}{2})}$$

$$= 1 + \frac{(1-(\frac{1}{2})^n)}{\frac{1}{2}}$$

$$= 1 + 2\left(1 - \frac{1}{2^n}\right)$$

$$< 1 + 2 = 3$$

$\therefore S_n < 3$, for each n

Thus $\{S_n\}$ is bounded.

Then clearly $\{S_n\}$ is monotonically increasing sequence.

Then by known theorem, we have that $\{S_n\}$ is a convergent sequence.

Thus the definition makes sums & $S_n \to e$

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* **Theorem** :- Prove that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

Proof : Let $S_n = \sum_{k=0}^{n} \frac{1}{k!}$, $t_n = \left(1+\frac{1}{n}\right)^n$

By the binomial theorem,

$$t_n = 1 + \frac{1}{1!} + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(n-1)}{n}\right)$$

$$\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} = S_n$$

Therefore for each n we have that $t_n \leq S_n$

$\Rightarrow \limsup_{n\to\infty} t_n \leq \limsup_{n\to\infty} S_n = \lim_{n\to\infty} S_n = e$

Therefore $\limsup_{n\to\infty} t_n \leq e$

For an integer 'm' such that $n \geq m$

Consider $t_n = \left(1+\frac{1}{n}\right)^n$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{(n-1)}{n}\right)$$

$$\geq 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \dots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\cdots\left[1-\frac{(m-1)}{n}\right]$$

Now, fix 'm' and take $n \to \infty$ then we get that

$$\liminf_{n\to\infty} t_n \geq \liminf_{n\to\infty} \left(1+\frac{1}{1}+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{m!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{(m-1)}{n}\right)\right)$$

$$= 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{m!} = S_m$$

$\Rightarrow S_m \leq \liminf_{n\to\infty} t_n$ for all m

Now by ~~l~~eletting $m\to\infty$ we finally get that

$e = \lim_{m\to\infty} S_m$, $e \leq \liminf_{n\to\infty} t_n$

We have that $\limsup_{n\to\infty} t_n \leq e \leq \liminf_{n\to\infty} t_n$

$$\Rightarrow \limsup_{n\to\infty} t_n = \liminf_{n\to\infty} t_n = e$$

$$\Rightarrow \lim_{n\to\infty} t_n = e$$

$$\Rightarrow \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = e$$

** Imp

*** Theorem : e is irrational.

** proof : In a contrary way, assume that e is rational.

Then there exist two positive integer p, q such that $e = p/q$

$$\Rightarrow qe = p$$

$$\Rightarrow (q!)e \text{ is an integer}$$

By known result, we have that $0 < e - S_q < \frac{1}{q!\,q}$

multiply with $q!$

$$\Rightarrow 0 < q!(e - S_q) < \frac{1}{q}$$

consider $(q!)S_q = q!\left(1+1+\frac{1}{2!}+\cdots+\frac{1}{q!}\right)$

$= q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{q!}$

$= q! + q! + ((3)(4)\ldots(q)) + ((4)(5)\ldots(q)) + \ldots$
$\ldots + 1$

is an integer.

Therefore, $q!(e - S_q) = q!e - q!S_q$ is an integer

finally, we get that $0 < q!(e - S_q) < \frac{1}{q}$, $q \geq 1$

If $q = 1$, then $0 < 1(e-2) < \frac{1}{1} \Rightarrow 0 < (e-2) < 1$.

Since there is no integer $(e-2)$ with this property.

$S_q = \sum_{q=0}^{1} \frac{1}{0!} + \frac{1}{1!} = 1 + 1 = 2$

We have reached a contradiction.

If $q > 1$ then $0 < q!(e - S_q) < \frac{1}{q} < 1$

Since there is no integer between 0 and 1 (which is not equal to 0 (or) 1, We have a contradiction)

Thus, e is not a rational number.

Therefore, e is irrational number.

** Imp

Theorem: (**Root test**) State and prove.

Statement: Given $\sum a_n$, put $\alpha = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$,

then (a) if $\alpha < 1$, $\sum a_n$ converges

(b) if $\alpha > 1$, $\sum a_n$ diverges

(c) if $\alpha = 1$, the test gives no information.

proof: (a) Suppose $\alpha < 1$

choose β so that $\alpha < \beta < 1$ & an integer N such that $\sqrt[n]{|a_n|} < \beta$ for $n \geq N$

$(\sqrt[n]{|a_n|})^n < \beta^n$

$\Rightarrow (|a_n|^{1/n})^n < \beta^n$

i.e., $|a_n| < \beta^n$ for $n \geq N$

Since $0 < \beta < 1$, $\sum \beta^n$ converges

Since $\sum \beta^n$ converges & $|a_n| < \beta^n$ for all $n \geq N$
By comparision test we have that

$\sum a_n$ converges.

(b) Suppose $\alpha > 1$

since $\alpha = \sup E$

where E is the set of number α (in the extended real number system)

Such that $S_{n_k} \to \alpha$ for some subsequence $\{S_{n_k}\}$ of $\{S_n\}$

Therefore $\alpha > 1$, $S_{n_k} \to \alpha$

$\Rightarrow$ there exists a positive number M such that $S_{n_k} > 1$ for all $n \geq m$

$$\sqrt[n_k]{|a_n|} > 1 \text{ for all } n_k \geq M$$

$$|a_n| \geq 1 \text{ for all } n_k \geq M$$

If possible suppose $\sum a_n$ converges
Then by known theorem, we get that $a_n \to 0$
This implies that a positive integer $M_1 \ni$

$|a_n| < 1 \; \forall \; n \geq M$

Write $M_2 = \max\{M, M_1\}$

Now for $n \geq M_2$, we have $|a_n| > 1$ and $|a_n| < 1$, a contradiction.

Therefore, $\sum a_n$ doesnot converges

Therefore, $\sum a_n$ diverges if $\alpha > 1$

(c) the test cannot given information if $\alpha = 1$
for example, consider the series $\sum \frac{1}{n}$ & $\sum \frac{1}{n^2}$
for these two sequences $\lim\limits_{n \to \infty} \sup \sqrt[n]{\frac{1}{n}} = 1$ and

$\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = 1$

By known result, we have that $\Sigma \frac{1}{n}$ diverges.

By known theorem $\Sigma \frac{1}{n^2}$ converges.

$\therefore$ Therefore the test gives no information.

*** Result** :: Before 'e' is irrational

let $S_n = \sum_{k=0}^{n} \frac{1}{k!}$ then $0 < e - S_n < \frac{1}{(n!)n}$

Proof: Consider $0 < e - S_n = \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=n+1}^{\infty} \frac{1}{k!}$.

$$= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots$$

$$= \frac{1}{(n+1)!}\left[1 + \frac{1}{(n+2)} + \frac{1}{(n+3)(n+2)} + \cdots\right]$$

$$\cancel{\leq \frac{1}{(n+1)!}\left[1 + \frac{1}{(n+2)} + \cdots\right.}$$

$$< \frac{1}{(n+1)!}\left[\frac{1}{1-\left(\frac{1}{n+1}\right)}\right]$$

$$= \frac{1}{(n+1)!} \frac{(n+1)}{n} = \frac{1}{n!\,n}$$

$$\therefore 0 < e - S_n < \frac{1}{n!\,n}$$

***** Imp

*** Theorem** :- **Ratio test**

Statement: The series Σa_n

(a) Converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

(b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for $n \geq n_0$

where n_0 is some fixed integer

proof: (a) Suppose $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

choose β, such that $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < \beta < 1$

Now, there exist an integer $N \ni n \geq N$

We have $\left|\frac{a_{n+1}}{a_n}\right| < \beta$

$$|a_{n+1}| < \beta |a_n|$$

for this, we get that $|a_{n+2}| < \beta |a_{n+1}| < \beta \cdot \beta |a_n|$
$= \beta^2 |a_n|$

Similarly, $|a_{n+p}| \leq \beta^p |a_n|, \forall n \geq N$

Now for $n \geq N$ we have

$$|a_n| = |a_{N+(n-N)}| < \beta^{n-N} |a_N|$$
$$= (\beta^{-N} |a_N|) \beta^n$$

Since $0 < \beta < 1$ we have that $\Sigma \beta^n$ converges

Since $\beta^{-N} |a_N|$ is a constant, we have that

$\Sigma (\beta^{-N} |a_N|) (\beta^n$ converges)

By comparision that Σa_n converges

(b) Suppose $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for $n \geq n_0$

In a contrary way we have that $a_n \to 0$

Now there exists an integer $M \ni |a_n| < |a_{n_0}|$

Take $n > \max\{n_0, M\}$

Consider, $|a_n| = |a_{n_0+(n-n_0)}| \geq |a_{n_0}|$

a contradiction

$\therefore \Sigma a_n$ diverges

Theorem: For any sequence $\{c_n\}$ of a positive numbers.

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n}$$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

proof: First we prove that

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

Write $\alpha = \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$

If $\alpha = +\infty$ then the inequality is clear

Suppose α is infinite

choose a number $B \ni \beta > \alpha = \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$

$\Rightarrow$ there exists an integer N such that

$\frac{c_{n+1}}{c_n} \leq \beta$ for $n \geq N$

$\Rightarrow c_{n+1} \leq \beta c_n$ for $n \geq N$

Now, $c_{n+2} \leq \beta c_{n+1} \leq \beta \cdot \beta c_n = \beta^2 c_n$

In the same way, $c_{N+p} \leq \beta^p c_n$ for any p.

for $n \geq N$

Consider, $c_n = c_N + (n-N) \leq \beta^{n-N} c_N$

$$= (\beta^{-N} c_N)\beta^N$$

$$\sqrt[n]{c_n} \leq \sqrt[n]{\beta^{-N} c_N}\, \beta$$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \left[\sqrt[n]{\beta^{-N} c_N}\, \beta\right]$$

$$= \beta \limsup_{n\to\infty} \sqrt[n]{\beta^{-N} c_N} = \beta(1) = \beta$$

$\therefore \limsup_{n\to\infty} \sqrt[n]{c_n} \le \beta \quad \forall\ \beta > \alpha$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \alpha$$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

Now consider,

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n}$$

Write $\alpha = \liminf_{n\to\infty} \frac{c_{n+1}}{c_n}$

If $\alpha = -\infty$ then the inequality is true

Assume that α is finite

choose a number β such that $\beta < \alpha$

$$= \lim_{n\to\infty} \frac{c_{n+1}}{c_n}$$

$\Rightarrow$ There exists an integer N such that

$\beta < \frac{c_{n+1}}{c_n}$ for $n \ge N$

$\Rightarrow c_n\beta \le c_{n+1}$ for $n \ge N$

$\Rightarrow c_{n+2} \ge c_{n+1}\beta \ge c_n\beta\cdot\beta = c_n\beta^2$

In the same way, we get that $c_{N+p} = \beta^p c_N$ for any p

fix $n \ge N$

Consider, $c_n = c_{N+(n-N)} \ge c_N\beta^{n-N} = c_N\beta^{-N}\cdot\beta^n$

$\Rightarrow \sqrt[n]{c_N} \ge \sqrt[n]{c_N\beta^{-N}}\ \beta$

$\Rightarrow \liminf_{n\to\infty} \sqrt[n]{c_N} \ge \liminf_{n\to\infty} \sqrt[n]{c_N\beta^{-N}}\ \beta$

$$\Rightarrow \liminf_{n\to\infty} \sqrt[n]{c_n} \geq \beta \cdot \liminf_{n\to\infty} \left[\sqrt[n]{c_N \beta^N}\right]$$

$$\Rightarrow \liminf_{n\to\infty} \sqrt[n]{c_n} \geq \beta \cdot 1 = \beta$$

Therefore, $\liminf\limits_{n\to\infty} \sqrt[n]{c_n} = \beta \;\; \forall \beta < \alpha$

$$\Rightarrow \liminf_{n\to\infty} \sqrt[n]{c_n} \geq \beta$$

$$\Rightarrow \liminf_{n\to\infty} \sqrt[n]{c_n} \geq \alpha$$

$$\Rightarrow \liminf_{n\to\infty} \sqrt[n]{c_n} \geq \liminf_{n\to\infty} \frac{c_{n+1}}{c_n}$$

* Power series :-

Definition: Given a sequence $\{c_n\}$ of complex numbers, the series $\sum\limits_{n=0}^{\infty} c_n z^n$ is called a power series. The numbers c_n are called the coefficients of the series, z is a complex number.

Theorem: Given the power series $\sum c_n z^n$, put $\alpha = \limsup\limits_{n\to\infty} \sqrt[n]{|c_n|}$, $R = \frac{1}{\alpha}$ (If $\alpha = 0$, $R = +\infty$, if $\alpha = +\infty$, $R = 0$) then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$

proof: Put $a_n = c_n z^n$

Now $|a_n|^{1/n} = |c_n|^{1/n} |z|$

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \left(\limsup_{n\to\infty} \sqrt[n]{|c_n|}\right)|z| = \alpha|z| = \frac{1}{R}|z|$$

Now by root test, we have that if $\frac{|z|}{R} < 1$ (i.e, $|z| < R$) the series $\sum a_n = \sum c_n z^n$ converges.

if $\frac{|z|}{R} > 1$ (i.e, $|z| > R$) then $\sum a_n = \sum C_n z^n$ diverges.

*** Note :-**

R is called the radius of convergence of $C_n z^n$

Examples :-

(a) Consider the power series $\sum n^n z^n$. Here $C_n = n^n$ for each n, therefore $\alpha = \limsup_{n\to\infty} \sqrt[n]{n^n} = \limsup_{n\to\infty} n = \infty$. Therefore $R = 0$.

(b) Consider the powerseries $\sum \frac{z^n}{n!}$. Here $C_n = \frac{1}{n!}$ for each N. Therefore $\alpha = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = 1$

Therefore $R = +\infty$

(c) Consider $\sum z^n$ then $C_n = 1$ for each n.

$\therefore \alpha = \limsup_{n\to\infty} \sqrt[n]{1} = 1 \Rightarrow R = 1$ if $|z| = 1$ then $\sum z^n = \sum 1$ diverges.

(d) Consider $\sum \frac{z^n}{n!}$ Here $C_n = \frac{1}{n}$ for each n.

$$\therefore \alpha = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \frac{1}{n\sqrt{n}} = \frac{1}{\sqrt[n]{n}} = 1$$

$\therefore R = 1$

If $z = 1$ then $\sum \frac{z^n}{n} = \sum \frac{1}{n}$ diverges.

(e) Consider the Series $\sum \frac{z^n}{n^2}$ Here $C_n = \frac{1}{n^2}$

$$\therefore \alpha = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \limsup_{n\to\infty} \frac{1}{\sqrt[n]{n^2}} = 1$$

$\therefore R = 1$

if $z=1$ then $\sum \frac{z^n}{n^2} = \sum \frac{1}{n^2}$ converg.

Problem :-

Find the radius of convergence of each of following powerseries.

(a) $\sum n^n z^n$ (b) $\sum \frac{2^n}{n!} z^n$ (c) $\sum \frac{2^n}{n^2} z^n$

(d) $\sum \frac{n^3}{3^n} z^n$.

$R = \frac{1}{\alpha}$

Sol: (a) Write $c_n = n^n$

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \limsup_{n\to\infty} \sqrt[n]{n^n} = \limsup_{n\to\infty} n = \infty$$

$<R \to$ cgs
$>R \to$ dgs

Therefore $R = 0$

By known theorem, if $|z| < R = 0$ then $\sum n^n z^n$ converges, if $|z| > R = 0$ then $\sum n^n z^n$ diverges

(b) Write $c_n = \frac{2^n}{n!}$

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \limsup_{n\to\infty} \sqrt[n]{\frac{2^n}{n!}} = \limsup_{n\to\infty} \frac{2}{(n!)^{1/n}}$$

$$= \frac{2}{\limsup_{n\to\infty} (n!)^{1/n}}$$

$$= \frac{2}{1}$$

$$= 2$$

$\therefore R = 1/2$

Now by known theorem, $|z| < R = \frac{1}{2}$ then $\sum \frac{2^n}{n!} z^n$ converges.

if $|z| > R = \frac{1}{2}$ then $\sum \frac{2^n}{n!} z^n$ diverges.

(c) Write $c_n = \dfrac{2^n}{n^2}$

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \limsup_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}} = \limsup_{n\to\infty} \frac{2}{(n^2)^{1/n}}$$

$$= \frac{2}{\limsup_{n\to\infty} (n^2)^{1/n}} = \frac{2}{1} = 2$$

$\therefore \quad R = 1/2$.

Now by known theorem, $|z| < R = \frac{1}{2}$ then $\sum \frac{2^n}{n^2} z^n$ converges, if $|z| > R = \frac{1}{2}$ then, the $\sum \frac{2^n}{n^2} z^n$ diverges.

(d) Write $c_n = \dfrac{n^3}{3^n}$

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \limsup_{n\to\infty} \sqrt[n]{\frac{n^3}{3^n}} = \limsup_{n\to\infty} \frac{n^{3/n}}{3}$$

$$= \frac{\lim_{n\to\infty} n^{3/n}}{3} = \frac{1}{3}$$

$\therefore \quad R = 3$

By known theorem,
if $|z| < R = 3$ then $\sum \frac{n^3}{3^n} z^n$ converges,
if $|z| > R = 3$ then $\sum \frac{n^3}{3^n} z^n$ diverges.

*** Imp

* Summation by Parts :- statement & prove

Theorem: Given two sequences $\{a_n\}, \{b_n\}$ put $A_n = \sum_{k=0}^{n} a_k$ if $n \geq 0$, put $A_{-1} = 0$ then if $0 \leq p \leq q$

We have $$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_q$$

proof: $A_n - A_{n-1} = \sum_{k=0} a_k - \sum_{k=0} a_k = a_n$

$$\therefore \sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n$$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$

$A_n b_n + A_q b_q$

$$= \left[\sum_{n=p}^{q-1} A_n b_n + A_q b_q\right] + \left[-A_{p-1} b_p - \sum_{n=p}^{q-1} A_n b_{n+1}\right]$$

$$= \sum_{n=p}^{q-1} [A_n b_n - A_n b_{n+1}] - A_{p-1} b_p + A_q b_q$$

$$= \sum_{n=p}^{q-1} A_n [b_n - b_{n+1}] - A_{p-1} b_p + A_q b_q$$

* Theorem :: Suppose

(a) The partial series A_n of $\sum a_n$ forms a bounded sequence.

(b) $b_0 \geq b_1 \geq b_2 \geq \dots$ (c) $\lim_{n \to \infty} b_n = 0$

then $\sum a_n b_n$ converges.

proof :: Since $\{A_n\}$ is bounded there exists M such that $|A_n| \leq M \; \forall M$

Let $\epsilon > 0$

Since $b_n \to 0$ there exist an integer N such that $b_n < \frac{\epsilon}{2M}$ for all $n \geq N$

Fix $N \leq p \leq q$

Consider $\left|\sum_{n=p}^{q} a_n b_n\right| = \left|\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right|$

$$\leq |A_n| \sum_{n=p}^{q} (b_n - b_{n+1}) + |A_q|(b_q) + A_{p-1}(b_p|$$

$$\leq M[(b_p - b_{p+1}) + (b_{p+1} - b_{p+2}) + \cdots + (b_{q-1} - b_q) + M b_q + M b_p$$

$$= 2Mb_p < 2M \frac{\epsilon}{2M} = \epsilon$$

Therefore, $\left|\sum_{n=p}^{q} a_n b_n\right| \leq \epsilon$ for $p \geq q \geq N$.

By Cauchy criterian for series, we have that $\sum a_n b_n$ converges.

***
* Theorem :- **Leibnitz theorem** :-
***
Suppose (a) $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$

(b) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ $(m = 1, 2, 3, \ldots)$

(c) $\lim_{n \to \infty} c_n = 0$

then $\sum c_n$ converges.

proof: Take $a_n = (-1)^{n+1}$, $b_n = |c_n|$ for $n = 1, 2, 3, \cdots$

Now, $\{b_n\}$ is a decreasing sequence.

Since $\lim_{n \to \infty} c_n = 0$, we have $\lim_{n \to \infty} |c_n| = 0$ & so $\lim_{n \to \infty} b_n = 0$

Write $A_n = \sum_{k=0}^{n} a_k$ for $n = 0, 1, 2, \ldots$

Now, $A_0 = a_0 = (-1)^{0+1} = (-1)^1 = -1$

$A_1 = a_0 + a_1 = -1 + 1 = 0$

$A_2 = a_0 + a_1 + a_2 = -1 + 1 - 1 = -1$

Continue this process, we observe that

$A_n = 0$ (or) -1

$\therefore |A_n| \leq 1$ for all n & so $\{A_n\}$ is bounded sequence.

Now, by known theorem, we have that $\sum a_n b_n$ converges.

$\Rightarrow \sum (-1)^{n+1} |c_n|$ converges.

If n is odd, $c_n = c_{2m-1} \geq 0$ and if n is even

$c_n = c_{2m} \leq 0$

therefore, $\sum (-1)^{n+1} |c_n| = \sum c_n$ converges.

* <u>Theorem</u> :: Suppose the radius of converges of $\sum c_n z^n$ is 1 and suppose $c_0 \geq c_1 \geq c_2 \geq \cdots$, $\lim_{n\to\infty} c_n = 0$ then $\sum c_n z^n$ converges at every point on the circle at $|z| = 1$ except possibly at $z = 1$.

<u>proof</u>: Put $a_n = z^n$, $b_n = c_n$

Write $A_n = \sum_{k=0}^{n} a_k$

Now $|A_n| = \left|\sum_{k=0}^{n} a_k\right| = |1 + z + z^2 + \cdots + z^n|$

$$= \left|\frac{1 - z^{n+1}}{1 - z}\right| = \left|\frac{1 - z^{n+1}}{1 - z}\right| \leq \frac{|1 + z^{n+1}|}{1 - z}$$

Now suppose, $|z| = 1$ and $z \neq 1$

Then $|A_n| \leq \frac{1 + |z^{n+1}|}{|1 - z|} = \frac{1+1}{|1-z|} = \frac{2}{|1-z|} = M$

Therefore the sequence $\{A_n\}$ of partial sums bounded.

Now, by the known theorem.

We get $\sum c_n z^n$ convergent

* Definition :-

Absolute convergence :- The series Σa_n is said to converge absolutely if the series $\Sigma |a_n|$ converges.

*** Theorem :- (Imp) If Σa_n converges absolutely then Σa_n converges.

proof :- Suppose Σa_n converges absolutely i.e, $\Sigma |a_n|$ converges.

but $\epsilon > 0$

Since $\Sigma |a_n|$ converges by cauchy criterian, $\exists$ an integer M such that

$$\left|\sum_{k=m}^{n} a_k\right| \leq \sum_{k=m}^{n} |a_k| < \epsilon \text{ for } n \geq m \geq M$$

Now, $$\left|\sum_{k=m}^{n} a_k\right| \leq \sum_{k=m}^{n} |a_k| < \epsilon \text{ for } n \geq m \geq M$$

Now by cauchy criterian, we get that Σa_n converges.

* Definition :-

If Σa_n converges but $\Sigma |a_n|$ diverges, we say that Σa_n converges non absolutely

Ex :- Write $a_n = \frac{(-1)^{n+1}}{n}$.

Now, we show that Σa_n converges non absolute

To show that Σa_n converges.

We use theorem (Leibnitz theorem) clearly

$|a_1| = 1 > \frac{1}{2} = |a_n| > \frac{1}{3} = |a_3| > \ldots\ldots$

$$a_{2m-1} = \frac{(-1)^{2m-1+1}}{2m-1} = \frac{1}{2m-1} \geq 0$$

$$a_{2m} = \frac{(-1)^{2m+1}}{2m} = \frac{-1}{2m} \leq 0$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0$$

$\therefore$ By Leibnitz theorem, the theore series Σa_n converges.

But $\Sigma|a_n| = \Sigma \left| \frac{(-1)^{n+1}}{n} \right| = \Sigma \frac{1}{n}$ diverges.

* Addition and Multiplication of Series :

Theorem : If $\Sigma a_n = A$, $\Sigma b_n = B$, then the $\Sigma(a_n + b_n) = A + B$ $\Sigma c a_n = cA$, for any fixed c.

proof : Write $A_n = \sum_{k=0}^{n} a_k$ and $B_n = \sum_{k=0}^{n} b_k$

Now $A + B = \Sigma a_n + \Sigma b_n = \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n = \lim_{n\to\infty} (A_n + B_n)$

$$\lim_{n\to\infty} \left(\sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k\right)$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} (a_k + b_k) = \sum_{k=0}^{\infty} (a_k + b_k)$$

Now take the constant c.

Consider $cA = c\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} (cA_n)$

$$= \lim_{n\to\infty} c\left(\sum_{k=0}^{n} a_k\right) = \lim_{n\to\infty} (cA_n)$$

$$= \lim_{n\to\infty} c\left(\sum_{k=0}^{n} a_k\right) = \lim_{n\to\infty} \sum_{k=0}^{n} c a_k$$

$$= \sum_{k=0}^{n} c a_k .$$

Definition:

Given $\sum a_n$ and $\sum b_n$

Write $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for $0, 1, 2, \ldots$

$\sum c_n$ is called the product of $\sum a_n$ and $\sum b_n$. This product is also called cauchy product.

* Note: Consider $\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right)$

$= (a_0 + a_1 z + a_2 z^2 + \ldots)(b_0 + b_1 z + b_2 z^2 + \ldots)$

$= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \ldots$

$= c_0 + c_1 z + c_2 z^2 + \ldots$

If we take $z = 1$, then we get

* $\sum c_n = (\sum a_n)(\sum b_n)$

*** Theorem: Suppose $\boxed{\text{Mertens theorem}}$

(a) $\sum_{n=0}^{\infty} a_n$ converges absolutely

(b) $\sum_{n=0}^{\infty} a_n = A$

(c) $\sum_{n=0}^{\infty} b_n = B$

(d) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, $n = 0, 1, 2, 3, \ldots$ then

$\sum_{n=0}^{\infty} c_n = AB$.

proof: Let $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$ and

$d_n = \sum_{k=0}^{n} c_k$

$B_n = \beta_n + B$

put $\beta_n = B_n - B$

consider $d_n = c_0 + c_1 + \cdots + c_n$

$= a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0)$

$= (a_0 b_0 + a_0 b_1 + a_0 b_2 + \cdots + a_0 b_n) + a_1 (b_0 + b_1 + \cdots + b_{n-1}) + \cdots + a_n b_0$

$= a_0 \sum_{k=0}^{n} b_k + a_1 \sum_{k=0}^{n-1} b_k + \cdots + a_n \sum_{k=0}^{0} b_k$

$= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0$

$= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \cdots + a_n (B + \beta_0)$

$= a_0 B + a_1 B + \cdots + a_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0$

$= B(a_0 + a_1 + \cdots + a_n) + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0$

since $A = \sum a_k$ we have that $\quad \lim_{n\to\infty} c s_n = cs$

$\lim_{n\to\infty} A_n = A$ and so $\lim_{n\to\infty} A_n B = AB$

consider $\lim_{n\to\infty} d_n = \lim_{n\to\infty} (A_n B + \gamma_n)$

$= \lim_{n\to\infty} A_n B + \lim_{n\to\infty} \gamma_n$

$= AB + \lim_{n\to\infty} \gamma_n$

Now we show that $\lim_{n\to\infty} \gamma_n = 0$

from (a) $\sum a_n$ converges absolutely

put $\alpha = \sum_{n=0}^{\infty} |a_n|$.

let $\epsilon > 0$

put $\epsilon_1 = \dfrac{\epsilon}{1+|\alpha|} = \dfrac{\epsilon}{1+\alpha}$

consider $\lim \beta_n = \lim (B_n - B) = \lim\limits_{n\to\infty} B_n - \lim\limits_{n\to\infty} B$

$= B - B = 0$

Therefore, $\exists$ an integer N such that $|\beta_n| < \epsilon_1$ for $n \geq N$. fix $n \geq N$

consider $|\gamma_n| = |a_0\beta_n + a_1\beta_{n-1} + \dots + a_n\beta_0|$

$\leq |\beta_0 a_n + \beta_1 a_{n-1} + \dots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-(N+1)} + \dots \beta_n a_0|$

$< |\beta_0 a_n + \beta_1 a_{n-1} + \dots + \beta_N a_{n-N}| + \epsilon_1 (|a_{n-(N+1)}| + \dots + |a_0|)$

$< |\beta_0||a_n| + |\beta_1||a_{n-1}| + \dots + |\beta_N||a_{n-N}| + \epsilon_1 \alpha$

$\Rightarrow \limsup\limits_{n\to\infty} |\gamma_n| < \limsup\limits_{n\to\infty} (|\beta_0||a_n| + |\beta_1||a_{n-1}| + \dots + |\beta_N||a_{n-N}|) + \epsilon_1 \alpha$

$= |\beta_0| \lim\limits_{n\to\infty} |a_n| + |\beta_1| \lim\limits_{n\to\infty} |a_{n-1}| + \dots + |\beta_N| \lim\limits_{n\to\infty} |a_{n-N}| + \epsilon_1 \alpha$

$= \epsilon_1 \alpha$

$= \left(\frac{\epsilon}{1+|\alpha|}\right)\alpha < \epsilon$

therefore $\limsup\limits_{n\to\infty} |\gamma_n| < \epsilon$

since ϵ is arbitary

$\limsup\limits_{n\to\infty} |\gamma_n| = 0$

$\liminf\limits_{n\to\infty} |\gamma_n| = 0$

therefore, $\lim\limits_{n\to\infty} \gamma_n = 0$

Thus. $\lim\limits_{n\to\infty} d_n = AB + \lim\limits_{n\to\infty} \gamma_n = AB + 0 = AB$

$\therefore$ Therefore $\boxed{\sum_{n=0}^{\infty} c_n = AB}$

* Rearrangement :-

Definition :- Let $\{k_n\}$, $n=1,2,3,\dots$ be a sequence in which every positive integer appe once and only once (that is $\{k_n\}$ is a bijection from N to N)

Suppose $\{a_n\}$ is a sequence. Then $\{a_{n_k} \mid n_k \in N\} = \{a_n \mid n \in N\}$

Now, write $a_n' = a_{k_n}$ for all $n \in N$.

Then $\sum a_n'$ is called a rearrangement of $\sum a_n$.

Example :

Consider the series, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

This series satisfies the properties.

(a) $|c_1| \geq |c_2| \geq \dots$

(b) $c_{2m-1} \geq 0$ & $c_{2m} \leq 0$ for $m = 1, 2, \dots$

The given series is convergent

Suppose

Consider one of its rearrangements

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots$$

Consider, $-\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$

$$= \left(-\frac{1}{4} + \frac{1}{5}\right) + \left(-\frac{1}{6} + \frac{1}{7}\right) + \dots \leq 0$$

Therefore, $s - \left(1 - \frac{1}{2} + \frac{1}{3}\right)$

$$= \left(-\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots\right) \leq 0$$

$$\Rightarrow s \leq 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

Suppose s_n' is the n^{th} partial sum of series

$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\cdots$

Since $\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{2k}>0$ [always true $\forall k$]

We have that $s'_3 < s'_6 < s'_9 < \cdots$

Therefore, $\limsup_{n\to\infty} s_n > s'_3 = \frac{5}{6}$

In the above, we see that $s < \frac{5}{6}$

Therefore, $\Sigma a'_n > \frac{5}{6}$ & $\Sigma a_n < \frac{5}{6}$

Therefore a rearrangement of a given series and the given series may not converge to the same point.

** Imp.

* **Theorem**: Let Σa_n be a series of a real numbers which converges but not absolutely. Suppose $-\infty \le \alpha \le \beta \le \infty$ then $\exists$ a rearrangement $\Sigma a'_n$ with partial sums s'_n such that

$$\liminf_{n\to\infty} s'_n = \alpha, \quad \limsup_{n\to\infty} s'_n = \beta$$

proof: Let $p_n = \frac{|a_n|+a_n}{2}$ & $q_n = \frac{|a_n|-a_n}{2}$ for $n=1,2,\ldots$

then $p_n - q_n = a_n$ & $p_n + q_n = |a_n|$

Also $p_n \geq 0$ and $q_n \geq 0$

Step-1: To show $\Sigma p_n, \Sigma q_n$ both diverges.

If Σp_n & Σq_n both converges then

$\Sigma(p_n+q_n) = \Sigma|a_n|$ converges.

a contradiction to hypothesis

If Σp_n converges and Σq_n diverges.

Σa_n diverges, a contradiction

Similarly, if Σp_n diverges and Σq_n converges then $\Sigma a_n = \Sigma p_n - \Sigma q_n$ diverges, a contradiction

$\therefore$ Σp_n and Σq_n both diverges.

Step - ii :

let $P_1, P_2, P_3 \ldots$ denote the non-negative terms of Σa_n in the order in which they occur

let $Q_1, Q_2, \ldots$ be absolute values of negative terms of Σa_n also in their original order.

Note that ΣP_n and Σp_n differ only by zero terms.

Similarly, ΣQ_n differ from Σq_n by zero terms

Since Σp_n, Σq_n diverges, We have that

$\Sigma Q_n = \Sigma q_n$, $\Sigma P_n = \Sigma p_n$ diverges.

Step - iii :

choose real valued sequences $\{\alpha_n\}$, $\{\beta_n\}$ such that $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\alpha_n < \beta_n$, $\beta_1 > 0$.

let m_1, k_1 be the smallest integers such that

$$P_1 + \ldots + P_{m_1} > \beta_1$$

$$P_1 + \ldots + P_{m_1} - Q_1 \ldots Q_{k_1} < \alpha_1$$

let m_2, k_2 be the smallest integer such that

$$- + \ldots + P_{m_1} - Q_1 \ldots Q_{k_1} + P_{m_1+1} + \ldots + P_{m_2} > \beta_2$$

$$- + P_{m_1} - Q_1 \ldots Q_{k_1} + P_{m_1+1} + \ldots + P_{m_2} - Q_{k_1+1} \ldots$$

$$\ldots Q_{k_2} < \alpha_2$$

Continue in this way, we get a series

$$- P_{m_1} - Q_1 \ldots Q_{k_1} + P_{m_1+1} + \ldots + P_{m_2} - Q_{k_1+1} \ldots Q_{k_2} + \ldots$$

Step - IV :

Suppose $x_1 = P_1 + \dots + P_m$, $y_1 = P_1 + \dots + P_{m_1} - Q_1 \dots Q_{k_1}$

$x_2 = P_1 + \dots + P_m - Q_1 - \dots - Q_{k_1} + P_{m+1} + \dots + P_{m_2}$,

$y_2 = P_1 + \dots + P_{m_1} - Q_1 \dots Q_{k_1} + P_{m_1+1} + \dots P_{m_2} - \dots - Q_{k_1+1} \dots Q_{k_2}$.

and so on

clearly, $x_n \to \beta$, $y_n = \alpha$

Therefore, $\liminf_{n\to\infty} S_n' = \lim_{n\to\infty} y_n = \alpha$, $\limsup_{n\to\infty} S_n' = \lim_{n\to\infty} = \beta$

where S_n' is the n^{th} partial sum of the $\sum a_n'$

Theorem :

If $\sum a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum a_n$ converges and they all converge to the same sum.

proof : Let $\sum a_n'$ be a rearrangement with partial sums S_n.

Given $\epsilon > 0$

There exists an integer N such that $m \geq n \geq N$

$\Rightarrow \sum_{i=n}^{m} |a_i| \leq \epsilon$

$\sum P_n$ and $\sum q_n$ both diverges

Since $\sum a_n'$ is a rearrangement, We may assu that $\sum a_n' = \sum a_{k_n}$

choose $p \ni$ the integers $1, 2, \dots N$ are all contained in the set $k_1, k_2, \dots, k_p$

Consider $S_n - S_n' = (a_1 + \dots + a_n) - (a_1' + \dots + a_n')$

If $n > p$ then $a_1, \dots, a_N$ occur in both the parts of right hand side and so they cancel. Then we get that

$$|S_n - S_n'| \le \sum_{i=n}^{m} |a_i| \le \epsilon$$

This is true for all $n > p$

This is show that $\{S_n - S_n'\}$ is convergent

Since $\{S_n\}$ is convergent and since

$$S_n' = S_n - (S_n - S_n')$$

We have that $\{S_n'\}$ is convergent.

i.e, $\sum a_n'$ converges.

Since $|S_n - S_n'| \le \epsilon$ for every $n > p$ & E is arbitrary, We conclude that $\{S_n\}$ & $\{S_n'\}$ converges to the same sum.

Thus, $\sum a_n$ & $\sum a_n$ converges to same point.

* Problems.

① Find the Upper & lower limits of the sequence $\{S_n\}$ defined by $S_1 = 0$, $S_{2m} = \frac{S_{2m-1}}{2}$,

$S_{2m+1} = \frac{1}{2} + S_{2m}$

Sol: $S_1 = 0$, $S_2 = \frac{S_{2-1}}{2} = \frac{S_1}{2} = \frac{0}{2} = 0$

$$S_3 = \frac{1}{2} + S_2 = \frac{1}{2}$$

$$S_4 = \frac{S_3}{2} = \frac{1}{4} = \frac{1}{2^2}$$

$$S_5 = \frac{1}{2} + S_4 = \frac{1}{2} + \frac{1}{2^2}$$

Consider the sequence $\{S_{2m+1}\}$ i.e, $S_1, S_3, \dots$

$S_1 = 0,\ S_3 = \frac{1}{2}, \ldots$ in general $S_{2m+1} = \sum_{k=1}^{m} \frac{1}{2^k}$

$$\lim_{n\to\infty} S_{2m+1} = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = 1$$

$\therefore \{S_{2m+1}\}$ converges to 1.

Now consider the sequence $\{S_{2n}\}$ i.e, $S_2, S_4, \ldots$

In general, $S_{2m} = S_{2m+1} - \frac{1}{2}$

$$\lim_{n\to\infty} S_{2m} = S_{2m} - \frac{1}{2}$$

$$\lim_{n\to\infty} S_{2m} = \lim_{n\to\infty}\left(S_{2m+1} - \frac{1}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

$\therefore \{S_{2m}\}$ converges to $1/2$

It is clear that if any subsequence converges to a then $a = \frac{1}{2}$ (or) 1

$\therefore E =$ the set of all subsequential limits

$\therefore$ The lower limits of $\{S_n\}$ is $1/2$ and the upper limit of $\{S_n\}$ is 1.

(2) If $S_1 = \sqrt{2}$ and $S_{n+1} = \sqrt{2+\sqrt{S_n}}$ $(n = 1, 2, \ldots)$ prove that $\{S_n\}$ converges and that $S_n > 2$ for $n = 1, 2, 3, \ldots$

Sol: Given that $S_1 = \sqrt{2}$, $S_2 = \sqrt{2+\sqrt{S_1}} = \sqrt{2+\sqrt{2}}$

$\Rightarrow S_2 > S_1$

Assume that $S_n > S_{n-1}$

Consider $S_n > S_{n-1} \Rightarrow \sqrt{S_n} > \sqrt{S_{n-1}}$

$\Rightarrow 2+\sqrt{S_n} > 2+\sqrt{S_{n-1}}$

$\Rightarrow \sqrt{2+\sqrt{S_n}} > \sqrt{2+\sqrt{S_{n-1}}}$

$\Rightarrow S_{n+1} > S_n$

$\therefore S_{n+1} > S_n$ for $n = 1, 2, 3, \ldots$

$\therefore \{S_n\}$ is monotonically increasing

clearly $S_n > 0$ for all $n = 1, 2, 3, \ldots$

$\therefore S_n < 2$ for all n

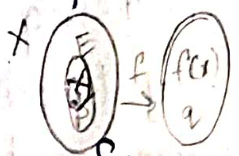
therefore, $\{S_n\}$ is bounded monotonically increasing sequence.

By known theorem, we have that it is convergent.

Unit - III Continuity

* Limit of functions:

<u>Definition</u>: Let (X, d_X), (Y, d_Y) be two metric spaces and $E \subseteq X$, $f: X \to Y$ and $q \in Y$, let 'p' be a limit point of E. We write $f(x) \to q$ as $x \to p$ (or) $\lim\limits_{x \to p} f(x) = q$ (or) limit $f(x) = q_Y$, if it satisfies the following property.



given, $\epsilon > 0$ there exist corresponds a $\delta > 0 \ni$ $0 < d_X(x, p) < \delta$

$\Rightarrow d_Y(f(x), q) < \epsilon$ where $x \in E$

***

<u>Theorem</u>: Let X and Y be two metric spaces $E \subseteq X$ and $f: E \to Y$ and p be a limit point of E then $\lim\limits_{x \to p} f(x) = q \Leftrightarrow \lim\limits_{n \to \infty} f(P_n) = q$ for every sequence $\{P_n\}$ in $E \ni P_n \neq p$ and $\lim\limits_{n \to \infty} P_n = P$.

$P_n \to P$

proof: Given that $\lim\limits_{x \to p} f(x) = q \to ①$

choose any $\epsilon > 0$ corresponding to this ϵ $\exists$ a $\delta > 0$ $\ni d_Y(f(x), q) < \epsilon$ where $0 < d_X(x, p) < \delta \to ②$

choose sequence $\{P_n\}$ which converges to p ($P_n \neq$ corresponding to the $\delta > 0$ $\exists$ a positive integer N $\ni d_X(P_n, P) < \delta \ \forall\ n \geq N$

$\Rightarrow d_Y(f(P_n), q) < \epsilon \ \forall\ n \geq N$

$\therefore \lim\limits_{n \to \infty} f(P_n) = q$

Converseely suppose that $\lim\limits_{n \to \infty} f(P_n) =$

claim: $\lim\limits_{x \to p} f(x) = q$

In a contrary way, suppose that $\lim_{x \to p} f(x) = q$ is not true.

If $\lim_{x \to p} f(x) = q$ is false then $\exists\ \epsilon > 0$ $\ni \delta > 0$ there is an $x \in E$ for which $d_Y(f(x), q) \geq \epsilon$

and $0 < d_X(x, p) < \delta$

Let us choose sequence $\{P_n\} \ni d_X(P_n, P) < \delta_n$

where $\delta_n = \frac{1}{n}$ for $n \in N$

Now, $d_X(P_n, P) < \delta_n \Rightarrow d_X(P_n, P) < \frac{1}{n}$

choose a positive integer $M \ni M\delta > 1$

Now, $\delta > \frac{1}{M}$ for each $n \geq M$

We have $d_X(P_n, P) < \frac{1}{n} < \frac{1}{M} < \delta$

So, we have a sequence $\{P_n\}$ which lies in $0 < d_X(P_n, P) < \delta$ and satisfying $d_Y(f(P_n), q) \geq \epsilon$

This shows that $f(P_n)$ doesnot converges to q.

[Conversely su $\lim_{n \to \infty} f(P_n)$

This is contradiction

So, $\lim_{x \to p} f(x) = q$.

* **Corollary**: If f has a limit point then the limit is unique.

Proof: Assume q & q' be two limit points of the function.

Now, we have to show that $q = q'$

let us assume that $q \neq q'$

Then $d(q, q') > 0$

choose $\epsilon = \frac{1}{2} d(q, q') > 0$

$\lim_{x \to p} f(x) = q$ $\epsilon > 0$ $\exists$ $\delta_1 > 0 \ni d_Y(f(x), q) < \epsilon$

where ever $0 < d_X(x, p) < \delta_1$

let $\epsilon > 0$ $\exists$ $\delta_2 > 0 \ni d_Y(f(x), q') < \epsilon$ wherever $0 < d_X(x, p) < \delta_2$.

let $\delta = \min\{\delta_1, \delta_2\}$

If $0 < d_X(x, p) < \delta$ then $d_Y(f(x), q) < \epsilon$ &
$d_Y(f(x), q') < \epsilon$

Now, $d_Y(q, q') \leq d_Y(q, f(x)) + d_Y(f(x), q')$

$$< \epsilon + \epsilon$$
$$< 2\epsilon$$
$$< 2 \cdot \frac{1}{2} d_Y(q, q')$$
$$< d_Y(q, q')$$

This is a contradiction.

Hence $q = q'$

Definition: Suppose f and g are two complex functions in E into Y then we define.

(i) $(f+g)x = f(x) + g(x)$

(ii) $(f-g)(x) = f(x) - g(x)$

(iii) $(fg)x = f(x) \cdot g(x)$

(iv) $\left(\frac{f}{g}\right)x = f(x) / g(x)$ if $g(x) \neq 0$

(v) $(\lambda f)(x) = \lambda f(x)$ for every real number $x \in E$

Theorem: Suppose $E \subseteq X$, a metric space. P is a limit point of E. f, g are complex functions on E and $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$ then prove that

(a) $\lim_{x \to p} f(x) \pm g(x) = A \pm B$

(b) $\lim_{x \to p} (fg)(x) = A \cdot B$ and

(c) $\lim_{x \to p} (f/g)(x) = A/B$ if $B \neq 0$

proof: Let $\{P_n\}$ be a sequence from E which converges to P [Here $P_n \neq p \ \forall n$]

Now, $\lim_{n \to \infty} f(p_n) = A$ and $\lim_{n \to \infty} g(P_n) = B$ where

$P_n \neq p$ and $\lim_{n \to \infty} P_n = P$.

By the definition

$$\lim_{n \to \infty} \{f(P_n) \pm g(P_n)\} = \lim_{n \to \infty} f(p_n) \pm \lim_{n \to \infty} g(P_n)$$
$$= A \pm B$$

$$\lim_{n \to \infty} \{f(p_n) \cdot g(P_n)\} = \lim_{n \to \infty} f(p_n) \cdot \lim_{n \to \infty} g(P_n) = A \cdot B$$

$$\lim_{n \to \infty} \frac{f(P_n)}{g(P_n)} = \frac{\lim_{n \to \infty} f(p_n)}{\lim_{n \to \infty} g(P_n)} = \frac{A}{B} \text{ if } B \neq 0.$$

* **Note**: From known theorem, we get that

(i) $\lim_{x \to p} (f(x) + g(x)) = \lim_{x \to p} f(x) + \lim_{x \to p} g(x)$

(ii) $\lim_{x \to p} (f(x) \cdot g(x)) = \lim_{x \to p} f(x) \cdot \lim_{x \to p} g(x)$

(iii) $\lim_{x \to p} [f(x)/g(x)] = \lim_{x\to p} f(x) \Big/ \lim_{x\to p} g(x)$.

*** Definition :**

Suppose X and Y are metric spaces wi[th] metrices d_X, d_Y respectively. $E \subseteq X$, $p \in E$ an[d] $f: E \to Y$ then f is said to be a continuous function at p, if for every $\epsilon > 0$ there corresponds a $\delta > 0$ such that $d_Y(f(x), f(y)) < \epsilon$ whenever $x \in E$ with $d_X(x, p) < \delta$. If f is conti[n]uous at every point of E, then we say that f is continuous on E.

*** Theorem :**

Let X and Y are metric space. $E \subseteq X$, p is a limit point of E then f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$

proof: Part - I

Suppose f is continuous at p.
Now we show that $\lim_{x\to p} f(x) = f(p)$

Since f is continuous, for each $\epsilon > 0$, $\exists\ \delta > 0$ $\ni d_Y(f(x), f(p)) < \epsilon$ whenever, $d_X(x, p) < \delta$ for $x \in E$

Write $q = f(p)$ then $q \in Y$ and $d_Y(f(x), q) < \epsilon$

for all $x \in E$ for which $0 < d_X(x, p) < \delta$

$\Rightarrow \lim_{x \to p} f(x) = q = f(p)$

$\therefore \lim_{x \to p} f(x) = f(p)$

Conversely suppose that $\lim_{x \to p} f(x) = f(p)$

Now, we show that f is continuous.

let $\epsilon > 0$

Since $\lim_{x \to p} f(x) = f(p)$ there exist a $\delta > 0$

such that $d_Y(f(x), f(p)) < \epsilon$

whenever $0 < d_X(x, p) < \delta \rightarrow$ ①

If $x = p$ then $d_Y(f(x), f(p)) = d_Y(f(p), f(p))$

$= 0 < \epsilon \rightarrow$ ②

from ① & ②

$d_Y(f(x), f(p)) < \epsilon$ when $d_X(x, p) < \delta$

$\therefore f$ is continuous at p.

*** Imp

** Theorem: Suppose X, Y, Z at metric space. $E \subseteq X$, $f: E \to Y$. g maps the range of f [that is $f(E)$] into Z. $h: E \to Z$ such that $h(x) = g(f(x))$ for all $x \in E$. If f is continuous at $p \in E$ and g is continuous at $f(p)$, then h is continuous at p.

range → only mapped ele in codom

Proof: To show that h is continuous, aim

Take $\epsilon > 0$ such that $d_Z(g(y), g(f(p))) < \epsilon$ if $d_Y(y, f(p)) < \delta$, $y \in f(E) \rightarrow$ ①

Since f is continuous at p, and $\epsilon' > 0$ there corresponds a $\delta > 0$ such that

$d_X(x, p) < \delta,\ x \in E \Rightarrow d_Y(f(x), f(p)) < \epsilon' \rightarrow ②$

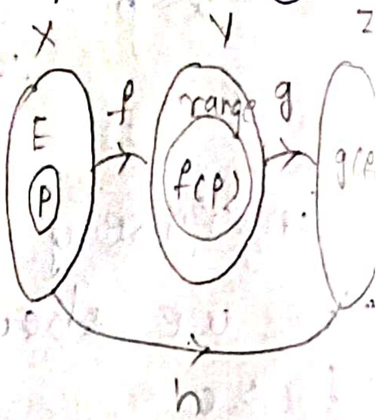
Now $x \in E$, $d_X(x, p) < \delta$

$\Rightarrow d_Y(f(x), f(p)) < \epsilon'$ (by ②)

$\Rightarrow d_Z(g(f(x)), g(f(p))) < \epsilon$ by ①

$\Rightarrow d_Z(h(x), h(p)) < \epsilon$

$\therefore$ h is continuous at p.

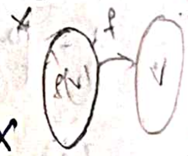


$h(x) = g(f(x))$

***

Imp

** Theorem: A mapping $f: X \to Y$ is continuous on X iff $f^{-1}(V)$ is open in X for every open set V in Y.



proof: Suppose f is continuous on X

Let V be an openset in Y. To show $f^{-1}(V)$ is open

We have to show that, every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$.

Suppose, $p \in X$, and $f(p) \in V$ that is

$p \in f^{-1}(V) = \{x \mid f(x) \in V\}$

Now, we show that p is an interior point of $f^{-1}(V)$.

Since V is open there exists an $\epsilon > 0$ such that

$\{y \in Y \mid d_Y(y, f(p)) < \epsilon\} \subseteq V$ is contained in

Since f is continuous,

$\Rightarrow$ There exists $\delta > 0$ such that $d_Y(f(x), f(p))$ whenever $d_X(x, p) < \delta$

$\Rightarrow \{x \in X \mid d_X(x, p) < \delta\} \subseteq f^{-1}(V)$

$\Rightarrow p$ is an interior point of $f^{-1}(V)$ $\quad x \in N_{r}(p)$, p is int

that is $f^{-1}(V)$ is open in X $\quad d(x,y)<r$

Conversely suppose $f^{-1}(V)$ is open in X $\quad y \in N_r(x)$, $y \in x \subseteq E$

for any openset V in Y

Now, we have to show that f is continuous on X.

let $p \in X$ and $\epsilon > 0$

$$V = \{ y \in Y \mid d_Y(f(p), y) < \epsilon \} \rightarrow y \in V$$

Since V is an openset in Y. $\rightarrow$ let $p \in X$, $f(p) \in Y$

$\rightarrow y = f(p)$

Since $(d_Y(f(p), f(p))) = 0 < \epsilon$, we have $f(p) \in V$

$\therefore$ By converse hypothesis, $f^{-1}(V)$ is open &

$p \in f^{-1}(V)$ $\quad p \in X$, $x \in X$

$\Rightarrow$ There exists $\delta > 0$ such that

$\{ x \in X \mid d_X(x, p) < \delta \} \subseteq f^{-1}(V)$

Now $x \in X$ such that $d_X(x, p) < \delta$ $\quad y = f(x)$

$\Rightarrow x \in f^{-1}(V)$

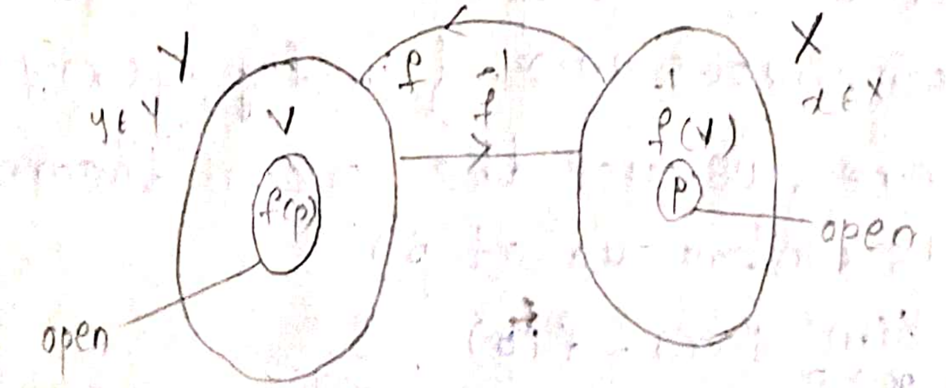
$\Rightarrow f(x) \in V$ $\quad f(p) \in Y$

$x \in N_\delta(p) \subseteq f^{-1}(V)$ and $V = \{ f(x) \in Y \mid d_Y(f(p), f(x)) < \epsilon \}$

(by the definition of V)

$\Rightarrow d_Y(f(p), f(x)) < \epsilon$

This shows that f is continuous.



** corollary : A mapping f of a metric space X into a metric space Y is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y.

Imp

proof: Suppose that f is continuous.

Given that C is closed set in Y.

That implies C' is open in Y.

$f^{-1}(C')$ is open in X (by known theorem)

So, $f^{-1}(C')'$ is closed in X

$\therefore f^{-1}(C)$ is closed in X.

Conversely suppose that $f^{-1}(C)$ is closed in X

To show that f is continuous.

It is enough to show that $f^{-1}(V)$ is open if V is open in Y.

Suppose V is open in Y.

$\Rightarrow V'$ is closed in Y

$\Rightarrow f^{-1}(V')$ is closed in X

$\Rightarrow f^{-1}(V')'$ is open

$\Rightarrow f^{-1}(V)$ is open in X

$\therefore f$ is the continuous.

Hence the theorem is proved.

* Theorem : Let f and g be complex continuous functions on a metric space X then $f+g$, fg and f/g are continuous on X. {in f/g, $g(x) \neq 0$}

proof: Here, we use the known theorem.

(i.e f is continuous at p)

iff $\lim_{x \to p} f(x) = f(p)$

let $p \in X$

① $\lim_{x\to p}(f+g)x = \lim_{x\to p}(f(x)+g(x)) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x)$

[$\because f, g$ are continuous at p] $= f(p) + g(p)$

$= (f+g)p$

② $\lim_{x\to p}(f\cdot g)(x) = \lim_{x\to p} f(x)\cdot g(x)$

$= \lim_{x\to p} f(x) \cdot \lim_{x\to p} g(x)$ [$\because f, g$ are continuous at p]

$= f(p)\cdot g(p)$

$= fg(p)$

③ $\lim_{x\to p}(f/g)(x) = \lim_{x\to p}\frac{f(x)}{g(x)} = \frac{\lim_{x\to p} f(x)}{\lim_{x\to p} g(x)} = \frac{f(p)}{g(p)} = \frac{f}{g}(p)$

Hence, $f+g$, fg, f/g are continuous at all elements $p \in X$.

Hence these are continuous.

*** Theorem : Let $f_1, f_2, \ldots f_k$ be real functions on a metric space X and Y be a mapping of X into R^k defined by $f(x) = (f_1(x), f_2(x), \ldots f_k(x))$ $(x \in K)$. then f is continuous iff each of the functions $f_1, f_2, \ldots$ is continuous.

(b) If f and g are continuous mapping of X into R^k, then $f+g$, fg are continuous.

proof: (a) Suppose $f_i = (f_1, f_2, \ldots f_k)$ is continuous on X.

Now, we have to show that f_i is continuous for $1 \le i \le k$.

Let p be any arbitrary point of X.

Since f is continuous at p.

$\Rightarrow$ given any $\epsilon > 0$ $\exists$ a $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ whenever $|x - p| < \delta$

By definition of metric in R^k, we have

$$|f(x) - f(p)| = \left| \sum_{i=1}^{k} (f_i(x) - f_i(p))^2 \right|^{1/2} < \epsilon \text{ whenever } |x-p| < \delta$$

$$[f_j(x) - f_j(p)]^2 \le \sum_{i=1}^{k} [f_i(x) - f_i(p)]^2$$

$$|f_j(x) - f_j(p)| = \left\{ (f_j(x) - f_j(p))^2 \right\}^{1/2} \le \left[\sum_{i=1}^{k} (f_i(x) - f_i(p))^2 \right]^{1/2}$$

whenever $|x - p| < \delta$ for $j = 1, 2, \ldots, k$

$\Rightarrow$ f_j is continuous at the point p

This is true for all $1 \le i \le k$. $\Rightarrow$ f_i is also continuous at a point p.

<u>Converse</u> :- let $\epsilon > 0$ then $\epsilon/\sqrt{k} > 0$

Since each f_j is continuous at p $\exists$ $\delta_i > 0$ $\ni$

$$|f_i(x) - f_i(p)| < \frac{\epsilon}{\sqrt{k}} \rightarrow (1)$$

Whenever $|f_j(x) - f_j(p)| < \frac{\epsilon}{\sqrt{k}}$ where $|x - p| < \delta$

$$\Rightarrow |f(x) - f(p)| = \left\{ \sum_{j=1}^{k} |f_j(x) - f_j(p)|^2 \right\}^{1/2}$$

from (1),
$$\le \left(\left(\frac{\epsilon}{\sqrt{k}}\right)^2 + \cdots + \left(\frac{\epsilon}{\sqrt{k}}\right)^2 \right)^{1/2}$$

$$= \left(\frac{k\epsilon^2}{k}\right)^{1/2} = \epsilon \text{ when } |x - p| < \delta$$

$\therefore f$ is continuous at p . ie real

(b) Given f, g are continuous. $\} < \phi(p)$

By known theorem, $\therefore$ above theorem

$f+g$, fg are continuous.

Continuity and Compactness

* **Definition**: By an open cover of a set E in a metric space X, we mean a collection $\{G_\alpha\}$ of open sets in X such that $E \subseteq \bigcup G_\alpha$

* **Definition**: A subset K of a metric space X is said to be compact every open cover of K contains a finite subcover (i.e) if $K \subseteq \bigcup G_\alpha$ then $\exists\ \alpha_1, \alpha_2, \ldots \alpha_n \ni K \subseteq (G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n})$

* **Note**: Every finite set is always compact.

proof: Let E be the finite set

Let $\{G_i\}$, $i \in I$ be an open cover.

Now, $E = \{x_1, x_2, \ldots x_n\} \subseteq \bigcup G_i$, $x_i \in \bigcup G_i$

$\exists\ \alpha_i \in I$ such that $x_i \in G_{\alpha_i}$ for $1 \leq i \leq n$

Now, $x_1 \in G_{\alpha_1}, x_2 \in G_{\alpha_2}, \ldots x_n \in G_{\alpha_n}$

$x_1, x_2 \ldots x_n \in G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}$

$E \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$ (This is finite subcover)

Hence every open cover for E has a finite subcover.

This shows that E is compact.

* Definition : A mapping f of a set E into R' is said to be bounded if there is a real number M such that $|f(x)| \le M \; \forall x \in E$,

$-M \le f(x) \le M$ $\qquad f : X \to R$, $f(x)$

Imp

*** Theorem :- Suppose f is a continuous mapping of a compact metric space X into a metric space R^k then $f(X)$ is compact.

proof : Given that f is continuous mapping and $f(X) = \{f(x) / x \in X\}$

Let $\{V_\alpha\}$ be an open cover of $f(X)$

Since f is continuous on X, $f^{-1}(V_\alpha)$ is also an open for each α. (because each V_α is open)

$\Rightarrow f(X) \subseteq \cup_\alpha V_\alpha$ [open cover def]

$\Rightarrow X \subseteq \cup_\alpha (V_\alpha) f^{-1}$

$\Rightarrow \{f^{-1}(V_\alpha)\}$ forms an open cover for X.

Since X is compact, $\exists\, \alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$X \subseteq f^{-1}(V_{\alpha_1}) \cup f^{-1}(V_{\alpha_2}) \cup \ldots \cup f^{-1}(V_{\alpha_n})$

$f(X) \subseteq (V_{\alpha_1} \cup V_{\alpha_2} \cup \ldots \cup V_{\alpha_n})$ which is a finite subcover.

Hence $f(X)$ is compact.

Imp

*** theorem :- If f is continuous mapping of a compact metric space X into R^k then $f(x)$ is closed and bounded. thus f is bounded.

proof : By known theorem (above theorem)

$f(x)$ is compact in R^k ie real

In R^k, we known theorem a subset $E \subset \phi(P)$ compact, iff E is closed and bounded. [Heine borel]

By this fact, we have $f(x)$ is closed and bounded.

Hence f is bounded.

Theorem: Suppose f is a continuous real function on a compact metric space X and $M = \sup_{p \in X} f(p)$ and $m = \inf_{p \in X} f(p)$, then $\exists$ points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$

Proof: By known theorem, we get $f(x)$ is closed and bounded.

By known theorem (i.e, $E \subseteq R$, E is bounded above)

$y = \sup$ then $y \in \bar{E}$ $\exists$

$M = \sup f(x) \in \overline{f(x)}$

Since $f(x)$ is closed then $\overline{f(x)} = f(x)$

Hence $M = \sup f(x) \in \overline{f(x)} = f(x)$ and

$M = \inf f(x) \in \overline{f(x)} = f(x)$.

Imp

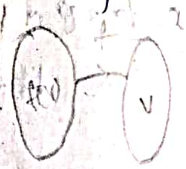
*** Theorem: Suppose f is continuous 1-1 mapping of a compact metric space X onto a metric space Y, then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x$ $(x \in X)$ is a continuous mapping of Y onto X.

Proof: We know that the mapping $f: X \to Y$ is a continuous iff $f^{-1}(V)$ is open in X for

* every open set v in Y.

Therefore to show that $f: Y \to X$ is a continuous:

It is enough to show that $f(v)$ is open in Y if v is open in X.



Since v is open:

$\Rightarrow v^c$ is closed in x

Since every closed subset of a compact set is compact then v^c is compact.

Since v^c is compact

$f(v^c)$ is closed in Y.

$\Rightarrow (f(v^c))^c$ is open in Y

$\Rightarrow f(v)$ is open.

$\Rightarrow f^{-1}$ is continuous.

Uniformly continuous

* **Definition:** Let f be a mapping of a metric space x into a metric space Y.

We say that f is uniformly continuous on x if for every $\exists\ \delta > 0 \ni d_Y(f(p), f(q))$ $\forall\ p, q \in X$ for which $d_X(p, q) < \delta$.

*** Imp

** **Theorem:** Let f be a continuous mapping of compact metric space X into metric space Y then f is uniformly continuous on X.

proof: Let given $\epsilon > 0$. let $p \in X$

Since f is continuous $\exists$ a positive real number $\phi(p)$ such that $q \in X$, $d_X(p,q) < \phi(p)$

$$\Rightarrow d_Y(f(p), f(q)) < \epsilon/2$$

Define $J(p) = \{ q \in X \mid d_X(p,q) < \frac{1}{2}\phi(p) \}$ nbd

Now, $p \in J(p)$ & $J(p)$ is an open set

So, $J(p)$ is non-empty set. interior pt

for each $p \in X$, we get an open set $p \in J(p) \subseteq X$

Now, $X = \bigcup_{p \in X} J(p)$ and so $\{J(p)\}$, $p \in X$ is an open cover of X.

Since X is compact, X has a finite subcover that is there exist $P_1, P_2, \ldots P_n \in X$ such that

$$X \subseteq J(P_1) \cup J(P_2) \cup \ldots \cup J(P_n)$$

Take $\delta = \frac{1}{2} \min\{\phi(P_1), \phi(P_2), \ldots \phi(P_n)\}$ $\exists$ i $(1 \leq i \leq n)$

Such that $p \in J(P_i)$ (because p is nbd)

$$\Rightarrow d_X(P, P_i) < \frac{1}{2}\phi(P_i)$$

Now $d_X(P, P_i) < \frac{1}{2}\phi(P_i) \Rightarrow d_Y(f(p), f(P_i)) < \frac{\epsilon}{2}$

also $d_X(q, P_i) \leq d_X(q, p) + d_X(P, P_i)$

$$< \delta + \frac{1}{2}\phi(P_i)$$

$$< \frac{1}{2}\phi(P_i) + \frac{1}{2}\phi(P_i) = \phi(P_i)$$

$\therefore d_Y(f(p), f(q)) \leq d_Y(f(p), f(P_i)) + d_Y(f(q), f(P_i))$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$< \epsilon$$

Define $J(P)$

$\neq \phi$

open cover compact

sub cover

δ

Hence f is uniformly continuous on X.

$d_X(P_i, q)$

$d_Y(f(P), f(q))$

** Theorem : let E be a non compact set in R then

① there exist a continuous function on E which is not bounded

② $\exists$ a continuous and bounded function on E which has no maximum.

③ If E is bounded then more there exist a continuous function on E which is not uniform continuous.

Proof: Part-1 :

Given that E is not compact set in R.

Suppose E is bounded.

If E is also closed then it is both closed & bounded which implies E is compact.

This is a contradiction.

Therefore, E is not closed.

That is $E \nsubseteq E'$

So $\exists\ x_0 \in \overline{E} - E$

Now, x_0 is a limit point of E

Consider function $f(x) = \dfrac{1}{x - x_0}$ $(x \in E)$

We know that this is continuous function E but.

$\therefore$ We get $\exists$ a continuous function on E which is not bounded.

Now, we show that $f(x)$ is not uniform continuous:

let $\epsilon > 0$ and $\delta > 0$ be an arbitrary
choose $x \in E \ni |x - x_0| < \delta$
We can select a point 't' near to x_0
such that $|f(t) - f(x)| \geq \epsilon$ where as
$|t - x| < \delta$
This means f is not uniformly continuous
on E.
Hence f is continuous function.
which is not uniformly continuous.
Therefore, we get if E is bounded then
$\exists$ a continuous on E which is not uniformly
continuous.
Let function on E which is not uniformly
continuous.
Let us define g as $g(x) = \frac{1}{x + (x - x_0)^2}$ is the
continuous on E and is bounded.
Since $0 < g(x) < 1$
It is clear that $\sup_{x \in E} g(x) = 1$
Therefore, we get there exist a continuous
and bounded functions which has no maximum.
Suppose E is bounded.
Define $f(x) = x$ for all $x \in E$
clearly, it is an unbounded function & continuous
$\therefore$ We get there exist a continuous function
on E, which is not bounded.
Now, we define a function $h(x)$ as

$$h(x) = \frac{x^2}{1+x^2} \quad (x \in E)$$

clearly, it is continuous and bounded $(0 \leq h(x) \leq 1)$

Now, we show that it has no maximal values.

$$h(x) = \frac{x^2}{1+x^2}, \; x \in E$$

Since $0 < h(x) < 1$

$\therefore \sup\limits_{x \in E} h(x) = 1$ and $h(x) < 1 \; \forall x \in E$

$\Rightarrow$ "h" no maximum values on E

Hence, we get $\exists$ a continuous and the bounded function on E which has no maximum

Continuity and Connectedness

Definition: Two subsets A and B of a metric space X are said to be seperated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty (that is no point of A is lies in the closure of B & no point of B lies in the closure of A)

A set $E \subset X$ is said to be connected if E is not an union of two non-empty seperated sets. $E \neq A \cup B$ [E is connected]

Theorem :- If f is continuous mapping of a metric space X into a metric Y and if E is a connected subset of X, then $f(E)$ is connected. Imp

proof: In a contrary way, assume $f(E)$ is not connected.

This implies $f(E)$ can be represented as the union of two seperated sets A and B (subsets of $f(E)$]

That is $f(E) = A \cup B$ where A and B are non-empty sets.

$$\ni \bar{A} \cap B = A \cap \bar{B} = \phi$$

(By definition of connected ness) G, H not empty, sepera, ϕ φ

Let us define $G = E \cap f^{-1}(A)$ & $H = E \cap f^{-1}(B)$

then $G \cup H = [E \cap f^{-1}(A)] \cup [E \cap f^{-1}(B)]$

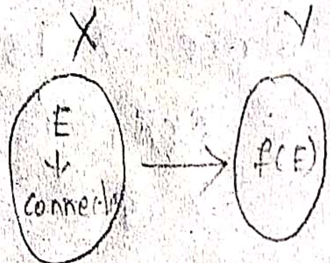
$= E \cap \{f^{-1}(A) \cup f^{-1}(B)\}$

$= E \cap \{f^{-1}(A \cup B)\}$

$= E \cap \{f^{-1}(f(E))\}$

$= E \cap E$

$= E$



$G \cup H = E$
$G \cap H \neq \phi$
$f(G) \neq \phi$
$f(H) \neq \phi$
$f(\bar{G} \cap H) \neq \phi$
$f(G \cap \bar{H}) = \phi$

Here $\boxed{G \cup H = E}$

Consider, $G \cap H = \{E \cap f^{-1}(A)\} \cap \{E \cap f^{-1}(B)\}$

$= E \cap \{f^{-1}(A \cap B)\}$

$= E \cap f^{-1}(\phi)$ [since $A \cap B = \phi$]

$= E \cap \phi$

$= \phi$

Since $G = E \cap f^{-1}(A)$ we have

$f(G) = f(E \cap f^{-1}(A))$

$= f(E) \cap f(f^{-1}(A))$

$= f(E) \cap A$ $\quad [\because A \subset f(E)]$

$= A$

$\neq \phi$

$f(G) \neq \phi \Rightarrow G \neq \phi$

Similarly, we can prove H is nonempty

Now, we show that $\overline{G} \cap H = \phi$ & $G \cap \overline{H} = \phi$

To show $\overline{G} \cap H = \phi$:

We know that $A \subseteq \bar{A}$ (since $\bar{A}$ is closure of $X \Rightarrow f^{-1}(A) \subseteq f^{-1}(\bar{A})$

Now, $G = E \cap f^{-1}(A)$ $\quad [\because f^{-1}(A) \subseteq E]$

$= f^{-1}(A) \subseteq f^{-1}(\bar{A})$

Since $\bar{A}$ is a closed set, f is continuous

$\Rightarrow f(\bar{A})$ is a closed set

$\Rightarrow G \subseteq f^{-1}(A)$ $\quad [\rightarrow E$ is closed $\Leftrightarrow \overline{E} = E]$

$\Rightarrow \overline{G} \subseteq f^{-1}(\bar{A})$

$f(\overline{G}) \subseteq \bar{A} \rightarrow (1)$

Now $H = E \cap f^{-1}(B)$

$f(H) = f(E) \cap B$ $\quad [\because B \subseteq f(E)]$

$= B$

Consider $f(\overline{G} \cap H) = f(\overline{G}) \cap f(H)$

$\subseteq \bar{A} \cap B$

$f(\overline{G} \cap H) = \phi$

Therefore $G \cap H = \phi$

To show that $G \cap \bar{H} = \phi$ $[\because B \subseteq \bar{B}]$

$f^{-1}(B) \subset f^{-1}(\bar{B})$

Since $H = E \cap f^{-1}(B)$

$\subseteq f^{-1}(\bar{B})$ & $f^{-1}(\bar{B})$ is closed

(since f is continuous & $\bar{B}$ is closed.)

$\Rightarrow \bar{H} \subset f^{-1}(\bar{B})$ $\quad H \subset f^{-1}(B)$

$f(\bar{H}) \subseteq f f^{-1}(\bar{B}) \Rightarrow f(\bar{H}) \subseteq \bar{B}$

$G = E \cap f^{-1}(A)$ $\quad [\because A \subseteq f(E)]$

$f(G) = f(E) \cap A = A$

Consider $f(G \cap \bar{H}) = f(G) \cap f(\bar{H})$

$\subseteq A \cap \bar{B}$

$= \phi$

Hence G & H are seperated sets.

That implies E is not connected. $(\because E = G \cup H)$

That is a contradiction to the hypothesis

So, $f(E)$ must be a connected set

*** Imp

*Theorem : Intermediate value theorem :

**

Statement : Let f be a continuous real function on the $[a,b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$ then $\exists$ a point $x \in (a,b)$ such that $f(x) = c$.

Proof : Let f be a continuous mapping on $[a,b]$

We know that $[a,b]$ is a connected,

So, $(f[a,b])$ is connected (by above thr)

Now, $f(a), f(b) \in f([a,b])$

$\Rightarrow (f(a), f(b)) \subseteq f([a,b])$

[A subset E of R is connected iff $x, y \in E \Rightarrow (x,y) \subseteq E$]

Let c be a number such that $f(a) < c < f(b)$

$\Rightarrow c \in [(f(a), f(b))] \subseteq f[a,b]]$

$\Rightarrow c \in f([a,b]) = \{f(x) \mid x \in [a,b]\}$

$\exists$ a point $x \in [a,b]$ such that $f(x) = c$

Since $f(a) < c < f(b)$, we have $a \neq x \neq b$

Hence $\exists\ x \in (a,b)$ such that $f(x) = c$

~~$x \in (a,b)$~~

$f(a) < f(x) < f(b)$

$a < x < b$

$a \neq x \neq b$

Discontinuites

* **Definition**: If x is a point in the domain of definition of the function 'f' at which f is not continuous, it is said that f is discontinuous at x (or) f has a discontinuity at x.

Definition. Let f be defined on (a,b), $x \in (a,b)$ such that $a \leq x < b$ then $f(x^+) = \alpha$ if $f(t_n) \to \alpha$ as $n \to \infty$ $\forall \{t_n\} \in (x,b)$ such that $t_n \to x$

Similarly $f(x^-)$ for $a < x \leq b$ where $\{t_n\} \in (a,x)$ then $\lim_{t \to x} f(t)$ exist if and only if

$$f(x^+) = f(x^-) = \lim_{t \to x} f(t)$$

Definition: Let f be defined on (a,b) if f is discontinuous at a point x & if $f(x^+)$ and $f(x^-)$ exist then f is discontinuity of 1st kind (or) simply discontinuity) otherwise the discontinuity is said to be of second kind.

Monotonic function

*Definition: Let f be a real function on (a,b) then

① f is said to be monotonically increasing on (a,b) if $a<x<y<b \Rightarrow f(x) \le f(y)$

② f is said to be monotonically decreasing on (a,b) if $a<x<y<b \Rightarrow f(x) \ge f(y)$

*Theorem: Let f be monotonically increasing on (a,b) then

(i) $f(x^+)$ and $f(x^-)$ exist at every point of x of (a,b) more precisely.

$$\sup_{a<t<x} f(t) = f(x^-) \le f(x) < f(x^+) = \inf_{x<t<b} f(t)$$

(ii) further more if $a<x<y<b$ then $f(x^+) \le f(y^-)$

Proof: Since f is monotonically increasing on (a,b)

We have $a<x<y<b$

$\Rightarrow f(x) \le f(y)$

Consider the set of all number $f(t)$ where $a<t<x$.

this is bounded above by $f(x)$

[Since f is monotonically increasing, $f(x)$ is an upper bound of set of all $f(t)$ whenever $a<t<x$]

By the completeness axiom the set must have supremum.
Let it is be A
We have to show that $A = f(x^-)$
Choose an $\epsilon > 0$ then $\exists\ \delta > 0 \ni a < x - \delta < x$ and $A - \epsilon < f(x - \delta) \leq A + \epsilon \rightarrow$ ①
Since f is monotonically increasing we have (by ① also)

$$x - \delta < t < x$$

$$A - \epsilon < f(x - \delta) < f(t) \leq A + \epsilon$$

$$A - \epsilon < f(t) < A + \epsilon$$

$$|f(t) - A| < \epsilon \Rightarrow A = f(x^-)$$

$$x - \delta < t < x$$

$$\lim_{t \to x} f(t) = A$$

$$f(x^-) = A$$

Show that $f(x^+) = \underset{x < t < b}{\text{Inf}}(f(t))$
Since f is monotonically increasing on (a,b)
We have $a < x < y < b$

$$\Rightarrow f(x) \leq f(y)$$

Consider, $\{f(t) / x < t < b\}$
Since f is monotonically increasing, this set is bounded below by $f(x)$.
By the completeness axiom on the set must have infimum.

Let it is be B.

We have to show that $B = f(x^-)$

choose an $\epsilon > 0$ then $\exists\ \delta > 0 \rightarrow x < x+\delta < b$

and $B - \epsilon < f(x+\delta) < B+\epsilon \rightarrow$ (2)

since f is monotonically increasing we have (by (2) also)

$$x < t < x+d$$

$$B-\epsilon < f(t) < f(x+d) < B+\epsilon$$

$$B-\epsilon < f(t) < B+\epsilon$$

$$|f(t) - B| < \epsilon$$

with $x < t < x+\delta$

$$\lim_{t \to x} f(t) = B$$

$$f(x^+) = B$$

$$f(x^-) = \sup_{a<t<x} f(t) \le f(x) \rightarrow (B$$

(since $f(x)$ is an upper bound)

$$f(x^+) = \inf_{x<t<b} f(t) \geq f(x) \rightarrow (B$$

(since $f(x)$ is a lower bound)

$$\therefore f(x^-) \le f(x) \le f(x^+)$$

(2) let $a < x < y < b$, $f(x^+) = \inf\limits_{x<t<b} (f(t))$

$$= \inf_{x<t<y} f(t) \rightarrow (5)$$

$$f(y) = \sup_{a<t<y} f(t)$$

$$= \sup_{x<t<y} f(t)$$

$$= \sup_{x<t<y} f(t) \rightarrow (4)$$

from (2) & (4)

$$f(x^+) = \inf_{x<t<y} f(t) \leq \sup_{x<t<y} f(t) = f(y^-)$$

$$\therefore f(x^+) \leq f(y^-)$$

Note: The same results holds for monotonic decreasing function.

*Corollary :- Monotonic function have no discontinuities of the second kind.

proof: Suppose f is monotonic function, then By the known theorem, $f(x^+)$, $f(x^-)$ exist.

If f is discontinuous at any point p then since $f(p^+)$, $f(p^-)$ exist we have that it is discontinuity of first kind.

Hence there is no discontinuity of second kind for monotonic functions.

problem: Consider R^k for each $i (1 \leq i \leq k)$ we define $\phi : R^k \rightarrow R$ as follows i.e. if $x_1, x_2, \ldots$ are the coordinates of the point $x \in R^k$ the function ϕ_i defined by $\phi_i(x) = x_i$ (i.e. $\phi(x_1, x_2, x_3, \ldots x_k) = x_i$) then each ϕ_i is continuous function.

Sol: Take $\epsilon > 0$ write $\delta = \epsilon > 0$

Now, we have to show that $d(x, y) < \delta$

$\Rightarrow d(\phi_i(x), \phi_i(y)) < \epsilon$

Suppose $d(x, y) < \delta$

$d((x_1, x_2, \ldots x_k), (y_1, y_2, \ldots y_k)) < \delta$

$[(x_1 - y_1)^2 + \ldots + (x_k - y_k)^2]^{1/2} < \delta$

consider, $|x_i - y_i| \leq [(x_i - y_i)^2]^{1/2}$

$|x_i - y_i| \leq [(x_1 - y_1)^2 + \ldots + (x_k - y_k)^2]^{1/2} < \delta$

$\Rightarrow |\phi_i(x) - \phi_i(y)| < \delta = \epsilon$

$\therefore d(\phi_i(x), \phi_i(y)) < \epsilon$

$\therefore \phi_i$ is continuous.

problem: Define $f(x) = \begin{cases} x & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational} \end{cases}$

1) Show that f is continuous at $x = 0$

2) f is discontinuous at every point a such that $a \neq 0$

Sol: 1) To show that f is continuous at $x = 0$

let $\epsilon > 0$, write $\delta = \epsilon$

We have to show that $|x - 0| < \delta$

$\Rightarrow |f(x) - f(0)| < \epsilon$

If x is rational then $|f(x) - f(0)| = |x - 0| < \delta = \epsilon$

Hence for every $x \ni |x - 0| < \delta$ we have $|f(x) - f(0)| < \epsilon$

Thus show that f is continuous at $x = 0$.

$\Rightarrow$ let $0 \neq a \in \mathbb{R}$ clearly $|a| > 0$, write $\epsilon = \frac{|a|}{2}$

If f is continuous at a then $\exists\ \delta > 0 \ni |x - a| < \delta$

$\Rightarrow |f(x) - f(a)| < \epsilon$

Take $\delta_1 = \min\{\delta, \epsilon/2\}$

Then $\delta_1 < \delta$ & $|x-a| < \delta_1 < \delta$

$\Rightarrow |f(x) - f(a)| < \epsilon$

Case-i: Suppose a is rational number.

Let y be an irrational number such that

$a - \delta_1 < y < a + \delta_1$

Now $|a-y| < \delta_1$

$\Rightarrow |f(a) - f(y)| < \epsilon$

$\Rightarrow |a| = |a - 0| = \frac{|a|}{2}$

$\Rightarrow |a| < \frac{|a|}{2}$

This is a contradiction.

therefore, f is not continuous at a if $a \neq 0$ & a is a rational number.

Case-ii: Suppose a is a irrational number.

Let y be a rational number such that

$a - \delta_1 < y < a + \delta_1$

Now $|a-y| < \delta_1$

$|f(a) - f(y)| < \epsilon$

$|y| = |0 - y| = |f(0) - f(y)| < \epsilon = \frac{|a|}{2}$

$\Rightarrow |y| < \frac{|a|}{2} \rightarrow (2)$

$\Rightarrow a - \delta_1 < y < a + \delta_1$ $\quad (\because \delta_1 \leq \epsilon/2)$

$\Rightarrow a - \epsilon/2 < a - \delta_1 < y < a + \delta_1 < a + \frac{\epsilon}{2}$

$\Rightarrow a - \epsilon/2 < y < a + \epsilon/2$

Since 'a' is positive

then y is positive and $|a| = a$ and $|y| = y$

$|a - \epsilon/2| = a - \epsilon/2 < y$

$|y| < \frac{|a|}{2}$ (by ②)

Now, $a/2 < 3a/4 = \left|\frac{3a}{4}\right| = \left|a - \frac{|a|}{4}\right| = |a - \epsilon/2|$

$< \frac{|a|}{2} < \frac{a}{2}$

$\therefore a/2 < \frac{a}{2}$

$\therefore$ This is a contradiction.

$\therefore$ In this case f is not continuous at 'a'

Now suppose 'a' is negative.

then y is negative and $|a| = -a$ and $|y| = -y$

Since $a - \epsilon/2 < y < a + \epsilon/2$

We have $y < a + \epsilon/2$

$\Rightarrow -y > -(a + \epsilon/2)$

$\Rightarrow = -a - \epsilon/2$

$\Rightarrow |a| - \epsilon/2 = -a - \epsilon/2 < -y = |y| < \frac{|a|}{2}$ [by ②]

$|a| - \frac{|a|}{4} < \frac{|a|}{2}$

$\Rightarrow \frac{3|a|}{4} < \frac{|a|}{2}$

This is a contradiction.

Hence in this case f is not continuous at 'a'

from case (i) & (ii) we get that f is not continuous at all points 'a' such that $a \neq 0$

** $f(x) = \begin{cases} x+2; & -3 < x < -2 \\ -x-2; & -2 \le x < 0 \\ x+2; & 0 \le x < 1 \end{cases}$, Then f has simple discontinuity at $x=0$

and continuous at any point of $(-3, 1)$

Sol: At $x=0$, $f(0^+) = \lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0^+} (x+2) = 0+2 = 2$

At $f(0^-) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x-2) = 0-2 = -2$

Hence f has a simple discontinuity at $x=0$

Let $a \in (-3,1)$ and $a \neq 0$

Suppose $a=-2$ then $f(-2^+) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (-x-2)$

$= +2-2 = 0$

$f(-2^-) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (+x+2) = -2+2 = 0$

and $f(-2) \overset{-x-2 = -(-2)-2 = 0}{=} 2-2 = 0$

$\therefore f(2^+) = f(-2^-) = f(-2)$

this show that f is continuous at $a=-2$

Since $f(x) = x+2$ on $(-3,-2)$ we have f is continuous on $(-3,-2)$.

Since $f(x) = -x-2$ on $[-2,0]$ we have that f is continuous on $[-2,0)$

Since $f(x) = x+2$ on $[0,1)$

We have that f is continuous on $(0,1)$

$\therefore$ f is continuous at all points $a \in (-3,1)$ except $a=0$

* $f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x=0 \end{cases}$ then show that

(1) $f(0^-)$ is not exist

(2) $f(0^+)$ is not exist and

(3) f has discontinuity of second kind.

Sol: Consider the sequence $\frac{2}{n\pi}$

We know that $\frac{2}{n\pi} \to 0$ as $n \to \infty$

(1) consider two sequences of $\frac{2}{n\pi}$ defined by

$a_n = \frac{2}{(4n+1)\pi}$

$b_n = \frac{2}{(4n+3)\pi}$

clearly $a_n \to 0$ as $n \to \infty$ & $b_n \to 0$ as $n \to \infty$.

let $f(a_n) = \sin\left(\frac{1}{a_n}\right)$

$= \sin\left(\frac{(4n+1)\pi}{2}\right)$

$= \sin\left(\frac{4n\pi}{2} + \frac{\pi}{2}\right) = \sin\left(2n\pi + \frac{\pi}{2}\right) = \sin\frac{\pi}{2}$

$= 1$

$\therefore f(a_n) = 1$ for each n

Hence $\{f(a_n)\}$ converges to 1

Let $f(b_n) = \sin\left(\frac{1}{b_n}\right)$

$= \sin\left(\frac{4n+3}{2}\right)\pi$

$= \sin\left(\frac{4n\pi}{2} + \frac{3\pi}{2}\right)$

$= \sin\left(2n\pi + \frac{3\pi}{2}\right)$

$= \sin\left(2n\pi + \pi + \frac{\pi}{2}\right)$

$= \sin\left(\pi + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$

$\therefore f(b_n) = -1$ for each n.

Hence $\{f(b_n)\}$ converges to -1

Now, $\{a_n\}, \{b_n\}$ are two sequences which tends to 0^{+} but $\{f(a_n)\}$ & $f(b_n)\}$ do not tend to same limit

Hence $f(0^{-})$ do not exist

(2) Write $s_n = \frac{-2}{(4n+1)\pi}$ and $t_n = \frac{-2}{(4n+3)\pi}$

Now $s_n \to 0^{-}$ and $t_n \to 0^{-}$ as $n \to \infty$

$f(s_n) = \sin\left(\frac{1}{s_n}\right)$

$= \sin\left(-\frac{(4n+1)\pi}{2}\right)$

$= \sin\left(\frac{4n\pi}{2} + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$

$f(t_n) = \sin\left(\frac{1}{t_n}\right)$

$= \sin\left(-\frac{(4n+3)\pi}{2}\right)$

$= -\sin\left(2n\pi + \pi + \frac{\pi}{2}\right)$

$= -\sin\frac{\pi}{2} = 1$

Now $E \subseteq f^{-1}(f$

$\bar{E} \subseteq \overline{f^{-1}(f}$

$\bar{E} \subseteq f^{-1}(\bar{f}$

$f(\bar{E}) \subseteq \overline{f(E}$

Now $s_n \to 0$, $t_n \to 0^{-}$ & $f(s_n)$, $f(t_n)$ do not tend to same limit.

Hence $f(0^{-})$ does not ~~limit~~ exist

(3) Since $f(0^{+})$ & $f(0^{-})$ do not exist, we can say that $f(x)$ has discontinuity of second kind.

* Suppose f is a real function defined on R^1 which satisfies $\lim [f(x+h) - f(x-h)] = 0$ for every $x \in R^1$ does the simply that f is a continuous.

Sol: Consider $f: R \to R$ defined by $f(x) = \begin{cases} |x| & \text{if} \\ 1 & \text{if} \end{cases}$

clearly f is discontinuous at $x = 0$.

But it satisfies the required property namely $\lim_{h\to 0}[f(a+h)-f(a-h)]=0$ where $a=0$

If $a=0$ then $f(a+h)-f(a-h)=f(h)-f(h)$

$=|h|-|-h| = |h|+|h|=0$

$\lim_{h\to 0}[f(a+h)-f(a-h)]=0$

<u>conclusion</u> :- Hence the condition $\lim_{h\to 0}[f(x+h)-f(x-h)]=0$ donot imply that f is continuous at x.

** $f: X\to Y$ is continuous and $E\subset X$ then show that $f(\bar{E})\subseteq \overline{f(E)}$

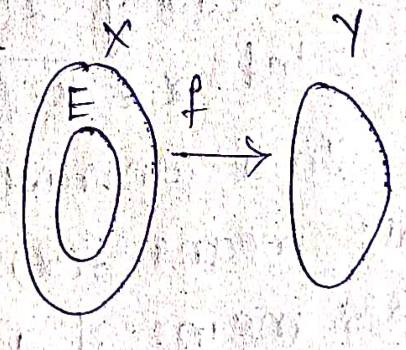
<u>Sol</u>: Let $E\subseteq X$

$f: X\to Y$ is continuous, every set is subset of itself.

$f(E)\subseteq f(E)$

$E\subseteq f^{-1}(f(E)) \to ①$

$A\subset \bar{A}$



We know that $f(\bar{E})\subseteq \overline{f(E)}$

$E\subseteq f^{-1}(\overline{f(E)})$

$\bar{E}\subseteq f^{-1}(\overline{f(E)}) \to ②$

Since f is continuous and $\overline{f(E)}$ is a closed set.

We have that $f^{-1}(\overline{f(E)})$ is also closed set [By corollary, f is continuous iff $f^{-1}(C)$ is closed from <u>any</u> closed set C] and

$f^{-1}(\overline{f(E)}) = \overline{f^{-1}(\overline{f(E)})}$ $\quad A=\bar{A}$ [closed]

Now, $\bar{E}\subseteq f^{-1}(\overline{f(E)})$

$\Rightarrow \bar{E} \subseteq f^{-1}(\overline{f(E)})$

$\Rightarrow f(\bar{E}) \subseteq f(f^{-1}(\overline{f(E)}))$ $[\because f(f^{-1}(\overline{f(E)})) \subseteq \overline{f(E)}]$

$\Rightarrow f(\bar{E}) \subseteq \overline{f(E)}$

* <u>Definition</u> :- Let f be a continuous real func on a metric space X, write $Z(f) = \{p \in X \mid f(p) =$

Thus $Z(f)$ is called the zeroset of f.

<u>Result</u>:- Let f be a continuous real function o a metric space X. let $Z(f)$ (the zero set) be the set of all $p \in X$ at which $f(p) = 0$ prove that $Z(f)$ is closed.

<u>Sol</u>. To show that $Z(f)$ is closed

It is enough to show $\overline{Z(f)} = Z(f)$

clearly $Z(f) \subseteq \overline{Z(f)}$

let $x \in \overline{Z(f)}$

In a contrary way, suppose $x \notin Z(f)$

This means $f(x) \neq 0$ & x is a limit point $Z(f)$

This means $\exists$ a sequence $\{P_n\}$ in $Z(f) \ni P_n \to$

Since f is continuous, we get that $f(P_n) \to f($

Now, $f(P_n) = 0$ (since $P_n \in Z(f)$) for each n.

$\therefore f(P_n) \to 0$

Now, we have $f(P_n) \to f(x)$, $f(P_n) \to 0$

this means $f(x) = 0$

this is a contradiction

Hence $x \in Z(f)$

$\overline{Z(f)} = Z(f)$ which shows that $Z(f)$ is clo

* Definition :- A subset E of a metric space X is said to be dense of $\bar{E} = X$ (equivalently every element of X is a point of E (or) a limit point of E.

** Result :- Let f and g be continuous mappings of a metric space X into a metric space Y & E be a dense subset in (X) then

1) P.T $f(E)$ is dense in $f(X)$

2) If $g(p) = f(p)$ $\forall$ $p \in X$ then P.T $g(p) = f(p)$ $\forall$ $p \in X$.

* **Note** :- In otherwords show that a condition mapping is determined by its values on a dense subset E of its domain X.

Proof : ① Since E is dense subset of X we have that $\bar{E} = X$

$\Rightarrow f(\bar{E}) = f(X) \rightarrow$ ①

By known theorem $f(\bar{E}) \subseteq \overline{f(E)}$

$\therefore$ from ①

$f(X) = f(\bar{E}) \subseteq \overline{f(E)} \subseteq f(E) \cup k$

Where k = the set of all limits points of $f(E)$

$\Rightarrow f(X) \subseteq f(E) \cup k$.

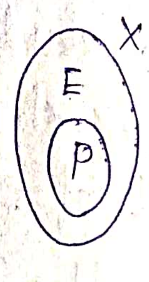
Every element of $f(X)$ is an element of $f(E)$ or an element of k (i.e a limit point of $f(E)$)

Hence $f(E)$ is dense in $f(X)$

② Suppose $f(p) = g(p)$ $\forall$ $p \in E$, let $x \in X$, if $x \in E$ $\rightarrow$ ①

$f(x) = g(x)$

Suppose $x \in X$, $x \notin E$ then x is a limit point of E.



(Since E is dense in X and $x \notin E$)

This means $\exists$ a sequence $x_n \in E \ni x_n \to x$

Since $x_n \in E$ we have that $f(x_n) = g(x_n)$ by (1

$$\Rightarrow (f-g)(x_n) = 0$$

Since f, g are continuous functions

$(f-g)$ is also continuous functions.

Since $x_n \to x$, $f-g$ is a continuous we have that

$$(f-g)(x_n) \to (f-g)(x)$$

Since $(f-g)(x_n) \to f(x_n) - g(x_n) = 0$ for each n

We have that $(f-g)(x_n) \to 0$

Hence $(f-g)(x) = 0$

This implies $f(x) - g(x) = 0$

$$f(x) = g(x)$$

Hence $f(x) = g(x)$ $\forall x \in X$.

**

* Show that the function $f(x)$ defined on $\mathbb{R}$ is discontinuity of first kind where Imp

$$f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Sol: If $x > 0$ then $f(x) = \frac{|x|}{x} = \frac{x}{x} = 1$

If $x < 0$ then $f(x) = \frac{-x}{x} = -1$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (1) = 1$ and

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-1) = -1$

$\therefore \lim_{x\to 0^+} f(x) = 1 \neq -1 = \lim_{x\to 0^-} f(x)$ (and these are not equal to $f(0) = a$)

Hence f is not continuous at $x = 0$

Hence f has discontinuity of 1st kind.

** Show that the function $f(x)$ defined on $\mathbb{R}$ is discontinuous at $x=0$, show that $\lim_{x\to 0^+} f(x) = +\infty$ and $\lim_{x\to 0^-} f(x) = -\infty$ where $f(x) = \begin{cases} 1/x \ ; & \text{if } x \neq 0 \\ 0 \ ; & \text{if } x = 0 \end{cases}$

Imp

Sol:- To show that $\lim_{x\to 0^+} f(x) = +\infty$

Take $\{\frac{1}{n}\}$

clearly $\frac{1}{n} \to 0^+$

Then $f(\frac{1}{n}) = \left(\frac{1}{1/n}\right) = n \to \infty$ as $n \to \infty$

$\therefore \lim_{x\to 0^+} f(x) = +\infty$

To show $\lim_{x\to 0^-} f(x) = -\infty$

Take $\left\{\frac{1}{(-n)}\right\}$

clearly $\frac{1}{(-n)} \to 0^-$

Now, $f(-\frac{1}{n}) = \frac{1}{1/-n} = -n$ as $-\infty$ as $n \to \infty$

$\lim_{x\to 0^-} f(x) = -\infty$

Hence $f(0^+) \neq f(0^-)$ and $f(0^+) \neq f(0) \neq f(0^-)$

Hence f has a discontinuity of second kind at $x=0$

* Discontinuity of first kind :-

Let $f:[a,b]\rightarrow R$ and $x \in [a,b]$; f is said to have the discontinuity of first kind at x. if $f(x^{+})$, $f(x^{-})$ are finite unequals.

* Discontinuity of second kind :-

Let $f:[a,b]\rightarrow R$ & $x \in [a,b]$ f is said to have discontinuity of second kind at x if any one of $f(x+)$ (or) $f(x^{-})$ is either "infinite" nor "not exist"

Unit - 4 Differentiation

* The derivation of a real function:

Let $f:[a,b]\to R$ and $x\in[a,b]$, Suppose $a<t<b$ and $t\neq x$ if the $f'(x)=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$ exists. It is called the derivative of f. It is denoted by $f'(x)$ (or) $\frac{df}{dx}$. In this case we say that f is differentiable at x. We say that f is differentiable on a subset E of R if f is differentiable at every point of E.

** Theorem: Let f be a defined on $[a,b]$. If f is differentiable at a point $x\in[a,b]$ then f is continuous at x.

Proof: Suppose f is differentiable at a point $x\in[a,b]$.

Now we show that f is continuous at x.

Let $t\in[a,b] \ni t\neq x$.

Consider $f(t)-f(x)=\frac{f(t)-f(x)}{(t-x)}(t-x)$ then

$$\lim_{t\to x} f(t)-f(x)=\lim_{t\to x}\frac{f(t)-f(x)}{(t-x)}\cdot\lim_{t\to x}(t-x)$$

$$=f'(x)\cdot 0=0$$

$\lim_{n\to\infty} S_n = l$

$\in$ $S_n - l < \epsilon$

S_n is continu

$$\lim_{t\to x} f(t)=f(x)$$

$\therefore f$ is continuous at $x\in[a,b]$

Note: The converse of the above theorem need not be true. for example, define $f(x)=|x|$ on $[-1,1]$. Thus, $f(x)$ is continuous at $x=0$ but not differentiable.

*Verification: To show $f(x)$ is continuous at $x=0$, we have to show $\lim_{x\to 0} f(x) = f(0)$

let $\epsilon > 0$, $\delta = \epsilon$

Suppose $d(x,0) < \delta$ then $d(f(x), f(0)) = |f(x) - f(0)|$

$$= ||x| - |0||$$

$$= |x - 0|$$

$$= d(x,0)$$

$$< \delta$$

$\therefore d(f(x), f(0)) < \epsilon$

Hence f is continuous at $x=0$.

Now to show that f is not differentiable

let $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \frac{|x|-0}{x-0} = \frac{x}{x} = 1$

and $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{|x|}{x} = \frac{-x}{x} = -1$

$\therefore$ Right limit $\neq$ left limit

Hence limit does not exist.

This shows that $f(x)$ is not differentiable.

*** Imp

* Theorem: Suppose f and g are defined on $[a,b]$ and are differentiable at $x \in [a,b]$ then $f+g$, fg, f/g are differentiable at x

(i) $(f+g)'(x) = f'(x) + g'(x)$

(ii) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

(iii) $(f/g)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$, $[g(x) \neq 0]$

proof: Given that f and g are differentiable at $x \in [a, b]$

① Write $h = f + g$

Now, we have to show that $h'(x) = f'(x) + g'(x)$

$$h'(x) = \lim_{t \to x} \frac{h(t) - h(x)}{t - x}$$

$$= \lim_{t \to x} \frac{(f+g)(t) - (f+g)(x)}{(t - x)}$$

$$= \lim_{t \to x} \frac{f(t) + g(t) - f(x) - g(x)}{(t - x)}$$

$$= \lim_{t \to x} \frac{f(t) - f(x)}{(t - x)} + \lim_{t \to x} \frac{g(t) - g(x)}{(t - x)}$$

$$h'(x) = f'(x) + g'(x)$$

$$\Rightarrow (f+g)'(x) = f'(x) + g'(x).$$

② Write $h = fg$

Now, we have to show that $h'(x) = f'(x)g(x) + f(x)g'(x)$

Let $h(t) - h(x) = (fg)(t) - (fg)(x)$

$$= f(t)g(t) - f(x)g(x) - f(t)g(x) + f(t)g(x)$$

$$\frac{h(t) - h(x)}{t - x} = \frac{f(t)[g(t) - g(x)]}{t - x} + \frac{g(x)[f(t) - f(x)]}{t - x}$$

Now, $h'(x) = \lim_{t \to x} \frac{h(t) - h(x)}{t - x}$

$$h'(x) = \lim_{t \to x} \left[\frac{f(t)[g(t) - g(x)]}{t - x}\right] + \lim_{t \to x} \left[\frac{g(x)[f(t) - f(x)]}{t - x}\right]$$

$$= \lim_{t \to x} [f(t)] \cdot \lim_{t \to x} \frac{g(t) - g(x)}{t - x} + g(x) \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

$\therefore h'(x) = f(x)g'(x) + g(x)f'(x)$.

③ Let us take $h = \frac{f}{g}$

then we show that $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Let $h(t) - h(x) = \frac{f}{g}(t) - \frac{f}{g}(x)$

$$= \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}$$

$$= \frac{f(t)g(x) - f(x)g(t)}{g(t)g(x)}$$

$$= \frac{f(t)g(x) - f(x)g(x) + f(x)g(x) - f(x)g}{g(t)g(x)}$$

$$\lim_{t\to x} \frac{h(t)-h(x)}{t-x} = \frac{\cancel{\lim_{t\to x}}\ g(x)\left[\frac{f(t)-f(x)}{t-x}\right] - f(x)\left[\frac{g(t)-g(x)}{t-x}\right]}{\lim_{t\to x} g(x)g(t)}$$

then $h'(x) = \lim_{t\to x} \dfrac{h(t)-h(x)}{t-x}$

$$= \lim_{t\to x}\left[\frac{g(x)\left[\frac{f(t)-f(x)}{t-x}\right] - f(x)\left[\frac{g(t)-g(x)}{t-x}\right]}{g(x)\cdot g(t)}\right]$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Hence proved.

* Show that $f(x) = |x|$ in $[-1, 1]$ is continuou

Sol: Let $\epsilon > 0$

write $\epsilon = \delta$

Suppose $|x-y|<\delta$ for x,y both are negative (or) both are positive then

$$|f(x)-f(y)| = ||x|-|y||$$
$$= |x-y| < \delta = \epsilon$$

If one of the x and y is positive and other is ≤ 0 then $|f(x)-f(y)| = ||x|-|y|| < |x-y| < \delta = \epsilon$

$|f(x)-f(y)| < \epsilon$ whenever $|x-y|<\delta$

*** **chain rule theorem** : Imp
**

Statement : Suppose f is continuous on $[a,b]$. $f'(x)$ exists at some point $x \in [a,b]$. g is defined on the interval I which contains the range of f and g is differentiable at the point $f(x)$. If $h(t) = g(f(t))$, $a \leq t \leq b$ then h is differentiable at x and $h'(x) = g'(f(x)) \cdot f'(x)$.

Proof: Let $y = f(x)$

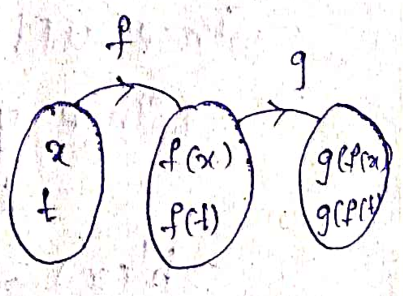
Write $u(t) = \dfrac{f(t)-f(x)}{t-x} - f'(x) \rightarrow ①$

$$v(s) = \frac{g(s)-g(y)}{s-y} - g'(y) \rightarrow ②$$

Now, $\lim\limits_{t\to x} u(t) = \lim\limits_{t\to x}\left[\dfrac{f(t)-f(x)}{t-x} - f'(x)\right]$

$$= \lim_{t\to x}\frac{f(t)-f(x)}{t-x} - \lim_{t\to x} f'(x)$$

$$= f'(x) - f'(x) = 0$$



$\lim\limits_{t\to x} u(t) = 0$

Similarly $\lim\limits_{s\to y} v(s) = 0$

Also, we have from ① & ②

$$\frac{f(t)-f(x)}{t-x} = f'(x) + u(t) \rightarrow ③ \text{ and}$$

$$\frac{g(s)-g(y)}{s-y} = g'(y) + v(s) \rightarrow ④$$

Take $s = f(t)$ & $y = f(x)$

Since $h(t) = g(f(t))$, $a \le t \le b$

$$h(t) - h(x) = g(f(t)) - g(f(x))$$
$$= g(s) - g(y) \quad \text{from } ④$$
$$= (s-y)[g'(y) + v(s)]$$
$$= [f(t) - f(x)][g'(y) + v(s)]$$
$$= (t-x)[f'(x) + u(t)][g'(y) + v(s)] \quad \text{from } ③$$

$$\frac{h(t)-h(x)}{t-x} = [f'(x) + u(t)][g'(y) + v(s)]$$

Taking limits on b.s at $t \to x$ then we get

$$h'(x) = \lim_{t \to x} \frac{h(t)-h(x)}{t-x}$$
$$= \lim_{t \to x} [f'(x) + u(t)][g'(y) + v(s)]$$
$$= (f'(x) + 0)(g'(y) + 0) \quad \text{by } ①$$
$$= f'(x) \cdot g'(y)$$

Since $u(t) \to 0$ as $t \to x$, $v(s) \to 0$ as $s \to y$

(Since $y = f(x)$)

$$\therefore h'(x) = f'(x) \cdot g'(f(x))$$

$\therefore h'(x) = g'(f(x)) f'(x)$ Hence proved.

* Mean value theorem

* **Local Maximum**: Let 'f' be a real function defined on a metric space X, we say f has a local maximum at a point $p \in X$ if $\exists\ \delta > 0$ $f(q) \le f(p)\ \forall\ q \in X$ with $d(p,q) < \delta$ then we say that f has a local maximum.

* **Local minimum**: Let 'f' be a real function defined on a metric space X, we say f has a local minimum at a point $p \in X$, if $\exists\ \delta > 0 \ni$ $f(q) \ge f(p)\ \forall\ q \in X$, with $d(p,q) < \delta$ then we say that f has a local minimum.

*** **Theorem**: Let 'f' be defined on $[a,b]$. If f has a local maximum at a point $x \in (a,b)$ and if $f'(x)$ exists, then $f'(x) = 0$. Imp

proof: choose a $\delta > 0$

By known theorem, $a < x - \delta < x < x + \delta < b$

and $\Rightarrow f(x) \ge f(t)$ whenever $d(t,x) < \delta$

Since $t < x$, we have $t - x < 0$

Also, $f(t) - f(x) < 0$

let $x - \delta < t < x$

$$\therefore \frac{f(t) - f(x)}{t - x} \ge 0 \quad [\because t - x < 0,\ f(t) - f(x) < 0]$$

$\frac{-ve}{-ve} = +ve > 0$

Now, $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \ge 0 \Rightarrow f'(x) \ge 0$

Similarly, $x < t < x + \delta$, then $\frac{f(t) - f(x)}{t - x} \le 0$

$f(t) \le f(x)$ $\quad t > x \Rightarrow t - x > 0$

$f(t) - f(x) \le 0$ $[\because f(t) - f(x) \le 0$ & $t - x \ge 0]$ $\quad \frac{-ve}{+ve} = -ve < 0$

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \leq 0 \Rightarrow f'(x) \leq 0$$

We proved that $f'(x) \geq 0$, $f'(x) \leq 0$

$$\therefore f'(x) = 0$$

***

* **Theorem**: Let f be defined on $[a, b]$. If f has a local minimum at a point $x \in (a, b)$ and if $f'(x)$ exist then $f'(x) = 0$. Imp.

Proof: choose a $\delta > 0$

By known theorem,

$a < x - \delta < x < x + \delta < b$ and $f(x) \leq f(t)$ whenever $d(t, x) < \delta$

Since $t < x$ we have $t - x < 0$

Also $f(t) \leq f(x) \Rightarrow f(t) - f(x) \leq 0$

Now, we have $\dfrac{f(t) - f(x)}{t - x} \geq 0$ $\quad \dfrac{-ve}{-ve} = +ve$

Now, $f'(x) = \lim_{t \to x} \dfrac{f(t) - f(x)}{t - x} \geq 0$

$$\Rightarrow f'(x) \geq 0$$

Similarly, $x < t < x + \delta$ then $\dfrac{f(t) - f(x)}{t - x} \leq 0$

$t - x > 0$ $\quad \left(\dfrac{-ve}{+ve} = -ve\right.$

[$\because$ Since $t - x > 0$ and $f(t) - f(x) \leq 0$

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \leq 0$$

$$\Rightarrow f'(x) \leq 0$$

We proved that $f'(x) \geq 0$, $f'(x) \leq 0$ then $\boxed{f'(x) =}$

*** Cauchy Mean value (or) Generalise
** value theorem : Imp

Statement : If f and g are continuous real functions on [a, b] which are differentiable (a, b) then there is a point $x \in (a, b)$ in which $(f(b) - f(a)) g'(x) = [g(b) - g(a)] f'(x)$ (or)

$$\left[\frac{f(b) - f(a)}{g(b) - g(b)} = \frac{f'(x)}{g'(x)}\right]$$

define
1) $h(a) = h(b)$
2) $h'(x) = 0$
in 3 cases

$f'(x) = 0$
$h'(x) = 0$

proof: Given that,
f and g are continuous real function on [a, b] and which are differentiable (a, b).

Define $h(t) = [f(b) - f(a)] g(t) - [g(b) - g(a)] f(t)$ → ①
for $a \leq b$

Since $g(x), f(x)$ are continuous and differentiable
We have that then h is also continuous & differentiable on [a, b] and (a, b)

clearly, $h(a) = [f(b) - f(a)] g(a) - [g(b) - g(a)] f(a)$

$= f(b) g(a) - f(a) g(a) - g(b) f(a) + g(a) f(a)$

$h(a) = f(b) g(a) - g(b) f(a)$

Also $h(b) = [f(b) g(a) - g(b) f(a)]$

$\therefore h(a) = h(b)$ → ②

Now, we shall prove it in 3 cases.
It is enough to prove that $h'(x) = 0 \ \forall x \in (a, b)$

Case-i : Suppose h is a constant function.
clearly $h'(x) = 0 \ \forall x \in (a, b)$

case-ii: Suppose h is not a constant function and $h(t) > h(a)$ for some $t \in (a, b)$

Since h is continuous on $[a, b]$, it attains its maximum value at some point $x \in (a, b)$

Then h has its local maximum at x.

$\therefore$ By known theorem $h'(x) = 0$

case-iii: Suppose h is not a constant function and $h(t) < h(a)$ for some $t \in (a, b)$

Since h is continuous on $[a, b]$, it has a local minimum at some point $x \in (a, b)$

$\therefore$ By known theorem, $h'(x) = 0$

$\therefore$ In all cases, we get that $h'(x) = 0$

Now, $h(x) = [f(b) - f(a)] g(x) - [g(b) - g(a)] f(x)$

$$h'(x) = [f(b) - f(a)] g'(x) - [g(b) - g(a)] f'(x)$$

since $h'(x) = 0$

$$0 = [f(b) - f(a)] g'(x) - [g(b) - g(a)] f'(x)$$

$$\Rightarrow [f(b) - f(a)] g'(x) = [g(b) - g(a)] f'(x)$$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)}$$

*** Imp

** Legrange's mean value theorem :

** statement : Let f be a real continuous function on $[a, b]$ which is differentiable in (a, b) then there is a point $x \in (a, b)$ which

$$f(b) - f(a) = (b - a) f'(x)$$

proof :

Define the function h as

$h(t) = (f(b)-f(a))t - (b-a)f(t) \rightarrow (1)$

for $a \le t \le b$ which is continuous $[a,b]$ and is differentiable in (a,b)

and $h(a) = (f(b)-f(a))a - (b-a)f(a)$

$t=a$ $\quad = af(b) - af(a) - bf(a) + \cancel{af(a)}$

$= af(b) - bf(a)$

Similarly, $h(b) = af(b) - bf(a)$

$t=b$ $\quad \therefore h(a) = h(b)$

We get $h'(x) = 0$ for some $x \in (a,b)$ [by known theorem]

$h(x) = [f(b)-f(a)]x - (b-a)f(x)$

then $h'(x) = [f(b)-f(a)](1) - (b-a)f'(x)$

$0 = [f(b)-f(a)] - (b-a)f'(x)$

$(b-a)f'(x) = f(b)-f(a)$

$$f'(x) = \frac{f(b)-f(a)}{(b-a)}$$

$\therefore$ Hence f is a real continuous function on $[a,b]$.

* Theorem : Suppose f is a differentiable on (a,b)

(a) If $f'(x) \ge 0$ $\forall x \in (a,b)$ then f is monotinically increasing

(b) If $f'(x) = 0$ $\forall x \in (a,b)$ then f is constant

(c) If $f'(x) \le 0$ $\forall x \in (a,b)$ then f is monotonically decreasing.

proof: Let x_1, x_2 are two arbitary points in (a,b)

i.e, $(a < x_1 < x_2 < b)$ Legrange's mean value

By known theorem, the interval $[x_1, x_2]$

$\exists$ a point $x \in (x_1, x_2) \ni \dfrac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x) \rightarrow (1)$

(a) Given $f'(x) \geq 0$

$$\Rightarrow \frac{f(x_2)-f(x_1)}{x_2-x_1} \geq 0$$

$$\Rightarrow f(x_2)-f(x_1) \geq 0$$

$$\Rightarrow f(x_2) \geq f(x_1)$$

$$\Rightarrow x_1 \leq x_2$$

$$\Rightarrow f(x_1) \leq f(x_2)$$

It shows that f is monotonically increasin

(c) Suppose $f'(x) \leq 0$

By (1), we have $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} \leq 0$

$$\Rightarrow f(x_2)-f(x_1) \leq 0$$

$$\Rightarrow f(x_2) \leq f(x_1)$$

Therefore $x_1 \leq x_2$

$$\Rightarrow f(x_2) \leq f(x_1)$$

$$\Rightarrow f(x_1) \geq f(x_2)$$

$\therefore f$ is monotonically decreasing

(b) If $f'(x)=0$ then by (1) $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} = 0$

$$\Rightarrow f(x_2)-f(x_1) = 0$$

$\Rightarrow f(x_1) = f(x_2)$

It shows that f is a constant fn (t_2)

** The continuity of derivates: Imp

** Theorem: Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$ then there is a point $x \in (a,b) \ni f'(x) = \lambda$

proof: Given $\lambda \in (f'(a), f'(b))$

$\Rightarrow f'(a) - \lambda < 0$ & $f'(b) - \lambda > 0$

~~But~~ put $g(t) = f(t) - \lambda t$ for $t \in (a,b)$

Take λt as a constant function

Then given that f is differentiable

We know that λt is differentiable.

Therefore, $g'(t) = f'(t) - \lambda$

$g'(a) = f'(a) - \lambda < 0$

$g'(b) = f'(b) - \lambda > 0$

Since $g'(a) < 0$, g is decreasing at 'a'

$\Rightarrow \exists$ some point $t_1 \in (a,b) \ni g(a) > g(t_1)$

$\Rightarrow g(a) > g(t_1) \geq \min\limits_{x \in [a,b]} (g(x))$

$\Rightarrow g(a) \neq \min\limits_{x \in [a,b]} g(x) \rightarrow (1)$

Similarly $g'(b) > 0$

$\Rightarrow g$ is increasing at 'b'

$\Rightarrow \exists$ some point $t_2 \in (a,b) \ni g(t_2) < g(b)$

$\Rightarrow \min\limits_{x \in [a,b]} g(x) \leq g(t_2) < g(b)$

$\Rightarrow \min\limits_{x \in (a,b)} g(x) \neq g(b) \rightarrow (2)$

$\because$ Every differentiable function is continuous
We have g is continuous.
By known theorem, the function g attains its minimum at $x \in [a, b]$
By ① & ②, we have $a \neq x \neq b$ (i.e, $x \in (a, b)$)

$\therefore$ $g'(x) = 0$ at $x \in [a, b]$

$\Rightarrow f'(x) = \lambda$

$\exists\, x \in (a, b) \ni f'(x) = \lambda$

$g'(x) = f'(x) - \lambda$
$0 = f'(x) - \lambda$
$f'(x) = \lambda$

** Imp
Theorem: Suppose f is real differentiable function on $[a, b]$ & suppose $f'(a) > \lambda > f'(b)$ then there is a point $x \in (a, b) \ni f'(x) = \lambda$

[derivative of continuit

Proof: Given $\lambda \in (f'(a), f'(b))$

$\Rightarrow f'(a) - \lambda > 0$ & $f'(b) - \lambda < 0$

put $g(t) = f(t) - \lambda t$ for $t \in (a, b)$
Take λt as a constant function.
Then given that f is differentiable.
We know that λt is differentiable.
Therefore, $g'(t) = f'(t) - \lambda$

$g'(a) = f'(a) - \lambda > 0$

$g(a) > 0$ $\quad g'(b) = f'(b) - \lambda < 0$

g is increasing at a

$\Rightarrow \exists$ some point $t_1 \in (a, b) \ni g(a) < g(t_1)$

$\Rightarrow g(a) < g(t_1) \leq \max\limits_{s \in [a, b]} g(s) = g(x)$,

$\Rightarrow g(a) \neq g(x) \rightarrow$ ①

Similarly, $g'(b) < 0$

$\Rightarrow g$ is decreasing at b.

$\Rightarrow \exists$ some point $t_2 \in (a,b) \ni g(b) > g(t_2)$

$\Rightarrow g(b) > g(t_2) \leq \max_{s \in [a,b]} g(s) = g(x)$,

$\Rightarrow g(b) \neq g(x) \rightarrow (2)$

Since every differentiable function is continuous.

We have g is continuous.

By known theorem, the function g attains its maximum at $x \in [a,b] \ni g(x) = \max_{s \in [a,b]} g(s)$

By (1) & (2), we have $a \neq x \neq b$ (ie, $x \in (a,b)$)

$\therefore g'(x) = 0$ at $x \in [a,b]$

$\Rightarrow f'(x) = \lambda$

$\exists x \in (a,b) \ni f'(x) = \lambda$.

$g'(x) = f'(x) - \lambda$
$0 = f'(x) - \lambda$
$f'(x) = \lambda$

* Corollary: If f is differentiable on $[a,b]$ then f' cannot have any simple discontinuities on $[a,b]$.

proof: Write $g = f'$

Suppose $g = f'$ have a discontinuity of first kind then

$\exists x$ in $[a,b] \ni g(x^+), g(x^-)$ both exists & not equal.

Now, we first shown that $g(x^+) = g(x)$

In a contrary way, assume that $g(x^+) \neq g(x)$

W.L.G, We assume that $g(x^+) > g(x)$

Write $\epsilon = g(x^+) - g(x)$

$\frac{\epsilon}{2} + \frac{\epsilon}{2} = g(x^+) - g(x)$

Then $g(x^+) - \frac{\epsilon}{2} = g(x) + \frac{\epsilon}{2} \rightarrow (1)$

Write, $x_n = x + \frac{1}{n}$ for each $n \in N$

clearly, $x_n \to x^+$

$S_n \to l$

$|S_n - l| <$

$l - \epsilon < S_n < l$

So, $\exists\ m \ni n \geq m$

$\Rightarrow |g(x_n) - g(x^+)| < \frac{\epsilon}{2}$

$g(x^+) - \frac{\epsilon}{2} < g(x_n) < g(x^+) + \frac{\epsilon}{2}$

Now, $g(x) < g(x) + \frac{\epsilon}{2} = g(x^+) - \frac{\epsilon}{2} < g(x_n)$

$g(x) < g(x^+) - \frac{\epsilon}{2} < g(x_n)$

By the above theorem $\exists\ t_n \in (x, x_n) = (x, x + \frac{1}{n})$

$\ni g(t_n) = g(x^+) - \frac{\epsilon}{2}$

This is true for all $n \geq m$

$\therefore g(t_n) \to g(x^+) - \frac{\epsilon}{2}$

This is a contradiction.

Since $x < t_n < x + \frac{1}{n} \Rightarrow t_n \to x^+ \Rightarrow g(t_n) \to g(x^+)$

In the same way, we get a contradiction if we assume that $g(x^+) < g(x)$

Hence $g(x^+) < g(x)$

Hence $g(x^+) = g(x)$

In the same way, we can prove that

$g(x^-) = g(x)$

***
** Taylor's theorem :- Imp

Statement :- Suppose f is a real function on $[a,b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a,b]$. $f^{(n)}(t)$ exists for any $t \in [a,b]$. let α, β be distinct points of $[a,b]$ and defined $p(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$. Then there exists a point $x \in (\alpha, \beta)$ such that $f(\beta) = p(\beta) + \frac{f^{n}(x_n)}{n!}(\beta-\alpha)^n$

proof:

Consider $g(t) = f(t) - p(t) - M(t-\alpha)^n$, $[t \in [a,b]]$ $\rightarrow ①$

Where M is the number defined by $f(\beta) = p(\beta) + M(\beta - \alpha)^n$

$$(\text{or})\quad M = \frac{f(\beta) - p(\beta)}{(\beta-\alpha)^n} \rightarrow ②$$

$$p(t) = \sum_{k=0}^{n-1} \frac{f^{k}(\alpha)}{k!}(t-\alpha)^k$$

last so $p^{n-1}(t)$ means $(n-1)$th derivative

$$= f(\alpha) + \frac{f'(\alpha)(t-\alpha)}{1!} + \frac{f''(\alpha)(t-\alpha)^2}{2!} + \dots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(t-\alpha)^{n-1} \rightarrow ③$$

The n^{th} derivative of $g(t)$ is given by

$$g'(t) = f'(t) - p'(t) - M \cdot n(t-\alpha)^{n-1}$$

$$g^2(t) = f^2(t) - p^2(t) - Mn(n-1)(t-\alpha)^{n-2}$$

2 ↓ double tim

Continuing in this way, we get

$$g^n(t) = f^n(t) - p^{(n)}(t) - M(n(n-1)(n-2)\dots 3.2.1)(t-\alpha)^{n-n}$$

$$g^n(t) = f^n(t) - p^n(t) - Mn! \text{ for } (a<t<b)$$

Since $p^n(t) = 0$ [Since $p(t)$ is a polynomial of degree $(n-1)$]

3rd derivative x^2

nth derivative p^{n-1} = ...

Then, we have $g^n(t) = f^n(t) - Mn!$ $\rightarrow$ (4)

Now we show that there exists a point $x \in ($...

$\Rightarrow g^n(x) = 0$ then the proof is complete. (1st theorem)

Taking the derivative on both sides of (3), we get:

$$p^{(k)}(t) = f^{(k)}(\alpha) + \text{terms containing } (t-\alpha)$$

Let $p^{(k)}(\alpha) = f^{(k)}(\alpha)$, $k = 0, 1, 2, \ldots (n-1)$

[since $t - \alpha = 0$ when $t = \alpha$]

$$g(\alpha) = f(\alpha) - p(\alpha) - M(\alpha - \alpha)^n \quad [\text{from } (1)]$$

$$g(\alpha) = f(\alpha) - p(\alpha) - 0$$

In (1), take derivates and substituting $t = \alpha$, we get

$$g(\alpha) = f(\alpha) - p(\alpha)$$

$$g'(\alpha) = f'(\alpha) - p'(\alpha) = 0$$

$$g''(\alpha) = f''(\alpha) - p''(\alpha) = 0$$

$$g^{(n-1)}(\alpha) = f^{(n-1)}(\alpha) - p^{(n-1)}(\alpha) = 0$$

since by (α, β) $g(\alpha) = g(\beta)$ $g^n(\alpha)$

then from (1), we get

$$g(\beta) = f(\beta) - p(\beta) - M(\beta - \alpha)^n \quad [\text{from } (1)]$$

$$= f(\beta) - p(\beta) - \frac{f(\beta) - p(\beta)}{(\beta - \alpha)^n} (\beta - \alpha)^n = 0 \rightarrow (5)$$

Now, $g(\alpha) = 0$, $g(\beta) = 0$

$x_1 \in (\alpha, \beta)$

By mean value theorem $\exists\ \alpha < x_1 < \beta \ni$

$$g'(x_1) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha}$$

$$= \frac{0 - 0}{\beta - \alpha} = 0$$

Since $g'(\alpha) = 0 = g'(\beta)$, by mean value theorem
$\exists\, x_2 \in (\alpha, \beta) \Rightarrow g''(x_2) = 0$
In the same way $\exists\, x_n \in g^{n}(x_n) = 0$, $x_n \in (\alpha, \beta)$
Now, by substituting $t = x_n$ in ④, we get

$$0 = g^{n}(x_n) = f^{n}(x_n) - M n!$$

$$\Rightarrow M = \frac{f^{n}(x_n)}{n!}$$

$\boxed{f(\beta) = M + p(\beta)}$

By substituting these in ②, we get

$$f(\beta) = p(\beta) + \frac{f^{n}(x_n)}{n!}(\beta - \alpha)^n.$$

Hence we take x as x_n //.

Hence, statement - -

* Derivatives of Higher order :-

Definition : If f has a derivative f' on an interval and if f' is itself differentiable, we denote the derivative of f' by f'' and call f'' is second derivative of f containing in this manner, we obtain the function $f, f', f'', \ldots f^{(n)}$, $f^{(n)}$ is called the nth derivative of f.

Note : If 'g' is a polynomial of n^{th} degree then $g^{(n+1)}(x) = 0$.