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PAPER: M 202, REAL ANALYSIS - II

M. Sc. I YEAR, SEMESTER - II

PREPARED BY

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DEPARTMENT OF MATHEMATICS,

Riemann-Stieltjes integral

Definition and existence of the integral:

Definition:-

let $[a,b]$ be a given interval by a partition P of $[a,b]$ we mean a finite set of points $x_0, x_1, x_2, \dots, x_n$ where $a = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_{n-1} \leq x_n = b$ the length of the sub-interval is $x_i - x_{i-1}$ and is denoted by Δx_i

let f be a bounded real function defined on $[a,b]$ corresponding to each partition 'P' of $[a,b]$

Ex: let $S_n = 2^n$, $\forall n \in Z^+$

LB of $\{S_n\} = 2$
GLB of $\{S_n\} = 2$

Let $S_n = (-1)^n$, $\forall n \in Z^+$

$S_1 = (-1)^1 = -1$,

Range of $\{S_n\} = \{-1, 1, -1, 1, \dots\}$
$= \{-1, 1\}$

we put $M_i = \sup f(x)$ LUB

$m_i = \inf f(x)$ GLB

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

The upper Riemann integral of f on $[a,b]$ is defined by $\inf U(P,f)$

LUB is upper Riemann integral

we denoted by $\overline{\int_a^b} f(x)dx = \inf U(P,f)$ → upper bounds Infimum

The lower Riemann integral of f on $[a,b]$ is defined by $\sup L(P,f)$

we denoted by $\underline{\int_a^b} f(x)dx = \sup L(P,f)$ Lower bounds supremum

If $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$ then f is said to be Riemann integrals over $[a,b]$

we denote the common value by $\int_a^b f(x)dx$

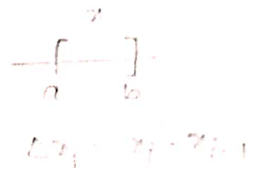
we write $f \in R$; ie., R denotes the set of Riemann integral functions

Result:-

If $f:[a,b] \rightarrow \mathbb{R}$ is bounded and there exists two numbers m, M such that $m \leq f(x) \leq M$ for all $x \in [a,b]$ then

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

Proof:- let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a,b]$



$$m_i = \overset{GLB}{\inf} f(x) \text{ for all } x \in [x_{i-1}, x_i]$$

$$M_i = \overset{LUB}{\sup} f(x) \text{ for all } x \in [x_{i-1}, x_i]$$

we know that, $\overset{LB}{m} \leq \overset{GLB}{m_i} \leq \overset{LUB}{M_i} \leq \overset{UB}{M}$

$$\sum_{i=1}^{n} m \Delta x_i \leq \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i \leq \sum_{i=1}^{n} M \Delta x_i$$

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

Definition:-

Riemann - Stieltjes integral :-

let α be a monotonically increasing function on $[a,b]$. let f be a bounded function on $[a,b]$

let f be a bounded real function defined on $[a,b]$ correspon- -ding to each partition P of $[a,b]$ we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

Clearly, $\Delta\alpha_i \geq 0$

$$M_i = \sup f(x) \text{ for all } x \in [x_{i-1}, x_i]$$

$$m_i = \inf f(x) \text{ for all } x \in [x_{i-1}, x_i]$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

we defined $\overline{\int_a^b} f\,d\alpha = \inf U(P,f,\alpha)$ is called Upper Riemann stieltjes integral of f with respect to α over $[a,b]$

$\underline{\int_a^b} f\,d\alpha = \sup L(P,f,\alpha)$ is called lower Riemann stieltjes integral of f with respect to α over $[a,b]$

If $\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$ then f is said to be Riemann steltjes integrable on $[a,b]$ with respect to α and denoted it by $f \in \mathcal{R}(\alpha)$

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha$$

Definition:-

A partition P^* of $[a,b]$ is said to be a refinement of a Partition P of $[a,b]$ if $P \subset P^*$

Definition:-

A partition P^* of $[a,b]$ is said to be a common refinement of P_1 and P_2 if $P^* = P_1 \cup P_2$ where P_1, P_2 are also partitions of $[a,b]$

Theorem:- If P^* is a refinement of P then

$$L(P,f,\alpha) \leq L(P^*,f,\alpha)$$

$$U(P^*,f,\alpha) \leq U(P,f,\alpha)$$

Proof:- Suppose P^* contains only one point other than P and let it be x^*

$P = \{a = x_0, x_1, \ldots x_{j-1}, x_j, \ldots, x_n = b\}$ be a partition of $[a,b]$

$$P^* = \{a = x_0, x_1, \ldots, x_{j-1}, x^*, x_j, \ldots, x_n = b\}$$

$$m_j = \inf f(x) \text{ for all } x \in [x_{j-1}, x_j]$$

$$w_1 = \inf f(x) \text{ for all } x \in [x_{j-1}, x^*]$$

$$w_2 = \inf f(x) \text{ for all } x \in [x^*, x_j]$$

We know that, $m_j \le W_1$, $m_j \le W_2$ (4)

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$= \sum_{i=1}^{j-1} m_i \Delta\alpha_i + m_j \Delta\alpha_j + \sum_{i=j+1}^{n} m_i \Delta\alpha_i$$

$$= \sum_{i=1}^{j-1} m_i \Delta\alpha_i + m_j [\alpha(x_j) - \alpha(x_{j-1})] + \sum_{i=j+1}^{n} m_i \Delta\alpha_i$$

$$L(P^*,f,\alpha) = \sum_{i=1}^{j-1} m_i \Delta\alpha_i + W_1[\alpha(x^*) - \alpha(x_{j-1})] + W_2[\alpha(x_j) - \alpha(x^*)] + \sum_{i=j+1}^{n} m_i \Delta\alpha_i$$

$$L(P^*,f,\alpha) - L(P,f,\alpha) = W_1[\alpha(x^*) - \alpha(x_{j-1})] + W_2[\alpha(x_j) - \alpha(x^*)] - m_j[\alpha(x_j) - \alpha(x_{j-1})]$$

$$= W_1[\alpha(x^*) - \alpha(x_{j-1})] + W_2[\alpha(x_j) - \alpha(x^*)] - m_j[\alpha(x^*) - \alpha(x_{j-1})] - m_j[\alpha(x_j) - \alpha(x^*)]$$

$$= (W_1 - m_j)[\alpha(x^*) - \alpha(x_{j-1})] + [W_2 - m_j][\alpha(x_j) - \alpha(x^*)]$$

$$\ge 0$$

$$\Rightarrow L(P^*,f,\alpha) - L(P,f,\alpha) \ge 0$$

$$\Rightarrow L(P^*,f,\alpha) \ge L(P,f,\alpha)$$

i.e., $L(P,f,\alpha) \le L(P^*,f,\alpha)$

If P^* contains k points more than P we repeat this reasoning k times and arrive the required condition

$$L(P,f,\alpha) \le L(P^*,f,\alpha)$$

(ii) $M_j = \sup f(x)$ for all $x \in [x_{j-1}, x_j]$

$W_1 = \sup f(x)$ for all $x \in [x_{j-1}, x^*]$

$W_2 = \sup f(x)$ for all $x \in [x^*, x_j]$

we know that $W_1 \le M_j$, $W_2 \le M_j$

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$= \sum_{i=1}^{j-1} M_i \Delta\alpha_i + M_j \Delta\alpha_j + \sum_{i=j+1}^{n} M_i \Delta\alpha_i$$

$$= \sum_{i=1}^{j-1} M_i \Delta\alpha_i + M_j[\alpha(x_j) - \alpha(x_{j-1})] + \sum_{i=j+1}^{n} M_i \Delta\alpha_i$$

$$U(P^*,f,\alpha) = \sum_{i=1}^{j-1} M_i \Delta\alpha_i + W_1[\alpha(x^*) - \alpha(x_{j-1})] + W_2[\alpha(x_j) - \alpha(x^*)] + \sum_{i=j+1}^{n} M_i \Delta\alpha_i$$

$$U(P^*,f,\alpha) - U(P,f,\alpha) = W_1[\alpha(x^*) - \alpha(x_{j-1})] + W_2[\alpha(x_j) - \alpha(x^*)] - M_j[\alpha(x_j) - \alpha(x_{j-1})]$$

$$= W_1[\alpha(x^*) - \alpha(x_{j-1})] + W_2[\alpha(x_j) - \alpha(x^*)] - M_j[\alpha(x^*) - \alpha(x_{j-1})] - M_j[\alpha(x_j) - \alpha(x^*)]$$

$$= (W_1 - M_j)[\alpha(x^*) - \alpha(x_{j-1})] + (W_2 - M_j)[\alpha(x_j) - \alpha(x^*)]$$

$$\Rightarrow U(P^*,f,\alpha) - U(P,f,\alpha) \leq 0$$

$$\therefore U(P^*,f,\alpha) \leq U(P,f,\alpha)$$

If P^* contains k point more than P we repeat this reasoning k times and arrive the required condition

$$U(P^*,f,\alpha) \leq U(P,f,\alpha)$$

Imp **Theorem:-** Prove that

$$\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$$

Proof: Let P^* be a common refinement of P_1 and P_2

Since P^* is a refinement of P_1

$$L(P_1,f,\alpha) \leq L(P^*,f,\alpha)$$

and P^* is a refinement of P_2

$$\Rightarrow U(P^*,f,\alpha) \leq U(P_2,f,\alpha)$$

for any partition P, we have $L(P,f,\alpha) \leq U(P,f,\alpha)$

$$\Rightarrow L(P^*,f,\alpha) \leq U(P^*,f,\alpha)$$

$$L(P_1,f,\alpha) \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P_2,f,\alpha)$$

$$\Rightarrow L(P_1,f,\alpha) \leq U(P_2,f,\alpha)$$

If P_2 is fixed and take supremum over all P_1

$$\sup L(P_1,f,\alpha) \leq U(P_2,f,\alpha)$$

$$\underline{\int_a^b} f\,d\alpha \leq U(P_2,f,\alpha)$$

take infimum over all P_2

$$\underline{\int_a^b} f\,d\alpha \le \inf U(P_2, f, \alpha)$$ [∵ by upper riemann stieltjes integral]

$$\Rightarrow \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$$

Theorem:-

(3) $f \in R(\alpha)$ on $[a,b]$ iff for every $\epsilon > 0$ there exists a partition P such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

Proof:

Sufficient condition:-

Suppose for every $\epsilon > 0$ $\exists$ a partition P such that

$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

claim:- $f \in R(\alpha)$ on $[a,b]$

we know that $$\underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$$

$$\underline{\int_a^b} f\,d\alpha \ge L(P,f,\alpha) \Rightarrow -\underline{\int_a^b} f\,d\alpha \le -L(P,f,\alpha)$$

$$\overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha)$$

$$\overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \le U(P,f,\alpha) - L(P,f,\alpha)$$

$$0 \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \epsilon \qquad \left[\because \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha\right]$$

Since $\epsilon > 0$ is an arbitrary

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

$\therefore f \in R(\alpha)$ on $[a,b]$

Necessary conditions:-

Suppose that $f \in R(\alpha)$ on $[a,b]$ and let $\epsilon > 0$

then $\int_a^b f d\alpha = \underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha$

Claim:- $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

choose two partitions P_1 and P_2

such that $U(P_2,f,\alpha) - \int_a^b f d\alpha < \epsilon/2$ ——— (1)

$$\int_a^b f d\alpha - L(P_1,f,\alpha) < \epsilon/2 \quad \text{——— (2)}$$

Let P be a common refinement of P_1 and P_2

$$\Rightarrow P = P_1 \cup P_2$$

$$L(P_1,f,\alpha) \leq L(P,f,\alpha)$$

$$U(P,f,\alpha) \leq U(P_2,f,\alpha) < \epsilon/2 + \int_a^b f d\alpha \quad (\text{from (1)})$$

$$< \epsilon/2 + \epsilon/2 + L(P_1,f,\alpha) \quad (\because \text{from (2)})$$

$$\Rightarrow U(P,f,\alpha) < \epsilon + L(P,f,\alpha)$$

$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

Theorem:-

(a) If $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ for some partition P and some ϵ then $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ holds (with the same ϵ) for any refinement of P

(b) If $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ for holds for $P = \{a = x_0, x_1, \ldots x_n = b\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$ then $\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon$

(c) If $f \in R(\alpha)$ and the hypothesis of (b) holds then

$$\left| \sum_{i=1}^{n} f(t_i) \Delta\alpha_i - \int_a^b f d\alpha \right| < \epsilon$$

Proof:- (a) let P^* be a refinement of P

$$L(P,f,\alpha) \le L(P^*,f,\alpha)$$

$$-L(P,f,\alpha) \ge -L(P^*,f,\alpha)$$

$$U(P^*,f,\alpha) \le U(P,f,\alpha)$$

$$U(P^*,f,\alpha) - L(P^*,f,\alpha) \le U(P,f,\alpha) - L(P,f,\alpha)$$

$$< \epsilon$$

$$\therefore U(P^*,f,\alpha) - L(P^*,f,\alpha) < \epsilon$$

(b) let $\epsilon > 0$

$P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a,b]$ and

$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

$$M_i = \sup f(x), \forall x \in [x_{i-1}, x_i]$$

$$m_i = \inf f(x), \forall x \in [x_{i-1}, x_i]$$

we know that, $m_i \le f(x) \le M_i$, $\forall x \in [x_{i-1}, x_i]$

let s_i, t_i be two arbitrary points in $[x_{i-1}, x_i]$

then $m_i \le f(t_i), f(s_i) \le M_i$

$$\begin{bmatrix} -m_i \ge -f(t_i) \\ -f(t_i) \le -m_i \end{bmatrix}$$

Now $|f(s_i) - f(t_i)| \le M_i - m_i$

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i$$

$$\le \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$$

$$= \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$= U(P,f,\alpha) - L(P,f,\alpha)$$

$$< \epsilon$$

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon$$

(c) Suppose $f \in R(\alpha)$

let $\epsilon > 0$, $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a,b]$ and

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

let $t_i \in [x_{i-1}, x_i]$

$$\Rightarrow m_i \le f(t_i) \le M_i$$

$$\Rightarrow \sum_{i=1}^{n} m_i \Delta\alpha_i \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \le \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$\Rightarrow L(P,f,\alpha) \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \le U(P,f,\alpha) \quad \text{—— ①}$$

since $f \in R(\alpha)$, $\sup L(P,f,\alpha) = \int_a^b f\,d\alpha = \inf U(P,f,\alpha)$

$$L(P,f,\alpha) \le \int_a^b f\,d\alpha \le U(P,f,\alpha)$$

$$-U(P,f,\alpha) \le -\int_a^b f\,d\alpha \le -L(P,f,\alpha) \quad \text{—— ②}$$

from ① and ②

$$-[U(P,f,\alpha) - L(P,f,\alpha)] \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha$$

$$\le U(P,f,\alpha) - L(P,f,\alpha)$$

$$\Rightarrow -\epsilon \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha < \epsilon$$

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \epsilon$$

Theorem :-

If f is continuous on $[a,b]$ then $f \in R(\alpha)$

Proof :- Suppose f is continuous on $[a,b]$

we shall prove that $f \in R(\alpha)$

It is enough to prove that for $\epsilon > 0$ $\exists$ a partition P of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

since α is monotonically increasing,

$\alpha(a) < \alpha(b)$

$\Rightarrow \alpha(b) - \alpha(a) > 0$

for given $\epsilon > 0$, choose η such that $\eta[\alpha(b) - \alpha(a)] < \epsilon$

Since f is continuous on $[a,b]$, f is uniformly continuous on $[a,b]$

for given $\eta > 0$ $\exists$ $\delta > 0$ $\ni |f(s) - f(t)| < \eta$, $\forall$ $s,t \in [a,b]$

for which $|s-t| < \delta$

Let $P = \{a = x_0, x_1, ---, x_n = b\}$ be a partition on $[a,b]$

such that $\Delta \alpha_i$

Let $M_i = \sup f(x)$, $\forall x \in [x_{i-1}, x_i]$

$m_i = \inf f(x)$, $\forall x \in [x_{i-1}, x_i]$

then $\exists$ two points p & q in $[x_{i-1}, x_i]$

$\ni M_i = f(p)$, $m_i = f(q)$

$M_i - m_i < \eta$

$\Rightarrow M_i - m_i < \eta$

Now

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha_i$$

$$< \eta[\alpha(b) - \alpha(a)]$$

$$< \epsilon$$

$\therefore U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

$\therefore f \in R(\alpha)$

Theorem:-

If f is monotonic on $[a,b]$ and if α is continuous on $[a,b]$ then $f \in R(\alpha)$

Proof: Suppose f is monotonic on $[a,b]$ and α is continuous on $[a,b]$

Claim: $f \in R(\alpha)$

Let $\epsilon > 0$ be given

Since α is monotonically increasing $\alpha(a) < \alpha(b)$

choose any positive integer $n > 1 \ni \alpha(a) < \dfrac{\alpha(b)-\alpha(a)}{n} + \alpha(a) < \alpha(b)$

Since α is continuous, $\exists\, x_1 \in (a,b)$ $(a = x_0)$

Intermediate value theorem: $f(a) < c < f(b)$ $x \in (a,b)$ $\ni f(x) = c$

$$\alpha(x_1) = \frac{\alpha(b)-\alpha(a)}{n} + \underset{= x_0}{\alpha(a)}$$

$(\alpha(x_1) - \alpha(a) = \Delta\alpha_1)$

$\alpha(x_1) - \alpha(x_0) = \Delta\alpha_1$

$$\Delta\alpha_1 = \frac{\alpha(b)-\alpha(a)}{n}$$

Similarly $\exists\, x_2 \in (a,b)$

$$\alpha(x_2) = \frac{\alpha(b)-\alpha(a)}{n} + \alpha(x_1)$$

$$\Delta\alpha_2 = \frac{\alpha(b)-\alpha(a)}{n}$$

continuing in this way,

for any positive integer, $\Delta\alpha_n = \dfrac{\alpha(b)-\alpha(a)}{n}$

Let P be a partition on $[a,b]$

i.e., $P = \{x_0, x_1, x_2, \dots, x_n\}$

Suppose f is monotonically increasing then

$M_i = \sup f(x), \ \forall [x_{i-1}, x_i]$

$m_i = \inf f(x), \ \forall [x_{i-1}, x_i]$

Now, $U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$

$$= \frac{\alpha(b)-\alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$= \frac{\alpha(b)-\alpha(a)}{n} [f(x_n) - f(x_0)]$$

$< \epsilon$, if n is taken large

$\therefore U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

$\therefore f \in R(\alpha)$

Theorem:-

Suppose $f \in R(\alpha)$ on $[a,b]$, $m \le f \le M$, ϕ is continuous on $[m,M]$ and $h(x) = \phi(f(x))$ on $[a,b]$ then $h \in R(\alpha)$ on $[a,b]$. ☆

Proof: Given that ϕ is continuous on $[m,M]$

$\Rightarrow \phi$ is uniformly continuous on $[m,M]$

for $\epsilon > 0 \ \exists \ \delta > 0 \ni |\phi(s) - \phi(t)| < \epsilon$, $\forall s,t \in [m,M]$

for which $|s-t| < \delta$

without loss of generality assume that $\delta < \epsilon$ (by known theorem)

Since $f \in R(\alpha)$ on $[a,b]$, for $\delta^2 > 0 \ \exists$ a partition

$P = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a,b] \ni U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$

let $M_i = \sup$ of $f(x)$, $\forall x \in [x_{i-1}, x_i]$

$m_i = \inf$ of $f(x)$, $\forall x \in [x_{i-1}, x_i]$

$M_i^* = \sup$ of $h(x)$, $\forall x \in [x_{i-1}, x_i]$

$m_i^* = \inf$ of $h(x)$, $\forall x \in [x_{i-1}, x_i]$

We divide the numbers $1, 2, \ldots, n$ into the classes

$$A = \{ i \mid 1 \le i \le n,\ M_i - m_i < \delta \}$$

$$B = \{ i \mid 1 \le i \le n,\ M_i - m_i \ge \delta \}$$

for $i \in A$, $M_i - m_i < \delta$ (uniformly continuous def)

$$\Rightarrow |f(p) - f(q)| < \delta,\ p, q \in [x_{i-1}, x_i]$$

$$\Rightarrow |\phi(f(p)) - \phi(f(q))| < \epsilon$$

$$\Rightarrow |h(p) - h(q)| < \epsilon$$

$$\Rightarrow M_i^* - m_i^* < \epsilon$$

for $i \in B$, $M_i^* - m_i^* = \sup \phi(x) - \inf \phi(x)$ $\forall\, x \in [m, M]$ (by result)

$$= \sup \phi(x) + \sup(-\phi(x))$$

$$\le \sup |\phi(x)| + \sup |\phi(x)|$$

$$= 2k \ ; \text{ where } k = \sup |\phi(x)|$$

$$\therefore M_i^* - m_i^* \le 2k, \text{ for } i \in B$$

for $i \in B$, $\delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B} (M_i - m_i) \Delta\alpha_i$

$$\le \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$$

$$= U(P, f, \alpha) - L(P, f, \alpha)$$

$$< \delta^2$$

$$\delta \sum_{i \in B} \Delta\alpha_i < \delta^2$$

$$\sum_{i \in B} \Delta\alpha_i < \delta$$

Now, $U(P, h, \alpha) - L(P, h, \alpha) = \sum_{i=1}^{n} (M_i^* - m_i^*) \Delta\alpha_i$

$$= \sum_{i \in A} (M_i^* - m_i^*) \Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta\alpha_i$$

$$< \epsilon \sum_{i=1}^{n} \Delta\alpha_i + 2k\delta$$

$$< \epsilon [\alpha(b) - \alpha(a)] + 2k\epsilon$$

$$= \epsilon [\alpha(b) - \alpha(a) + 2k]$$

$$U(P, h, \alpha) - L(P, h, \alpha) < \epsilon'$$

$$\therefore h \in R(\alpha)$$

Theorem:-

Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$ & α is continuous at every point at which f is discontinuous then $f \in R(\alpha)$

Proof: let $\epsilon > 0$ be given

$$M = \sup |f(x)|, \ x \in [a,b]$$

let E be the set of points which f is discontinuous E is finite and α is continuous at every point of E

let $E = \{c_1, c_2, ---, c_n\}$

Enclose these points in 'n' non over lapping intervals $[u_1, v_1], [u_2, v_2] --- [u_n, v_n]$ such that $\sum_{i=1}^{n} [\alpha(v_i) - \alpha(u_i)] < \epsilon$

$$[a, u_1] \cup (u_1, v_1) \cup [v_1, u_2] \cup (u_2, v_2) \cup --- \cup [v_n, b]$$

This can be written as

$$[a,b] = [a, u_1] \cup (u_1, v_1) \cup [v_1, u_2] \cup (u_2, v_2) \cup --- \cup [v_n, b]$$

Remove the segments $(u_1, v_1), (u_2, v_2) \dots (u_n, v_n)$ from $[a, b]$

we get a set $k = [a, u_1] \cup [v_1, u_2] \cup [v_2, u_3] \cup \dots \cup [v_n, b]$

Therefore, k is a finite union of closed sets

$\Rightarrow k$ is compact

Since the discontinuities $c_1, c_2, \dots, c_n$ of E are not in k

f is continuous on k

$\Rightarrow f$ is uniformly continuous on k

for $\epsilon > 0$ there exists $\delta > 0$ such that $|f(s) - f(t)| < \epsilon \ \forall s, t \in k$

for which $|s - t| < \delta$

Now, form a partition $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ of $[a, b]$

as follows

each u_j occurs in P. each v_j occurs in P

No point of any segment (u_j, v_j) occurs in P

let $M_i = \sup f(x) \ \forall x \in [x_{i-1}, x_i]$

$m_i = \inf f(x) \ \forall x \in [x_{i-1}, x_i]$

If x_{i-1} is not one of u_j then $\Delta x_i < \delta$

ie., x_{i-1} is one of 'a' (or) v_j

Then $[x_{i-1}, x_i] = [a, u_1]$ (or) $[v_j, u_{j+1}]$

Since f is uniformly continuous on $[a, u_1]$ (or) $[v_j, u_{j+1}]$

for $\epsilon > 0 \ \exists \ \delta > 0 \ni |f(s) - f(t)| < \epsilon \ \forall s, t \in [x_{i-1}, x_i]$

for which $|s - t| < \delta$

$\Rightarrow M_i - m_i < \epsilon$

If x_{i-1} is one of U_j then $M_i - m_i \le 2M$

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

where $A = \{i / x_{i-1} \text{ is one of } U_j\}$

$B = \{i / x_{i-1} \text{ is not one of } U_j\}$

$$< 2M \sum_{i=1}^{n} \Delta\alpha_i + \epsilon \sum_{i=1}^{n} \Delta\alpha_i$$

$$< 2M\epsilon + \epsilon[\alpha(b) - \alpha(a)]$$

$$< \epsilon$$

$\therefore U(P,f,\alpha) - L(P,f,\alpha) < \epsilon'$

Theorem:-

$\therefore f \in R(\alpha)$

* **Properties of Integral** :-

If $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$ on $[a,b]$ then $f_1 + f_2 \in R(\alpha)$ and

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$

Proof: let $f = f_1 + f_2$

and $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition on $[a,b]$

Let m_i, M_i be the Inf and sup of f on $[x_{i-1}, x_i]$

m_i', M_i' be the Inf and sup of f_1 on $[x_{i-1}, x_i]$

m_i'', M_i'' be the Inf and sup of f_2 on $[x_{i-1}, x_i]$

then $f(x) = f_1(x) + f_2(x)$

$\le M_i' + M_i''$

$\Rightarrow M_i' + M_i''$ be the upper bound of $f(x)$

since M_i is the least upper bound of f (sup)

Similarly, $m_i' + m_i'' \le m_i$ GLB

LUB

$\Rightarrow m_i' + m_i'' \le m_i \le M_i \le M_i' + M_i''$

$$\Rightarrow \sum_{i=1}^{n} (m_i' + m_i'')\Delta\alpha_i \le \sum_{i=1}^{n} m_i \Delta\alpha_i \le \sum_{i=1}^{n} M_i \Delta\alpha_i \le \sum_{i=1}^{n} (M_i' + M_i'')\Delta\alpha_i$$

$$\Rightarrow \sum_{i=1}^{n} m_i' \Delta\alpha_i + \sum_{i=1}^{n} m_i'' \Delta\alpha_i \le \sum_{i=1}^{n} m_i \Delta\alpha_i \le \sum_{i=1}^{n} M_i \Delta\alpha_i \le \sum_{i=1}^{n} M_i' \Delta\alpha_i + \sum_{i=1}^{n} M_i'' \Delta\alpha_i$$

$$\Rightarrow L(P, f_1, \alpha) + L(P, f_2, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P, f_1, \alpha) + U(P, f_2, \alpha)$$

Since $f_1 \in R(\alpha)$, $f_2 \in R(\alpha)$

$$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \epsilon/2$$

$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \epsilon/2$$

Now,

$$U(P, f, \alpha) - L(P, f, \alpha) \le U(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_1, \alpha) + L(P, f_2, \alpha)$$

$$< \epsilon/2 + \epsilon/2$$

$$= \epsilon$$

$$\therefore U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$\therefore f = f_1 + f_2 \in R(\alpha)$$

Since $f \in R(\alpha)$

$$\Rightarrow \underline{\int_a^b} f\, d\alpha = \int_a^b f\, d\alpha = \overline{\int_a^b} f\, d\alpha$$

Consider,

$$\int_a^b f\, d\alpha \le U(P, f, \alpha)$$

$$\le U(P, f_1, \alpha) + U(P, f_2, \alpha)$$

$$< L(P, f_1, \alpha) + L(P, f_2, \alpha) + \epsilon/2 + \epsilon/2$$

$$< \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha + \epsilon$$

Since $\epsilon > 0$ is arbitrary

$$\int_a^b f\, d\alpha \le \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha \quad \text{—— (1)}$$

~~Replace~~ Multiplying with (-1) on both sides

$$-\left[\int_a^b f\, d\alpha\right] \ge -\int_a^b f_1\, d\alpha - \int_a^b f_2\, d\alpha$$

$$\Rightarrow -\int_a^b f\,d\alpha \geq -\left[\int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha\right]$$

$$\Rightarrow \int_a^b f\,d\alpha \geq \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha \quad \text{—— (2)}$$

from (1) & (2)

$$\int_a^b f\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$$

$$\therefore \int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha.$$

Theorem:-

If $f \in R(\alpha)$ and c is a constant then $cf \in R(\alpha)$ and

$$\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$$

Proof: Suppose that $f \in R(\alpha)$ and c is a constant

Claim: $cf \in R(\alpha)$ and $\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$

since $f \in R(\alpha)$, for $\epsilon > 0$ $\exists$ a partition $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ of $[a,b]$ $\ni$ $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

let M_i, m_i be the Supremum and Infimum of 'f' of $[x_{i-1}, x_i]$

If $c>0$ then cM_i, cm_i be the sup & inf of cf on $[x_{i-1}, x_i]$

$$U(P,cf,\alpha) = \sum_{i=1}^{n} cM_i\Delta\alpha_i = c\sum_{i=1}^{n} M_i\Delta\alpha_i = cU(P,f,\alpha)$$

$$L(P,cf,\alpha) = \sum_{i=1}^{n} cm_i\Delta\alpha_i = c\sum_{i=1}^{n} m_i\Delta\alpha_i = cL(P,f,\alpha)$$

$$U(P,cf,\alpha) - L(P,cf,\alpha) = c\left[U(P,f,\alpha) - L(P,f,\alpha)\right]$$
$$< c\epsilon$$

since $\epsilon > 0$ is arbitrary, $cf \in R(\alpha)$ if $c > 0$

If $c < 0$ then cM_i, cm_i be the sup & inf of cf on $[x_{i-1}, x_i]$

$$U(P,cf,\alpha) = \sum_{i=1}^{n} c\, m_i \Delta\alpha_i = c \sum_{i=1}^{n} m_i \Delta\alpha_i = c\, L(P,f,\alpha)$$

$$L(P,cf,\alpha) = \sum_{i=1}^{n} c\, M_i \Delta\alpha_i = c \sum_{i=1}^{n} M_i \Delta\alpha_i = c\, U(P,f,\alpha)$$

$$U(P,cf,\alpha) - L(P,cf,\alpha) = -c\left[U(P,f,\alpha) - L(P,f,\alpha)\right]$$
$$< (-c)\epsilon$$

since $\epsilon > 0$ is arbitrary, $cf \in R(\alpha)$ if $c < 0$

$$\therefore cf \in R(\alpha)$$

Now $\int_a^b cf\,d\alpha = \overline{\int_a^b} cf\,d\alpha = \underline{\int_a^b} cf\,d\alpha$

If $c > 0$, $\overline{\int_a^b} cf\,d\alpha = \inf U(P,cf,\alpha)$

$$= \inf c\,U(P,f,\alpha) \quad \left(\because \overline{\int_a^b} f\,d\alpha = \inf U(P,f,\alpha)\right)$$

$$= c\overline{\int_a^b} f\,d\alpha$$

$$= c\int_a^b f\,d\alpha$$

$$\therefore \int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha, \text{ if } c > 0$$

If $c < 0$, put $c = -k$ $(k > 0)$

cf is equal to $k(-f)$

$$\int_a^b cf\,d\alpha = \int_a^b k(-f)\,d\alpha$$

$$= -k\int_a^b f\,d\alpha$$

$$= c\int_a^b f\,d\alpha$$

$$\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha \quad , \text{ if } c<0$$

$$\therefore \int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$$

where c is a constant

Theorem:-

If $f\in R(\alpha)$ on $[a,b]$ and $a<c<b$ then $f\in R(\alpha)$ on $[a,c]$ and on $[c,b]$ and also $\int_a^c f\,d\alpha + \int_c^b f\,d\alpha = \int_a^b f\,d\alpha$

Proof: Suppose that $f\in R(\alpha)$ on $[a,b]$ and $a<c<b$

Claim:- $f\in R(\alpha)$ on $[a,c]$ and on $[c,b]$ and also

$$\int_a^c f\,d\alpha + \int_c^b f\,d\alpha = \int_a^b f\,d\alpha$$

Since $f\in R(\alpha)$ on $[a,b]$,

for $\epsilon>0$ $\exists$ a partition $P=\{a=x_0,x_1,x_2,\ldots,x_n=b\}$ of $[a,b]$

$\Rightarrow U(P,f,\alpha)-L(P,f,\alpha)<\epsilon$

let P' be a refinement of P

$P'=\{a=x_0,x_1,x_2,\ldots,x_j,\ c,\ x_{j+1},\ldots,x_n=b\}$

then $U(P', f, \alpha) - L(P', f, \alpha) < \epsilon$

Also, $L(P, f, \alpha) \leq L(P', f, \alpha)$ (previous theorem)

$U(P', f, \alpha) \leq U(P, f, \alpha)$

Let $P_1 = \{a = x_0, x_1, \ldots, x_j, c\}$

$P_2 = \{c, x_{j+1}, \ldots, x_n = b\}$

then P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$

$L(P, f, \alpha) \leq L(P', f, \alpha)$

$= L(P_1, f, \alpha) + L(P_2, f, \alpha)$

$L(P, f, \alpha) \leq L(P_1, f, \alpha) + L(P_2, f, \alpha)$

$\sup L(P, f, \alpha) \leq \sup L(P_1, f, \alpha) + \sup L(P_2, f, \alpha)$ (definition) (lower)

$$\underline{\int_a^b} f \, d\alpha \leq \underline{\int_a^c} f \, d\alpha + \underline{\int_c^b} f \, d\alpha \quad \text{---} \; (1)$$

$U(P, f, \alpha) \geq U(P', f, \alpha)$

$= U(P_1, f, \alpha) + U(P_2, f, \alpha)$

$\inf U(P, f, \alpha) \geq \inf U(P_1, f, \alpha) + \inf U(P_2, f, \alpha)$ (definition) (upper)

$$\overline{\int_a^b} f \, d\alpha \geq \overline{\int_a^c} f \, d\alpha + \overline{\int_c^b} f \, d\alpha \quad \text{---} \; (2)$$

$U(P', f, \alpha) - L(P', f, \alpha) = U(P_1, f, \alpha) + U(P_2, f, \alpha) - [L(P_1, f, \alpha) + L(P_2, f, \alpha)]$

$U(P_1, f, \alpha) + U(P_2, f, \alpha) - L(P_1, f, \alpha) - L(P_2, f, \alpha)$

$< \epsilon$

$\Rightarrow U(P_1, f, \alpha) - L(P_1, f, \alpha) < \epsilon/2$

and $U(P_2, f, \alpha) - L(P_2, f, \alpha) < \epsilon/2$

$\Rightarrow f \in R(\alpha)$ on $[a, c]$

$\Rightarrow f \in R(\alpha)$ on $[c, b]$

from ① & ②

$$\int_a^b f\,d\alpha \le \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

$$\int_a^b f\,d\alpha \ge \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

$$\therefore \int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

Theorem:-

If $f \in R(\alpha)$ on $[a,b]$ and if $|f(x)| \le M$ on $[a,b]$ then

$$\left| \int_a^b f\,d\alpha \right| \le M[\alpha(b) - \alpha(a)]$$

Proof: Given that, $f \in R(\alpha)$ on $[a,b]$ and $|f(x)| \le M$ on $[a,b]$

Since $f \in R(\alpha)$, $\int_a^b f\,d\alpha$ exists

$|f(x)| \le M \Rightarrow -M \le f(x) \le M$ on $[a,b]$

$$\Rightarrow \int_a^b -M\,d\alpha \le \int_a^b f\,d\alpha \le \int_a^b M\,d\alpha$$

$$\Rightarrow -M \int_a^b (1)\,d\alpha \le \int_a^b f\,d\alpha \le M \int_a^b (1)\,d\alpha$$

$$\Rightarrow -M[\alpha(b) - \alpha(a)] \le \int_a^b f\,d\alpha \le M[\alpha(b) - \alpha(a)]$$

$$\Rightarrow \left| \int_a^b f\,d\alpha \right| \le M[\alpha(b) - \alpha(a)]$$

Theorem:-

If $f_1 \in R(\alpha)$, $f_2 \in R(\alpha)$ on $[a,b]$ and also $f_1(x) \le f_2(x)$ on $[a,b]$ then

$$\int_a^b f_1 d\alpha \le \int_a^b f_2 d\alpha$$

Proof: Suppose that $f_1 \in R(\alpha)$, $f_2 \in R(\alpha)$ on $[a,b]$, $f_1(x) \le f_2(x)$ on $[a,b]$

claim:- $\int_a^b f_1 d\alpha \le \int_a^b f_2 d\alpha$

Since $f_1 \in R(\alpha)$, $f_2 \in R(\alpha)$ on $[a,b]$ and $f_1(x) \le f_2(x)$ on $[a,b]$

Now $f_2 - f_1 \in R(\alpha)$ on $[a,b]$, $f_2 - f_1 \ge 0$

Let P be a partition on $[a,b]$

then $\inf(f_2 - f_1) \ge 0$, for any subinterval of $[a,b]$

$L(P, f_2 - f_1, \alpha) \ge 0$

$\sup L(P, f_2 - f_1, \alpha) \ge 0$

$$\Rightarrow \underline{\int_a^b} (f_2 - f_1)\, d\alpha \ge 0$$

$$\Rightarrow \int_a^b (f_2 - f_1)\, d\alpha \ge 0$$

$$\Rightarrow \int_a^b f_2\, d\alpha - \int_a^b f_1\, d\alpha \ge 0$$

$$\Rightarrow \int_a^b f_2\, d\alpha \ge \int_a^b f_1\, d\alpha$$

$$\therefore \int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha$$

Theorem:-

If $f \in R(\alpha_1)$, $f \in R(\alpha_2)$ then $f \in R(\alpha_1 + \alpha_2)$ and $\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$

Proof: Suppose that $f \in R(\alpha_1)$, $f \in R(\alpha_2)$

claim:- $f \in R(\alpha_1 + \alpha_2)$ and $\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$

Since $f \in R(\alpha_1)$

for $\epsilon > 0$ $\exists$ a partition P_1 of $[a,b]$ $\ni$ $U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) < \epsilon/2$

Since $f \in R(\alpha_2)$

for $\epsilon > 0$ $\exists$ a partition P_2 of $[a,b]$ $\ni$ $U(P_2,f,\alpha_2) - L(P_2,f,\alpha_2) < \epsilon/2$

Let P be a common refinement of P_1 and P_2

then $U(P,f,\alpha_1) - L(P,f,\alpha_1) < \epsilon/2$

$U(P,f,\alpha_2) - L(P,f,\alpha_2) < \epsilon/2$

Let $\alpha = \alpha_1 + \alpha_2$

$$
\begin{aligned}
U(P,f,\alpha_1+\alpha_2) &= U(P,f,\alpha) \\
&= \sum_{i=1}^{n} M_i \Delta\alpha_i \\
&= \sum_{i=1}^{n} M_i \Delta\alpha_{1i} + \sum_{i=1}^{n} M_i \Delta\alpha_{2i} \\
&= U(P,f,\alpha_1) + U(P,f,\alpha_2)
\end{aligned}
$$

Similarly,

$L(P,f,\alpha_1+\alpha_2) = L(P,f,\alpha_1) + L(P,f,\alpha_2)$

Consider

$$
\begin{aligned}
U(P,f,\alpha_1+\alpha_2) - L(P,f,\alpha_1+\alpha_2) &= U(P,f,\alpha_1) + U(P,f,\alpha_2) - L(P,f,\alpha_1) - L(P,f,\alpha_2) \\
&= U(P,f,\alpha_1) - L(P,f,\alpha_1) + U(P,f,\alpha_2) - L(P,f,\alpha_2) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \frac{2\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

then

$U(P,f,\alpha_1+\alpha_2) - L(P,f,\alpha_1+\alpha_2) < \epsilon$

$\therefore f \in R(\alpha_1+\alpha_2)$

Since P_1 and P_2 are the partitions of $[a,b]$

and $f \in R(\alpha_1)$, $f \in R(\alpha_2)$ on $[a,b]$

$U(P_1,f,\alpha_1) - \int_a^b f\,d\alpha_1 < \epsilon/2$

$U(P_2,f,\alpha_2) - \int_a^b f\,d\alpha_2 < \epsilon/2$

Since P is a common refinement of P_1 and P_2

$$U(P,f,\alpha_1) - \int_a^b f\,d\alpha_1 < \epsilon/2$$

$$U(P,f,\alpha_2) - \int_a^b f\,d\alpha_2 < \epsilon/2$$

$$L(P,f,\alpha) \leq U(P,f,\alpha)$$

$$\begin{aligned} \sup L(P,f,\alpha) &\leq \sup U(P,f,\alpha) \\ &= \sup U(P,f,\alpha_1) + \sup U(P,f,\alpha_2) \\ &= U(P,f,\alpha_1) + U(P,f,\alpha_2) \end{aligned}$$

$$\int_a^b f\,d\alpha \leq \int_a^b f\,d\alpha_1 + \epsilon/2 + \int_a^b f\,d\alpha_2 + \epsilon/2$$

$$\int_a^b f\,d\alpha \leq \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 + \epsilon$$

Since $\epsilon > 0$ is arbitrary

$$\int_a^b f\,d\alpha \leq \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 \quad \text{——— (1)}$$

Since P_1 and P_2 are the partition of $[a,b]$

and $f \in R(\alpha_1)$, $f \in R(\alpha_2)$ on $[a,b]$

$$\int_a^b f\,d\alpha_1 - L(P_1,f,\alpha_1) < \epsilon/2$$

$$\int_a^b f\,d\alpha_2 - L(P_2,f,\alpha_2) < \epsilon/2$$

Since P is a common refinement of P_1 and P_2

$$\int_a^b f\,d\alpha_1 - L(P,f,\alpha_1) < \epsilon/2$$

$$\int_a^b f\,d\alpha_2 - L(P,f,\alpha_2) < \epsilon/2$$

$$U(P,f,\alpha) \geq L(P,f,\alpha)$$

$\inf U(P,f,\alpha) \geq \inf L(P,f,\alpha)$

$= L(P,f,\alpha_1) + L(P,f,\alpha_2)$

$$\int_a^b f\,d\alpha \geq -\epsilon/2 + \int_a^b f\,d\alpha_1 - \epsilon/2 + \int_a^b f\,d\alpha_2$$

$$\int_a^b f\,d\alpha \geq \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 - \epsilon$$

Since $\epsilon > 0$, is arbitrary

$$\int_a^b f\,d\alpha \geq \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 \quad \text{---} (2)$$

from (1) and (2)

$$\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$$

since $\alpha = \alpha_1 + \alpha_2$

$$\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$$

Theorem:-

If $f \in R(\alpha)$ and 'c' is a positive constant then $f \in R(c\alpha)$

and $\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha$

Proof: Suppose that $f \in R(\alpha)$ and c is a positive constant

Claim: $f \in R(c\alpha)$ and $\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha$

Since $f \in R(\alpha)$

for $\epsilon > 0$ $\exists$ a partition $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ of $[a,b]$

$\ni U(P,f,\alpha) - L(P,f,\alpha) < \epsilon/c$

let $M_i = \sup$ of $f(x)$, $\forall x \in [x_{i-1}, x_i]$

$m_i = \inf$ of $f(x)$, $\forall x \in [x_{i-1}, x_i]$

$$U(P, f, c\alpha) = \sum_{i=1}^{n} M_i \Delta c\alpha_i$$
$$= c \sum_{i=1}^{n} M_i \Delta \alpha_i$$
$$= c\, U(P, f, \alpha)$$

$$L(P, f, c\alpha) = \sum_{i=1}^{n} m_i \Delta c\alpha_i$$
$$= c \sum_{i=1}^{n} m_i \Delta \alpha_i$$
$$= c\, L(P, f, \alpha)$$

$$U(P, f, c\alpha) - L(P, f, c\alpha) = c\,U(P, f, \alpha) - c\,L(P, f, \alpha)$$
$$= c[U(P, f, \alpha) - L(P, f, \alpha)]$$
$$< c\left(\frac{\epsilon}{c}\right)$$
$$< \epsilon$$

$\therefore U(P, f, c\alpha) - L(P, f, c\alpha) < \epsilon$

$\therefore f \in R(c\alpha)$

Consider, $$\int_a^b f\, d(c\alpha) = \inf U(P, f, c\alpha)$$
$$= \inf c\, U(P, f, \alpha)$$
$$= c \cdot \inf U(P, f, \alpha)$$
$$= c \cdot \int_a^b f\, d\alpha$$

$$\therefore \int_a^b f\, d(c\alpha) = c \cdot \int_a^b f\, d\alpha$$

Theorem:-

If $f \in R(\alpha)$, $g \in R(\alpha)$ on $[a,b]$ then (i) $fg \in R(\alpha)$

(ii) $|f| \in R(\alpha)$ and $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$

Proof: Given that $f \in R(\alpha)$, $g \in R(\alpha)$ on $[a,b]$

let $\phi(t) = t^2$

since $\phi(t)$ is a polynomial, it is continuous

$$\phi[f(x)] = [f(x)]^2$$

By known theorem, $f^2 \in R(\alpha)$

since $f \in R(\alpha)$ and $g \in R(\alpha)$

$f-g \in R(\alpha)$ and $f+g \in R(\alpha)$

$(f-g)^2 \in R(\alpha)$ & $(f+g)^2 \in R(\alpha)$

$\Rightarrow (f+g)^2 - (f-g)^2 \in R(\alpha)$

$\Rightarrow 4fg \in R(\alpha)$

$\Rightarrow fg \in R(\alpha)$

(b) let $\phi(t) = |t|$

Then ϕ is continuous on $[a,b]$

$$\phi[f(x)] = |f(x)|$$

By known theorem, $|f| \in R(\alpha)$

choose $c = \pm 1$, so that $c\int_a^b f\,d\alpha \geq 0$

$$\left|\int_a^b f\,d\alpha\right| = c\int_a^b f\,d\alpha$$

$$= \int_a^b c f\,d\alpha$$

$$\leq \int_a^b |f|\,d\alpha$$

$$\therefore \left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$$

Z

** **Definition:** The unit step function I is defined by

$$I(x) = \begin{cases} 0 \ ; & (x \leq 0) \quad (\text{negative, zero}) \\ 1 \ ; & (x > 0) \quad (\text{Positive}) \end{cases}$$

Imp
*** **Theorem:**

If $a < s < b$, f is bounded on $[a,b]$, f is continuous at s and $\alpha(x) = I(x-s)$ then $\int_a^b f\,d\alpha = f(s)$

Proof: Given that $a < s < b$, f is bounded on $[a,b]$. claim: $\int_a^b f\,d\alpha = f(s)$

f is continuous at s and $\alpha(x) = I(x-s)$.

let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a,b]$

and $x_1 = s \Rightarrow a = x_0,\ x_1 = s,\ x_3 = b$

$\Rightarrow a < s < x_2 < b$

$$\alpha(x_0) = I(x_0 - s) = 0$$

$$\alpha(x_1) = I(x_1 - s) = 0$$

$$\alpha(x_2) = I(x_2 - s) = 1$$

$$\alpha(x_3) = I(x_3 - s) = 1$$

$$U(P,f,\alpha) = \sum_{i=1}^{3} M_i \Delta\alpha_i$$

$$U(P,f,\alpha) = M_1[\alpha(x_1) - \alpha(x_0)] + M_2[\alpha(x_2) - \alpha(x_1)] + M_3[\alpha(x_3) - \alpha(x_2)]$$

$$U(P,f,\alpha) = M_1[0-0] + M_2[1-0] + M_3[1-1]$$

$$U(P,f,\alpha) = M_2$$

where M_1, M_2, M_3 are sup of $[x_0,x_1]$, $[x_1,x_2]$ and $[x_2,x_3]$ respectively

$$L(P,f,\alpha) = \sum_{i=1}^{3} m_i \Delta\alpha_i$$

$$L(P,f,\alpha) = m_1[\alpha(x_1) - \alpha(x_0)] + m_2[\alpha(x_2) - \alpha(x_1)] + m_3[\alpha(x_3) - \alpha(x_2)]$$

$$L(P,f,\alpha) = m_1(0-0) + m_2(1-0) + m_3(1-1)$$

$$L(P,f,\alpha) = m_2$$

where m_1, m_2, m_3 are ~~infimum~~ infimum of $[x_0,x_1]$, $[x_1,x_2]$ and $[x_2,x_3]$ respectively

let $u, v \in [x_1,x_2] \Rightarrow f(u) = M_2$ and $f(v) = m_2$

since f is continuous at s, for $\epsilon > 0$ $\exists$ $\delta > 0$ such that $|f(x_2) - f(s)| < \epsilon$ for some $|x_2 - s| < \delta$

Since $u, v \in [x_1, x_2]$

$|u - s| \leq |x_2 - s| < \delta$ and $|v - s| \leq |x_2 - s| < \delta$

$\Rightarrow |f(u) - f(s)| < \epsilon$ and $|f(v) - f(s)| < \epsilon$

$\Rightarrow |M_2 - f(s)| < \epsilon$ and $|m_2 - f(s)| < \epsilon$

$\Rightarrow M_2 \to f(s)$ and $m_2 \to f(s)$ $\quad (\because |S_n - l| < \epsilon \Rightarrow S_n \to l)$

$\Rightarrow \inf U(P, f, \alpha) = f(s) = \sup L(P, f, \alpha)$

$$\Rightarrow \int_a^b f\,d\alpha = f(s)$$

Theorem:-

Suppose $c_n \geq 0$ for $1, 2, \ldots$ $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a,b) and $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$. let f be continuous on $[a,b]$ then $\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$

Proof: Given that $c_n \geq 0$ for $1, 2, \ldots$, $\sum c_n$ converges $\{s_n\}$ is a sequence of distinct point in (a,b) and $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$

Suppose f is continuous on $[a,b]$

let $a_n = c_n I(x - s_n) = \begin{cases} 0; & \text{if } x \leq s_n \\ c_n; & \text{if } x > s_n \end{cases}$

Since $\sum c_n$ converges, by comparison test, $\sum a_n$ converges $\alpha(a) = 0$ if $a < s_1 < s_2 < s_3 < \cdots < s_n \leq b$

$$\alpha(b) = \sum_{n=1}^{\infty} c_n$$

$\therefore$ α is monotonically increasing

let $\epsilon > 0$ be given and choose N so that $\sum_{n=N+1}^{\infty} c_n < \epsilon$

Put $\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n)$

$$\alpha_2(x) = \sum_{n=N+1}^{\infty} c_n I(x - s_n)$$

$$\alpha = \alpha_1 + \alpha_2$$

$$\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$$

Now
$$\int_a^b f\,d\alpha_1 = \int_a^b f\,d\left(\sum_{n=1}^{N} c_n I(x-s_n)\right)$$
$$= \sum_{n=1}^{N} c_n \int_a^b f\,d\left(I(x-s_n)\right) \quad \text{(above relation)}$$
$$= \sum_{n=1}^{N} c_n f(s_n) \quad \text{(A)}$$

$$\alpha_2(b) - \alpha_2(a) = \sum_{n=N+1}^{\infty} c_n < \epsilon$$

Since f is continuous on $[a,b]$, f is bounded on $[a,b]$
$\Rightarrow$ there exists $M > 0$ such that $|f(x)| \leq M \ \forall \ x \in [a,b]$

$$\left|\int_a^b f\,d\alpha_2\right| \leq \int_a^b |f|\,d\alpha_2$$
$$\leq M\left[\alpha_2(b) - \alpha_2(a)\right]$$
$$\leq M\epsilon$$

Consider,
$$\left|\int_a^b f\,d\alpha - \sum_{n=1}^{N} c_n f(s_n)\right|$$
$$= \left|\int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 - \int_a^b f\,d\alpha_1\right|$$
$$= \left|\int_a^b f\,d\alpha_2\right|$$
$$\leq M\epsilon$$

This is true for all $\epsilon > 0$

If we let $N \to \infty$

$$\lim_{N \to \infty} \sum_{n=1}^{N} c_n f(s_n) = \int_a^b f\,d\alpha$$

$$\Rightarrow \sum_{n=1}^{\infty} c_n f(s_n) = \int_a^b f\,d\alpha$$

Int

Theorem: Assume α increases monotonically and $\alpha' \in R$ on $[a,b]$. let f be a bounded real function on $[a,b]$ then $f \in R(\alpha)$ if and only if $f\alpha' \in R$. In that case $\int_a^b f d\alpha = \int_a^b f\alpha' dx$ ** (2)

Proof: Given that α increases monotonically $\alpha' \in R$ on $[a,b]$ and f be bounded real function on $[a,b]$

we have to show that $f \in R(\alpha)$ if and only if $f\alpha' \in R$. and $\int_a^b f d\alpha = \int_a^b f\alpha' dx$

suppose $f \in R(\alpha)$

Since $\alpha' \in R$, for $\epsilon > 0$ $\exists$ a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that

$U(P,\alpha') - L(P,\alpha') < \epsilon$ —— ①

since α' exists, α is differentiable on (a,b) and α is continuous on $[a,b]$

By mean value theorem,

$\exists$ $t_i \in [x_{i-1}, x_i]$ such that $\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)(x_i - x_{i-1})$

$$\Rightarrow \Delta\alpha_i = \alpha'(t_i)\Delta x_i \quad \text{—— ②}$$

let $M_i = \sup \alpha'(x)$ for all $x \in [x_{i-1}, x_i]$

$m_i = \inf \alpha'(x)$ for all $x \in [x_{i-1}, x_i]$

$x_{i-1} \le s_i \le t_i < x_i$

$x_i \le s_i \le t_i < M_i$

$\alpha'(s_i) \le M_i$, $m_i \le \alpha'(t_i)$ for some $s_i, t_i \in [x_{i-1}, x_i]$

$$|\alpha'(s_i) - \alpha'(t_i)| \le M_i - m_i$$

$$\Rightarrow \sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i \le \sum_{i=1}^{n} (M_i - m_i)\Delta x_i$$

$$= U(P,\alpha') - L(P,\alpha')$$

$$< \epsilon \quad \text{—— ③}$$

Since f is bounded on $[a,b]$ $\exists$ $M > 0 \ni |f(x)| \le M$ $\forall x \in [a,b]$

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i \quad (\because \text{by ②})$$

Consider, $\left| \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right|$

$$= \left| \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right|$$

$$= \left| \sum_{i=1}^{n} f(s_i)[\alpha'(t_i) - \alpha'(s_i)]\Delta x_i \right|$$

$$\le M \sum_{i=1}^{n} |\alpha'(t_i) - \alpha'(s_i)\Delta x_i|$$

$\leq M\epsilon$

$\Rightarrow \left| \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right| \leq M\epsilon$

Let $\bar{M}_i = \sup f\alpha'(x)$ for all $x \in [x_{i-1}, x_i]$

$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq M\epsilon + \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i$

$\leq M\epsilon + U(P, f, \alpha')$

This is true for all $s_i \in [x_{i-1}, x_i]$

$U(P, f, \alpha) \leq M\epsilon + U(P, f, \alpha')$

$\Rightarrow |U(P, f, \alpha) - U(P, f, \alpha')| \leq M\epsilon$

Since, $U(P, \alpha') - L(P, \alpha') < \epsilon$ is true for any refinement P' of P

$|U(P, f, \alpha) - U(P, f, \alpha')| \leq M\epsilon$ also true for such refinement

$|\inf U(P, f, \alpha) - \inf U(P, f, \alpha')| \leq M\epsilon$

since $\epsilon > 0$ is arbitrary

$$\bar{\int_a^b} f\,d\alpha = \bar{\int_a^b} f\alpha'\,dx$$

$$\underline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\alpha'\,dx$$

Since $f \in R(\alpha)$, $\bar{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha$

$$\Rightarrow \underline{\int_a^b} f\alpha'\,dx = \bar{\int_a^b} f\alpha'\,dx$$

$$\Rightarrow f\alpha' \in R$$

$$\Rightarrow \int_a^b f\,d\alpha = \int_a^b f\alpha'\,dx$$

Conversely suppose that, $f\alpha' \in R$ and $\int_a^b f\,d\alpha = \int_a^b f\alpha'\,dx$

$f\alpha' \in R \Rightarrow \bar{\int_a^b} f\alpha'\,dx = \underline{\int_a^b} f\alpha'\,dx = \int_a^b f\alpha'\,dx$

$$\Rightarrow \bar{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha$$

$$\Rightarrow f \in R(\alpha)$$

Theorem :-

Change of variable :-

Suppose ϕ is strictly increasing continuous that maps an interval $[A,B]$ onto $[a,b]$ suppose α is monotonically increasing on $[a,b]$ and $f\in R(\alpha)$ on $[a,b]$. Define β and g on $[A,B]$ by $\beta(y)=\alpha(\phi(y))$, $g(y)=f(\phi(y))$ then $g\in R(\beta)$ and $\int_A^B g\,d\beta=\int_a^b f\,d\alpha$

Proof: Given that, $\phi:[A,B]\longrightarrow[a,b]$ is strictly increasing continuous and onto monotonically increasing on $[a,b]$ and $f\in R(\alpha)$ on $[a,b]$

β and g are defined by β, $g:[A,B]\longrightarrow\mathbb{R}$ by $\beta(y)=\alpha(\phi(y))$

$g(y)=f(\phi(y))$

since $f\in R(\alpha)$, for $\epsilon>0$ $\exists$ a partition $P=\{a=x_0,x_1,\ldots x_n=b\}$ of $[a,b]$ such that $U(P,f,\alpha)-L(P,f,\alpha)<\epsilon$

since $\phi:[A,B]\longrightarrow[a,b]$ is onto

for every partition $P=\{x_0,x_1,\ldots,x_n\}$ of $[a,b]$ there exist a partition $Q=\{y_0,y_1,\ldots,y_n\}$ of $[A,B]$ such that $\phi(y_i)=x_i$

let $M_i'=\sup g(y)$ $\forall$ $y\in[y_{i-1},y_i]$

$m_i'=\inf g(y)$ $\forall$ $y\in[y_{i-1},y_i]$

$M_i=\sup f(x)$ $\forall$ $x\in[x_{i-1},x_i]$

$m_i=\inf f(x)$ $\forall$ $x\in[x_{i-1},x_i]$

$\Rightarrow M_i'=M_i\ ,\ m_i'=m_i$

$$\begin{aligned}\Delta\beta_i&=\beta(y_i)-\beta(y_{i-1})\\&=\alpha(\phi(y_i))-\alpha(\phi(y_{i-1}))\\&=\alpha(x_i)-\alpha(x_{i-1})\\&=\Delta\alpha_i\end{aligned}$$

$$U(\phi, g, \beta) = \sum_{i=1}^{n} M_i' \Delta\beta_i$$

$$= \sum_{i=1}^{n} M_i \Delta\alpha_i = U(P, f, \alpha)$$

$$L(\phi, g, \beta) = \sum_{i=1}^{n} m_i' \Delta\beta_i$$

$$= \sum_{i=1}^{n} m_i \Delta\alpha_i = L(P, f, \alpha)$$

$$U(\phi, g, \beta) - L(\phi, g, \beta) = U(P, f, \alpha) - L(P, f, \alpha)$$

$$< \epsilon$$

$$\therefore g \in R(\beta)$$

$$\int_A^B g\,d\beta = \underline{\int_A^B} g\,d\beta = \overline{\int_A^B} g\,d\beta$$

$$\overline{\int_A^B} g\,d\beta = \inf U(\phi, g, \beta)$$

$$= \inf U(P, f, \alpha)$$

$$= \overline{\int_a^b} f\,d\alpha$$

$$\overline{\int_A^B} g\,d\beta = \int_a^b f\,d\alpha$$

since $g \in R(\beta)$ $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$

Integration and differentiation

(37)

*Theorem:-

Integration and differentiation:-

let $f \in R$ on $[a,b]$ for $a \le x \le b$, put $F(x) = \int_a^x f(t)\,dt$ then F is continuous on $[a,b]$, furthermore if f is continuous at a point x_0 of $[a,b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$ **

Proof:- Given that $f \in R$ on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$ for $a \le x \le b$

Also given that f is continuous at x_0 of $[a,b]$

since $f \in R$ on $[a,b]$, f is bounded on $[a,b]$

$\Rightarrow$ there exists $M > 0$ such that $|f(x)| \le M \ \forall x \in [a,b]$

let $a \le x < y \le b$ with $|x-y| < \delta = \frac{\epsilon}{M}$

$$F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_x^a f(t)\,dt + \int_a^y f(t)\,dt$$

$$= \int_x^y f(t)\,dt$$

$$|F(y) - F(x)| = \left| \int_x^y f(t)\,dt \right|$$

$$\le \int_x^y |f(t)|\,dt$$

$$\le M(y-x)$$

$$\le M\delta$$

$$= M \frac{\epsilon}{M}$$

$$= \epsilon$$

F is uniformly continuous on $[a,b]$

$\Rightarrow F$ is continuous on $[a,b]$

Given that f is continuous at x_0

for $\epsilon > 0 \ \exists \ \delta > 0 \ni |f(t) - f(x_0)| < \epsilon$ whenever $|t - x_0| < \delta$

Consider, $\left| \dfrac{F(t)-F(x_0)}{t-x_0} - f(x_0) \right|$

$$= \left| \frac{\int_{x_0}^{t} f(u)\,du}{t-x_0} - \frac{\int_{x_0}^{t} f(x_0)\,du}{t-x_0} \right|$$

$$= \frac{1}{|t-x_0|} \left| \int_{x_0}^{t} (f(u)-f(x_0))\,du \right|$$

$$\leq \frac{1}{|t-x_0|} \int_{x_0}^{t} |f(u)-f(x_0)|\,du$$

$$\leq \frac{\epsilon}{|t-x_0|} \int_{x_0}^{t} du$$

$$= \frac{\epsilon}{|t-x_0|} (t-x_0)$$

$$= \epsilon$$

$$\left| \frac{F(t)-F(x_0)}{t-x_0} - f(x_0) \right| < \epsilon$$

This is true for all $\epsilon > 0$

$$\lim_{t \to x_0} \frac{F(t)-F(x_0)}{t-x_0} = f(x_0)$$

$$\Rightarrow F'(x_0) = f(x_0)$$

~~Int~~

The fundamental theorem of calculus.

Theorem :-

If $f \in R$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ $\ni$ ~~Alt~~ $F' = f$ then $\int_a^b f(x)\,dx = F(b)-F(a)$

Proof: Since $f \in R$ on $[a,b]$ for $\epsilon > 0$ $\exists$ a partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a,b]$ $\ni$ $U(P,f) - L(P,f) < \epsilon$

Since F is differentiable on $[a,b]$, F is differentiable on each sub interval $[x_{i-1}, x_i]$

By mean value theorem,

$\exists\ t_i \in (x_{i-1}, x_i)$ such that $F(x_i) - F(x_{i-1}) = F'(t_i)(x_i - x_{i-1})$

$$F(x_i) - F(x_{i-1}) = F'(t_i)\Delta x_i \quad (\because F' = f)$$

$$F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$$

$$\sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

$$F(b) - F(a) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

Let $M_i = \sup f(x)\ \forall\ x \in [x_{i-1}, x_i]$

$m_i = \inf f(x)\ \forall\ x \in [x_{i-1}, x_i]$

$m_i \le f(t_i) \le M_i \quad \forall\ t_i \in [x_{i-1}, x_i]$

$$\sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} f(t_i)\Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(P,f) \le F(b) - F(a) \le U(P,f) \quad \text{--- (1)}$$

since $f \in R$ on $[a,b]$

$$\int_a^b f\,dx = \underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$$

$$\int_a^b f\,dx = \overline{\int_a^b} f\,dx = \inf U(P,f) \le U(P,f)$$

$$\int_a^b f\,dx = \underline{\int_a^b} f\,dx = \sup L(P,f) \ge L(P,f)$$

$$L(P,f) \le \int_a^b f\,dx \le U(P,f)$$

$$-U(P,f) \le -\int_a^b f\,dx \le -L(P,f) \quad \text{--- (2)}$$

(2) + (1)

$$-(U(P,f) - L(P,f)) \le F(b) - F(a) - \int_a^b f\,dx \le U(P,f) - L(P,f)$$

$$-\epsilon \le F(b) - F(a) - \int_a^b f\,dx \le \epsilon$$

$$\Rightarrow \left| F(b) - F(a) - \int_a^b f\,dx \right| < \epsilon$$

This is true $\forall\ \epsilon$

$$\int_a^b f\,dx = F(b) - F(a)$$

Integration by parts

*** **Theorem :- (Integration by parts).** -A

(17) Suppose F and G are differentiable functions on $[a,b]$ $F' = f \in R$ and $G' = g \in R$ then $\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$

Proof: let $H(x) = F(x)G(x)$

$\Rightarrow H'(x) = F(x)G'(x) + F'(x)G(x)$

Since F and G are differentiable on $[a,b]$

F and G are continuous on $[a,b]$

$\Rightarrow F \in R$ and $G \in R$

$\Rightarrow FG' \in R$ and $F'G \in R$

$\Rightarrow FG' + F'G \in R$

$\Rightarrow H' \in R$

By fundamental theorem of calculus,

$$\int_a^b (F(x)G'(x) + F'(x)G(x))\,dx = F(b)G(b) - F(a)G(a)$$

$$\Rightarrow \int_a^b F(x)G'(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x)\,dx \qquad [F' = f,\ G' = g]$$

$$\Rightarrow \int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$$

Integration of vector valued functions:

Definition:- let $\bar{f} = (f_1, f_2, \ldots, f_k)$ be a mapping of $[a,b]$ into R^k where each f_i is a real valued function on $[a,b]$. If α increases monotonically on $[a,b]$ we say that $\bar{f} \in R(\alpha)$, if $f_i \in R(\alpha)$, for $i = 1, 2, \ldots, k$ and we define

$$\int_a^b \bar{f}\,d\alpha = \left(\int_a^b f_1\,d\alpha, \int_a^b f_2\,d\alpha, \ldots, \int_a^b f_k\,d\alpha\right)$$

*** **Theorem :-** If $\bar{f}$ and $\bar{F}$ map $[a,b]$ into R^k if $\bar{f} \in R$ on $[a,b]$ and $\bar{F}' = \bar{f}$

(2) then $\int_a^b \bar{f}(t)\,dt = \bar{F}(b) - \bar{F}(a)$.

Proof: let $\bar{f} = (f_1, f_2, \ldots, f_k)$

$\bar{F} = (F_1, F_2, \ldots, F_k)$

since $\bar{f} \in R$ on $[a,b] \Rightarrow f_i \in R$ on $[a,b]$ for $1 \le i \le k$

$\bar{F}' = \bar{f} \Rightarrow F_i' = f_i$ for $1 \le i \le k$

Then by fundamental theorem of calculus,

$$\int_a^b f_i(t)\,dt = F_i(b) - F_i(a)$$

$$\int_a^b \bar{f}(t)\,dt = \left(\int_a^b f_1(t)\,dt, \int_a^b f_2(t)\,dt, \ldots, \int_a^b f_k(t)\,dt \right)$$

$$= (F_1(b) - F_1(a), F_2(b) - F_2(a), \ldots, F_k(b) - F_k(a))$$

$$= (F_1(b), F_2(b), \ldots, F_k(b)) - (F_1(a), F_2(a), \ldots, F_k(a))$$

$$= \bar{F}(b) - \bar{F}(a)$$

$$\int_a^b \bar{f}(t)\,dt = \bar{F}(b) - \bar{F}(a)$$

Theorem:-

If $\bar{f}$ maps $[a,b]$ into $\mathbb{R}^k$ and $\bar{f} \in R(\alpha)$ for some monotonically increasing function α on $[a,b]$ then $|\bar{f}| \in R(\alpha)$ & $\left| \int_a^b \bar{f}\,d\alpha \right| \le \int_a^b |\bar{f}|\,d\alpha$ ☆

Proof: let $\bar{f} = (f_1, f_2, \ldots, f_k)$

Given that $\bar{f} : [a,b] \to \mathbb{R}^k$ and $\bar{f} \in R(\alpha)$ for some monotonically increasing function α on $[a,b]$

Since $\bar{f} \in R(\alpha) \Rightarrow f_i \in R(\alpha)$ for $1 \le i \le k$

$$\Rightarrow f_i^2 \in R(\alpha)$$

$$\Rightarrow \sum_{i=1}^{k} f_i^2 \in R(\alpha)$$

let $g = f_1^2 + f_2^2 + \cdots + f_k^2 \in R(\alpha)$

$\Rightarrow g$ is bounded

$\Rightarrow 0 \le g(x) \le M$

let $\phi(t) = \sqrt{t}$, $t \in [0, M]$

Since ϕ is a polynomial, ϕ is continuous on $[0, M]$

$$\phi(g(x)) = \sqrt{g}(x)$$

$$\Rightarrow \sqrt{g} \in R(\alpha)$$

$$\Rightarrow |\bar{f}| \in R(\alpha)$$

Now, we have to prove that $\left|\int_a^b \bar{f}\, d\alpha\right| \leq \int_a^b |\bar{f}|\, d\alpha$

Put $\bar{y} = (y_1, y_2, \ldots, y_k)$

let $y_i = \int_a^b f_i\, d\alpha$

$$\bar{y} = \int_a^b \bar{f}\, d\alpha$$

Consider, $|\bar{y}|^2 = \sum_{i=1}^{k} y_i^2$

$$= \sum_{i=1}^{k} y_i \int_a^b f_i\, d\alpha$$

$$= \int_a^b \sum_{i=1}^{k} y_i f_i\, d\alpha$$

By schwartz inequality

$$\sum_{i=1}^{k} y_i f_i \leq |\bar{y}||\bar{f}|$$

$$\int_a^b \sum_{i=1}^{k} y_i f_i\, d\alpha \leq \int_a^b |\bar{y}||\bar{f}|\, d\alpha$$

$$|\bar{y}|^2 \leq \int_a^b |\bar{y}||\bar{f}|\, d\alpha$$

$$|\bar{y}| \leq \int_a^b |\bar{f}|\, d\alpha$$

$$\therefore \left|\int_a^b \bar{f}\, d\alpha\right| \leq \int_a^b |\bar{f}|\, d\alpha$$

Rectifiable Curve:-

Definition:- A continuous mapping γ of an interval $[a,b]$ into $\mathbb{R}^k$ is called a curve in $\mathbb{R}^k$. If γ is one-one, γ is called an arc. If $\gamma(a) = \gamma(b)$, γ is said to be a closed curve. let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a,b]$ and let γ be a curve on $[a,b]$

write $\Lambda(P,\gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$

The i^{th} term in the sum is the distance between the points $\gamma(x_{i-1})$ & $\gamma(x_i)$

$\therefore \Lambda(P,\gamma)$ is the length of the polynomial path with vertices $\gamma_0, \gamma_1, \ldots, \gamma_n$

The length of the curve γ is defined as

$\Lambda(\gamma) = \sup \Lambda(P,\gamma)$ where the supremum is taken over all partitions of $[a,b]$

If $\Lambda(\gamma) < \infty$ we say that γ is rectifiable

Imp

Theorem:

If γ' is continuous on $[a,b]$ then γ is rectifiable and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt$$

Proof: Given that γ' is continuous on $[a,b]$

$\Rightarrow \gamma' \in R$ on $[a,b]$

let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a,b]$ and $\gamma' \in R$ on $[x_{i-1}, x_i]$

$$\Rightarrow \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\, dt \right| \leq \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt$$

$$\Rightarrow |\gamma(x_i) - \gamma(x_{i-1})| \leq \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt$$

$$\Rightarrow \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt$$

$$\Rightarrow \Lambda(P,\gamma) \leq \int_a^b |\gamma'(t)|\, dt$$

This is true for every partition P of $[a,b]$

$\int_a^b |\gamma'(t)|\, dt$ is an upperbound of $\{\Lambda(P,\gamma) / P \text{ be a partition of } [a,b]\}$

$$\Lambda(\gamma) = \sup\{\Lambda(P,\gamma) / P \text{ be a partition of } [a,b]\}$$

$$\leq \int_a^b |\gamma'(t)|\, dt$$

$$\Lambda(\gamma) \leq \int_a^b |\gamma'(t)|\,dt$$

$\Rightarrow \gamma$ is rectifiable

Since γ' is continuous on $[a,b]$

$\Rightarrow \gamma'$ is uniformly continuous on $[a,b]$ and $\Delta x_i < \delta \ \forall i$

$\epsilon > 0 \ \exists \ \delta > 0 \ni |\gamma'(t) - \gamma'(s)| < \epsilon \ \forall \ t, s \in [a,b]$ for which $|t-s| < \delta$

let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a,b]$ and $\Delta x_i < \delta \ \forall x$

If $t \in [x_{i-1}, x_i]$ then $|\gamma'(t)| - |\gamma'(x_i)| \leq |\gamma'(t) - \gamma'(x_i)| < \epsilon$

$$|\gamma'(t)| \leq |\gamma'(x_i)| + \epsilon$$

$$\int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt \leq \int_{x_{i-1}}^{x_i} (|\gamma'(x_i)| + \epsilon)\,dt$$

$$= \int_{x_{i-1}}^{x_i} |\gamma'(x_i)|\,dt + \epsilon\,\Delta x_i$$

$$= \left| \int_{x_{i-1}}^{x_i} (\gamma'(x_i) - \gamma'(t) + \gamma'(t))\,dt \right| + \epsilon\,\Delta x_i$$

$$\leq \left| \int_{x_{i-1}}^{x_i} (\gamma'(x_i) - \gamma'(t))\,dt \right| + \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\,dt \right| + \epsilon\,\Delta x_i$$

$$\leq \epsilon\,\Delta x_i + \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\,dt \right| + \epsilon\,\Delta x_i$$

$$\Rightarrow \int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt \leq 2\epsilon\,\Delta x_i + \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\,dt \right|$$

$$\Rightarrow \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt \leq \sum_{i=1}^{n} 2\epsilon\,\Delta x_i + \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\,dt \right|$$

$$\Rightarrow \int_a^b |\gamma'(t)|\,dt \leq 2\epsilon(b-a) + \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$$

$$\leq 2\epsilon(b-a) + \Lambda(P, \gamma)$$

Since $\epsilon > 0$ is arbitrary,

$$\int_a^b |\gamma'(t)|\,dt \leq \Lambda(P, \gamma) = \Lambda(\gamma)$$

$\rightarrow \int_a^b |\gamma'(t)|dt \le \Lambda(\gamma)$

$\Lambda(\gamma) = \int_a^b |\gamma'(t)|dt$

Problems:

(1) If $F(x)=0$ for all irrational x, $F(x)=1$ $\forall$ rational x, prove that $F \notin R$ on $[a,b]$ for any $a<b$.

Given that $F:[a,b] \rightarrow \mathbb{R}$ where $F(x) = \begin{cases} 1 & \forall \text{ rational} \\ 0 & \forall \text{ irrational} \end{cases}$

let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a,b]$

Put $m_i = \inf f(x)$ $\forall x \in [x_{i-1}, x_i]$

$M_i = \sup f(x)$ $\forall x \in [x_{i-1}, x_i]$

then $m_i = 0$, $M_i = 1$

$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = b-a$

$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0\,\Delta x_i = 0$

Since this is true for every partition of $[a,b]$

we have $\overline{\int_a^b} f\,dx = \inf U(P,f) = b-a$

$\underline{\int_a^b} f\,dx = \sup L(P,f) = 0$

$\therefore \underline{\int_a^b} f\,dx \neq \overline{\int_a^b} f\,dx$

$\Rightarrow f \notin R$ on $[a,b]$

2) If $f(x) = a$ for all irrational x, $f(x) = b$ $\forall$ rational x, prove that $f \notin R$ on $[a,b]$

S Given $f:[a,b] \rightarrow \mathbb{R}$ where $f(x) = \begin{cases} b & \forall \text{ rational } x \\ a & \forall \text{ irrational } x \end{cases}$

let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a,b]$

(45)

Put $m_i = \inf f(x) \ \forall x \in [x_{i-1}, x_i]$

$M_i = \sup f(x) \ \forall x \in [x_{i-1}, x_i]$

then $m_i = a$ and $M_i = b$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} b \Delta x_i = b(b-a)$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} a \Delta x_i = a(b-a)$$

Since this is true for every partition of $[a,b]$ we have

$$\overline{\int_a^b} f\,dx = \inf U(P,f) = b(b-a)$$

$$\underline{\int_a^b} f\,dx = \sup L(P,f) = a(b-a)$$

$$\therefore \overline{\int_a^b} f\,dx \neq \underline{\int_a^b} f\,dx$$

$\Rightarrow f \notin R$ on $[a,b]$

③ Suppose $f \geq 0$, f is continuous on $[a,b]$ and $\int_a^b f(x)\,dx = 0$ prove that $f(x) = 0 \ \forall x \in [a,b]$

Sol Suppose $f \geq 0$, f is continuous on $[a,b]$ and

$$\int_a^b f(x)\,dx = 0 \quad \text{—— ①}$$

Claim:- $f(x) = 0 \ \forall x \in [a,b]$

Suppose $f(x_0) \neq 0$ for some $x_0 \in [a,b]$

since $f \geq 0$ and $f(x_0) \neq 0$

$\Rightarrow f(x_0) > 0$

Put $\epsilon = f(x_0/2) > 0$

Since f is continuous on $[a,b]$ and $x_0 \in [a,b]$

$\Rightarrow f$ is continuous at x_0

for every $\epsilon > 0 \ \exists \ \delta > 0 \ni |f(x) - f(x_0)| < \epsilon \ \forall x \in [a,b]$

for which $|x - x_0| < \delta$

$\Rightarrow f(x_0) - f(x) < \epsilon$ for which $x_0 - \delta < x < x_0 + \delta$

$\Rightarrow f(x_0) - f(x) < \frac{f(x_0)}{2} \quad \forall \; x_0 - \delta < x < x_0 + \delta$

$\Rightarrow f(x) > \frac{f(x_0)}{2} \quad \forall \; x_0 - \delta < x < x_0 + \delta$

$\Rightarrow$ Now $\int_a^b f(x)dx = \int_a^{x_0-\delta} f(x)dx + \int_{x_0-\delta}^{x_0+\delta} f(x)dx + \int_{x_0+\delta}^b f(x)dx$

$$\geq \int_{x_0-\delta}^{x_0+\delta} f(x)dx$$

$$> \int_{x_0-\delta}^{x_0+\delta} \frac{f(x_0)}{2} dx$$

$$= \frac{f(x_0)}{2} \int_{x_0-\delta}^{x_0+\delta} dx$$

$$= \frac{f(x_0)}{2} 2\delta$$

$$= f(x_0)\delta > 0$$

$\therefore \int_a^b f(x)dx > 0$ which is a contradiction to ①

$\therefore f(x) = 0 \quad \forall \; x \in [a,b]$

imp

④ Suppose f is a real continuously differentiable function on $[a,b]$,

** $f(a) = f(b) = 0$ and $\int_a^b f^2(x)dx = 1$ prove that $\int_a^b x f(x) \cdot f'(x) dx = \frac{-1}{2}$ and that $\int_a^b [f'(x)]^2 dx \int_a^b x^2 f^2(x) dx > \frac{1}{4}$

Sol: Suppose f is a real continuosly differentiable function on $[a,b]$

$f(a) = f(b) = 0$ and $\int_a^b f^2(x)dx = 1$ — ①

Put $I = \int_a^b x f(x) f'(x) dx$

then $I = [x f^2(x)]_a^b - \int_a^b [x f'(x) + f(x)] f(x) dx$

$= bf^2(b) - af^2(a) - \int_a^b x f(x) f'(x) dx - \int_a^b f^2(x)dx$

$\Rightarrow I = 0 - 0 - I - 1$

$\Rightarrow I + I = -1$

$2I = -1$

$I = -1/2$

$\therefore \int_a^b x f(x) f'(x)\,dx = \frac{-1}{2}$

By schwarz inequality we have

$\left(\int_a^b x f(x) f'(x)\,dx\right)^2 \le \int_a^b (x f(x))^2\,dx \int_a^b (f'(x))^2\,dx$

$\Rightarrow \int_a^b x^2 f^2(x)\,dx \int_a^b (f'(x))^2\,dx \ge \left(\frac{-1}{2}\right)^2$

$\Rightarrow \int_a^b x^2 f^2(x)\,dx \int_a^b (f'(x))^2\,dx > \frac{1}{4}$

5) Let γ_1 be a curve in $\mathbb{R}^k$ defined on $[a,b]$. Let ϕ be continuous 1-1 mapping of $[c,d]$ onto $[a,b] \ni \phi(c) = a$ and define $\gamma_2(s) = \gamma_1(\phi(s))$. Prove that γ_2 is an arc a closed curve or a rectifiable curve iff the same is true of γ_1 prove that γ_1 and γ_2 have the same length.

Proof: Let $\phi(d) = b$

Define $\gamma_2 : [c,d] \to \mathbb{R}^k$ as $\gamma_2(s) = \gamma_1(\phi(s))\ \forall s \in [c,d]$

Then γ_2 is a curve in $\mathbb{R}^k$

suppose γ_1 is an arc

i.e., γ_1 is 1-1

Let $s,t \in [c,d] \ni \gamma_2(s) = \gamma_2(t)$

$\Rightarrow \gamma_1(\phi(s)) = \gamma_1(\phi(t))$

$\Rightarrow \phi(s) = \phi(t)$ $\quad (\because \gamma_1 \text{ is } 1\text{-}1)$

$\Rightarrow s = t$ $\quad (\because \phi \text{ is } 1\text{-}1)$

$\therefore \gamma_2$ is 1-1 and hence an arc

suppose γ_1 is closed

i.e., $\gamma_1(a) = \gamma_1(b)$

Now $\gamma_2(c) = \gamma_1(\phi(c))$

$= \gamma_1(a) = \gamma_1(b) = \gamma_1(\phi(d)) = \gamma_2(d)$

$\therefore \gamma_2$ is closed

Suppose γ_1 is rectifiable

i.e., $\Lambda(\gamma_1) < \infty$

Let $P = \{c = x_0, x_1, \ldots, x_n = d\}$ be a partition of $[c,d]$

$$\Lambda(P, \gamma_2) = \sum_{i=1}^{n} |\gamma_2(x_i) - \gamma_2(x_{i-1})| \text{ and}$$

$$\Lambda(\gamma_2) = \sup\{\Lambda(P,\gamma_2) / P \text{ is a partition of } [c,d]\}$$

$$= \sup\{\Lambda(P, \gamma_1 \circ \phi) \mid P \text{ is a partition of } [c,d]\}$$ $\gamma_2 = \gamma_1 \circ \phi$

$< \infty$ [$\because \gamma_1$ is rectifiable]

$\therefore \gamma_2$ is rectifiable

Conversely, suppose that γ_2 is an arc

i.e., γ_2 is 1-1

Let $t_1, t_2 \in [a,b] \ni \gamma_1(t) = \gamma_1(t_2)$

Since $\phi : [c,d] \longrightarrow [a,b]$ is onto and $t_1, t_2 \in [a,b]$

$\Rightarrow \exists\, s_1, s_2 \in [c,d] \ni \phi(s_1) = t_1$ and $\phi(s_2) = t_2$

Now $\gamma_1(t_1) = \gamma_2(t_2)$

$\Rightarrow \gamma_1(\phi(s_1)) = \gamma_2(\phi(s_2))$

$\Rightarrow \gamma_1(s_1) = \gamma_2(s_2)$

$\Rightarrow s_1 = s_2$

$\Rightarrow \phi(s_1) = \phi(s_2)$

$\Rightarrow t_1 = t_2$

$\therefore \gamma_1$ is 1-1

$\Rightarrow \gamma_1$ is an arc

Suppose γ_2 is closed

i.e., $\gamma_2(c) = \gamma_2(d)$

Consider, $\gamma_1(a) = \gamma_2(\phi(c))$

$= \gamma_2(c)$

$\Rightarrow \gamma_2(d) = \gamma_1(\phi(d)) = \gamma_1(b)$

$\therefore \gamma_1(a) = \gamma_1(b)$

$\therefore \gamma_1$ is closed

Suppose γ_2 is rectifiable then $\Lambda(\gamma_2) < \infty$

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$

Then $\Lambda(P, \gamma_1) = \sum_{i=1}^{n} |\gamma_2(x_i) - \gamma(x_{i-1})|$

Now $\Lambda(\gamma_1) = \sup\{\Lambda(P, \gamma_1) / P \text{ is a partition of } [a,b]\}$

$< \infty$

$\therefore \gamma_1$ is rectifiable

Consider, $\Lambda(\gamma_2) = \int_c^d |\gamma_2'(t)| dt$

$= \int_c^d |\gamma_1(\phi(t))'| dt$

$= \int_c^d |\gamma_1'(\phi(t))| \phi'(t)| dt$

$= \int_a^b |\gamma_1'(s)| ds$

$= \Lambda(\gamma_1)$

$\therefore \gamma_1$ and γ_2 have the same length

Unit-II

Sequences & Series of the functions

Definition:- Suppose $\{f_n\}$, $n=1,2,\dots$ is a sequence of functions defined on E and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$ then we define a function f by $f(x) = \lim\limits_{n\to\infty} f_n(x)$ $(x \in E)$ we say that sequence $\{f_n\}$ converges on E of f point wise.

Definition:- If $\sum f_n(x)$ converges for every $x \in E$ and if we define $f(x) = \sum\limits_{n=1}^{\infty} f_n(x)$ then the function f is called a sum of the series $\sum f_n$.

***Note:**

Limit cannot be interchanged in a double sequence.

Ex) for $m = 1,2,\dots$, $n = 1,2,\dots$ define $S_{m,n} = \dfrac{m}{m+n}$

claim:. $\lim\limits_{m\to\infty} \lim\limits_{n\to\infty} S_{m,n} \neq \lim\limits_{n\to\infty} \lim\limits_{m\to\infty} S_{m,n}$

If m is fixed and taking limit as $n \to \infty$

$$\lim_{n\to\infty} S_{m,n} = \lim_{n\to\infty} \frac{m}{m+n} = 0$$

$$\lim_{m\to\infty} \lim_{n\to\infty} S_{m,n} = \lim_{m\to\infty} 0 = 0$$

If n is fixed and taking limit as $m \to \infty$

$$\lim_{m\to\infty} S_{m,n} = \lim_{m\to\infty} \frac{m}{m+n} = 1$$

$$\lim_{n\to\infty} \lim_{m\to\infty} S_{m,n} = \lim_{n\to\infty} 1 = 1$$

***Note:** If the sequence $\{f_n\}$ converges to f then the sequence $\{f_n'\}$ need not be converges to f'.

Eg: Let $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$.

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$f(x) = \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}}$$

$$f(x) = 0$$

$$f'(x) = 0$$

$$f_n'(x) = \sqrt{n} \cos nx$$

In particular, $f_n'(0) = \sqrt{n} \cos 0 = \sqrt{n} \to \infty$ as $n \to \infty$

$\{f_n'(x)\}$ does not converges to f' even though $f_n \to f$

* **Note:** Converges series of continuous functions may have a discontinuous sum.

Eg: let $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$, x is real & $n = 0, 1, 2, \ldots$

then prove that f is discontinuous at $x = 0$

Sol: $f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \dfrac{x^2}{(1+x^2)^n}$

$$= x^2 + \frac{x^2}{(1+x^2)} + \frac{x^2}{(1+x^2)^2} + \cdots$$

$$= x^2 + \frac{x^2}{(1+x^2)}\left(1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \cdots\right)$$

$$= x^2 + \frac{x^2}{(1+x^2)}\left(\frac{1}{1-\frac{1}{1+x^2}}\right)$$

$$= x^2 + \frac{x^2}{1+x^2}\left(\frac{1+x^2}{1+x^2-1}\right)$$

$$= x^2 + \frac{x^2}{1+x^2}\left(\frac{1+x^2}{x^2}\right)$$

$$f(x) = x^2 + 1$$

$$f(x) = \begin{cases} x^2+1 & ; \text{ if } x \neq 0 \\ 0 & ; \text{ if } x = 0 \end{cases}$$

Note:-

limit of the integral need not be equal to the integral of the limit.

Eg: let $f_n(x) = n^2 \cdot x(1+x^2)^n$, $0 < x < 1$, $n = 1, 2, \ldots$

Sol: $f_n(x) = n^2 \cdot x(1-x^2)^n = \dfrac{n^2 \cdot x}{(1-x^2)^{-n}} = \dfrac{n^2 x}{\left(\frac{1}{(1-x^2)}\right)^n} = \dfrac{n^2 x}{\left(1+\frac{x^2}{(1-x^2)}\right)^n}$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(\frac{n^2 x}{\left(1+\frac{x^2}{1-x^2}\right)^n}\right) = x \cdot \lim_{n\to\infty} \frac{n^2}{\left(1+\frac{x^2}{1-x^2}\right)^n} = x \cdot 0 = 0$$

$$\lim_{n\to\infty} f_n(x) = 0$$

$$\Rightarrow \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0$$

Now, $\displaystyle\int_0^1 f_n(x)\,dx = \int_0^1 n^2 x(1-x^2)^n\,dx$

$$= n^2\int_0^1 x(1-x^2)^n\,dx = n^2\int_0^1 \frac{-2x}{-2}(1-x^2)^n\,dx$$

$$= \cancel{n^2\left(\frac{(1-x^2)^{n+1}}{n+1}\left(\frac{+1}{2}\right)\right)_0^1} = \frac{n^2}{-2}\int_0^1 (1-x^2)^n(-2x)\,dx$$

$$= \left(\frac{-n^2}{2}\right)\left[\frac{(1-x^2)^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow \int_0^1 f_n(x)\,dx = \frac{n^2}{2(n+1)}$$

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\left(\frac{n^2}{2(n+1)}\right) = \infty$$

$$\int_0^1 \lim_{n\to\infty} f(x)\,dx \neq \lim_{n\to\infty} f(x)\,dx$$

Uniform Convergence :- A sequence of function $\{f_n\}$, $n=1,2,\dots$ converges uniformly on E to a function f if for every $\epsilon>0$ there is an integer N such that $n\geq N$ implies $|f_n(x)-f(x)|\leq\epsilon$ for all $x\in E$

The series $\sum f_n(x)$ converges uniformly on E if the sequence $\{S_n\}$ of Partial sum defined by $S_n(x)=\sum_{i=1}^{n} f_i(x)$ converges uniformly on E

Note:
Every uniformly convergent sequence is pointwise convergent.

*** ~~Theorem~~ Int :-

Cauchy Criterian for uniform Convergence :-

The sequence function $\{f_n\}$ defined on E converges uniformly on E iff for every $\epsilon>0$ $\exists$ an integer $N \ni m\geq N$, $n\geq N$, $x\in E$ implies $|f_n(x)-f_m(x)|\leq\epsilon$.

Proof. Suppose $\{f_n\}$ converges uniformly on E
let f be the limit function

Then for every $\epsilon > 0$ $\exists$ an integer $N \ni n \geq N, x \in E$

$\Rightarrow |f_n(x) - f(x)| < \epsilon/2$

Also for $m \geq N$, $|f_m(x) - f(x)| < \epsilon/2$

Consider, $|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)|$

$$\leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$$

$$\leq \epsilon/2 + \epsilon/2$$

$$= \epsilon$$

$\Rightarrow |f_n(x) - f_m(x)| \leq \epsilon$ if $n \geq N$ and $x \in E$

Conversely, suppose for every $\epsilon > 0$ $\exists$ an integer $N \ni m \geq N, x \in E$

$\Rightarrow |f_n(x) - f_m(x)| \leq \epsilon$

We know that, every cauchy sequence in $\mathbb{R}$. converges to some limit.

$\Rightarrow \{f_n\}$ converges to some limit f

let $\epsilon > 0$ be given. choose $N \ni m \geq N, n \geq N$

$x \in E \Rightarrow |f_n(x) - f_m(x)| \leq \epsilon$

fix n and let $m \to \infty$

$\Rightarrow |f_n(x) - \lim_{m \to \infty} f_m(x)| \leq \epsilon$

$\Rightarrow |f_n(x) - f(x)| \leq \epsilon$ $\forall x \in E$ and $n \geq N$

$\{f_n\}$ converges uniformly on E to a function f

~~Theorem~~ Int

Suppose $\lim_{n \to \infty} f_n(x) = f(x)$ $(x \in E)$ put $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ then

$f_n \to f$ uniformly on E iff $M_n \to 0$ as $n \to \infty$

Proof: Suppose $f_n \to f$ uniformly on E then

for given $\epsilon > 0$ $\exists$ an integer $N \ni n \geq N$

$\Rightarrow |f_n(x) - f(x)| \leq \epsilon$ $\forall x \in E$, $M_n = \sup_{x \in E} |f_n(x) - f(x)| \leq \epsilon$

for all $n \geq N \Rightarrow M_n \leq \epsilon$ $\forall n \geq N$

$\Rightarrow M_n \to 0$ as $n \to \infty$

Conversely, suppose $M_n \to 0$ as $n \to \infty$

for $\epsilon > 0$ $\exists$ an integer $N \ni |M_n - 0| \leq \epsilon \; \forall \, n \geq N$

$\Rightarrow M_n \leq \epsilon \quad \forall \, n \geq N$

$\Rightarrow \sup |f_n(x) - f(x)| \leq \epsilon \quad \forall \, n \geq N$ and $x \in E$

$\Rightarrow |f_n(x) - f(x)| \leq \sup |f_n(x) - f(x)| \leq \epsilon \quad \forall \, n \geq N$ and $x \in E$.

$\Rightarrow |f_n(x) - f(x)| \leq \epsilon \; \forall \, n \geq N$ and $x \in E$

$\Rightarrow f_n \to f$ uniformly on E

Int

Theorems:-

Weierstrass M-test:-

Statement:- Suppose $\{f_n\}$ is a sequence of functions defined on E and suppose $|f_n(x)| \leq M_n$ $(x \in E, n = 1, 2, 3, \ldots)$ then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Proof: Given that $\{f_n\}$ is a sequence of functions defined on E and

$|f_n(x)| \leq M_n$ $(x \in E, n = 1, 2, 3, \ldots)$

Suppose $\sum M_n$ converges

$\Rightarrow$ The sequence of partial sums $\{t_n\}$ of $\sum M_n$ converges

where, $t_n = \sum_{i=1}^{n} M_i$

$\{t_n\}$ is a cauchy sequence.

for $\epsilon > 0$ $\exists$ an integer N such that $|t_n - t_m| \leq \epsilon \quad \forall \, n, m \geq N$

$\Rightarrow \left| \sum_{i=m+1}^{n} M_i \right| \leq \epsilon \quad \forall \, m, n \geq N$

Consider, $\left| \sum_{i=m+1}^{n} f_i(x) \right| \leq \sum_{i=m+1}^{n} |f_i(x)|$

$\leq \sum_{i=m+1}^{n} M_i$

$\leq \epsilon \quad \forall \, n, m \geq N$

$\Rightarrow \left| \sum_{i=m+1}^{n} (f_i(x)) \right| \leq \epsilon$ for $n, m \geq N$

$\Rightarrow \left| S_n(x) - S_m(x) \right| \leq \epsilon \quad \forall\, n, m \geq N$

where $S_n(x) = \sum_{i=1}^{n} f_i(x)$

i.e., the sequence $\{S_n\}$ of partial sums of $\sum f_n$ satisfies cauchy conditions

$\Rightarrow \sum f_n$ converges uniformly on E

Uniform Convergence and Continuity

*Int

Uniform convergence and continuity :-

Statement:- Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E and suppose that $\lim_{t\to x} f_n(t) = A_n \; (n = 1, 2, \ldots)$ then $\{A_n\}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$. In other words, the conclusion is that $\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$.

Proof: Given that $f_n \to f$ uniformly on E

Then for $\epsilon > 0$ $\exists\, N \ni |f_n(t) - f_m(t)| \leq \epsilon \quad \forall\, n, m \geq N,\ t \in E$

Given that $\lim_{t\to x} f_n(t) = A_n$

$$\left| \lim_{t\to x} f_n(t) - \lim_{t\to x} f_m(t) \right| \leq \epsilon \quad \forall\, m, n \geq N,\ t \in E$$

$$\Rightarrow |A_n - A_m| \leq \epsilon \quad \forall\, n, m \geq N$$

$$\Rightarrow \{A_n\} \text{ converges}$$

let $\lim_{n\to\infty} A_n = A$

Then for $\epsilon > 0$ $\exists$ an integer N_1 such that $|A_n - A| \leq \epsilon/3 \quad \forall\, n \geq N_1$

Since $f_n \to f$ uniformly on E

for each $\epsilon > 0$ $\exists\, N_2 \ni |f_n(t) - f(t)| \leq \epsilon/3 \quad \forall\, n \geq N_2$

let $N = \max\{N_1, N_2\}$

for this N choose a neighbourhood V of x such that

$|f_n(t) - A_n| \leq \epsilon/3 \quad \forall\, t \in V \cap E$ and $t \neq x$

Consider,

$$|f(t)-A| = |f(t)-f_n(t)+f_n(t)-A_n+A_n-A|$$

$$\leq |f(t)-f_n(t)| + |f_n(t)-A_n| + |A_n-A|$$

$$\leq \epsilon/3 + \epsilon/3 + \epsilon/3$$

$$= \epsilon \quad \forall\, t\in V\cap E \text{ and } t\neq x$$

$$|f(t)-A| \leq \epsilon \quad \forall\, t\in V\cap E \text{ and } t\neq x$$

$$\lim_{t\to x} f(t) = A$$

But $\lim_{n\to\infty} A_n = A$

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$$

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$

***** Imp Theorem:-**

If $\{f_n\}$ is a sequence of continuous on E and if $f_n\to f$ uniformly on E then f is continuous on E.

Proof: Given that $\{f_n\}$ is a sequence of continuous function on E

Let $x\in E$

$\Rightarrow f_n$ is continuous at x

$$\lim_{t\to x} f_n(t) = f_n(x)$$

Since $f_n\to f$ uniformly on E

$$\Rightarrow \lim_{n\to\infty} f_n(t) = f(t)$$

Now,
$$\lim_{t\to x} f(t) = \lim_{t\to x}\lim_{n\to\infty} f_n(t)$$

$$= \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$

$$= \lim_{n\to\infty} f_n(x)$$

$$\lim_{t\to x} f(t) = f(x)$$

$\Rightarrow f$ is continuous at x

$\Rightarrow f$ is continuous on E

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Theorem:

Suppose k is compact.

(a) $\{f_n\}$ is a sequence of continuous functions on k

(b) $\{f_n\}$ converges pointwise to a continuous function f on k.

(c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in k$, $n = 1, 2, \ldots$ then $f_n \to f$ uniformly on k.

Proof: Define $g_n(x) = f_n(x) - f(x)$

since $\{f_n\}$ is a sequence of continuous function on k

$f_n \to f$ pointwise and f is continuous

g_n is continuous and $g_n \to 0$ pointwise

Since $f_n(x) \geq f_{n+1}(x)$, $n = 1, 2, \ldots$

$\Rightarrow f_n(x) - f(x) \geq f_{n+1}(x) - f(x)$

$\Rightarrow g_n(x) \geq g_{n+1}(x)$

we shall prove that $g_n \to 0$ uniformly on k

Let $\epsilon > 0$

Let $k_n = \{x \in k / g_n(x) \geq \epsilon\}$ where $k_n \geq k_{n+1}$

Since g_n is continuous

k_n is a closed subset of a compact set k

then k_n is compact

If each k_n is non-empty then $\bigcap_{n=1}^{\infty} k_n \neq \phi$

there exists x such that $g_n(x) \geq \epsilon$ which is a contradiction to the

fact that $g_n \to 0$

$\exists$ N such that k_N empty

k_N is empty for all $n \geq N$

i.e., $g_n(x) < \epsilon$ $\forall$ $n \geq N$ and $x \in k$

$\Rightarrow g_n \to 0$ uniformly on k

$\Rightarrow f_n \to f$ uniformly on k

Definition: Suppose X is a metric space let $\mathbb{C}(X)$ denotes the set of all complex valued functions defined on X and are continuous bounded on X of each $f \in \mathbb{C}(X)$, $\|f\| = \sup\limits_{x \in X} |f(x)|$

Theorem:-

The set $\mathbb{C}(X)$ of all complex valued continuous bounded functions on X is a complex metric space with metric defined by

$$d(f,g) = \|f-g\| = \sup_{x \in X} |f(x) - g(x)|$$

Proof: let $f,g \in \mathbb{C}(X)$

(i) $d(f,g) = \|f-g\| = \sup\limits_{x \in X} |f(x)-g(x)| \geq 0$

(ii) $d(f,g) = 0 \Rightarrow \|f-g\| = 0$

$$\Rightarrow \sup_{x \in X} |f(x)-g(x)| = 0$$

$$\Rightarrow f(x) - g(x) = 0 \quad \forall x \in X$$

$$\Rightarrow f(x) = g(x) \text{ for all } x \in X$$

$$\Rightarrow f = g$$

$$d(f,g) = \|f-g\|$$

$$= \sup_{x \in X} |f(x)-g(x)|$$

$$= \sup_{x \in X} |g(x)-f(x)|$$

$$= \|g-f\|$$

$$= d(g,f)$$

(iii) let $f,g,h \in \mathbb{C}(X)$

$$|f(x)-h(x)| \leq |f(x)-g(x)| + |g(x)-h(x)|$$

$$\leq \|f-g\| + \|g-h\|$$

$$\|f-h\| \leq \|f-g\| + \|g-h\|$$

$$d(f,h) \leq d(f,g) + d(g,h)$$

$\mathbb{C}(X)$ is a metric space

Now we have to prove that every Cauchy's sequence in $C(X)$ converges to some limit function $f \in C(X)$

let $\{f_n\}$ be a cauchy's sequence in $C(X)$

for $\epsilon > 0$ $\exists$ an integer $N \ni \|f_n - f_m\| < \epsilon$ $\forall$ $m, n \geq N$

$\Rightarrow \sup |f_n(x) - f_m(x)| < \epsilon$ $\forall$ $m, n \geq N$ and $x \in X$

$\Rightarrow |f_n(x) - f_m(x)| \leq \sup |f_n(x) - f_m(x)| < \epsilon$ $\forall$ $m, n \geq N$ $\forall$ $x \in X$

$\Rightarrow |f_n(x) - f_m(x)| < \epsilon$ $\forall$ $m, n \geq N$ and $x \in X$

by cauchy's criterian for uniform convergence

$\Rightarrow \{f_n\}$ converges uniformly to some function f

Now we have to prove that $f \in C(X)$

since $f_n \in C(X) \Rightarrow f_n$ is a continuous function

let $x \in X$

$\Rightarrow f_n$ is continuous at x

$$\Rightarrow \lim_{t \to x} f_n(t) = f_n(x)$$

since $f_n \to f$ uniformly

$$\lim_{n \to \infty} f_n(x) = f(x)$$

$$\lim_{t \to x} f(x) = \lim_{t \to x} \lim_{n \to \infty} f_n(t)$$

By known theorem,

$$\lim_{t \to x} f(t) = \lim_{t \to x} \lim_{n \to \infty} f_n(t)$$

$$= \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

$$= \lim_{n \to \infty} f_n(x)$$

$$= f(x)$$

$$\lim_{t \to x} f(t) = f(x)$$

$\Rightarrow f$ is continuous at x

$\Rightarrow f$ is continuous on X

Since $f_n \in \mathcal{C}(x)$

$\Rightarrow f_n$ is bounded on X

and since $\exists\, n \ni |f(x)-f_n(x)| < 1 \quad \forall x \in X$

$$|f(x)| \le |f(x)-f_n(x)| + |f_n(x)|$$

$$< 1 + |f_n(x)| \quad \forall x \in X$$

f is bounded on X

f is complex valued and continuous bounded function on X

$\Rightarrow f \in \mathcal{C}(X)$

$\mathcal{C}(X)$ is a complete metric space.

*** Int

Theorem:-

Uniform convergence and integration:

Statement:

Let α be monotonically increasing on $[a,b]$. suppose $f_n \in R(\alpha)$ on $[a,b]$ for $n=1,2,\dots$ and suppose $f_n \to f$ uniformly on $[a,b]$ then $f \in R(\alpha)$ on $[a,b]$ and $\int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha$

Proof: let $\epsilon > 0$ be given

choose η such that $\eta[\alpha(b)-\alpha(a)] < \epsilon/3$

Since, $f_n \to f$ uniformly on $[a,b]$

for $\eta > 0$ $\exists$ an integer $N \ni |f_n(x)-f(x)| \le \eta \quad \forall n \ge N$ & $x \in [a,b]$

$\Rightarrow f_n(x) - \eta \le n \quad \forall n \ge N$ & $x \in [a,b]$

let M_n & M be supremum of f_n and f respectively on $[a,b]$

$$M_n - \eta \le M \le M_n + \eta$$

since $f_n \in R(\alpha)$ on $[a,b]$ for $\epsilon > 0$ $\exists$ a partition $P = \{a = x_0, x_1, \dots, x_n = b\}$ of $[a,b]$ such that $U(P,f_n,\alpha) - L(P,f_n,\alpha) < \epsilon/3$

Since $U(P,f_n,\alpha) - L(P,f_n,\alpha) < \epsilon/3$ holds $\forall x \in [a,b]$

it also holds for $x \in [x_{i-1}, x_i]$

let M_{n_i} and M_i be supremum of f_n and f respectively on $[x_{i-1}, x_i]$

$$\Rightarrow M_{n_i} - \eta \le M_i \le M_{n_i} + \eta$$

Now $U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$

$$\leq \sum_{i=1}^{n} (M_{n_i} + \eta)\, \Delta\alpha_i$$

$$= \sum_{i=1}^{n} M_{n_i} \Delta\alpha_i + \sum_{i=1}^{n} \eta \Delta\alpha_i.$$

$$= U(P, f_n, \alpha) + \eta\,[\alpha(b) - \alpha(a)]$$

$$< U(P, f_n, \alpha) + \epsilon/3$$

$$U(P,f,\alpha) < U(P,f_n,\alpha) + \epsilon/3$$

Similarly, $L(P,f,\alpha) > L(P,f_n,\alpha) - \epsilon/3$

Now $U(P,f,\alpha) - L(P,f,\alpha) < U(P,f_n,\alpha) + \epsilon/3 - L(P,f_n,\alpha) + \epsilon/3$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3$$

$$= \epsilon$$

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

$$\Rightarrow f \in R(\alpha)$$

Consider, $\left| \int_a^b f\,d\alpha - \int_a^b f_n\,d\alpha \right|$

$$= \left| \int_a^b (f - f_n)\,d\alpha \right|$$

$$\leq \int_a^b |f - f_n|\, d\alpha(x)$$

$$< \int_a^b \eta\, d\alpha$$

$$= \eta\,[\alpha(b) - \alpha(a)]$$

$$< \epsilon/3$$

$$\Rightarrow \left| \int_a^b f\,d\alpha - \int_a^b f_n\,d\alpha \right| < \epsilon/3$$

$$\Rightarrow \int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha$$

Corollary: If $f_n \in R(\alpha)$ on $[a,b]$ and if $f(x) = \sum_{n=1}^{\infty} f_n(x)$ $(a \le x \le b)$ the series converges uniformly on $[a,b]$ then $\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\,d\alpha$ (63)

Proof: Given that, $f_n \in R(\alpha)$ on $[a,b]$

$f(x) = \sum_{n=1}^{\infty} f_n(x)$ the series converges uniformly on $[a,b]$

where $S_n = \sum_{i=1}^{n} f_i(x)$

Since each $f_n \in R(\alpha)$, $S_n = f_1 + f_2 + \dots + f_n \in R(\alpha)$

$\Rightarrow$ The sequence $\{S_n\}$ converges uniformly on $[a,b]$ and $S_n \in R(\alpha)$

By known theorem,

$$\int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b S_n\,d\alpha$$

$$= \lim_{n\to\infty} \int_a^b \sum_{i=1}^{n} f_i(x)\,d\alpha$$

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \int_a^b f_i(x)\,d\alpha \right)$$

$$\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,d\alpha$$

Theorem:-

Uniform Convergence and differentiation :-

(8) Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$ then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f and

$f'(x) = \lim_{n\to\infty} f_n'(x)$ $(a \le x \le b)$

Proof: Let $\epsilon > 0$ be given

Since $\{f_n(x_0)\}$ converges, for $\epsilon > 0$ $\exists$ an integer $N \ni |f_n(x_0) - f_m(x_0)| < \epsilon/2$ $\forall m,n \ge N_1$

Since $\{f_n'\}$ converges uniformly on $[a,b]$

for $\epsilon > 0$ $\exists$ an integer N_2 such that $|f_n'(x) - f_m'(x)| \le \frac{\epsilon}{2(b-a)}$ $\forall m,n \ge N_2$ & $x \in [a,b]$

Let $N = \max\{N_1, N_2\}$

$|f_n(x_0) - f_m(x_0)| < \epsilon/2$ and $|f_n'(x) - f_m'(x)| \le \frac{\epsilon}{2(b-a)}$ holds $\forall m,n \ge N$

By apply mean value theorem to the function $f_n - f_m$

$$\left|(f_n-f_m)(x)-(f_n-f_m)(t)\right| = \left|(f_n-f_m)'(z)\right| |x-t| \quad \text{where } z\in(x,t)$$

$$= \left|(f_n'-f_m')(z)\right| |x-t|$$

$$= \left|f_n'(z)-f_m'(z)\right| |x-t|$$

$$\leq \frac{\epsilon}{2(b-a)} |x-t| \quad \text{--- (i)}$$

$$\leq \epsilon/2$$

$$\left|(f_n-f_m)(x)-(f_n-f_m)(t)\right| \leq \epsilon/2$$

Consider, $|f_n(x)-f_m(x)|$

$$= |f_n(x)-f_m(x)+f_n(x_0)-f_n(x_0)+f_m(x_0)-f_m(x_0)|$$

$$= |(f_n(x)-f_m(x))-(f_n(x_0)-f_m(x_0))+(f_n(x_0)-f_m(x_0))|$$

$$\leq |(f_n-f_m)(x)-(f_n-f_m)(x_0)+f_n(x_0)-f_m(x_0)|$$

$$\leq |(f_n-f_m)(x)-(f_n-f_m)(x_0)| + |f_n(x_0)-f_m(x_0)|$$

$$\leq \epsilon/2+\epsilon/2$$

$$= \epsilon \quad \forall\ n,m\geq N \text{ and } x\in[a,b]$$

$\Rightarrow |f_n(x)-f_m(x)| \leq \epsilon \quad \forall\ n,m\geq N$ and $x\in[a,b]$

$\Rightarrow \{f_n\}$ converges uniformly on $[a,b]$

let $f(x) = \lim_{n\to\infty} f_n(x)$ —— (ii)

for some $x\in[a,b]$ define $\phi_n(t) = \dfrac{f_n(t)-f_n(x)}{(t-x)}$ and

$$\phi(t) = \frac{f(t)-f(x)}{t-x} \quad \forall\ t\in[a,b] \text{ and } t\neq x$$

Consider, $$|\phi_n(t)-\phi_m(t)| = \left|\frac{f_n(t)-f_n(x)-f_m(t)+f_m(x)}{(t-x)}\right|$$

$$= \left|\frac{(f_n-f_m)(t)-(f_n-f_m)(x)}{(t-x)}\right|$$

$$\leq \frac{\epsilon}{2(b-a)}\,\frac{|x-t|}{|t-x|}$$

$$\leq \epsilon \quad \forall\ m,n\geq N \text{ and } t\in[a,b]$$

$\Rightarrow \{\phi_n\}$ converges uniformly on $[a,b]$

$$\lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty}\left(\frac{f_n(t)-f_n(x)}{t-x}\right)$$

$$= \frac{1}{t-x}\left(\lim_{n\to\infty} f_n(t) - \lim_{n\to\infty} f_n(x)\right)$$

$$= \frac{1}{t-x}\left(f(t)-f(x)\right)$$

$$= \frac{f(t)-f(x)}{t-x} = \phi(t)$$

$$\therefore \lim_{n\to\infty} \phi_n(t) = \phi(t) \quad \text{--- } ②$$

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Now $$\lim_{t\to x} \phi_n(t) = \lim_{t\to x}\left(\frac{f_n(t)-f_n(x)}{t-x}\right)$$

$$= f_n'(x)$$

and $$\lim_{t\to x} \phi(t) = \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right) = f'(x)$$

$$\lim_{t\to x} \phi(t) = \lim_{t\to x}\lim_{n\to\infty} \phi_n(t) \quad (\text{from } ②)$$

$$= \lim_{n\to\infty}\lim_{t\to x} \phi_n(t)$$

$$\lim_{t\to x} \phi(t) = \lim_{n\to\infty} f_n'(x)$$

$$\Rightarrow f'(x) = \lim_{n\to\infty} f_n'(x)$$

Theorem:- $\exists$ a real continuous function on the real line which is no where differentiable.

Proof: Define $\phi: \mathbb{R}\to\mathbb{R}$ by $\phi(x) = |x-2n|$ where $2n-1 \le x \le 2n+1$

$\phi(m) = 0$ if m is an even integer

$\phi(m) = 1$ if m is an odd integer

also $\phi(x) = |x|$ $\forall$ $-1\le x\le 1$ & $\phi(x+2n) = \phi(x)$ $\forall$ $x\in\mathbb{R}$, $n\in\mathbb{Z}$

$\Rightarrow 0 \le \phi(x) \le 1$

$\Rightarrow \phi$ is bounded

Consider, $|\phi(s)-\phi(t)| = \big||s|-|t|\big| \le |s-t|$ —— ⊛

$\Rightarrow \phi$ is continuous on $\mathbb{R}$

let, $$f(x) = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \phi(4^n \cdot x)$$

$$\left|\left(\frac{3}{4}\right)^n \phi(4^n.x)\right| = \left(\frac{3}{4}\right)^n |\phi(4^n.x)| \leq \left(\frac{3}{4}\right)^n$$

also $\left(\frac{3}{4}\right)^n \longrightarrow 0$ as $n \to \infty$

By weierstrass M-test,

$\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \phi(4^n.x)$ converges uniformly on $\mathbb{R}$

Since ϕ is continuous on $\mathbb{R}$, f is continuous on $\mathbb{R}$

Now we shall prove that f is not differentiable

fix a real number x and positive integer m

Put $\delta_m = \pm\frac{1}{2}4^{-m}$

where the sign is so choose that no integer lies between $4^m x$ and $4^m(x+\delta_m)$

This is possible, since $4^m|\delta_m| = \frac{1}{2}$

Now define $\gamma_n = \dfrac{\phi(4^n(x+\delta_m)) - \phi(4^n x)}{\delta_m}$

If $n>m$

$$\phi(4^n(x+\delta_m)) = \phi\left(4^n x + \frac{1}{2}4^{n-m}\right)$$
$$= \phi(4^n x)$$

$\gamma_n = 0$ if $n>m$

if $0 \leq n \leq m$, $|\gamma_n| = \left|\dfrac{\phi(4^n(x+\delta_m)) - \phi(4^n(x))}{\delta_m}\right|$

$$\leq \left|\frac{4^n(x+\delta_m) - 4^n x}{\delta_m}\right|$$

$$= \frac{4^n\delta_m}{\delta_m} = 4^n$$

$|\gamma_n| \leq 4^n$ if $0 \leq n \leq m$

also $|\gamma_m| = 4^m$

Consider $\left|\dfrac{f(x+\delta_m) - f(x)}{\delta_m}\right| = \left|\dfrac{\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \phi(4^n(x+\delta_m)) - \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \phi(4^n.x)}{\delta_m}\right|$

$$= \left|\frac{\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\left(\phi(4^n(x+\delta_m)) - \phi(4^n.x)\right)}{\delta_m}\right|$$

$$= \left| \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \cdot \gamma_n \right|$$

$$= \left| \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \cdot \gamma_n \right|$$

$$\geq 3^m - \sum_{n=0}^{m-1} 3^n \geq 3^m - \frac{(3^m - 1)}{3-1}$$

$$= 3^m - \frac{(3^m-1)}{2} = \frac{3^m - 1}{2} \longrightarrow \infty$$

as $m \to \infty$

Rough

$\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \gamma_n = \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \gamma_n + \left(\frac{3}{4}\right)^m \gamma_m + \sum_{n=m+1}^{\infty}\left(\frac{3}{4}\right)^n \gamma_n$

$p = \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \gamma_n$ and $q = \left(\frac{3}{4}\right)^m \gamma_m$

But $\delta_m \to 0$ as $m \to \infty$

$$\lim_{\delta_m \to 0} \left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| \text{ is } \infty$$

f is not differentiable at x

Equicontinuous families of functions

Definition:-

Let $\{f_n\}$ be a sequence of functions defined on set E. we say that $\{f_n\}$ is pointwise bounded on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$ i.e., if $\exists$ a finite value function ϕ defined on $E \ni |f_n(x)| < \phi(x)$ $(x \in E, n = 1, 2, \ldots)$

we say that $\{f_n\}$ is uniformly bounded on E if $\exists$ a number $M \ni |f_n(x)| < M$ $(x \in E, n = 1, 2, \ldots)$

definition: A family $\mathcal{F}$ of complex function f defined on a set E in a metric space X is said to be equicontinuous on E if for every $\epsilon > 0$ $\exists$ $\delta > 0 \ni |f(x) - f(y)| < \epsilon$ whenever $d(x,y) < \delta$ $x \in E, y \in E$ & $f \in \mathcal{F}$

Theorem:-

If $\{f_n\}$ is a point wise bounded sequence of complex functions on a countable set E then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

Proof: Since E is countable

we can write $E = \{x_1, x_2, \ldots\}$

let $\{f_n\}$ be a pointwise bounded sequence on E

$\Rightarrow \{f_n(x)\}$ is bounded

let $\{f_n\}$ has a subsequence $\{f_{1k}\}$ such that $\{f_{1k}(x_1)\}$ is convergent

Now $\{f_n\}$ is bounded

$\Rightarrow$ any subsequence of $\{f_n\}$ bounded

$\Rightarrow \{f_{1k}\}$ is pointwise bounded on E

$\Rightarrow \{f_{1k}(x_2)\}$ is bounded

$\Rightarrow \{f_{1k}\}$ has a subsequence $\{f_{2k}\}$ such that $\{f_{2k}(x_2)\}$ is convergent

let $S_1 = \{f_{11}, f_{12}, \ldots\}$, $S_2 = \{f_{21}, f_{22}, \ldots\}, \ldots, S_n = \{f_{n1}, f_{n2}, \ldots\}$ and so on.

we have a sequence $\{S_i\}$ such that

(a) S_n is a subsequence of S_{n-1} for $n = 2, 3, \ldots$ and

(b) $\{f_{nk}(x_n)\}$ converges.

Put $S = \{f_{11}, f_{22}, \ldots\}$

Clearly, S is a subsequence of the given sequence $\{f_n\}$ and $\{f_{nn}(x_i)\}$ is a subsequence of the convergent sequence $\{f_{in}(x_i)\}$

Therefore, $\{f_{nn}(x_i)\}$ is convergent for each i

Therefore, $\{f_n\}$ has a subsequence $\{f_{nn}\}$ such that $\{f_{nn}(x_i)\}$ is convergent for all i.

Theorem:- If k is a compact metric space, if $f_n \in \mathcal{C}(k)$ for $i = 1, 2, \ldots$ and if $\{f_n\}$ convergence uniformly on k then $\{f_n\}$ is equicontinuous on k.

Proof: let $\epsilon > 0$

since $\{f_n\}$ converges uniformly on k $\exists$ an integer $N \ni n, m \geq N$

$\Rightarrow |f_n(x) - f_m(x)| \leq \epsilon/3 \quad \forall x \in k$ —— (1)

$\sup_{x \in k} |f_n(x) - f_m(x)| \leq \epsilon/3 \Rightarrow \|f_n - f_m\| \leq \epsilon/3$

Now $f_1, f_2, \ldots, f_N$ are continuous on the compact set k and

$\therefore$ uniformly continuous on k, then $\exists$ a $\delta_1 > 0 \ni d(x,y) < \delta_i$

for $x,y \in k \Rightarrow |f_i(x)-f_i(y)| < \epsilon/3$, for $i=1,2,\dots,N$ — ①

Put $\delta = \min\{\delta_1, \delta_2, \dots, \delta_N\}$

let $x,y \in k \ni d(x,y) < \delta$

If $n \in \{1,2,\dots,N\}$ then $d(x,y) < \delta \le \delta_n$

$\Rightarrow |f_n(x)-f_n(y)| < \epsilon/3$

If $n > N$ then,

$$|f_n(x)-f_n(y)| \le |f_n(x)-f_N(x)+f_N(x)-f_N(y)+f_N(y)-f_n(y)|$$
$$\le |f_n(x)-f_N(x)| + |f_N(x)-f_N(y)| + |f_N(y)-f_n(y)|$$
$$\le \|f_n - f_N\| + \epsilon/3 + \|f_N - f_n\|$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3$$
$$= \epsilon$$

$\therefore \{f_n\}$ is equicontinuous.

Theorem: If k is compact if $f_n \in \mathcal{C}(k)$ for $n=1,2,\dots$ and if $\{f_n\}$ is pointwise bounded and equicontinuous on k then

ⓐ $\{f_n\}$ is uniformly bounded on k

ⓑ $\{f_n\}$ contains uniformly convergent subsequence.

Proof: ⓐ let $\epsilon > 0$

since $\{f_n\}$ is equicontinuous $\exists\ \delta > 0 \ni$ whenever $x,y \in k$ and $d(x,y) < \delta$

we have $|f_n(x)-f_n(y)| < \epsilon \ \forall n$

Now $k \subseteq \bigcup_{P \in k} N_\delta(P)$ this is an open cover for k

Since k is compact $\exists$ a finite subcover for this

i.e., $\exists$ a finite set of points $P_1, P_2, \dots, P_r \in k \ni k \subseteq \bigcup_{i=1}^{r} N_r(P_i)$

Now, since $\{f_n\}$ is pointwise bounded on k & $\{f_n(P)\}$ $n=1,2,\dots$ is bounded.

$\therefore$ there exists reals $M_1, M_2, \dots, M_r$ (say) $\ni |f_n(P_i)| \le M_i$ for $i=1,2,\dots,r$ & $n=1,2,\dots$

Put $M = \max\{M_1, M_2, \dots, M_r\}$

let $x \in K \subseteq \bigcup_{i=1}^{r} N_\delta(P_i)$ then $x \in N_\delta(P_i)$ for some i

ie, $d(x, P_i) < \delta \Rightarrow |f_n(x) - f_n(P_i)| < \epsilon \ \forall n$

Consider, $|f_n(x)| \le |f_n(x) - f_n(P_i)| + |f_n(P_i)|$

$$< \epsilon + M_i$$
$$< \epsilon + M$$

$\therefore |f_n(x)| \le \epsilon + M \ \forall x \ \& \ n$

$\Rightarrow \{f_n\}$ is uniformly bounded

(b) Since K is compact it contains a countable dense set E (say)

Therefore, $\{f_n\}$ is pointwise bounded on K

Since E is countable,

$\{f_n\}$ has a subsequence $\{g_i\}$

$\Rightarrow \{g_i(x_i)\}$, $i = 1, 2, \dots$, converges for each $x \in E$,

Now, we claim the subsequence $\{g_i\}$ converges uniformly on K

let $\epsilon > 0$

since $\{f_n\}$ is equicontinuous on K

there exist an $\delta > 0 \ni x, y \in K \ \& \ d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon/3$

Now $K \subseteq \bigcup_{x \in E} N_\delta(x)$

since K is compact,

$\exists$ a finite set $x_1, x_2, \dots x_m \in E$ such that $K \subseteq \bigcup_{i=1}^{m} N_\delta(x_i)$

Now, $\{g_i(x_1)\}, \{g_i(x_2)\}, \dots, \{g_i(x_m)\}$ are all convergent sequences.

$\therefore$ There is an integer N such that $i, j \ge N$

$\Rightarrow |g_i(x_k) - g_j(x_k)| < \epsilon/3$ for $k = 1, 2, \dots, m$

fix $k \Rightarrow x \in N_\delta(x_k)$ for some $k \in [1, m]$

$\Rightarrow d(x, x_k) < \delta$

$|g_i(x) - g_j(x)| = |g_i(x) - g_i(x_k) + g_i(x_k) - g_j(x_k) + g_j(x_k) - g_j(x)|$

$\le |g_i(x) - g_i(x_k)| + |g_i(x_k) - g_j(x_k)| + |g_j(x_k) - g_j(x)|$

$< \epsilon/3 + \epsilon/3 + \epsilon/3$

$= \epsilon$

Now, we prove that for any $\epsilon > 0$ $\exists$ an integer $N \ni i, j \geq N$

$\Rightarrow |g_i(x) - g_j(x)| < \epsilon \;\; \forall x \in k$

$\therefore \{g_i\}$ converges uniformly on k.

Weierstrass theorem:- (approximation):-

Statement:- If f is a continuous complex function on $[a,b]$ there exist a sequence of polynomials $\{P_n\}$ such that $\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $[a,b]$.

If f is real then P_n may be taken real.

Proof: Case (i):-

First let us take $[0,1]$ in place of $[a,b]$

$\ni f(0) = f(1) = 0$ — (1)

Also define $f(x) = 0$ for x outside $[0,1]$

Then f is uniformly continuous on the whole real line.

Define $Q_n(x) = C_n(1-x^2)^n$, $n = 1, 2, \ldots$

where C_n is choosen so that $\int_{-1}^{1} Q_n(x)\,dx = 1$ — (2)

$$C_n = \frac{1}{\int_{-1}^{1} (1-x^2)^n} dx$$

Consider the function

$h(x) = (1-x^2)^n - 1 + nx^2$

$h(0) = 0$

$h'(x) = n(1-x^2)^{n-1}(-2x) + 2nx$

$h'(0) = 0$

$h'(1) = 2n > 0$

Since $0 \leq x \leq 1$

$h(x) \geq h(0)$

$\Rightarrow (1-x^2)^n - 1 + nx^2 \geq 0$

$\Rightarrow (1-x^2)^n \geq 1 - nx^2$

Now we have $\int_{-1}^{1}(1-x^2)^n dx$

$$\geq 2\int_0^{1/\sqrt{n}}(1-x^2)^n dx$$

$$\geq 2\int_0^{1/\sqrt{n}}(1-nx^2)dx$$

$$= 2\left[x-\frac{nx^3}{3}\right]_0^{1/\sqrt{n}}$$

$$= 2\left[\frac{1}{\sqrt{n}}-\frac{1}{3\sqrt{n}}-0\right]$$

$$= 2\cdot\frac{2}{3\sqrt{n}} > \frac{1}{\sqrt{n}}$$

$$1 = c_n\int_{-1}^{1}(1-x^2)^n dx$$

$$> c_n\frac{1}{\sqrt{n}}$$

$$\Rightarrow 1 > c_n\frac{1}{\sqrt{n}} \Rightarrow \sqrt{n} > c_n \Rightarrow c_n < \sqrt{n}$$

for any $\delta > 0$

$Q_n(x) \leq \sqrt{n}(1-\delta^2)^n$ for $\delta \leq |x| \leq 1$

$\Rightarrow Q_n(x) \to 0$ uniformly in $\delta \leq |x| \leq 1$

Put $P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)dt$, $0 \leq x \leq 1$

$$= \int_{-1}^{-x} f(x+t)Q_n(t)dt + \int_{-x}^{1-x} f(x+t)Q_n(t)dt + \int_{1-x}^{1} f(x+t)Q_n(t)dt$$

$-1 \leq t \leq -x$

$\Rightarrow x-1 \leq x+t \leq 0$, $x+t \in [x-1, 0]$

$\Rightarrow f(x+t) = 0$

The first integral on the right side vanishes

Similarly third integral is also equal to zero

$$P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)dt$$

$$= \int_0^1 f(u)Q(u-x)du$$

since $\{P_n\}$ is a sequence of polynomials with complex coefficients and f is a real valued function, P_n's are polynomials with real coefficients.

Now, we shall prove that $P_n \to f$ uniformly.

let $\epsilon>0$ be given

since f is uniformly continuous $\exists\ \delta>0 \ni |f(s)-f(t)| < \epsilon/2$ whenever $|s-t|<\delta$ $\forall\ s,t \in [0,1]$

since f is continuous, f is bounded

Put $M = \sup|f(x)|$

consider, $|P_n(x)-f(x)| = \left|\int_{-1}^{1} f(x+t)\,Q_n(t) - f(x)\right|$

$$= \left|\int_{-1}^{1}(f(x+t)-f(x))Q_n(t)dt\right|$$

$$\leq \int_{-1}^{1}|f(x+t)-f(x)|\,Q_n(t)\,dt$$

$$= \int_{-1}^{-\delta}|f(x+t)-f(x)|\,Q_n(t)dt + \int_{-\delta}^{\delta}|f(x+t)-f(x)|\,Q_n(t)dt + \int_{\delta}^{1}|f(x+t)-f(x)|\,Q_n(t)dt$$

$$\leq 2M\int_{-1}^{-\delta}Q_n(t)dt + \frac{\epsilon}{2}\int_{-\delta}^{\delta}Q_n(t)dt + 2M\int_{\delta}^{1}Q_n(t)dt$$

$$\leq 2M\sqrt{n}(1-\delta^2)^n\left(\int_{-1}^{-\delta}dt + \int_{\delta}^{1}dt\right) + \frac{\epsilon}{2}\delta\int_{-1}^{1}Q_n(t)dt$$

$$\leq 4M\sqrt{n}(1-\delta^2)^n + \delta\frac{\epsilon}{2}$$

$< \epsilon$ for all large enough n

$P_n \to f$ uniformly

case (ii): let f an arbitrary continuous function on $[0,1]$

Define $g(x) = f(x) - f(0) - x[f(1)-f(0)]$

Then g is continuous and $g(0) = g(1) = 0$

$\exists$ a sequence $\{P_n\}$ of polynomials $\ni P_n \to g$ uniformly

$f(x)-g(x) = f(0) + x[f(1)-f(0)]$

then $f-g$ is a polynomial [which is real if f is real]

and $P_n + (f-g) \to f$ uniformly

Case (iii):

Let f be a continuous function on an arbitrary interval $[a,b]$

we can assume that $a<b$

Define $\alpha:[0,1]\rightarrow[a,b]$ & $\beta:[a,b]\rightarrow[0,1]$

by $\alpha(x) = a+(b-a)x$ and $\beta(x) = \frac{-a}{b-a} + \frac{x}{b-a}$

Then both α and β are polynomials and they are inverses to each other $f\circ\alpha$ is a continuous function on $[0,1]$ and there exists sequence $\{P_n\}$ of polynomials such that $P_n \rightarrow f\circ\alpha$ uniformly.

Now, each $P_n\circ\beta$ is a polynomials and $P_n\circ\beta \rightarrow f$ uniformly.

Corollary:

For every interval $[-a,a]$ there is a real polynomials P_n such that $P_n(0)=0$ and such that $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on $[-a,a]$.

Proof: It is clear that $|x|$ is a real continuous function on $[-a,a]$

By weierstrass Approximation theorem,

$\exists$ a sequence $\{P_n^*\}$ of real polynomials $\ni \lim_{n\to\infty} P_n^*(x) = |x|$ uniformly on $[-a,a]$

In particular, $P_n^*(0) \longrightarrow 0$ as $n\rightarrow\infty$

Put $P_n(x) = P_n^*(x) - P_n^*(0)$

Then the sequence $\{P_n\}$ is a sequence of real polynomials with real coefficients such that $P_n(0)=0$ and $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on $[-a,a]$

Definition:-

A family of complex functions defined on a set E is said to be an algebra if

(i) $f+g \in A$

(ii) $f.g \in A$

(iii) $c.f \in A$ $\forall f\in A, g\in A$ and for all complex constraints c.

i.e., A is closed under Addition, Multiplication and scalar multiplication.

Definition :-

An algebra A is said to be uniformly closed if it has the Property that $f \in A$ whenever $f_n \in A$ $n = 1, 2, \ldots$ and $f_n \to f$ uniformly on E.

Definition :-

Let B be the set of all functions which all limits of uniformly convergent sequence of members of A. B is called the uniform closure of A.

3 **Theorem :-** Let B be the uniform closure of an algebra A of bounded functions then B is uniformly closed Algebra.

(10) **Proof :-** Let $f \in B$ and $g \in B$ then $\exists$ uniformly convergent sequences $\{f_n\}$ and $\{g_n\} \ni f_n \to f$, $g_n \to g$ where $f_n, g_n \in A$

Now we prove that B is an Algebra.

i.e., $f+g$, $f \cdot g$, $c \cdot f \in B$ where c is a complex constant

i.e., $f_n + g_n \to f+g$ uniformly

$f_n g_n \to fg$ uniformly

$cf_n \to cf$ uniformly

Since $f_n \to f$ uniformly.

for $\epsilon > 0$ $\exists$ an integer $N_1 \ni |f_n - f| \leq \epsilon/2$ $\forall$ $n \geq N_1$

Since $g_n \to g$ uniformly,

for $\epsilon > 0$ $\exists$ an integer $N_2 \ni |g_n - g| \leq \epsilon/2$ $\forall$ $n \geq N_2$

Let $N = \max\{N_1, N_2\}$

Then $|f_n - f| \leq \epsilon/2$ and $|g_n - g| \leq \epsilon/2$ $\forall$ $n \geq N$

Consider,

$$
\begin{aligned}
|(f_n + g_n) - (f+g)| &= |f_n - f + g_n - g| \\
&\leq |f_n - f| + |g_n - g| \\
&\leq \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}
$$

$f_n + g_n \to f+g$ uniformly

$f+g \in B$

Consider,

$$|f_ng_n - fg| = |f_ng_n - f_ng + f_ng - fg_n + fg_n + fg - fg - fg|$$
$$= |f_ng_n - f_ng + f_ng - fg_n - fg + fg + fg_n - fg|$$
$$= |f_n(g_n - g) - f(g_n - g) + f(g_n - g) + g(f_n - f)|$$
$$= |(f_n - f)(g_n - g) + f(g_n - g) + g(f_n - f)|$$
$$\leq |f_n - f||g_n - g| + |f||g_n - g| + |g||f_n - f|$$
$$\leq \epsilon/2 \cdot \epsilon/2 + |f| \epsilon/2 + |g| \epsilon/2$$
$$\leq \epsilon$$

$f_ng_n \to fg$ uniformly $\Rightarrow fg \in B$

Consider, $|cf_n - cf| = c|f_n - f|$
$$= c \cdot \epsilon/c$$
$$\leq \epsilon$$

$cf_n \to cf$ uniformly $\Rightarrow c.f \in B$

$\therefore$ B is an algebra

Let $\{f_n\}$ be a sequence of members of B converging uniformly to a function f

Now, we prove that $f \in B$

Since $f_n \to f$ uniformly

for $\epsilon > 0$ $\exists$ an integer $N_1 \ni |f_n(x) - f(x)| \leq \epsilon/2 \ \forall n \geq N_1$

Since $f_n \in B$, f_n is a limit of uniformly convergent sequence $\{g_n\}$ of A

for $\epsilon > 0$ $\exists$ an integer $N_2 \ni |g_n(x) - f_n(x)| \leq \epsilon/2 \ \forall n \geq N_2$

Let $N = \max\{N_1, N_2\}$

Consider, $|g_n(x) - f(x)| = |g_n(x) - f_n(x) + f_n(x) - f(x)|$
$$\leq |g_n(x) - f_n(x)| + |f_n(x) - f(x)|$$
$$\leq \epsilon/2 + \epsilon/2$$
$$= \epsilon$$

$\Rightarrow g_n \to f$ uniformly $\Rightarrow f \in B$

$\therefore$ B is uniformly closed

$\Rightarrow f \in B$

B is uniformly closed.

Definition:

Let A be a family of functions defined on a set E then A is said to be separate points on E if to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in A$ such that $f(x_1) \neq f(x_2)$.

Definition:-

In to each $x \in E$ there corresponds a function $g \in A \ni g(x) \neq 0$ then we say that A vanishes at no point of E.

Theorem:-

17 Suppose A is an Algebra of functions on a set E. A separates points on E & A vanishes at no point of E. Suppose x_1, x_2 are distinct of E & c_1, c_2 are constants (real if A is a real algebra). Then A contains a function $f \ni f(x_1) = c_1$, $f(x_2) = c_2$.

Proof: Given that A is an algebra of function on a set E

i.e., $f+g \in A$

$f \cdot g \in A$

$cf \in A \ \forall f, g \in A$, c is a constant

A separate points on E i.e., to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $g \in A \ni g(x_1) \neq g(x_2)$

A vanishes at no point of E

i.e., for $x_1, x_2 \in E$ there corresponds function $h, k \in A$

$\ni h(x_1) \neq 0$, $k(x_2) \neq 0$

Put $u = gk - g(x_1)k$

$v = gh - g(x_2)h$

Since A is an algebra $u, v \in A$

$u(x_1) = g(x_1)k(x_1) - g(x_1)k(x_1)$

$= 0$

$u(x_2) = g(x_2)k(x_2) - g(x_1)k(x_2)$

$= k(x_2)(g(x_2) - g(x_1)) \neq 0$

$$V(x_1) = g(x_1)h(x_1) - g(x_2)h(x_1)$$
$$= h(x_1)(g(x_1) - g(x_2))$$
$$\neq 0$$
$$V(x_2) = g(x_2)h(x_2) - g(x_2)h(x_2)$$
$$= 0$$

Define $f = \dfrac{c_1 V}{V(x_1)} + \dfrac{c_2 U}{U(x_2)} \Rightarrow f \in A$

$$f(x_1) = \frac{c_1 V(x_1)}{V(x_1)} + \frac{c_2 U(x_1)}{U(x_2)} = c_1$$

$$f(x_2) = \frac{c_1 V(x_2)}{V(x_1)} + \frac{c_2 U(x_2)}{U(x_2)} = c_2$$

$$f(x_2) = c_2$$

Stone-Weierstrass theorem :-

(Generalization of weierstrass approximation theorem):

let A be an algebra of real continuous function on a compact set k. If A seperates points on k and if A vanishes at no point of k, then the uniform closure B of A consists of all real continuous function on k.

Proof: Given that A be an algebra of real continuous function on a compact set k.

Also A separates points on k, A vanishes at no point of k

A vanishes at no point of k

B separates point on k & B vanishes at no point of k

we divide the proof into four steps.

Step (i):- If $f \in B$ then $|f| \in B$

let $f \in B$

let $a = \sup\limits_{x \in k} |f(x)|$

Since f is continuous on a compact set k, f is bounded and a is a real number.

By the corollary of weierstrass approximation theorem, there exists a sequence of polynomials $\{P_n\}$ such that $\lim_{n\to\infty} P_n(y) = |y|$ uniformly $-a \le y \le a$ cos $y \in [-a, a]$

Put $P_n(y) = C_0 + C_1 y + C_2 y^2 + \dots + C_n y^n$

let $g_n = C_0 + C_1 f + C_2 f^2 + \dots + C_n f^n$

$\Rightarrow g_n \in B$

$\Rightarrow \lim_{n\to\infty} g_n(x) = |f(x)|$ uniformly

$\Rightarrow |f| \in B$

Step-(ii):

If $f \in B$ & $g \in B$ then $\max(f,g) \in B$ and $\min(f,g) \in B$

$\max(f,g)$ is a function defined by

$h(x) = f(x)$ if $f(x) \ge g(x)$

$= g(x)$ if $f(x) < g(x)$

write $\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2}$

$\min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2}$

Since B is an algebra

for $f,g \in B$, $f+g, f-g \in B$

By step(i), $|f-g| \in B$

By interian we can extend this to any finite set of functions

If $f_1, f_2, \dots, f_n \in B$ then

$\max(f_1, f_2, \dots, f_n) \in B$

and $\min(f_1, f_2, \dots, f_n) \in B$

step(iii):- Give a real function f continuous on k a point $x \in k$ & $\epsilon > 0$ $\exists$ a function $g_x \in B$ such that $g_x(x) = f(x)$ and $g_x(t) > f(t) - \epsilon \; (t \in k)$

Let f be a real continuous function on k, a point $x \in k$ and $\epsilon > 0$ for each $y \in k$ $\exists$ a function $h_y \in B \ni h_y(x) = f(x)$ & $h_y(y) = f(y)$,

since h_y is continuous

$\exists$ open sets J_y containing $y \ni h_y(t) > f(t) - \epsilon$, $t \in J_y$ the family $\{J_y / y \in k\}$ is an open cover of k

since k is compact, $\exists$ points $y_1, y_2, \ldots, y_n \in k \ni k \subset \bigcup_{i=1}^{n} J_{y_i}$

let $g_x(x) = \max\{h_{y_1}, h_{y_2}, \ldots, h_{y_n}\}$

By step-(iii), $g_n \in B$

$$g_x(x) = \max\{h_{y_1}(x), h_{y_2}(x) \ldots h_{y_n}(x)\}$$
$$= \max\{f(x), f(x) \ldots f(x)\}$$
$$= f(x)$$

$\therefore g_x(x) = f(x)$ and $g_x(t) > f(t) - \epsilon$, $(t \in k)$

Step (iv): Given a real function f continuous on k and $\epsilon > 0$

$\exists$ a function $h \in B \ni |h(x) - f(x)| < \epsilon$, $(x \in k)$

Let us consider the function g_x, for each $x \in k$ constructed in step (iii)

By the continuity g_x, $\exists$ an open sets V_x containing 'x' $\ni g_x(t) < f(t) + \epsilon$ $(t \in V_x)$

The family $\{V_x / x \in k\}$ is an open cover of k

since k is compact $\exists$ points $x_1, x_2, \ldots, x_n \in k \ni k \subset \bigcup_{i=1}^{n} V_{x_i}$.

let $h = \min\{g_{x_1}, g_{x_2}, \ldots g_{x_n}\} \leq g_{x_i}$

$$\Rightarrow h(t) \leq g_{x_i}(t) < f(t) + \epsilon$$

$$\Rightarrow h(t) < f(t) + \epsilon$$

By step (iii), $h \in B$ and it follows that $h(t) > f(t) - \epsilon$

$$f(t) - \epsilon < h(t) < f(t) + \epsilon$$
$$\Rightarrow |h(t) - f(t)| < \epsilon$$
$$\Rightarrow f \in B$$

$\therefore$ B consists of all real continuous functions on k

Definition:-

An algebra $\mathcal{A}$ of complex functions is said to be self-adjoint. If for every $f \in \mathcal{A}$ is complex conjugate $\bar{f}$ also belong to $\mathcal{A}$.

Theorem:-

Suppose $\mathcal{A}$ is a self adjoint algebra of complex functions on a compact set k, $\mathcal{A}$ separates points on k then the uniform closure $\mathcal{B}$ of $\mathcal{A}$ consists. In otherwords. A is dense in $C(k)$.

Proof: Given that, $\mathcal{A}$ is a self adjoint algebra of complex functions on a compact set k then for every $f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$

A separate points on k

ie., for every $x_1, x_2 \in k$, $x_1 \neq x_2$ $\exists$ $f \in \mathcal{A} \Rightarrow f(x_1) \neq f(x_2)$

$\mathcal{A}$ vanishes at no points of k

ie., for every $x \in k$ there corresponds a function $g \in \mathcal{A} \ni g(x) \neq 0$

Let $A_{\mathbb{R}}$ be the set of all real continuous function on k which belongs to $\mathcal{A}$

If $f \in \mathcal{A}$ and $f = u + iv$ with u and v are real then $2u = f + \bar{f}$

since $\mathcal{A}$ is a self adjoint algebra

$f, \bar{f} \in \mathcal{A}$

$\Rightarrow f + \bar{f} \in \mathcal{A}$

$\Rightarrow u \in \mathcal{A}$

$\Rightarrow u \in \mathcal{A}_{\mathbb{R}}$

If $x_1 \neq x_2$ then there exists a function $f \in \mathcal{A}$

$\ni f(x_1) = 1$ and $f(x_2) = 0$

$\Rightarrow u(x_1) = 1$ and $u(x_2) = 0$

ie, for $x_1 \neq x_2$ $\exists$ a function $u \in A_{\mathbb{R}} \ni u(x_1) \neq u(x_2)$

$\Rightarrow A_{\mathbb{R}}$ separates points on k

since $\mathcal{A}$ vanishes at no point of k and for every $x \in k$ there corresponds a function $g \in \mathcal{A} \ni g(x) \neq 0$ there is a complex number λ such that $\lambda g(x) > 0$

If $f = \lambda g$ and $f = u+iv$

i.e., for $x \in k$ there corresponds a function $u \in A_{\mathbb{R}}$

$\Rightarrow A_{\mathbb{R}}$ vanishes at no point of k

$\Rightarrow A_{\mathbb{R}}$ is an algebra of real continuous function on a compact set k.

Also $A_{\mathbb{R}}$ separates point on k

and $A_{\mathbb{R}}$ vanishes at no point of k

Then by known theorem (Stones-weierstrass theorem)

The uniform of $A_{\mathbb{R}}$ consists of all real continuous on k and therefore lies in B

If f is a complex continuous function on k

$f = u+iv$

$f \in B$

$\therefore$ The uniform closure B of A consists of all complex continuous functions on k.

Problem:

(1) (6) If $\{f_n\}$ and $\{g_n\}$ converges uniformly on a set E prove that $\{f_n+g_n\}$ converges uniformly on E.

If in addition, $\{f_n\}$ and $\{g_n\}$ are sequence of bounded function

Prove that $\{f_n g_n\}$ converges uniformly on E.

Sol: Given that $\{f_n\}$ and $\{g_n\}$ converges uniformly on a set E

we claim that $\{f_n+g_n\}$ converges uniformly on E

Since $\{f_n\}$ converges uniformly on E

for $\epsilon>0$ $\exists$ an integer $N_1 \ni |f_n(x)-f_m(x)| \leq \epsilon/2$ —① $\forall\, n,m \geq N_1$ and $x \in E$

Since $\{g_n\}$ converges uniformly on E

for $\epsilon>0$ $\exists$ an integer $N_2 \ni |g_n(x)-g_m(x)| \leq \epsilon/2$ —② $\forall\, n,m \geq N_2$ & $x \in E$

Let $N = \max\{N_1, N_2\}$

Then eqn ① & ② holds $\forall\ n, m \geq N$

Consider,

$$|(f_n+g_n)(x)-(f_m+g_m)(x)| = |f_n(x)+g_n(x)-f_m(x)-g_m(x)|$$
$$\leq |f_n(x)-f_m(x)| + |g_n(x)-g_m(x)|$$
$$\leq \epsilon/2 + \epsilon/2$$
$$= \epsilon$$

$|(f_n+g_n)(x)-(f_m+g_m)(x)| \leq \epsilon \quad \forall\ n, m \geq N\ \&\ x \in E$

$\Rightarrow \{f_n+g_n\}$ converges uniformly on E

Consider, $f_n g_n - fg = (f_n-f)(g_n-g) + g(f_n-f) + f(g_n-g)$

Since $f_n \to f$ uniformly and $\{f_n\}$ is bounded

$\exists\ N \ni |f_N(x)-f(x)| \leq 1 \quad \forall\ x \in E$

$\Rightarrow |f(x)| \leq |f_N(x)| + 1$

Since f_N is bounded, $\exists\ k>0 \ni |f(x)| \leq k\ \forall\ x \in E$

$\Rightarrow |f(x)| \leq k+1$

$\Rightarrow |f(x)| \leq k_1$ where $k_1 = k+1$

$\Rightarrow f$ is bounded

Now, $|f_n g_n - fg| = |(f_n-f)(g_n-g) + g(f_n-f) + f(g_n-g)|$

$$\leq |(f_n-f)(g_n-g)| + |g(f_n-f)| + |f(g_n-g)|$$
$$\leq |f_n-f||g_n-g| + |g||f_n-f| + |f||g_n-g|$$
$$\leq \epsilon/2 \cdot \epsilon/2 + k_2\,\epsilon/2 + k_1\,\epsilon/2$$
$$= \epsilon^2/4 + \epsilon^2/4\,(k_1+k_2)$$
$$= \epsilon'$$

$\therefore |f_n g_n - fg| \leq \epsilon'$

$\therefore f_n g_n \to fg$ uniformly on E

(2) The convergence of the series $\sum_{k=1}^{\infty} \frac{x}{1+n^4x^2}$ for $x \in [0,\infty)$

sol Let $f_n(x) = \frac{x}{1+n^4x^2}$

$$\lim_{n\to\infty} f_n(x) > \lim_{n\to\infty} \frac{x}{1+n^4x^2} = 0$$

$$\Rightarrow f(x) = 0$$

$$M_n = \sup_{x\in[0,\infty)} |f_n(x)-f(x)| = \sup_{x\in[0,\infty)} |f_n(x)-0|$$

$$= \sup_{x\in[0,\infty)} |f_n(x)|$$

$$f_n'(x) = 0 \Rightarrow \frac{1-n^4x^2}{(1+n^4x^2)^2} = 0$$

$$\Rightarrow 1-n^4x^2 = 0$$

$$\Rightarrow x^2 = \frac{1}{n^4} \quad \Rightarrow x = \frac{1}{n^2}$$

$$f_n(x) = \frac{1/n^2}{1+n^4\left(\frac{1}{n^4}\right)} = \frac{1}{2n^2}$$

$$M_n = \sup_{x\in[0,\infty)} |f_n(x)| = \frac{1}{2n^2} \to 0 \text{ as } n\to\infty$$

$\Rightarrow$ The given series converges uniformly on $[0,\infty)$

$\Rightarrow$ The given series converges on $[0,\infty)$

3) For $n=1,2,3,\dots, x$ real, put $f_n(x) = \frac{x}{1+nx^2}$ show that $\{f_n\}$ converges uniformly to a function f and that the equation $f'(x) = \lim_{n\to\infty} f_n'(x)$ is correct if $x\neq 0$ but false if $x=0$

sol Given that $f_n(x) = \frac{x}{1+nx^2}$, $n=1,2,\dots, x$ real

for all x, $\lim_{x\to\infty} f_n(x) = \lim_{x\to\infty} \frac{x}{1+nx^2} = 0$

$$\Rightarrow f(x) = 0$$

$$M_n = \sup_{x\in\mathbb{R}} |f_n(x)-f(x)| = \sup_{x\in\mathbb{R}} |f_n(x)-0| = \sup_{x\in\mathbb{R}} |f_n(x)|$$

for $x \neq 0$, $|f_n(x)| = \left|\frac{x}{1+nx^2}\right| = \frac{|x|}{1+nx^2}$

$$\leq \frac{|x|}{2\sqrt{n}\,|x|} = \frac{1}{2\sqrt{n}}$$

$|f_n(x)| \leq \frac{1}{2\sqrt{n}}$ if $x \neq 0$

If $x=0$, $f_n(x) = 0 \leq \frac{1}{2\sqrt{n}}$

for all x, $|f_n(x)| \leq \frac{1}{2\sqrt{n}}$

$M_n = \sup_{x\in\mathbb{R}} |f_n(x)| = \frac{1}{2\sqrt{n}} \longrightarrow 0$ as $n \to \infty$

$\Rightarrow M_n \to 0$ as $n \to \infty$

$\therefore f_n \to f$ uniformly on $\mathbb{R}$

$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$

If $x \neq 0$, $\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)^2}$

$$= \lim_{n\to\infty} \frac{n\left(\frac{1}{n}-x^2\right)}{n^2\left(\frac{1}{n}+x^2\right)^2} = \lim_{n\to\infty} \frac{\frac{1}{n}-x^2}{n\left(\frac{1}{n}+x^2\right)^2} = 0$$

$\therefore \lim_{n\to\infty} f_n'(x) = f'(x)$ if $x \neq 0$

If $x=0$, $\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} 1 = 1 \neq f'(x)$

$\Rightarrow \lim_{n\to\infty} f_n'(x) \neq f'(x)$ if $x=0$

(4) Every uniformly convergent sequence of bounded function is uniformly bounded.

Sol: let $\{f_n\}$ be a sequence of bounded functions

then for each i, $|f_i(x)| \leq M_i \; \forall x$

$\therefore \{f_n(x)\}$ is uniformly convergent

for $\epsilon>0$ $\exists$ an integer $N \ni |f_n(x) - f_m(x)| < \epsilon \;\; \forall n,m \geq N \; \& \; \forall x \in E$

In particular, $|f_n(x)-f_N(x)| \le \epsilon \ \forall n \ge N \ \& \ \forall x$

Now for $n \ge N$

$$|f_n(x)| = |f_n(x) - f_N(x) + f_N(x)|$$
$$\le |f_n(x)-f_N(x)| + |f_N(x)|$$
$$\le \epsilon + M_N$$

Let $M = \max\{M_1, M_2, \ldots, M_{N-1}, \epsilon + M_N\}$

then $|f_n(x)| \le M \ \forall n \ \forall x$

$\Rightarrow \{f_n\}$ is uniformly bounded.

Unit-iii

Some special functions:

Power series:

Definition: The functions of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{—— (1)}$$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{——} \rightarrow (2)$$ are called analytic functions

If equation (1) Converges for all x in $(-R, R)$ for some $R>0$ (R may be $+\infty$). we say that f is expanded in a power series about the Point $x=a$.

Definition:- Given a sequence $\{c_n\}$ of complex numbers the series $\sum_{n=0}^{\infty} c_n z^n$ is called the power series, the numbers c_n are called the coefficients of the series z is a complex numbers.

Put $\alpha = \lim\limits_{n\to\infty} \sup \sqrt[n]{c_n}$ and $R = 1/\alpha$ then R is called the Radius of convergence $\sum c_n z^n$

Problem:

Find the radius of convergence of the following series

(a) $\sum n^n z^n$ (b) $\sum 1 \cdot z^n$

(a) $\alpha = \sup\limits_{n\to\infty} \lim \sqrt[n]{n^n} = \lim\limits_{n\to\infty} \sup (n^n)^{1/n}$

$= \lim\limits_{n\to\infty} \sup n = \infty$

$R = \frac{1}{\alpha} = \frac{1}{\infty} = 0$

(b) $\alpha = \lim\limits_{n\to\infty} \sup \sqrt[n]{1^n} = \lim\limits_{n\to\infty} \sup (1^n)^{1/n}$

$= \lim\limits_{n\to\infty} \sup 1 = 1$

$R = \frac{1}{\alpha} = \frac{1}{1} = 1$

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Theorem: Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x|<R$ and define $f(x) = \sum_{n=0}^{\infty} c_n x^n \, (|x|<R)$ then $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly on $[-R+\epsilon, R-\epsilon]$ no matter which $\epsilon>0$ is choosen the function f is continuous and differentiable in $(-R,R)$ and $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \, (|x|<R)$ 89

Proof: let $\epsilon>0$

for $|x| \le R-\epsilon$

we have $|c_n x^n| \le |c_n (R-\epsilon)^n|$ and $\sum c_n (R-\epsilon)^n$ is converges

[$\because$ Every power series converges absolutely in the interior of its interval of convergence.]

By Weierstrass M-test,

$\sum c_n x^n$ —(1) converges uniformly on $[-R+\epsilon, R-\epsilon]$

consider, $\sum_{n=1}^{\infty} n c_n x^{n-1}$ —(2)

since $\sqrt[n]{n} \to 1$ as $n \to \infty$

$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \limsup_{n\to\infty} \sqrt[n]{n|c_n|}$

Radius of convergence of (1) & (2) are same

so (1) & (2) have the same interval of convergence

The series $\sum_{n=1}^{\infty} n c_n x^{n-1}$ converges for $|x|<R$

Also the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$ converges uniformly on $[-R+\epsilon, R-\epsilon]$

So, by the theorem, on uniform convergences and differentiation,

f is differentiable on $[-R+\epsilon, R-\epsilon]$ is $f'(x) = \sum_{n=0}^{\infty} f_n'(x)$

for $|x| \le R-\epsilon$ where $f_n(x) = c_n x^n$

But given any x, $\exists$ $|x|<R$

we can find an $\epsilon>0 \ni |x|<R-\epsilon$

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \text{ for } |x|<R$$

Since $f'(x)$ exists, f is differentiable & continuous on $(-R,R)$

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Corollary: Under the hypotheses of the above theorem f has derivatives of all order in $(-R,R)$ which are given by $f^{(k)}(x)=\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\,c_n x^{n-k}$.
In particular $f^{(k)}(0)=k!\,c_k$ $(k=0,1,2,\dots)$

Proof: By above theorem, $f(x)=\sum_{n=0}^{\infty} c_n x^n$ for $|x|<R$

f is continuous & differentiable in $(-R,R)$ and

$$f'(x)=\sum_{n=1}^{\infty} n c_n x^{n-1} \text{ for } x\in(-R,R)$$

Since $f'(x)=\sum_{n=1}^{\infty} n c_n x^{n-1}$ for $x\in(-R,R)$

Since $f'(x)$ is differentiable $f''(x)=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$

continuing in this way we get

$$f^{(k)}(x)=\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n x^{n-k}$$

$$=k(k-1)\cdots(k-k+1)c_k x^{k-k}+\sum_{n=k+1}^{\infty} n(n-1)\cdots(n-k+1)c_n x^{n-k}$$

$$f^{(k)}(0)=k!\,c_k+0$$

$$f^{(k)}(0)=k!\,c_k \quad (k=0,1,2,\dots)$$

Theorem: Suppose $\sum c_n$ converges. put $f(x)=\sum_{n=0}^{\infty} c_n x^n\ (-1\le x\le 1)$ then

$$\lim_{x\to 1} f(x)=\sum_{n=0}^{\infty} c_n$$

Proof: let $s_n=c_0+c_1+\cdots+c_n$ and $s_{-1}=0$

$$\sum_{n=0}^{m} c_n x^n=\sum_{n=0}^{m}(s_n-s_{n-1})x^n$$

$$=(s_0-s_{-1})+(s_1-s_0)x+(s_2-s_1)x^2+\cdots+(s_{m-1}-s_{m-2})x^{m-1}+(s_m-s_{m-1})x^m$$

$$=s_0(1-x)+s_1(x-x^2)+s_2(x^2-x^3)+\cdots+s_{m-1}(x^{m-1}-x^m)+s_m x^m$$

$$=(1-x)[s_0+s_1x+s_2x^2+\cdots+s_{m-1}x^{m-1}]+s_m x^m$$

$$=(1-x)\sum_{n=0}^{m-1} s_n x^n+s_m x^m$$

for $|x|<1$, taking limit as $n\to\infty$

$$f(x)=(1-x)\sum_{n=0}^{\infty} s_n x^n+0$$

$$f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n$$

Suppose $s = \lim_{n\to\infty} s_n$

(by convergence definition)

for $\epsilon > 0$ $\exists$ an integer $N \ni |s_n - s| < \epsilon/2$ $\forall$ $n > N$

Consider, $|f(x) - s| = \left|(1-x)\sum_{n=0}^{\infty} s_n x^n - s\right|$

$$= \left|(1-x)\sum_{n=0}^{\infty} s_n x^n - (1-x)\sum_{n=0}^{\infty} x^n s\right|$$

$$= \left|(1-x)\sum_{n=0}^{\infty} (s_n - s)x^n\right|$$

$$= \left|(1-x)\sum_{n=0}^{N} (s_n - s)x^n + (1-x)\sum_{n=N+1}^{\infty} (s_n - s)x^n\right|$$

$|a+b| \le |a| + |b|$

$$\le \left|(1-x)\sum_{n=0}^{N} (s_n - s)x^n\right| + \left|(1-x)\sum_{n=N+1}^{\infty} (s_n - s)x^n\right|$$

$$< \left|(1-x)\sum_{n=0}^{N} (s_n - s)x^n\right| + \epsilon/2$$

we can make this relation ϵ for small values of $(1-x)$

ie, we can choose $\delta > 0 \ni (1-x) < \delta \Rightarrow |f(x) - s| < \epsilon$

Thus, $\lim_{x\to 1} f(x) = s = \sum_{n=0}^{\infty} c_n$

$$\lim_{x\to 1} f(x) = \sum_{n=0}^{\infty} c_n$$

Theorem:-

Given a double sequence $\{a_{ij}\}$, $i = 1,2,3,\ldots$, $j = 1,2,3,\ldots$ suppose that $\sum_{j=1}^{\infty} |a_{ij}| = b_i$ $(i = 1,2,3,\ldots)$ and $\sum b_i$ converges then $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$

Proof: let E be a countable set containing the points $x_0, x_1, x_2, \ldots$ and suppose $x_n \to x_0$ as $n \to \infty$

Define $f_i(x_0) = \sum_{j=1}^{\infty} a_{ij}$ $(i = 1,2,3,\ldots)$

$$f_i(x_n) = \sum_{j=1}^{n} a_{ij} \quad \text{--- (1)}$$

$$g(x) = \sum_{i=1}^{\infty} f_i(x) \quad \text{--- (2)}$$

Consider, $\left| f_i(x_0) - f_i(x_n) \right| = \left| \sum_{j=1}^{\infty} a_{ij} - \sum_{j=1}^{n} a_{ij} \right| = \left| \sum_{j=n+1}^{\infty} a_{ij} \right|$

$$\leq \sum_{j=n+1}^{\infty} |a_{ij}| \leq \sum_{j=1}^{\infty} |a_{ij}| = b_i < \epsilon$$

b_i converges, $b_i \to 0$ as $i \to \infty$

[ie., for $\epsilon > 0$ $\exists$ $n \in \mathbb{Z}^+$ $\ni$ $b_i - 0 < \epsilon$ $\forall i \geq N \Rightarrow b_i < \epsilon$]

$\therefore \left| f_i(x_0) - f_i(x_n) \right| < \epsilon \quad \forall i \geq N$

Since $x_n \to x_0$ as $n \to \infty$

each f_i is continuous at x_0

[$f: X \to Y$, f is continuous at x_0
iff $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$]

Since $|f_i(x)| \leq b_i$ for each $x \in E$

By weierstrass M-test,

$\sum_{i=1}^{\infty} f_i(x)$ converges uniformly

Since $g(x) = \sum_{i=1}^{\infty} f_i(x)$, g is continuous at x_0

($\because$ each f_i is continuous at x_0)

Now, $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} f_i(x_0)$

$$= g(x_0)$$

$$= \lim_{n \to \infty} g(x_n)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{\infty} f_i(x_n)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_{ij}$$

$$= \lim_{n \to \infty} \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{ij}$$

$$= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

Hence $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$

Theorem:- Suppose $f(x)=\sum_{n=0}^{\infty} c_n x^n$, the series converging in $|x|<R$. If $-R<a<R$ then f can be expanded in a power series about the point $x=a$ which converges in $|x-a|<R-|a|$ and $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ $(|x-a|<R-|a|)$

Proof: $f(x)=\sum_{n=0}^{\infty} c_n x^n$

$$=\sum_{n=0}^{\infty} c_n[a+(x-a)]^n \quad \text{— (2)}$$

$$=\sum_{n=0}^{\infty} c_n[{}^nC_0 a^n + {}^nC_1 a^{n-1}(x-a)+\dots+{}^nC_n(x-a)^n]$$

$$=\sum_{n=0}^{\infty} c_n\left[\sum_{m=0}^{n} {}^nC_m a^{n-m}(x-a)^m\right] \quad \text{— (1)}$$

Consider, $\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left|c_n\, {}^nC_m\, a^{n-m}(x-a)^m\right|$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}|c_n|\,{}^nC_m\,|a|^{n-m}|x-a|^m$$

$$=\sum_{n=0}^{\infty}|c_n|\left(|a|+|x-a|\right)^n \quad (\text{from } (*))$$

The above series converges for $|a|+|x-a|<R$ $\quad |x|<R$

i.e., $|x-a|<R-|a|$

By the above theorem, applied for eqn (1), we get

$$f(x)=\sum_{m=0}^{n}\left[\sum_{n=m}^{\infty} c_n\,{}^nC_m\,a^{n-m}\right](x-a)^m$$

If $m=0$, $\sum_{n=0}^{\infty} c_n\,{}^nC_0\,a^n=\sum_{n=0}^{\infty}c_n a^n=f(a)$

If $m=1$, $\sum_{n=1}^{\infty} c_n\,{}^nC_1\,a^{n-1}=\sum_{n=1}^{\infty}n c_n a^{n-1}=f'(a)$

Continuing this process k times, we get

$$\sum_{n=k}^{\infty} c_n\,{}^nC_k\,a^{n-k}=\frac{f^{(k)}(a)}{k!}$$

$$\therefore f(x)=\sum_{m=0}^{\infty}\frac{f^{(m)}(a)(x-a)^m}{m!}$$

$$\therefore f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Problem:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$ prove that f has derivatives of all order at $x=0$ and that $f^{(n)}(0) = 0$ for $n = 1, 2, \dots$

Sol: Since f is exponential, f is differentiable $\forall x \in R$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-1/x^2} - 0}{x} = \lim_{x \to 0} \frac{\frac{1}{x}}{e^{1/x^2}} = \lim_{x \to 0} \frac{1}{2} x e^{-1/x^2} = 0$$

$$f'(0) = 0$$

$$f'(x) = e^{-1/x^2} \frac{2}{x^3}$$

$$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{(2/x^3) e^{-1/x^2} - 0}{x}$$

$$= 2 \lim_{x \to 0} \frac{1/x^4}{e^{1/x^2}} = 2 \lim_{x \to 0} 2x^{-2} e^{-1/x^2} = 4 \lim_{x \to 0} \frac{1}{x^2} e^{-1/x^2} = 0$$

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(e^{-1/x^2} \frac{2}{x^3}\right)$$

$$= \left[e^{-1/x^2} 2(-3) x^{-4} + (-) e^{-1/x^2} \left(\frac{-2}{x^3}\right) \frac{2}{x^3} \right]$$

$$= \left[\frac{4}{x^6} + \frac{(-6)}{x^4} \right] e^{-1/x^2}$$

Similarly, for any $n \in \mathbb{Z}^+$, $f^{(n)}(x)$ is a linear combination of $x^{-\alpha} e^{-1/x^2}$ where $\alpha \in \mathbb{Z}^+$

$$\lim_{x \to 0} x^{-\alpha} e^{-1/x^2} = \lim_{x \to 0} \frac{1/x^\alpha}{e^{1/x^2}}$$

$$= \lim_{x \to 0} \frac{-\alpha x^{-\alpha-1}}{e^{1/x^2}\left(\frac{-2}{x^3}\right)} = \lim_{x \to 0} \frac{\alpha}{2} \left(\frac{x^{-\alpha+2}}{e^{1/x^2}} \right)$$

$$= \begin{cases} 0 & \text{if } -\alpha + 2 > 0 \\ \frac{\infty}{\infty} & \text{if otherwise} \end{cases}$$

$$= \lim_{x \to 0} \frac{\alpha}{2} \frac{(-\alpha+2) x^{-\alpha+1}}{e^{-1/x^2}\left(\frac{-2}{x^3}\right)}$$

$$\lim_{x\to 0} \frac{\alpha}{2^2} \frac{(\alpha-2)x^{-\alpha+4}}{e^{1/x^2}(-2/x^3)}$$

$\vdots \qquad \vdots$

$$\lim_{x\to 0} \frac{\alpha(\alpha-2)(\alpha-4)\cdots 1}{2^\alpha e^{1/x^2}} = 0$$

Hence $f^{(n)}(0) = 0$ for $n = 1, 2, \ldots$

Problem: If f is continuous on $[0,1]$ & $\int_0^1 f(x)x^n dx = 0$ $(n = 1,2,3,\ldots)$

(5) Prove that $f(x) = 0$ on $[0,1]$

Sol: By the hypothesis, $\int_0^1 f(x)dx = 0$, $\int_0^1 f(x)x\,dx = 0, \ldots$

If $P_n(x)$ is any polynomial, say $a_0 + a_1x + a_2x^2 + \cdots$

then we get $\int_0^1 f(x)P_n(x)dx = 0$

By weierstrass theorem,

$\exists$ sequence of polynomial $\{P_n\} \ni \lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $[0,1]$

$\lim_{n\to\infty} P_n(x)f(x) = f^2(x)$ uniformly on $[0,1]$

Since $\int_0^1 f(x)P_n(x)dx = 0$, $\int_0^1 f^2(x)dx = 0$

$(\because \int \lim f_n\, dx = \lim \int f_n\, dx$ if $f_n \to f$ uniformly$)$

$\Rightarrow f^2(x) \geq 0$ on $[0,1]$ & $\int_0^1 f^2(x)dx = 0$

$\Rightarrow f^2(x) = 0$ on $[0,1]$

$\Rightarrow |f^2(x)| = 0$ on $[0,1]$

$\Rightarrow |f(x)| = 0$ on $[0,1]$

$\Rightarrow f(x) = 0$ on $[0,1]$

Functions on Several Variables:

Definition:- A non-empty set $X \subset \mathbb{R}^n$ is said to be a vector space if $\bar{x} + \bar{y} \in X$ and $c\bar{x} \in X$ $\forall\, \bar{x}, \bar{y} \in X$ and $\forall$ scalars c.

where $\bar{x} = (x_1, x_2, \ldots, x_n)$, $\bar{y} = (y_1, y_2, \ldots, y_n)$

$x_i \in \mathbb{R}$, $y_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$

Defenition:-

If $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_k \in \mathbb{R}^n$ and $c_1, c_2, \ldots, c_k$ are scalars the vector $c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots + c_k\bar{x}_k$ is called a. l.c of $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_k$

Defenition:-

If $S \subset \mathbb{R}^n$ & E is the set of all linear combination of element of S, we say that S span E (or) E is the span of S.

Defenition:- The set consisting of vector $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_k$ is said to be independent if the relation $c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots + c_k\bar{x}_k \Rightarrow c_1 = c_2 = \ldots = c_k = 0$

otherwise $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\}$ is said to be dependent.

Defenition:-

If a vector space X contains an independent set of r vectors but contains no independent set of $(r+1)$ vectors.

we say that X has dimension r.

i.e., if the maximum number of independent vector in X is r then we say that dimension of $X = r$

Defenition:

An independent subset of a vector space X which spans X is called a basis of X

Note:

Suppose $E = \{\bar{x}_1, \bar{x}_2, \ldots \bar{x}_k\}$ is independent then every vector in the span of E is uniquelly expressed in the form $c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots + c_k\bar{x}_k$ $\forall$ scalar c_i

Sol: Suppose $\bar{x}$ is a vector in the span of E

suppose $\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots + c_k\bar{x}_k$ and $\bar{x} = d_1\bar{x}_1 + d_2\bar{x}_2 + \ldots + d_k\bar{x}_k$

$\Rightarrow (c_1 - d_1)\bar{x}_1 + (c_2 - d_2)\bar{x}_2 + \ldots + (c_k - d_k)\bar{x}_k = 0$

$\Rightarrow c_1 = d_1, c_2 = d_2, \ldots, c_k = d_k$

$\Rightarrow c_i = d_i \ \forall \ i = 1, 2, \ldots, k$

$\bar{x}$ can be uniquelly expressed as $c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots + c_k\bar{x}_k$

Theorem:

MID 5m Let r be a positive integer. If a vector space X spanned by a set r vectors then $\dim X \le r$. (II - 3M) *

Proof: Assume contrary, there is a vector space X which contains an independent set $Q = \{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{r+1}\}$ and which is spanned by set S_0 on of vectors $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_r$

Let $S_1 = \{\bar{y}_1, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_r\}$

Since, S_0 spans X and $\bar{y}_1 \in X$

$$\bar{y}_1 = c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots + c_r\bar{x}_r \quad \text{——} ①$$

Atleast, one of c_i's is non-zero otherwise $\bar{y}_1 = 0$

(since Q is independent, $\bar{y}_1 \neq 0$)

equation ① can be written as

$$c_i\bar{x}_i = \bar{y}_1 - (c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots + c_{i-1}\bar{x}_{i-1} + c_{i+1}\bar{x}_{i+1} + \ldots + c_r\bar{x}_r)$$

Since $c_i \neq 0$, $\bar{x}_i = \bar{c}_i^{\,\prime}\bar{y}_1 - \bar{c}_i^{\,\prime}(c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots + c_{i-1}\bar{x}_{i-1} + c_{i+1}\bar{x}_{i+1} + \ldots + c_r\bar{x}_r)$

from this we observe that the set S_2 containing of $\{\bar{y}_1, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{i-1}, \bar{x}_{i+1}, \ldots, \bar{x}_r\}$ also spans X

Repeating this process 'r' times (by putting one $\bar{y}_i$ & Removing one $\bar{x}_i$ successively)

we obtain the set S_{r+1} consisting of $\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_r\}$ also spans X

Since $\bar{y}_{r+1} \in X$ this can be written as

$$\bar{y}_{r+1} = d_1\bar{y}_1 + d_2\bar{y}_2 + \ldots + d_r\bar{y}_r$$

$$(1)\bar{y}_{r+1} - d_1\bar{y}_1 - d_2\bar{y}_2 - \ldots - d_r\bar{y}_r = \bar{0}$$

Since $1 \neq 0$

$Q = \{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_r, \bar{y}_{r+1}\}$ is dependent

which is a contradiction.

$\therefore$ If a vector space X is spanned by a set of r vectors, then $\dim X \le r$

Note:
standard basis for $\mathbb{R}^n = \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\}$ where $\bar{e}_1 = (1,0,0,\ldots,0)$, $\bar{e}_2 = (0,1,0,\ldots,0), \ldots, \bar{e}_n = (0,0,\ldots,1)$

Corollary:
$\dim \mathbb{R}^n = n$ (III - 3.11)

Proof: We know that $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\}$ be the standard basis of $\mathbb{R}^n$ where $\bar{e}_1 = (1,0,0,\ldots,0)$, $\bar{e}_2 = (0,1,0,\ldots,0) \ldots \bar{e}_n = (0,0,\ldots,1)$

Since $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\}$ spans $\mathbb{R}^n$

$\dim \mathbb{R}^n \leq n$ — ① (by above theorem)

Since $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \ldots, \bar{e}_n\}$ is linearly independent and $\dim \mathbb{R}^n$ is the maximum number of linearly independent vectors

$n \leq \dim \mathbb{R}^n$ — ②

from ① & ②, $\dim \mathbb{R}^n = n$

Theorem:
Suppose X is a vector space and $\dim X = n$

(a) A set E of 'n' vectors in X spans X iff E is independent.

(b) X has a basis and every basis consisting at 'n' vectors

(c) If $1 \leq r \leq n$ and $\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_r\}$ is an independent set in X, then X has a basis containing $\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_r\}$

Proof: Given that X is a vector space and $\dim X = n$

(a) let $E = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ be independent

let $\bar{y} \in X$

Consider, the set $\{\bar{y}, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$

Since $\dim X = n$, the maximum number of independent vectors in X is n.

then $\{\bar{y}, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ is linearly dependent

So, $\exists$ scalars $c_0, c_1, \ldots, c_n \ni c_0\bar{y} + c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots + c_n\bar{x}_n = 0$

Since, E is independent $c_0 \neq 0$

$$\bar{y} = -c_0^{-1}(c_1\bar{x}_1 + c_2\bar{x}_2 + \dots + c_n\bar{x}_n)$$

$\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$ spans X

$\Rightarrow$ E spans X

Conversely, suppose $E = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$ spans X.

Now we shall show that E is independent

If possible, assume that E is dependent then some $\bar{x}_i \in E$ can be written as

$$\bar{x}_i = c_1\bar{x}_1 + c_2\bar{x}_2 + \dots + c_{i-1}\bar{x}_{i-1} + c_{i+1}\bar{x}_{i+1} + \dots + c_n\bar{x}_n$$

$\therefore$ The set $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{i-1}, \bar{x}_{i+1}, \dots, \bar{x}_n\}$ spans X

By known theorem,

$$\dim X = n \leq n-1$$

which is a contradiction

$\therefore$ E is independent.

(b) Since $\dim X = n$ there is an independent set E consisting of n vectors

$\Rightarrow$ E spans X (∵ by (a))

$\Rightarrow$ E is a basis of X

Suppose B is any other basis of X consisting of "r" vectors

$\Rightarrow$ B spans X and $\dim X = n$

(by known theorem)

$\Rightarrow \dim X = n \leq r$

$\Rightarrow n \leq r$ — ①

Since $\dim X = n$ the maximum numbers of linearly independent vectors in X is n.

Since B is independent consisting of r vectors

$\Rightarrow r \leq n$ — ②

from ① & ②,

$$n = r$$

(c) Suppose $1 \le r \le n$ & $\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_r\}$ is an independent set.

Let $\{\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n\}$ be a basis of X

Consider the set $S_0 = \{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_r, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$

Since it contains more that n elements and $\dim X = n$

$\Rightarrow S_0$ is linearly dependent

$\Rightarrow$ one vector in S_0 can be expressed as linear combination of remaining vectors.

Removing the vector $\bar{x}_i$ from S_0

we get a set $S_1 = \{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_r, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{r-1}, \bar{x}_{i+1}, \ldots, \bar{x}_n\}$

spans X and linearly dependent

One of the x_i's say $\bar{x}_j$ is a linear combination of remaining vectors

Removing the vectors $\bar{x}_j$ from S_1,

we get another set S_2 which is also spans X and linearly independent.

Repeating this process 'r' times we get a set containing $\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_r\}$

which also spans X and linearly independent.

$\therefore X$ has a basis containing $\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_r\}$

*Definition: A mapping 'A' from a vector space X into a vector space Y is said to be a linear transformation if $A : X \to Y$

$$A(\bar{x} + \bar{y}) = A\bar{x} + A\bar{y}$$

$$A(c\bar{x}) = cA\bar{x}$$

$\forall\ \bar{x}, \bar{y} \in X$ and all scalars 'c'

Note:

(i) $A\bar{0} = \bar{0}$, if A is linear (T) linear transformation

$$A\bar{0} = A(\bar{0} + \bar{0})$$

$$\bar{0} + A\bar{0} = A\bar{0} + A\bar{0}$$ (Right cancellation law)

$$\bar{0} = A\bar{0}$$

$$A\bar{0} = \bar{0}$$

(iii) A linear transformation A of X into Y is completely determined by its action on any basis. (101)

Sol Suppose $\{\bar{x}_1, \bar{x}_2, --, \bar{x}_n\}$ is a basis of X

If $\bar{x} \in X$, then $\bar{x}$ can be expressed as a linear combination of $\{\bar{x}_1, \bar{x}_2, --\bar{x}_n\}$

$$\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n x_n$$

$$A\bar{x} = A(c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n)$$

$$= c_1 A\bar{x}_1 + c_2 A\bar{x}_2 + \cdots + c_n A\bar{x}_n \quad (\because A \text{ is linear transformation})$$

Thus $A\bar{x}$ is know; if $A\bar{x}_1, A\bar{x}_2, ---, A\bar{x}_n$ are known

Eg: Suppose A is a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$ given by $A(1,0) = (1,0,1)$, $A(0,1) = (1,1,0)$.

Sol Standard basis of $\mathbb{R}^2$ is $\{(1,0),(0,1)\}$

Every element $(x,y) \in \mathbb{R}^2$ can be expressed as $(x,y) = x(1,0) + y(0,1)$

$$A(x,y) = A(x(1,0) + y(0,1)) \quad [\because \text{linear Transformat.}]$$

$$= x A(1,0) + y A(0,1)$$

$$= x(1,0,1) + y(1,1,0)$$

$$= (x+y, y, x)$$

Definition:-

A linear transformation of X into itself is called a linear operator on X. $X \to X$

Definition:-

If A is a linear operator on X which is one-one & onto, we say that A is invertible. In this case, we can define an operator A^{-1} on X by required that $A^{-1}(A\bar{x}) = \bar{x}$, $\bar{x} \in X$

Theorem:- A linear operator A on a finite dimensional vector space X is one-one iff the range of A is all of X (I-5M)

Proof: Let A be a linear operator on a finite dimensional vector space X

we denote, the range of A by R(A)

we can prove that, A is one-one iff $R(A) = X$

let $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ be a basis of X

let $\bar{x} \in X$

then $\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n$

$$A\bar{x} = A(c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n)$$

$$= c_1 A\bar{x}_1 + c_2 A\bar{x}_2 + \cdots + c_n A\bar{x}_n$$

This shows that any element of range of A is a linear combination of $\{A\bar{x}_1 + A\bar{x}_2 + \cdots + A\bar{x}_n\}$

$\Rightarrow Q = \{A\bar{x}_1, A\bar{x}_2, \ldots, A\bar{x}_n\}$ spans $R(A)$

By a known theorem,

$R(A) = X$ iff Q is independent

suppose A is one-one, $c_1 A\bar{x}_1 + c_2 A\bar{x}_2 + \cdots + c_n A\bar{x}_n = \bar{0}$

$$A(c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n) = \bar{0} = A\bar{0}$$

$$c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n = 0 \quad (\because A \text{ is } 1\text{-}1)$$

$$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

$(\because \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ is independent$)$

$\Rightarrow \{A\bar{x}_1, A\bar{x}_2, \ldots, A\bar{x}_n\} = Q$ is independent $\quad \therefore R(A) = X$

Conversely, suppose Q is independent

let $\bar{x} \in X$ then $\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n$

Now, $A\bar{x} = \bar{0} \Rightarrow A(c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n) = \bar{0}$

$$\Rightarrow c_1 A\bar{x}_1 + c_2 A\bar{x}_2 + \cdots + c_n A\bar{x}_n = \bar{0}$$

$$\Rightarrow c_1 = c_2 = \cdots = c_n = \bar{0}$$

$$\Rightarrow \bar{x} = \bar{0}$$

$$A\bar{x} = \bar{0} \Rightarrow \bar{x} = \bar{0}$$

for any $\bar{x}, \bar{y} \in X$, suppose, $A\bar{x} = A\bar{y}$

$$\Rightarrow A\bar{x} - A\bar{y} = \bar{0}$$

$$\Rightarrow A(\bar{x} - \bar{y}) = \bar{0}$$

$$\Rightarrow \bar{x} = \bar{y}$$

$\therefore A$ is one-one

Definition:-

Let $L(X,Y)$ be the set of all linear transformation of a vector space X into a vector space Y.

If $A_1, A_2 \in L(X,Y)$ and if c_1, c_2 are scalars define $c_1A_1+c_2A_2$ by

$$(c_1A_1+c_2A_2)\bar{x} = c_1A_1\bar{x} + c_2A_2\bar{x} \quad , \bar{x}\in X$$

$c_1A_1+c_2A_2$ is a linear transformation :-

(i) $(c_1A_1+c_2A_2)(\bar{x}+\bar{y}) = c_1A_1(\bar{x}+\bar{y}) + c_2A_2(\bar{x}+\bar{y})$

$= c_1(A_1\bar{x}+A_1\bar{y}) + c_2(A_2\bar{x}+A_2\bar{y})$

$= (c_1A_1+c_2A_2)\bar{x} + (c_1A_1+c_2A_2)\bar{y}$

(ii) Now $(c_1A_1+c_2A_2)(d\bar{x}) = c_1A_1(d\bar{x}) + c_2A_2(d\bar{x})$

$= c_1 d A_1(\bar{x}) + c_2 d A_2(\bar{x})$

$= d[c_1A_1(\bar{x}) + c_2A_2(\bar{x})]$

$= d(c_1A_1+c_2A_2)\bar{x}$

$\therefore c_1A_1+c_2A_2 \in L(X,Y)$

* If $X=Y$ then we denote $L(X,X)$ by $L(X)$

Definition:-

If X,Y,Z are vector spaces, and if $A\in L(X,Y)$, $B\in L(Y,Z)$. we define their Product BA to be the composition of A and B

$$(BA)\bar{x} = B(A\bar{x}), \quad \bar{x}\in X$$

BA is a L.T :-

$[(BA)(\bar{x}+\bar{y}) = B(A(\bar{x}+\bar{y}))$

$= B(A\bar{x}+A\bar{y})$

$= B(A\bar{x})+B(A\bar{y})$

$= (BA)(\bar{x}) + (BA)(\bar{y})]$

Then $BA \in L(X,Z)$

Definition:-

If $A\in L(R^n,R^n)$ then $\|A\|$ is defined by

$$\|A\| = \sup\{|A\bar{x}| \,/\, \bar{x}\in R^n, |\bar{x}|\le 1\}$$

Result:

For $\bar{x} \in \mathbb{R}^n$ and $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ then $|A\bar{x}| \leq \|A\| \, |\bar{x}|$

Proof: Let $\bar{x} \in \mathbb{R}^n$

If $\bar{x} = \bar{0}$ then the more inequality is clear

If $\bar{x} \neq \bar{0}$, write $\bar{y} = \frac{\bar{x}}{|x|}$

$$|\bar{y}| = \frac{|\bar{x}|}{|\bar{x}|} = 1$$

$$|A\bar{y}| = \left|A \cdot \frac{\bar{x}}{|\bar{x}|}\right| = \frac{1}{|\bar{x}|} |A\bar{x}|$$

Now $|A\bar{x}| \leq \sup\{|A\bar{x}| / \bar{x} \in \mathbb{R}^n, |\bar{x}| \leq 1\} = \|A\|$

In particular, $|A\bar{y}| \leq \|A\|$

$$\Rightarrow \frac{1}{|\bar{x}|} |A\bar{x}| \leq \|A\|$$

$$\Rightarrow |A\bar{x}| \leq \|A\| \, |\bar{x}|$$

Theorem:

(a) If $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ then $\|A\| < \infty$ and A is uniformly continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$

(b) If $A, B \in L(\mathbb{R}^n, \mathbb{R}^n)$ & c is a scalar, then $\|A+B\| \leq \|A\| + \|B\|$, $\|cA\| = |c| \, \|A\|$ with the distance between A and B is defined as $\|A - B\|$, $L(\mathbb{R}^n, \mathbb{R}^n)$ is a metric space.

(c) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ & $B \in L(\mathbb{R}^m, \mathbb{R}^k)$ then $\|BA\| \leq \|B\| \, \|A\|$

Proof: (a) Let $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\}$ be a standard basis in $\mathbb{R}^n$

Let $x \in \mathbb{R}^n$ such that $|\bar{x}| \leq 1$

$\Rightarrow \bar{x}$ can be expressed as $\bar{x} = c_1\bar{e}_1 + c_2\bar{e}_2 + \cdots + c_n\bar{e}_n$

$$|\bar{x}| \leq 1 \Rightarrow |c_1\bar{e}_1 + c_2\bar{e}_2 + \cdots + c_n\bar{e}_n| \leq 1$$

$$\Rightarrow |c_1\bar{e}_1| + |c_2\bar{e}_2| + \cdots + |c_n\bar{e}_n| \leq 1$$

$$\Rightarrow |c_1||\bar{e}_1| + |c_2||\bar{e}_2| + \cdots + |c_n||\bar{e}_n| \leq 1$$

$$\Rightarrow |c_1| + |c_2| + \cdots + |c_n| \leq 1$$

$\Rightarrow |c_i| \leq 1$ for $1 \leq i \leq n$

$$|A\bar{x}| = \left|A \sum_{i=1}^{n} c_i \bar{e}_i\right|$$

$$\leq \sum_{i=1}^{n} c_i |A\bar{e}_i|$$

$$\leq \sum_{i=1}^{n} |A\bar{e}_i| \quad [\because |c_i| \leq 1]$$

$$< \infty$$

let $\epsilon > 0$ and choose $\delta = \frac{\epsilon}{\|A\|} > 0$

for $\bar{x}, \bar{y} \in \mathbb{R}^n$, $|\bar{x} - \bar{y}| < \delta$

$$|A\bar{x} - A\bar{y}| = |A(\bar{x} - \bar{y})|$$

$$\leq \|A\| |\bar{x} - \bar{y}| \quad (\because |A\bar{x}| = \|A\| |\bar{x}|)$$

$$< \|A\| \delta$$

$$= \|A\| \frac{\epsilon}{\|A\|}$$

$$= \epsilon$$

$|A\bar{x} - A\bar{y}| < \epsilon$ whenever $|\bar{x} - \bar{y}| < \delta$, $\forall \bar{x}, \bar{y} \in \mathbb{R}^n$

$\therefore$ A is uniformly continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$

(b) let $A, B \in L(\mathbb{R}^n, \mathbb{R}^n)$

Consider, $\|A+B\| = \sup\{|(A+B)\bar{x}| \,/\, \bar{x} \in \mathbb{R}^n, |\bar{x}| \leq 1\}$

$= \sup\{|A\bar{x} + B\bar{x}| \,/\, \bar{x} \in \mathbb{R}^n, |\bar{x}| \leq 1\}$

$\leq \sup\{|A\bar{x}| + |B\bar{x}| \,/\, \bar{x} \in \mathbb{R}^n, |\bar{x}| \leq 1\}$

$\leq \sup\{\|A\| |\bar{x}| + \|B\| |\bar{x}| \,/\, \bar{x} \in \mathbb{R}^n, |\bar{x}| \leq 1\}$

$\leq \sup\{\|A\| + \|B\| \,/\, \bar{x} \in \mathbb{R}^n, |\bar{x}| \leq 1\}$

$= \|A\| + \|B\|$

$\therefore \|A+B\| \leq \|A\| + \|B\|$

consider, $\|cA\| = \sup\{|c(A)\bar{x}| \,/\, \bar{x} \in \mathbb{R}^n, |\bar{x}| \leq 1\}$

$= \sup\{|c| |A\bar{x}| \,/\, \bar{x} \in \mathbb{R}^n, |\bar{x}| \leq 1\}$

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$$\|cA\| = |c| \sup\{ |A\bar{x}| / \bar{x} \in \mathbb{R}^n, |\bar{x}| \le 1 \}$$
$$= |c| \|A\|$$
$$\|cA\| = |c| \|A\|$$

The distance between A and B defined as

$$d(A,B) = \|A-B\|$$

(i) $d(A,B) = \|A-B\|$
$$= \sup\{ |A-B|\bar{x} / \bar{x} \in \mathbb{R}^n, |\bar{x}| \le 1 \} \ge 0$$
$$d(A,B) \ge 0$$

(ii) $d(A,B) = 0 \iff \|A-B\| = 0$
$$\iff A-B = 0$$
$$\iff A = B$$

(iii) $d(A,B) = \|A-B\|$
$$= \|(-1)(B-A)\|$$
$$= \|B-A\|$$
$$= d(B,A)$$

(iv) $d(A,B) = \|A-B\|$
$$\le \|(A-C)+(C-B)\|$$
$$= d(A,C) + d(C,B)$$
$$d(A,B) \le d(A,C) + d(C,B)$$

$\therefore$ d is a metric on $L(\mathbb{R}^n, \mathbb{R}^m)$

(c) let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$

Consider, $\|BA\| = \sup\{ |(BA)\bar{x}| / \bar{x} \in \mathbb{R}^n, |\bar{x}| \le 1 \}$
$$= \sup\{ |B(A\bar{x})| / \bar{x} \in \mathbb{R}^n, |\bar{x}| \le 1 \}$$
$$\le \sup\{ \|B\| |A\bar{x}| / \bar{x} \in \mathbb{R}^n, |\bar{x}| \le 1 \}$$
$$\le \sup\{ \|B\| \cdot \|A\| |\bar{x}| / \bar{x} \in \mathbb{R}^n, |\bar{x}| \le 1 \}$$
$$\le \sup\{ \|B\| \cdot \|A\| / \bar{x} \in \mathbb{R}^n, |\bar{x}| \le 1 \}$$
$$\le \|B\| \cdot \|A\|$$

$\therefore \|BA\| \le \|B\| \|A\|$

Theorem:

Let Ω be the set of all invertible linear operator on $\mathbb{R}^n$.

(a) If $A \in \Omega$ $B \in L(\mathbb{R}^n)$ & $\|B-A\|\|A^{-1}\| < 1$, then $B \in \Omega$ (I - 3M)

(b) Ω is an open set of $L(\mathbb{R}^n)$ and the mapping $A \to A^{-1}$ is continuous on Ω

Proof: let Ω be the set of all invertiable linear operator on $\mathbb{R}^n$

(a) If $A \in \Omega$, $B \in L(\mathbb{R}^n)$ and $\|B-A\|\|A^{-1}\| < 1$

then we have to prove that B is invertiable linear operator

To prove this it is enough to prove that B is one-one linear operator

Put $\|A^{-1}\| = \frac{1}{\alpha}$ and $\|B-A\| = \beta$

from the given condition,

$$\frac{\beta}{\alpha} < 1 \quad \Rightarrow \beta < \alpha$$

for every $\bar{x} \in \mathbb{R}^n$, $\alpha|\bar{x}| = \alpha|A^{-1}A\bar{x}|$ $\quad [\because |A\bar{x}| = \|A\||\bar{x}|]$

$$\alpha|\bar{x}| = \alpha\|A^{-1}\||A\bar{x}|$$

$$\alpha|\bar{x}| \le \alpha\frac{1}{\alpha}|A\bar{x}|$$

$$\alpha|\bar{x}| \le |A\bar{x}|$$

$$\alpha|\bar{x}| \le |(A-B+B)\bar{x}|$$

$$\alpha|\bar{x}| \le |(A-B)\bar{x}| + |B\bar{x}|$$

$$\alpha|\bar{x}| \le \|A-B\||\bar{x}| + |B\bar{x}|$$

$$\alpha|\bar{x}| \le \beta\bar{x} + |B\bar{x}|$$

$$\Rightarrow (\alpha-\beta)|\bar{x}| \le |B\bar{x}|$$

Since $\beta<\alpha$, $\alpha-\beta > 0$

$$\Rightarrow 0 < (\alpha-\beta)|\bar{x}| \le |B\bar{x}| \text{ if } |\bar{x}| \ne 0$$

$$\Rightarrow |B\bar{x}| \ne 0 \text{ if } |\bar{x}| \ne 0$$

for $\bar{x} \ne \bar{y}$

$$\Rightarrow |B\bar{x}| \ne |B\bar{y}|$$

$$\Rightarrow B\bar{x} \ne B\bar{y}$$

$\Rightarrow$ B is one-one

By a known theorem, $(R(A) = X$, iff A is one-one$)$

$\Rightarrow$ B is onto

$\therefore$ B is invertiable

$\therefore B \in \Omega$

(b) To show that Ω is open,

we have to show that every element A in Ω is an interior point

i.e., if $A \in \Omega$ $\exists$ a neighbourhood $S_\epsilon(A) \subset \Omega$

$$S_\epsilon(A) = \{ B \in L(\mathbb{R}^n) \mid \|B-A\| < \epsilon \}$$

Let $\epsilon > 0$ and $0 < \epsilon < \alpha$ where $\alpha = \frac{1}{\|A^{-1}\|}$

Now for any $B \in L(\mathbb{R}^n)$, $\|B-A\| < \epsilon$

$$\Rightarrow \|B-A\| < \alpha$$

$$\Rightarrow \|B-A\| \frac{1}{\alpha} < 1$$

$$\Rightarrow \|B-A\| \|A^{-1}\| < 1$$

from ⓐ,

$B \in \Omega$

$\Rightarrow S_\epsilon(A) \subset \Omega$

$\Rightarrow$ A is an interior point of Ω

Since A is arbitrary point of Ω

Every element of Ω is an interior point

$\therefore$ Ω is an open set

Now, we have to prove that the mapping $X : \Omega \longrightarrow \Omega$

defined by $X(A) = A^{-1}$ is continuous

we know that $(\alpha - \beta)|\bar{x}| \le |B\bar{x}|$

Replace $\bar{x}$ by $B^{-1}\bar{y}$

$$\Rightarrow (\alpha-\beta)|B^{-1}\bar{y}| \le |B \cdot B^{-1}\bar{y}| \le |\bar{y}|$$

So for any $\bar{y} \in \mathbb{R}^n$ and $|\bar{y}| \le 1$

$$(\alpha-\beta)|B^{-1}\bar{y}| \le 1$$

$$\Rightarrow |B^{-1}\bar{y}| \le \frac{1}{\alpha-\beta}$$

$$\|B^{-1}\| = \sup\{ |B^{-1}y| \mid \bar{y} \in \mathbb{R}^n, |\bar{y}| \le 1 \}$$

$$\le \sup\left\{ \frac{1}{\alpha-\beta} \,\middle|\, \bar{y} \in \mathbb{R}^n, |\bar{y}| \le 1 \right\}$$

$$\le \frac{1}{\alpha-\beta}$$

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$\therefore \|\vec{B}^1\| \le \frac{1}{\alpha-\beta}$

we have, $\vec{B}^1 - \vec{A}^1 = \vec{B}^1(A-B)\vec{A}^1$

$$\|\vec{B}^1 - \vec{A}^1\| \le \|\vec{B}^1\| \|A-B\| \|\vec{A}^1\|$$
$$\le \frac{1}{\alpha-\beta}\beta\cdot\frac{1}{\alpha}$$
$$\le \frac{\beta}{\alpha(\alpha-\beta)} \longrightarrow 0 \text{ as } B\longrightarrow A$$

$\Rightarrow \|\vec{B}^1 - \vec{A}^1\| \longrightarrow 0$ as $B \longrightarrow A$

$\Rightarrow \vec{B}^1 - \vec{A}^1 \longrightarrow 0$ as $B \longrightarrow A$

$\Rightarrow X(B) - X(A) \longrightarrow 0$ as $B \longrightarrow A$

$\therefore X$ is continuous on Ω

Matrices:

Let $\{\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots \bar{x}_n\}$ & $\{\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n\}$ be basis of vector space X and Y respectively then every $A \in L(X,Y)$ determines a set of numbers

$$a_{ij} \rightarrow A\bar{x}_j = \sum_{i=1}^{m} a_{ij}\bar{y}_i \quad (1 \le i \le m)$$

we can write these number in a rectangular array of m rows and n columns called an m by n matrix.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Let $\bar{x} \in X$

Then $\bar{x} = \sum_{j=1}^{n} c_j \bar{x}_j$

$$A\bar{x} = \sum_{j=1}^{n} c_j A\bar{x}_j$$
$$= \sum_{j=1}^{n} c_j \sum_{i=1}^{m} a_{ij}\bar{y}_i$$
$$= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_j a_{ij}\right)\bar{y}_i$$

Differentiation:

Definition:

Let $(a,b) \in \mathbb{R}^1$, $f:(a,b) \to \mathbb{R}$ and $x \in (a,b)$

If $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ exists then we say that f is differentiable at x and we denote this limit by $f'(x)$.

i.e., $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = f'(x)$, $f'(x)$ is real numbers

thus $f(x+h)-f(x) = f'(x)h + r(h)$ where the remainder $r(h)$ is small in the sense that $\lim\limits_{h\to 0} \dfrac{r(h)}{h} = 0$

Definition:

Let $(a,b) \in \mathbb{R}^1$, $\bar{f}:(a,b) \to \mathbb{R}^m$, $x \in (a,b)$. If $\lim\limits_{h\to 0} \dfrac{\bar{f}(x+h)-\bar{f}(x)}{h}$ exists then we say that $\bar{f}$ is differentiable at x and we denote this limit by $\vec{f}'(x)$

i.e., $\lim\limits_{h\to 0} \dfrac{\bar{f}(x+h)-\bar{f}(x)}{h} = \vec{f}'(x)$, $\vec{f}'(x)$ is real numbers, thus

$\bar{f}(x+h)-\bar{f}(x) = \vec{f}'(x)h + r(h)$ where $\dfrac{r(h)}{h} \to 0$ as $h \to 0$

we can regard $f'(x)$ as linear transformation of $\mathbb{R}^1$ into $\mathbb{R}^m$

Definition:

Suppose E is an open set in $\mathbb{R}^n$, $\bar{f}: E \to \mathbb{R}^n$ and $\bar{x} \in E$. If there exists a linear transformation

$$A: \mathbb{R}^n \to \mathbb{R}^m \ni \lim_{h\to 0} \left| \frac{\bar{f}(\bar{x}+\bar{h})-\bar{f}(\bar{x})-A\bar{h}}{|\bar{h}|} \right| = 0$$

then we say that $\bar{f}$ is differentiable at $\bar{x}$ and we write $\vec{f}'(\bar{x}) = A$

If $\bar{f}$ is differentiable at every $\bar{x} \in E$, then we say that $\bar{f}$ is differentiable on E.

Theorem: Suppose E is an open set in $\mathbb{R}^n$, $\bar{f}$ maps E into $\mathbb{R}^m$ and $\lim\limits_{\bar{h}\to 0} \dfrac{|\bar{f}(\bar{x}+\bar{h})-\bar{f}(\bar{x})-A\bar{h}|}{|\bar{h}|} = 0$ hold with $A = A_1$ and $A = A_2$ then $A_1 = A_2$

Proof: Given $\bar{x} \in E$ and $A_1, A_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$

$$\lim_{\bar{h}\to 0} \frac{|\bar{f}(\bar{x}+\bar{h})-\bar{f}(\bar{x})-A_1\bar{h}|}{|\bar{h}|} = 0 \text{ and}$$

$$\lim_{\bar{h}\to 0} \frac{|\bar{f}(\bar{x}+\bar{h}) - \bar{f}(\bar{x}) - A_2\bar{h}|}{|\bar{h}|} = 0$$

we have to show that $A_1 = A_2$

Put $B = A_1 - A_2$

then $|B\bar{h}| = |(A_1 - A_2)\bar{h}|$

$= |A_1\bar{h} - A_2\bar{h}|$

$= |\bar{f}(\bar{x}+\bar{h}) - \bar{f}(\bar{x}) - \bar{f}(\bar{x}+\bar{h}) + \bar{f}(\bar{x}) + A_1\bar{h} - A_2\bar{h}|$

$\leq |\bar{f}(\bar{x}+\bar{h}) - \bar{f}(\bar{x}) - A_2\bar{h}| + |\bar{f}(\bar{x}+\bar{h}) - \bar{f}(\bar{x}) - A_1\bar{h}|$

$$\frac{|B\bar{h}|}{|\bar{h}|} = \frac{\bar{f}(\bar{x}+\bar{h}) - \bar{f}(\bar{x}) - A_2\bar{h}}{|\bar{h}|} + \frac{|\bar{f}(\bar{x}+\bar{h}) - \bar{f}(\bar{x}) - A_1\bar{h}|}{|\bar{h}|} \longrightarrow 0 \text{ as } \bar{h}\to 0$$

$$\frac{B\bar{h}}{|\bar{h}|} \longrightarrow 0 \text{ as } \bar{h} \to 0$$

for fixed, $\bar{h} \neq 0$, $\dfrac{|B(t\bar{h})|}{|t\bar{h}|} \longrightarrow 0$ as $t \longrightarrow 0$ —— (1)

linearly of B shows that $B(t\bar{h}) = t\cdot B\bar{h}$

so that left hand side of (1) is independent of t

Thus for all $\bar{h} \in \mathbb{R}^n$, $B\bar{h} = 0$

$\Rightarrow (A_1 - A_2)\bar{h} = 0$

$\Rightarrow A_1 - A_2 = 0$

$\Rightarrow A_1 = A_2$

Hence the proof

Theorem: (Chain rule):-

Suppose E is an open set in $\mathbb{R}^n$, $\bar{f}$ maps E into $\mathbb{R}^m$, $\bar{f}$ is differentiable at $\bar{x}_0 \in E$, $\bar{g}$ maps an open set containing $\bar{f}(E)$ into $\mathbb{R}^k$ and $\bar{g}$ is differentiable at $\bar{f}(\bar{x}_0)$ then the mapping $\bar{F}$ of E into $\mathbb{R}^k$ defined by $\bar{F}(\bar{x}) = \bar{g}(\bar{f}(\bar{x}))$ is differentiable at $\bar{x}_0$ and $\bar{F}'(\bar{x}_0) = \bar{g}'(\bar{f}(\bar{x}_0))\bar{f}'(\bar{x}_0)$

Proof: Given that $\bar{f}$ is differentiable at $\bar{x}_0$
and $\bar{g}$ is differentiable at $\bar{f}(\bar{x}_0)$
where $\bar{x}_0 \in E$ an open subset of $\mathbb{R}^n$
we have to prove that $\bar{F}: E \to \mathbb{R}^k$ defined by $\bar{F}(\bar{x}) = \bar{g}(\bar{f}(\bar{x}))$ is differentiable at $\bar{x}_0$ and $\bar{F}'(\bar{x}_0) = \bar{g}'(\bar{f}(\bar{x}_0))\bar{f}'(x_0)$

Put $\bar{y}_0 = \bar{f}(\bar{x}_0)$, $A = \bar{f}'(x_0)$, $B = \bar{g}'(y_0)$

Define $U(\bar{h}) = \bar{f}(\bar{x}_0 + \bar{h}) - \bar{f}(\bar{x}_0) - A\bar{h}$, $\bar{h} \in \mathbb{R}^n$

$$V(\bar{k}) = \bar{g}(\bar{y}_0 + \bar{k}) - \bar{g}(\bar{y}_0) - B\bar{k}, \ \bar{k} \in \mathbb{R}^m$$

for which $\bar{f}(\bar{x}_0 + \bar{h})$ and $\bar{g}(\bar{y}_0 + \bar{k})$ are defined

then $|U(\bar{h})| = \epsilon(\bar{h})|\bar{h}|$ and $|V(\bar{k})| = \eta(\bar{k})|\bar{k}|$

where $\epsilon(h) \to 0$ as $\bar{h} \to 0$ and $\eta(\bar{k}) \to 0$ as $\bar{k} \to 0$

for given $\bar{h}$; put $\bar{k} = \bar{f}(\bar{x}_0 + \bar{h}) - \bar{f}(\bar{x}_0)$ —— ①

$$[k + \bar{f}(x_0) = \bar{f}(\bar{x}_0 + \bar{h})]$$

$$\begin{aligned}
|\bar{k}| &= |\bar{f}(\bar{x}_0 + \bar{h}) - \bar{f}(\bar{x}_0)| \\
&= |U(\bar{h}) + A\bar{h}| \\
&\leq |U(\bar{h})| + |A\bar{h}| \\
&\leq \epsilon(\bar{h}) \cdot |\bar{h}| + \|A\| \cdot |\bar{h}| \\
&= [\epsilon(\bar{h}) + \|A\|]\,|\bar{h}|
\end{aligned}$$

$$|k| \leq [\epsilon(\bar{h}) + \|A\|]\,|\bar{h}| \quad \text{—— ②}$$

Consider, $\bar{F}(\bar{x}_0 + \bar{h}) - \bar{F}(\bar{x}_0) - BA\bar{h}$

$$\begin{aligned}
&= \bar{g}(\bar{f}(\bar{x}_0 + \bar{h})) - \bar{g}(\bar{f}(\bar{x}_0)) - BA\bar{h} \\
&= \bar{g}(\bar{y}_0 + \bar{k}) - \bar{g}(y_0) - BA\bar{h} \\
&= V(\bar{k}) + B\bar{k} - BA\bar{h} \\
&= V(\bar{k}) + B(\bar{k} - A\bar{h}) \\
&= B[\bar{f}(\bar{x}_0 + \bar{h}) - \bar{f}(\bar{x}_0) - A\bar{h}] + V\bar{k}
\end{aligned}$$

$$= B\,U(\bar h) + V(\bar k)$$

$$|\bar F(\bar x_0+\bar h) - \bar F(\bar x_0) - BA\bar h| = |B\,U(\bar h) + V(\bar k)|$$
$$\le |B\,U(\bar h)| + |V(\bar k)|$$
$$\le \|B\|\,|U(\bar h)| + |V(\bar k)|$$
$$\le \|B\|\,\epsilon(\bar h)\,|\bar h| + \eta(\bar k)|\bar k|$$
$$\le \|B\|\,\epsilon(\bar h)|\bar h| + \eta(\bar k)[\,E(\bar h) + \|A\||\bar h|\,]$$

$$\Rightarrow \frac{|\bar F(\bar x_0+\bar h) - \bar F(\bar x_0 - BA\bar h)|}{|\bar h|} \le \|B\|\,\epsilon(\bar h) + \eta(\bar k)\,(E(\bar h)) + \|A\|$$

as $\bar h \to 0$, $E(\bar h) \to 0$

also as $\bar h \to 0$, $\bar k \to 0$

So that $\eta(k) \to 0$

$\Rightarrow \bar F$ is differentiable at $\bar x_0$ and $\bar F'(\bar x_0) = BA$

$$= \bar g'(\bar y_0)\,\bar f'(\bar x_0)$$

$$\therefore\ \bar F(\bar x_0) = \bar g'(f(x_0))\cdot \bar f'(x_0)$$

Hence the theorem.

Partial Derivatives:

let $E \subset \mathbb{R}^n$ be open and $\bar f : E \to \mathbb{R}^m$. let $\{\bar e_1, \bar e_2, \dots, \bar e_n\}$ be the standard basis of $\mathbb{R}^n$ and $\{\bar u_1, \bar u_2, \dots, \bar u_n\}$ be the standard basis of $\mathbb{R}^m$. let $\bar x \in E$, $\bar f(\bar x) \in \mathbb{R}^m$ and $\bar f(\bar x) = (f_1(\bar x_1), f_2(\bar x_2), \dots f_m(\bar x_n))$

$$= \sum_{i=1}^{n} f_i(\bar x)\cdot \bar u_i$$

Also $f_i(\bar x) = \bar f(\bar x)\bar u_i$

$$\Big[\ f(\bar x)\,\bar u_1 = (f_1(\bar x), f_2(\bar x), \dots, f_m(\bar x))(1,0,0,\dots,0)$$
$$= f_1(\bar x)\cdot 1 + f_2(\bar x)(0) + \dots + f_m(\bar x) = 0$$
$$= f_1(\bar x)\ \Big]$$

for $\bar x \in E$ $1 \le i \le m$, $1 \le j \le n$ we define

$$(D_j f_i)(\bar x) = \lim_{t\to 0} \frac{f_i(\bar x + t\bar e_j) - f_i(\bar x)}{t}$$

Provided the limit exists

$\therefore (D_j f_i)$ is the derivative of f_i with respect to x_j

It is also denoted by $\frac{\partial f_i}{\partial x_j}$ and $D_j f_i$ is called a partial derivative.

Theorem:-

Suppose $\bar{f}$ maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$ and $\bar{f}$ is differentiable at a point $\bar{x} \in E$ then the partial derivation $(D_j f_i)(\bar{x})$ exists and

$$\bar{f}'(\bar{x})\bar{e}_j = \sum_{j=1}^{m} (D_j f_i)(\bar{x})(\bar{u}_i) \quad (1 \le j \le n)$$

Proof. let $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\}$ and $\{\bar{u}_1, u_2, \ldots, \bar{u}_m\}$ be the standard basis for $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

Since $\bar{f}$ is differentiable at $\bar{x} \in E$

$$\bar{f}(\bar{x}+te_j) - \bar{f}(\bar{x}) = \bar{f}'(\bar{x}).t\bar{e}_j + \bar{r}(t\bar{e}_j) \quad \text{where } \frac{|\bar{r}(te_j)|}{t} \to 0 \text{ as } t \to 0$$

By linearity of $\bar{f}'(x)$

$$\bar{f}'(x)te_j = t\bar{f}'(\bar{x})e_j$$

$$\therefore \lim_{t\to 0} \frac{[\bar{f}(\bar{x})+te_j] - \bar{f}(\bar{x})}{t} = \bar{f}'(\bar{x})\bar{e}_j$$

Since $\bar{f}(\bar{x}) = \sum_{i=1}^{m} f_i(\bar{x})\bar{u}_i$

$$\lim_{t\to 0} \frac{\sum_{i=1}^{m} f_i(\bar{x}+t\bar{e}_j)u_i - \sum_{i=1}^{m} f_i(\bar{x})\bar{u}_i}{t} = \bar{f}'(\bar{x})\bar{e}_j$$

$\Rightarrow$ It follows that each quotient in this sum has a limit as $t \to 0$

So that each $(D_j f_i)(\bar{x})$ exists

$$\Rightarrow \sum_{i=1}^{m} (D_j f_i)(\bar{x})\bar{u}_i = \bar{f}'(\bar{x})\bar{e}_j \quad (1 \le j \le n)$$

Definition:-

If $E \subset \mathbb{R}^k$ then E is said to be convex if $\lambda\bar{x} + (1-\lambda)\bar{y} \in E$

where $\bar{x}, \bar{y} \in E$ and $0 < \lambda < 1$

$$\bar{x} = (x_1, x_2, \ldots, x_k), \ \bar{y} = (y_1, y_2, \ldots, y_k)$$

** Theorem: Suppose f maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, (115)

(2) $\bar{f}$ is differentiable in E and there is a real number M such that $\|f'(\bar{x})\| \leq M$ for every $\bar{x} \in E$ then $|\bar{f}(b) - \bar{f}(a)| \leq M|(\bar{b}-\bar{a})|$. (II - 4M)

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Proof: fix $\bar{a}, \bar{b} \in E$

Define $\gamma : [0,1] \to \mathbb{R}^n$ by $\gamma(t) = (1-t)\bar{a} + t\bar{b}$ $\forall t \in [0,1]$

Since $E \subset \mathbb{R}^n$ is convex, $\gamma(t) \in E$, $0 \leq t \leq 1$

Define $\bar{g} : [0,1] \to \mathbb{R}^m$ by $\bar{g}(t) = \bar{f}(\gamma(t))$

Since γ is differentiable on $[0,1]$ and $\gamma'(t) = \bar{b} - \bar{a}$

By the hypothesis, $\bar{f}$ is differentiable at $\gamma(t) \in E$

By chain rule, $\bar{g}$ is differentiable on $(0,1)$

$$\text{and } \bar{g}'(t) = \bar{f}'(\gamma(t))\gamma'(t)$$
$$= \bar{f}'(\gamma(t))(\bar{b}-\bar{a})$$

$$|\bar{g}'(t)| = |(\bar{f}'(\gamma(t)))(\bar{b}-\bar{a})|$$
$$\leq |\bar{f}'(\gamma(t))||\bar{b}-\bar{a}|$$
$$\leq \|\bar{f}'(\gamma(t))\| \, |\bar{b}-\bar{a}|$$
$$\leq M|\bar{b}-\bar{a}| \quad \text{——— (1)}$$

Since $\bar{g} : [0,1] \to \mathbb{R}^m$ is differentiable and continuous on $[0,1]$

By known theorem,

[Suppose $\bar{f}$ is a continuous mapping of $[a,b]$ into $\mathbb{R}^n$ and $\bar{f}$ is differentiable in (a,b) then $\exists\, x \in (a,b) \ni |\bar{f}(b) - \bar{f}(a)| \leq (b-a)|\bar{f}'(x)|$]

$$\exists\, t \in (0,1) \ni |\bar{g}(1) - \bar{g}(0)| \leq (1-0)|\bar{g}'(t)|$$

$$\Rightarrow |\bar{f}(\gamma(1)) - \bar{f}(\gamma(0))| \leq |\bar{g}'(t)|$$

$$\Rightarrow |\bar{f}(b) - \bar{f}(a)| \leq M|(\bar{b}-\bar{a})| \quad [\text{from (1)}]$$

Corollary:

If in addition $\bar{f}'(\bar{x})=0 \ \forall x \in E$ then $\bar{f}$ is constant.

Proof: Take $m=0$ in the above theorem

$\Rightarrow |\bar{f}(\bar{b})-\bar{f}(\bar{a})| \leq 0\,|\bar{b}-\bar{a}|$

$\Rightarrow |\bar{f}(\bar{b})-\bar{f}(\bar{a})| = 0$

$\Rightarrow \bar{f}(\bar{b}) = \bar{f}(\bar{a})$

$\therefore f$ is constant

Definition:

let $\bar{f}$ be a differentiable mapping of an open set E contained in $\mathbb{R}^n$ into $\mathbb{R}^n$ then $\bar{f}$ is said to be continuously differentiable in E if $\bar{f}'$ is a continuous mapping of E into $L(\mathbb{R}^n, \mathbb{R}^m)$

i.e., to every $\bar{x} \in E$ & to every $\epsilon>0$ there corresponds a $\delta>0 \ni \|\bar{f}'(y)-\bar{f}'(x)\|<\epsilon$ if $\bar{y} \in E$ & $|\bar{x}-\bar{y}|<\delta$. If this is so, we also say that $\bar{f}$ is a $\mathcal{C}'$-mapping (or) that $\bar{f} \in \mathcal{C}'(E)$.

Theorem:- suppose $\bar{f}$ maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$ then $\bar{f} \in \mathcal{C}'(E)$ iff the partial derivatives $D_j f_i$ exists and are continuous on E for $1 \leq i \leq m$, $1 \leq j \leq n$.

Proof: suppose $\bar{f} \in \mathcal{C}'(E)$

i.e., $\bar{f}'$ is a continuous mapping of E into $L(\mathbb{R}^n, \mathbb{R}^m)$

i.e., to every $\bar{x} \in E$ and to every $\epsilon>0$ there is corresponds a $\delta>0 \ni$ $\|\bar{f}'(y)-\bar{f}'(x)\| < \epsilon$ if $\bar{y} \in E$ and $|\bar{x}-\bar{y}|<\delta$

Since $\bar{f} \in \mathcal{C}'(E)$

By known theorem,

$(D_j f_i)$ exists and $\bar{f}'(\bar{x})e_j = \sum_{i=1}^{m} (D_j f_i)(\bar{x})(\bar{u}_i) \quad 1 \leq j \leq m$

$\bar{f}'(\bar{x})e_j = (D_j f_1)(\bar{x})(\bar{u}_1) + (D_j f_2)(\bar{x})(\bar{u}_2) + \cdots + (D_j f_m)(\bar{x})\bar{u}_m$

$(\bar{f}'(\bar{x})\bar{e}_j)\cdot\bar{u}_i = (D_j f_1)(\bar{x})\bar{u}_1\cdot\bar{u}_i + (D_j f_2)(\bar{x})\bar{u}_2\cdot\bar{u}_i + \cdots + (D_j f_m)(\bar{x})\bar{u}_m\cdot\bar{u}_i$

$= (D_j f_i)(\bar{x}) \quad \text{---} (1)$

equation ① is true for all i, j and $\forall\ \bar{x} \in E$

$(D_j f_i)(\bar{y}) - (D_j f_i)(\bar{x}) = (\bar{f}'(y)\bar{e}_j - \bar{f}'(\bar{x})\bar{e}_j) \cdot \bar{u}_i$

$|(D_j f_i)(\bar{y}) - (D_j f_i)(\bar{x})| = |\bar{f}'(\bar{y})\bar{e}_j - \bar{f}'(\bar{x})\bar{e}_j| \cdot |\bar{u}_i|$

$= |(\bar{f}'(\bar{y}) - \bar{f}'(\bar{x}))\bar{e}_j| \cdot 1$

$\leq \| \bar{f}'(\bar{y}) - \bar{f}'(\bar{x}) \| \cdot |\bar{e}_j|$

$= \| \bar{f}'(\bar{y}) - \bar{f}'(\bar{x}) \|$ $(\because |\bar{e}_j| = 1)$

$< \epsilon$ for $|\bar{x} - \bar{y}| < \delta$ $(\because \bar{f} \in \mathcal{C}'(E))$

$\therefore D_j f_i$ is continuous on E for $1 \leq i \leq m$, $1 \leq j \leq m$

Conversely, suppose that partial derivatives $D_j f_i$ exists and are continuous on E

We have to prove that $\bar{f}'$ is continuous

since $\bar{f}$ is completely determined by its m components $f_1, f_2, \ldots, f_m$ which are real valued function and since a vector valued function is continuous iff each of its components. ie., the real valued function are continuous.

It is enough if we prove that theorem for real valued function

let f be a real valued function

Let $\bar{x} \in E$ and $\epsilon > 0$

Since E is open, $\bar{x}$ is an interior point of E then

$\exists$ a open ball $S_{r_1}(\bar{x})$ with centre $\bar{x}$ and radius $r_1 \ni S_{r_1}(\bar{x}) \subset E$

Since each $D_j f_i$ is continuous

for $\epsilon > 0\ \exists\ \delta > 0 \ni |(D_j f_i)\bar{x} - (D_j f_i)\bar{y}| < \epsilon/2$ whenever $|\bar{x} - \bar{y}| < \delta$

let $r = \min\{\delta, r_1\}$

Consider the open ball $S = S_r(\bar{x})$

then $|(D_j f_i)\bar{x} - (D_j f_i)\bar{y}| < \epsilon/n$ whenever $\bar{y} \in S_r(\bar{x})$ —— ②

Suppose $\bar{h} \in \mathbb{R}^n$, $\bar{h} = (h_1, h_2, \ldots, h_n) = \sum_{j=1}^{n} h_j \bar{e}_j$ with $|\bar{h}| < r$

Put $\bar{V}_0 = 0$, $\bar{V}_1 = h_1\bar{e}_1$, $\bar{V}_2 = h_1\bar{e}_1 + h_2\bar{e}_2$, ... $\bar{V}_k = h_1\bar{e}_1 + \cdots + h_k\bar{e}_k$ $\forall\ 1 \leq k \leq n$

$$\text{then } f(\bar{x}+\bar{h})-f(\bar{x}) = f(\bar{x}+\bar{h})-f(\bar{x}+\bar{v}_{n-1}) + f(\bar{x}+\bar{v}_{n-1})+\cdots+f(\bar{x}+\bar{v}_1)+f(\bar{x}_1+\bar{v}_1)-f(\bar{x})$$

$$= \sum_{j=1}^{n}\{f(\bar{x}+v_j)-f(\bar{x}+\bar{v}_{j-1})\} \quad \text{--- (3)}$$

$$\bar{v}_K = \left|\sum_{j=1}^{k} h_j\bar{e}_j\right|$$

$$= \left|\sum_{j=1}^{n} h_j\bar{e}_j\right|$$

$$= |\bar{h}|$$

$$< r$$

$$\Rightarrow |\bar{v}_K| < r \text{ for } 1\le k\le n$$

Now $|\bar{x}+\bar{v}_{j-1}-\bar{x}| = |\bar{v}_{j-1}| < r$

$$\Rightarrow \bar{x}+\bar{v}_{j-1} \in S$$

Similarly, $\bar{x}+\bar{v}_j \in S$

Since S is convex,

$$(\bar{x}+\bar{v}_{j-1})(1-\theta_j)+(\bar{x}+\bar{v}_j)\theta_j = \bar{x}+\bar{v}_{j-1}-\bar{x}\theta_j-\bar{v}_{j-1}\theta_j+\bar{x}\cdot\theta_j+\bar{v}_j\theta_j$$

$$= \bar{x}+\bar{v}_{j-1}+(\bar{v}_j-\bar{v}_{j-1})\theta_j$$

$$= \bar{x}+\bar{v}_{j-1}+\theta_j h_j\bar{e}_j \in S \text{ for some } \theta_j\in(0,1)$$

By mean value theorem,

$$f(\bar{x}+\bar{v}_j)-f(\bar{x}+\bar{v}_{j-1}) = f(\bar{x}+\bar{v}_{j-1}+h_j\bar{e}_j)-f(\bar{x}+\bar{v}_{j-1})$$

$$= h_j(D_jf)(\bar{x}+\bar{v}_{j-1}+\theta_j h_j\bar{e}_j)$$

$$\sum_{j=1}^{n}[f(\bar{x}+\bar{v}_j)-f(\bar{x}+\bar{v}_{j-1})] = \sum_{j=1}^{n} h_j(D_jf)(\bar{x}+\bar{v}_{j-1}+\theta_j h_j\bar{e}_j)$$

$$\Rightarrow f(\bar{x}+\bar{h})-f(\bar{x}) = \sum_{j=1}^{n} h_j(D_jf)(\bar{x}+\bar{v}_{j-1}+\theta_j h_j\bar{e}_j)$$

$$\Rightarrow \left|f(\bar{x}+\bar{h})-f(x)-\sum_{j=1}^{n} h_j(D_jf)\bar{x}\right|$$

$$= \left|\sum_{j=1}^{n} h_j(D_jf)(\bar{x}+\bar{v}_{j-1}+\theta_j h_j\bar{e}_j)-\sum_{j=1}^{n} h_j(D_jf)\bar{x}\right|$$

$$= \left| \sum_{j=1}^{n} h_j (D_j f)(\bar{x} + \bar{V}_{j-1} + \theta_j h_j \bar{e}_j) - \bar{x} \right|$$

Since $\bar{x} + \bar{V}_{j-1} + \theta_j h_j \bar{e}_j \in S = S_r(\bar{x})$

$$\Rightarrow \left| f(\bar{x}+\bar{h}) - f(\bar{x}) - \sum_{j=1}^{n} h_j (D_j f)\bar{x} \right| \le \sum_{j=1}^{n} |h_j| \frac{\epsilon}{n}$$

$$\le |\bar{h}| \epsilon$$

$$\Rightarrow \frac{\left| f(\bar{x}+\bar{h}) - f(\bar{x}) - \sum_{j=1}^{n} h_j (D_j f)\bar{x} \right|}{|\bar{h}|} < \epsilon$$

$\Rightarrow f$ is differentiable at $\bar{x}$

$$f'(\bar{x})\bar{h} = \sum_{j=1}^{n} h_j (D_j f)\bar{x}$$

This says that f is differentiable at $\bar{x}$ and the matrix $[f'(x)]$ consist of the row.

$$[D_1(f)(\bar{x}) + (D_2 f)(\bar{x}) + \cdots + (D_n f)(\bar{x})]$$

Since each $(D_j f)(\bar{x})$ is continuous,

$f'(\bar{x})$ is continuous

$f \in C'(E)$

$\bar{f} \in C'(E)$

The Contradiction principle:

Def: let X be a metric space with metric 'd'. If ϕ maps X into X and if there is a number $c < 1$ such that $d(\phi(x), \phi(y)) \le c.d(x,y)$ $\forall x,y \in X$.

then ϕ is said to be a contradiction of X into X.

Theorem: If X is a complete metric space and if ϕ is a contradiction of X into X, then $\exists$ one and only one $x \in X$ such that $\phi(x) = x$.

Proof: Given that X is a complete metric space and

ϕ is a contradiction of X into X

then $\exists$ a number $c < 1 \ni d(\phi(x), \phi(y)) \le c.d(x,y)$ $\forall x,y \in X$

let $x_0 \in X$ be arbitrary

Define a sequence $\{x_n\}$ by $x_{n+1} = \phi(x_n)$, $n = 0, 1, 2, \ldots$ for $n \geq 1$

$$d(x_{n+1}, x_n) = d(\phi(x_n), \phi(x_{n-1}))$$
$$\leq c \cdot d(x_n, x_{n-1})$$
$$\leq c \cdot d(\phi(x_{n-1}), \phi(x_{n-2}))$$
$$\leq c \cdot c \cdot d(x_{n-1}, x_{n-2})$$
$$= c^2 \cdot d(x_{n-1}, x_{n-2})$$
$$\vdots$$
$$\leq c^n d(x_1, x_0)$$

$\therefore d(x_{n+1}, x_n) \leq c^n \cdot d(x_1, x_0)$ for $n \geq 1$

let m, n be two positive integers and $m > n$

Now $d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m+1}, x_m)$

$$\leq c^n d(x_1, x_0) + c^{n+1} d(x_1, x_0) + \ldots + c^{m-1} d(x_1, x_0)$$
$$= c^n \cdot d(x_1, x_0)(1 + c + c^2 + \ldots + c^{m-n-1})$$
$$< c^n \cdot d(x_1, x_0)(1 + c + c^2 + \ldots + c^{m-n-1} + \ldots)$$
$$= c^n \cdot d(x_1, x_0)\left(\frac{1}{1-c}\right) \longrightarrow 0 \text{ as } n \to \infty \quad (\because c < 1)$$

$\Rightarrow \{x_n\}$ is a cauchy sequence in X

Since, X is complete, $\{x_n\}$ converges to some $x \in X$

i.e., $\lim_{n \to \infty} x_n = x$

Since, every contradiction map is continuous

$$\lim_{n \to \infty} \phi(x_n) = \phi\left(\lim_{n \to \infty} x_n\right) = \phi(x)$$

But $\lim_{n \to \infty} \phi(x_n) = \lim_{n \to \infty} x_{n+1} = x$

$$\therefore \phi(x) = x$$

Now we prove that ϕ has unique fixed point

If possible $y \in X$ be another fixed point of ϕ, then $\phi(y) = y$ & $\phi(x) = x$

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(121)

$0 \le d(\phi(x), \phi(y)) \le c \cdot d(x,y)$

$\Rightarrow d(x,y) \le c \cdot d(x,y)$

$\Rightarrow d(x,y) = 0 \quad (\because c<1)$

$\Rightarrow x = y$

Hence ϕ has unique fixed point.

Imp
Inverse function theorem :-

(13) Suppose $\bar{f}$ is a $\mathscr{C}'$-mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $\bar{f}'(\bar{a})$ is invertiable for some $\bar{a} \in E$ and $\bar{b} = \bar{f}(\bar{a})$ then.

(a) $\exists$ open sets U and V in $\mathbb{R}^n$ $\ni \bar{a} \in U, \bar{b} \in V$ $\bar{f}$ is one to one on U and $\bar{f}(U) = V$

(b) If $\bar{g}$ is the inverse of $\bar{f}$ (which exists, by (a)) defined in V by $\bar{g}(\bar{f}(\bar{x})) = \bar{x}$ $(\bar{x} \in U)$, then $\bar{g} \in \mathscr{C}'(V)$.

Proof: Given that $\bar{f}$ is a $\mathscr{C}'$-mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $\bar{f}'(\bar{a})$ is invertiable for some $\bar{a} \in E$ and $\bar{b} = \bar{f}(\bar{a})$. — (1)

(a) put, $\bar{f}'(\bar{a}) = A$ and choose λ so that $2\lambda \|\bar{A}^{-1}\| = 1$

By hypothesis, $\bar{f}'(\bar{a})$ exists

$\Rightarrow \bar{A}^{-1}$ exists $\quad A^{-1}$ exists

Since $\bar{f}'$ is continuous at $\bar{a}$

for this, λ, $\exists$ $\delta > 0 \ni \|\bar{f}'(\bar{x}) - \bar{f}'(\bar{a})\| < \lambda$ whenever $|\bar{x} - \bar{a}| < \delta$

Put $U = \{\bar{x} \in E \mid |\bar{x} - \bar{a}| < \delta\}$

$U \subset E$

Then U is an open set and $\|\bar{f}'(\bar{x}) - A\| < \lambda$ — (2)

We associate to each $\bar{y} \in \mathbb{R}^n$ a function ϕ, defined by

$\phi(\bar{x}) = \bar{x} + \bar{A}^{-1}(\bar{y}) - \bar{f}(\bar{x})) \quad \forall \bar{x} \in E$ — (3)

A^{-1}

first we show that $\bar{f}(\bar{x}) = \bar{y} \Longleftrightarrow \phi(\bar{x}) = \bar{x}$

Note that $f(\bar{x})$ ...

Suppose $\bar{f}(\bar{x}) = \bar{y}$

then $\phi(\bar{x}) = \bar{x} + \bar{A}^{-1}(\bar{0})$ (by (3))

$= \bar{x} + \bar{0}$

$= \bar{x}$

$\therefore \bar{x}$ is a fixed of ϕ

Suppose $\bar{x}$ is a fixed point of ϕ

i.e., $\phi(\bar{x}) = \bar{x}$

Then $\bar{A}^{1}(\bar{y} - \bar{f}(\bar{x})) = \bar{0}$

$\Rightarrow \bar{f}(\bar{x}) = \bar{y}$

$\phi'(\bar{x}) = I - (\bar{A}^{1}\bar{f}'(\bar{x})) = \bar{A}^{1}(A - \bar{f}'(\bar{x}))$

$\Rightarrow \|\phi'(\bar{x})\| \leq \|\bar{A}^{1}\| \, \|A - \bar{f}'(\bar{x})\|$

$< \frac{1}{2\lambda} \cdot \lambda$

$= \frac{1}{2}$

$\Rightarrow \|\phi'(\bar{x})\| < \frac{1}{2} \quad (\bar{x} \in U)$

By known theorem,

$|\phi(\bar{x}_2) - \phi(\bar{x}_1)| \leq \frac{1}{2}|\bar{x}_2 - \bar{x}_1| \quad \forall \bar{x}_1, \bar{x}_2 \in U$

$\Rightarrow \phi$ has atmost a fixed point in U

$\Rightarrow \bar{f}(\bar{x}) = \bar{y}$ for atmost one $\bar{x} \in U$

$\Rightarrow \bar{f}$ is one-one

Since $\mathbb{R}^n$ is finite dimensional,

By a known theorem, $\bar{f}$ is onto

$\Rightarrow \bar{f}$ is invertiable

Put $V = \bar{f}(U)$

let $\bar{y}_0 \in V$ then $\exists \, \bar{x}_0 \in U \ni \bar{f}(\bar{x}_0) = \bar{y}_0$

let B be an open ball with centre at $\bar{x}_0$ and radius $r > 0$,

so small that its closure $\bar{B}$ lies in U

Now, we prove that $\bar{y} \in V$ whenever $|\bar{y} - \bar{y}_0| < r$

fix $\bar{y}$, $|\bar{y} - \bar{y}_0| < \lambda r$

with ϕ is in equation (2)

$|\phi(\bar{x}_0) - \bar{x}_0| = |\bar{A}^{1}(\bar{y} - \bar{f}(\bar{x}_0))|$

$\leq \|\bar{A}^{1}\| \, |\bar{y} - \bar{f}(\bar{x}_0)|$

$$|\phi(\bar{x}_0) - \bar{x}_0| = \|A^{-1}\| \, |\bar{y} - \bar{y}_0|$$
$$< \|A^{-1}\| \lambda r$$
$$< \frac{1}{2\lambda} \lambda r$$
$$= r/2$$

$$\Rightarrow |\phi(\bar{x}_0) - \bar{x}_0| < r/2$$

If $\bar{x} \in \bar{B}$ then $|\phi(\bar{x}) - \bar{x}_0| = |\phi(\bar{x}) - \phi(\bar{x}_0) + \phi(\bar{x}_0) - \bar{x}_0|$

$$\leq |\phi(\bar{x}) - \phi(\bar{x}_0)| + |\phi(\bar{x}_0) - \bar{x}_0|$$
$$\leq \frac{1}{2}|\bar{x} - \bar{x}_0| + \frac{r}{2}$$
$$\leq \frac{r}{2} + \frac{r}{2}$$
$$= r$$

$$\Rightarrow |\phi(\bar{x}) - \bar{x}_0| \leq r$$
$$\Rightarrow \phi(\bar{x}) \in \bar{B}$$

$\Rightarrow \phi$ is a mapping from $\bar{B}$ to $\bar{B}$

from ③, $|\phi(\bar{x}_2) - \phi(\bar{x}_1)| \leq \frac{1}{2}|\bar{x}_2 - \bar{x}_1|$ if $\bar{x}_1, \bar{x}_2 \in \bar{B}$

Thus ϕ is a contradiction, ~~of~~ principle of $\bar{B}$ into $\bar{B}$

since $\bar{B}$ is a closed subset of $\mathbb{R}^n$, it is complete

($\because$ Every closed subset of a complete metric space is complete)

By a known theorem,

ϕ has a fixed point, say $\bar{x} \in \bar{B}$

i.e., $\phi(\bar{x}) = \bar{x}$

$$\Rightarrow \bar{f}(\bar{x}) = \bar{y}$$
$$\Rightarrow \bar{y} = \bar{f}(\bar{x}) \in \bar{f}(\bar{B})$$
$$\Rightarrow \bar{y} \in \bar{f}(\bar{B}) \subseteq \bar{f}(U) = V$$
$$\Rightarrow \bar{y} \in V$$

$\Rightarrow V$ is open

b) Now, we have to prove that the inverse of $\bar{f}$ is continuously differentiable (124)

let $\bar{g}$ be the inverse of $\bar{f}$ defined on V by $\bar{g}(\bar{f}(\bar{x})) = \bar{x} \ \forall \bar{x} \in U$

let $\bar{y} \in V$ and $\bar{y}+\bar{k} \in V$

then $\exists \ \bar{x} \in U$ and $\bar{x}+\bar{h} \in U \ni \bar{f}(\bar{x}) = \bar{y}$ & $\bar{f}(\bar{x}+\bar{h}) = \bar{y}+\bar{k}$

consider, $\phi(\bar{x}+\bar{h}) - \phi(\bar{x}) = \left[\bar{x}+\bar{h}+\bar{A}^{-1}(\bar{y}-\bar{f}(\bar{x}+\bar{h}))\right] - \left[\bar{x}+\bar{A}^{-1}(\bar{y}-\bar{f}(\bar{x}))\right]$

$$= \bar{h} - \bar{A}^{-1}\left[(\bar{f}(\bar{x}+\bar{h}) - \bar{f}(\bar{x}))\right]$$

$$= \bar{h} - \bar{A}^{-1}\left[(\bar{y}+\bar{k}) - \bar{y}\right]$$

$$= \bar{h} - \bar{A}^{-1}(\bar{k})$$

Now, $|\bar{h} - \bar{A}^{-1}(\bar{k})| = |\phi(\bar{x}+\bar{h}) - \phi(\bar{x})|$

$$\leq \frac{1}{2}|\bar{x}+\bar{h}-\bar{x}|$$

$$\Rightarrow |\bar{h} - \bar{A}^{-1}(\bar{k})| < \frac{1}{2}|\bar{h}|$$

$$\Rightarrow \quad |\bar{A}^{-1}(\bar{k})| \geq \frac{1}{2}|\bar{h}|$$

$$\Rightarrow |\bar{h}| \leq 2|\bar{A}^{-1}(\bar{k})|$$

$$\leq 2\|\bar{A}^{-1}\| \cdot |\bar{k}|$$

$$= \frac{|\bar{k}|}{\lambda}$$

$$\Rightarrow |\bar{h}| \leq \frac{|\bar{k}|}{\lambda}$$

$$\Rightarrow \quad \frac{1}{|\bar{k}|} \leq \frac{1}{\lambda \cdot |\bar{h}|}$$

Now, $\|\bar{f}'(\bar{x}) - A\| \cdot \|\bar{A}^{-1}\| < \lambda \cdot \frac{1}{2\lambda} = \frac{1}{2} < 1$

By known theorem, $\bar{f}'(\bar{x})$ is invertible

let it be T

i.e., $T = \left[\bar{f}'(\bar{x})\right]^{-1}$

Now $\bar{g}(\bar{y}+\bar{k}) - \bar{g}(\bar{y}) - T\bar{k} = \bar{g}(\bar{f}(\bar{x}+\bar{h}) - \bar{f}(\bar{x})) - T\bar{k}$

$$= \bar{x}+\bar{h}-\bar{x} - T\bar{k}$$

$$= \bar{h} - T\bar{k}$$

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$$= T\left[(\bar{y}+\bar{k}) - \bar{y} - T^{-1}\bar{h}\right]$$

$$= -T\left[\bar{f}(\bar{x}+\bar{h}) - \bar{f}(\bar{x}) - T^{-1}\bar{h}\right]$$

$$\frac{|\bar{g}(\bar{y}+\bar{k}) - \bar{g}(\bar{y}) - T\bar{k}|}{|\bar{k}|} \leq \frac{\|T\|\,|\bar{f}(\bar{x}+\bar{h}) - \bar{f}(\bar{x}) - T^{-1}\bar{h}|}{|\bar{k}|}$$

$$\leq \frac{\|T\|}{\lambda} \cdot \frac{|\bar{f}(\bar{x}+\bar{h}) - \bar{f}(\bar{x}) - T^{-1}(\bar{h})|}{|\bar{h}|}$$

As $\bar{k} \to 0$ we have $\bar{h} \to 0$

Then, R.H.S of eqn (5) $\to 0$

$\Rightarrow$ L.H.S $\to 0$

$\Rightarrow T - \frac{1}{\bar{g}}(\bar{y})$

$T = \left[\bar{f}'(\bar{x})\right]^{-1}$

$= \left[\bar{f}'(\bar{g}(y))\right]^{-1}, \bar{y} \in V$

Then $\bar{g}$ is a continuous mapping of V onto U
(since $\bar{g}$ is differentiable)

that $\frac{1}{f}$ is a continuous mapping of U into the set Ω of all invertible elements of $L(\mathbb{R}^n)$

By known theorem,

The inversion is continuous of Ω onto Ω

$\Rightarrow \bar{g}' \in \mathcal{C}'(V)$

Hence the theorem.

Theorem: let $\bar{f}$ is a $\mathcal{C}'$-mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and if $\bar{f}'(\bar{x})$ is invertible for every $\bar{x} \in E$ then $\bar{f}(w)$ is an open subset of $\mathbb{R}^n$ for every open set $w \in E$.

Proof: Given that $\bar{f}$ is a $\mathcal{C}'$-mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and $\bar{f}'(\bar{x})$ is invertible for every $\bar{x} \in E$

b) Let W be an open subset of E

Let $\bar{y} \in \bar{f}(W)$ then $\bar{y} = \bar{f}(\bar{x})$ for some $\bar{x} \in W$

Since W is open, $\exists$ a neighbourhood U of $\bar{x}$ such that $U \subset W$

By inverse function theorem,

$\exists$ open sets U and V in $\mathbb{R}^n \ni \bar{x} \in U$, $\bar{y} = \bar{f}(\bar{x}) \in V$

$\bar{f}$ is one-to-one on U and $\bar{f}(U) = V$ since $\bar{y} \in V$ and V is open

| an open sphere $S_\epsilon(\bar{y}) \subseteq V = \bar{f}(U) \subset \bar{f}(W)$

$\Rightarrow \bar{y}$ is an interior point of $\bar{f}(W)$ and

hence $\bar{f}(W)$ is open set of $\mathbb{R}^n$

Unit-IV

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Notation:- If $\bar{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\bar{y} = (y_1, y_2, \dots, y_m) \in \mathbb{R}^m$ let us write $(\bar{x}, \bar{y})$ for the point (or vector) $(x_1, x_2, \dots, x_n, y_1, \dots, y_m) \in \mathbb{R}^{n+m}$ then the entry in $(\bar{x}, \bar{y})$ ie., $\bar{x} \in \mathbb{R}^n$ and second entry in $(\bar{x}, \bar{y})$ ie., $\bar{y} \in \mathbb{R}^m$

Every $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ can be split in to two linear transformation A_x and A_y defined by $A_x \bar{h} = A(\bar{h}, \bar{0})$, $A_y \bar{k} = A(\bar{0}, \bar{k})$ for any $\bar{h} \in \mathbb{R}^n$, $\bar{k} \in \mathbb{R}^m$ then $A_x \in L(\mathbb{R}^n)$, $A_y \in L(\mathbb{R}^m, \mathbb{R}^n)$ and $A(\bar{h}, \bar{k}) = A(\bar{h}, \bar{0}) + A(\bar{0}, \bar{k})$

$$= A_x \bar{h} + A_y \bar{k}$$

Theorem:-

If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and if A is invertible then there corresponds to every $\bar{k} \in \mathbb{R}^n$ a unique $\bar{h} \in \mathbb{R}^n$ such that $A(\bar{h}, \bar{k}) = \bar{0}$ thus $\bar{h}$ can be computed from $\bar{k}$ by the formula $\bar{h} = -(A_x)^{-1} \cdot A_y \bar{k}$.

Proof: Suppose $A(\bar{h}, \bar{k}) = 0$

we have $A(\bar{h}, \bar{k}) = A(\bar{h}, \bar{0}) + A(\bar{0}, \bar{k}) = A_x \bar{h} + A_y \bar{k}$

$$A_x \bar{h} + A_y \bar{k} = \bar{0}$$

$$A_x \bar{h} = -A_y \bar{k}$$

$$\bar{h} = -(A_x)^{-1} A_y \bar{k}$$

$A(\bar{h}_1, \bar{k}) = \bar{0}$ and $A(\bar{h}_2, \bar{k}) = \bar{0}$

$\Rightarrow A(\bar{h}_1, \bar{k}) = A(\bar{h}_2, \bar{k})$

$\Rightarrow A_x \bar{h}_1 + A_y \bar{k} = A_x \bar{h}_2 + A_y \bar{k}$

$\Rightarrow A_x \bar{h}_1 = A_x \bar{h}_2$

$\Rightarrow A_x(\bar{h}_1 - \bar{h}_2) = 0 \Rightarrow \bar{h}_1 - \bar{h}_2 = \bar{0}$

$\Rightarrow \bar{h}_1 = \bar{h}_2$

Theorem:-

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The Implicit function theorem:- (II - 5M)

(12) Let $\bar{f}$ be a C'-mapping of an open set $E \subset \mathbb{R}^{n+m}$ into $\mathbb{R}^n$ such that $\bar{f}(\bar{a}, \bar{b}) = \bar{0}$ for some point $(\bar{a}, \bar{b}) \in E$ put $A = \bar{f}'(\bar{a}, \bar{b})$

and assume that A_x is invertible. then $\exists$ open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^n$ with $(\bar{a},\bar{b}) \in U$ and $\bar{b} \in W$ having the following property:

To every $\bar{y} \in W$ corresponds a unique $\bar{x} \ni (\bar{x},\bar{y}) \in U$ and $\bar{f}(\bar{x},\bar{y}) = \bar{0}$

if this $\bar{x}$ is defined to be $\bar{g}(\bar{y})$ then $\bar{g}$ is a C'-mapping of W into $\mathbb{R}^n$,

$g(\bar{b}) = \bar{a}$, $(\bar{f}(\bar{g}(\bar{y})),\bar{y}) = \bar{0} \; (\bar{y} \in W)$ and $\bar{g}'(\bar{b}) = -(A_x)^{-1} A_y$

Proof: Define $\bar{F}: E \to \mathbb{R}^{n+m}$ by $\bar{F}(\bar{x},\bar{y}) = (\bar{f}(\bar{x},\bar{y}),\bar{y}) \;\; \forall (\bar{x},\bar{y}) \in E$

Since $\bar{f}$ is C' mapping, $\bar{F}$ is also C'-mapping

first we prove that $\bar{F}'(\bar{a},\bar{b})$ is an invertible element of $L(\mathbb{R}^{n+m})$

$$\bar{f}(\bar{a}+\bar{h},\bar{b}+\bar{k}) = \bar{f}(\bar{a}+\bar{h},\bar{b}+\bar{k}) - \bar{f}(\bar{a},\bar{b})$$
$$= \bar{f}'(\bar{a},\bar{b})(\bar{h},\bar{k}) + r(\bar{h},\bar{k})$$
$$= A(\bar{h},\bar{k}) + r(\bar{h},\bar{k})$$

where $r(\bar{h},\bar{k})$ is the remainder in the definition of $\bar{f}'(\bar{a},\bar{b})$

Now, $\bar{F}(\bar{a}+\bar{h},\bar{b}+\bar{k}) - \bar{F}(\bar{a},\bar{b}) = \bar{f}((\bar{a}+\bar{h},\bar{b}+\bar{k}),\bar{b}+\bar{k}) - (\bar{f}(\bar{a},\bar{b}),\bar{b})$

$$= (\bar{f}(\bar{a}+\bar{h},\bar{b}+\bar{k}) - \bar{f}(\bar{a},\bar{b}),\; \bar{b}+\bar{k}-\bar{b})$$
$$= (A(\bar{h},\bar{k}) + r(\bar{h},\bar{k}),\bar{k}) = (A(\bar{h},\bar{k}) + r(\bar{h},\bar{k}),\; \bar{k}+\bar{0})$$
$$= (A(\bar{h},\bar{k}),\bar{k}) + (r(\bar{h},\bar{k}),\bar{0})$$

$\Rightarrow \bar{F}'(\bar{a},\bar{b})$ is a linear operator on $\mathbb{R}^{n+m}$ that maps $(\bar{h},\bar{k})$ to $(A(\bar{h},\bar{k}),\bar{k})$

If this image vector is $\bar{0}$, then $A(\bar{h},\bar{k}) = \bar{0}$ and $\bar{k} = \bar{0}$, we have

$$A(\bar{h},\bar{k}) = A_x\bar{h} + A_y\bar{k} \Rightarrow A_x\bar{h} + A_y\bar{k} = \bar{0}$$
$$\Rightarrow A_x\bar{h} = \bar{0}, \quad A_y\bar{k} = \bar{0}$$
$$\Rightarrow \bar{h} = \bar{0}, \quad \bar{k} = \bar{0}$$

$\therefore \bar{F}'(\bar{a},\bar{b})(\bar{h},\bar{k}) = (\bar{0},\bar{0}) \Rightarrow (\bar{h},\bar{k}) = (\bar{0},\bar{0})$

$\Rightarrow \bar{F}'(\bar{a},\bar{b})$ is one-one

$\Rightarrow \bar{F}'(\bar{a},\bar{b})$ is onto

$\Rightarrow \bar{F}'(\bar{a},\bar{b})$ is invertible

Since $\bar{F}'$ is a C'-mapping and $\bar{F}'(\bar{a},\bar{b})$ is invertible we can apply, inverse function theorem to F

$\exists$ open sets U and V in $\mathbb{R}^{n+m}$ $\ni$ $(\bar{a},\bar{b}) \in U$ and $\bar{F}(\bar{a},\bar{b}) = (\bar{f}(\bar{a},\bar{b}),\bar{b})$

$= (\bar{a},\bar{b}) \in V$

$\bar{F}$ is one-to-one on U and $\bar{F}(U) = V$

Let $W = \{\bar{y} \in \mathbb{R}^m / (\bar{0},\bar{y}) \in V\}$

W is non-empty (since $(\bar{0},\bar{b}) \in V$)

Since V is open, W is open

If $\bar{y} \in W$ then $(\bar{0},\bar{y}) \in V = \bar{F}(U)$

So that $(\bar{0},\bar{y}) = \bar{F}(\bar{x},\bar{y})$ for some $(\bar{x},\bar{y}) \in U$

$\Rightarrow (\bar{0},\bar{y}) = (\bar{f}(\bar{x},\bar{y}),\bar{y})$

$\Rightarrow \bar{f}(\bar{x},\bar{y}) = \bar{0}$ for some $(\bar{x},\bar{y}) \in U$

we claim that $\bar{x}$ is unique

Suppose with the same $\bar{y}$ that $(\bar{z},\bar{y}) \in U$ and $\bar{f}(\bar{z},\bar{y}) = \bar{0}$

Now, $\bar{F}(\bar{z},\bar{y}) = (\bar{f}(\bar{z},\bar{y}),\bar{y}) = (\bar{0},\bar{y}) = (f(\bar{x},\bar{y}),\bar{y}) = \bar{F}(\bar{x},\bar{y})$

$\Rightarrow = \bar{x}$ ($\because \bar{F}$ is invertible)

Define $\bar{g}: W \to \mathbb{R}^n \ni (\bar{g}(\bar{y}),\bar{y}) \in U$ and $\bar{f}(\bar{g}(\bar{y}),\bar{y}) = \bar{0}$

Then $\bar{F}(\bar{g}(\bar{y}),\bar{y}) = (\bar{f}(\bar{g}(\bar{y}),\bar{y}),\bar{y}) = (\bar{0},\bar{y})$ —— ①

In particular, $\bar{F}(\bar{g}(\bar{b}),\bar{b}) = (\bar{0},\bar{b})$

But $\bar{F}(\bar{a},\bar{b}) = (\bar{f}(\bar{a},\bar{b}),\bar{b}) = (\bar{0},\bar{b})$

$\Rightarrow \bar{F}(\bar{a},\bar{b}) = \bar{F}(\bar{g}(\bar{b}),\bar{b})$

$\Rightarrow \bar{g}(\bar{b}) = \bar{a}$ ($\because$ F is one-one)

Since F is invertible, let $\bar{G}$ be the inverse of $\bar{F}'$ then $\bar{G} \in \mathcal{C}'$

from ①, $(\bar{g}(\bar{y}),\bar{y}) = \bar{G}(\bar{0},\bar{y})$

Since $\bar{G}$ is $\mathcal{C}'$-mapping, $\bar{g} \in \mathcal{C}'$

Define $\phi(\bar{y}) = (\bar{g}(\bar{y}),\bar{y})$

$\bar{g}(\bar{y}+\bar{k}) - \bar{g}(\bar{y}) = \bar{g}'(\bar{y})\bar{k} + r(\bar{k})$ —— (*)

where $\frac{r(\bar{k})}{k} \to 0$ as $\bar{k} \to \bar{0}$

Now, $\phi(\bar{y}+\bar{k}) - \phi(\bar{y}) = (\bar{g}(\bar{y}+\bar{k}), \bar{y}+\bar{k}) - (\bar{g}(\bar{y}), \bar{y})$

$= (\bar{g}(\bar{y}+\bar{k}) - \bar{g}(\bar{y}), \bar{y}+\bar{k}-\bar{y})$

$= (\bar{g}'(\bar{y})(\bar{k}) + r(\bar{k}), \bar{k})$

$= (\bar{g}'(\bar{y})\bar{k}, \bar{k}) + (r(\bar{k}), \bar{0})$

This shows that $\phi(\bar{y})$ is a linear mapping of $\mathbb{R}^m$ into $\mathbb{R}^{n+m}$

$\Rightarrow \phi'(\bar{y})\bar{k} = (\bar{g}'(\bar{y})(\bar{k}), \bar{k})$ $\forall\, \bar{y} \in W$ and $\bar{k} \in \mathbb{R}^m$ —— (*)

Now, $\bar{f}(\phi(\bar{y})) = \bar{f}(\bar{g}(\bar{y}), \bar{y}) = \bar{0}$

By chain rule, $\bar{f}'(\phi(\bar{y}))\cdot\phi(\bar{y}) = \bar{0}$ —— (2)

Take $\bar{y} = \bar{b}$ we get $\phi(\bar{b}) = (\bar{g}(\bar{b}), \bar{b}) = (\bar{a}, \bar{b})$

and $\bar{f}'(\phi(\bar{b})) = \bar{f}'(\bar{a}, \bar{b}) = A$

From (2), $A\cdot\phi'(\bar{b}) = \bar{0}$

or any $\bar{k} \in \mathbb{R}^m$, $A\cdot\phi'(\bar{b})\bar{k} = \bar{0}$

$\Rightarrow A(\bar{g}'(\bar{b})\cdot\bar{k}, \bar{k}) = \bar{0}$ $\quad$ ($\because$ by (*))

$\Rightarrow A_x\cdot\bar{g}'(\bar{b})\bar{k} + A_y\bar{k} = \bar{0}$

This is true for every $\bar{k}$, $\Rightarrow A_x\bar{g}'(\bar{b}) + A_y = \bar{0}$

$\Rightarrow A_x\bar{g}'(\bar{b}) = -A_y$

$\Rightarrow \bar{g}'(\bar{b}) = -(A_x)^{-1}A_y$

The Rank Theorem

Def: Suppose X and Y are vector spaces and $A \in L(X,Y)$. The nullspace of A denoted by $N(A)$ is defined as the set $N(A) = \{x \in X \mid Ax = 0\}$

The range of A is defined as the set $R(A) = \{Ax \mid x \in X\}$

Note:

① $N(A)$ is a vector space in X

Pf clearly $N(A) \subseteq X$

let $x_1, x_2 \in N(A)$ and c_1, c_2 are scalars

Then $Ax_1 = 0$ and $Ax_2 = 0$

Now $A(c_1x_1 + c_2x_2) = c_1Ax_1 + c_2Ax_2 = c_1(0) + c_2(0)$

$\Rightarrow c_1x_1 + c_2x_2 \in N(A)$

② $R(A)$ is a vector space in y.

Pf clearly $R(A) \subseteq y$

let $Ax_1, Ax_2 \in R(A)$ and c_1, c_2 are scalars

$\Rightarrow x_1, x_2 \in X \Rightarrow c_1x_1 + c_2x_2 \in X$ $[\because A \in L(x,y)]$

Then $c_1A_1x_1 + c_2Ax_2 = A(c_1x_1 + c_2x_2)$

$\Rightarrow c_1A_1x_1 + c_2A_2x_2 \in R(A)$

Def: let $A \in L(x,y)$. The rank of A is define as the dimension of $R(A)$.

Notes: ① The invertible elements of $L(\mathbb{R}^n)$ are precisely those whose rank is n.

② If $A \in L(x,y)$ and A has rank 0, Then $Ax = 0$ $\forall x \in A$

hence $N(A) = X$

Def: let x be a vector space. An operator $P \in L(X)$ is said to be a projection in X if $P^2 = P$.

Note: If P is a projection in X, then every $x \in X$ has a unique representation of the form $x = x_1 + x_2$ where $x_1 \in R(P)$, $x_2 \in N(P)$.

Pf let $P \in L(X)$ be projection

let $x \in X$

Put $x_1 = Px$, $x_2 = x_2 - x_1$

clearly, $x = x_1 + x_2$, $x_1 \in R(P)$, $x_2 \in N(P)$

$\left[\because Px_2 = P(x - x_1) = P(Px) - Px_1 = Px_1 - Px_1 = 0\right]$

Suppose $x = x_1 + x_2 = y_1 + y_2$

where $x_1, y_1 \in R(P)$ & $x_2, y_2 \in N(P)$

Now $Px = P(x_1 + x_2) = Px_1 + Px_2 = Px_1 + 0 = x_1$

$\left[\because x_2 \in N(P) \Rightarrow Px_2 = 0 \quad \therefore x_1 \in R(P) \Rightarrow x_1 = Px \text{ for some } x \in X \Rightarrow Px_1 = P(Px) = P^2x = Px = x_1\right]$

Similarly, we have $Px = y_1$

So $x_1 = y_1$ Hence $x_2 = y_2$

2) If X is a f.d.v.s and if X_1 is a v.s in X,
Then there is a projection P in X with $R(P) = X_1$ (15)

3) Every $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ is differentiable on $\mathbb{R}^n$ and $A'(x) = A$ $\forall x \in \mathbb{R}^n$

Sol: Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$

let $x \in \mathbb{R}^n$ and $r(h) = A(x+h) - Ax - Ah$

$= A(x+h-x-h)$ $(\because A$ is linear$)$

$= A(0) = 0$

So $\lim_{h\to 0} \frac{|r(h)|}{|h|} = \lim_{h \to 0} 0 = 0$

ie, $\lim_{h\to 0} \left| \frac{A(x+h) - Ax - Ah}{h} \right| = 0$

So A is differentiable at x and $A'(x) = A$

Theorem: Rank theorem: (I-4M)

Suppose m, n, r are non-negative integers $m \geq r$, $n \geq r$, F is a $\mathcal{C}'$ mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$ and $F'(x)$ has rank r for every $x \in E$. Fix $a \in E$, put $A = F'(a)$, let Y_1 be the range of A, and let P be a projection in $\mathbb{R}^m$, whose range is Y_1. Let Y_2 be the null space of P.

Then there are open sets U and V in $\mathbb{R}^n$ with $a \in U$, $U \subset E$ and there is a 1-1 $\mathcal{C}'$-mapping H of V onto U (whose inverse is also of class $\mathcal{C}'$) such that $F(H(x)) = Ax + \phi(Ax)$ $(x \in V)$ where ϕ is a $\mathcal{C}'$ mapping of the open set $A(V) \subset Y_1$ into Y_2.

Proof:

Case (i): Suppose $r = 0$

Let $x \in E$

Then rank of $F'(x) = r = 0$

$\Rightarrow \dim R(F'(x)) = 0$

$\Rightarrow R(F'(x)) = \{0\}$

$\Rightarrow F'(x) = 0$

Thus, $F'(x) = 0$ $\forall x \in E$

Since $a \in E$ and E is open

$\Rightarrow a$ is an interior point of E

$\Rightarrow a \in U$ for some open ball $U \subseteq E$

Since balls are convex, U is convex and $F'(x) = 0 \;\; \forall x \in U$

by corollary, F is constant on U

Take $V = U$

Let $H: V \to U$ be the identity mapping

i.e., $Hx = x \;\; \forall x \in V$

clearly, $H \in \mathscr{C}'(V)$ and $H^{-1} \in \mathscr{C}'(V)$

Now $A = F'(a) = 0$ $(\because a \in E)$

So, $Y_1 = R(A) = \{0\}$

Now P is the projection in $\mathbb{R}^m$ with range $Y_1 = \{0\}$

and hence $P = 0$

So, $Y_2 =$ null space of $P = \mathbb{R}^m$

Now $A(V) = \{0\} \subseteq Y_1$

Define $\phi: A(V) \to Y_2$ by $\phi(Ax) = F(a) \;\; \forall Ax \in A(V)$

clearly, $\phi \in \mathscr{C}'(A(V))$ for any $x \in V$

$F(H(x)) = F(x) = F(a)$

and $Ax + \phi(Ax) = 0 + \phi(0) = F(a)$

Hence for all $x \in V$, $F(Hx) = Ax + \phi(Ax)$

case (ii): suppose $r > 0$

Since $Y_1 =$ range of $A \subseteq \mathbb{R}^m$

and since rank of $F'(a)$ is r, we have that $\dim Y_1 = r$

Let $\{y_1, y_2, \ldots, y_r\}$ be a basis of Y_1

since $Y_1 = A(\mathbb{R}^n)$, for each $i = 1, 2, \ldots, r$ then

choose $z_i \in \mathbb{R}^n$ $(1 \le i \le r)$ such that $Az_i = y_i$ $(1 \le i \le r)$

Define $S: Y_1 \to \mathbb{R}^n$ by $S(c_1 y_1 + c_2 y_2 + \cdots + c_r y_r) = c_1 z_1 + c_2 z_2 + \cdots + c_r z_r$

for any scalars $c_1, c_2, \ldots, c_r$

then $ASy_i = Az_i = y_i$ for $1 \le i \le r$

So, for any $y \in Y_1$, $ASy = y$ —— (1)

i.e., $AS: Y_1 \to \mathbb{R}^m$ is the inclusion mapping

Define $G: E \to \mathbb{R}^n$ by $Gx = x + Sp(F(x) - Ax)$

Now $G'(a) = I$, the identity operator of $\mathbb{R}^n$

clearly $G'(a)$ is invertible

By inverse mapping theorem, there exist open sets $U (\subseteq E)$ and $V \subseteq \mathbb{R}^n$ with $a \in U$ ∋ G is a 1-1 mapping of U onto V whose inverse H is also class $\mathscr{C}$.

By shrinking U and V if necessary, we can assume that V is convex and for any $x \in V$, $H'(x)$ is invertible for any $x \in \mathbb{R}^m$

$(PA)x = Ax$ and hence $Ax \in A(\mathbb{R}^n) = \text{Range of } A = Y_1$

and P is a projection in $\mathbb{R}^m$ with range Y_1 and hence $PA = A$ —— (2)

for any $x \in \mathbb{R}^m$, $PAx \in Y_1$ and hence $ASPAx = ASAx$ $[\because (2)]$

$= Ax$ $[\because (1)]$

$\Rightarrow ASPA = A$ —— (3)

for any $x \in E$, $AG(x) = A(x + Sp(F(x) - Ax))$ (by (1) & (3)) $(\because$ (3) & $PF(x) \in Y_1)$

$= Ax + AspF(x) - AspAx = Ax + PF(x) - Ax = PF(x)$ —— (4)

for any $x \in V$, $H(x) \in U \subseteq E$ and hence $PF(H(x)) = AG(H(x)) = Ax$ $(\because G = H^{-1})$ —— (5)

Define $\varphi: V \to \mathbb{R}^m$ and $\varphi(x) = F(H(x)) - Ax$

for any $x \in V$,

$P\varphi(x) = PF(H(x)) - PAx$ $[\because F \text{ is linear}]$

$= Ax - Ax = 0$

So, the range of $\varphi \subseteq$ The null space of $P = Y_2$

Thus $\varphi: V \to Y_2$ $[P(\varphi(x)) = 0, \varphi(x) \in N(P)]$

Clearly, for any $x \in V$

$\varphi'(x) = F'(H(x))H'(x) - A'(x)$

$= F'(H(x))H'(x) - A$ —— (6)

Since $H'(x) \in L(\mathbb{R}^n, \mathbb{R}^n)$, $F'(H(x)) \in L(\mathbb{R}^n, \mathbb{R}^m)$

we have $F'(H(x))H'(x) \in L(\mathbb{R}^n, \mathbb{R}^m)$

Since $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ we have that $\varphi'(x)$ is continuous for every $x \in V$

So, $\varphi'(x)$ is a $\mathscr{C}'$-mapping of V

Clearly, $A(V) \subseteq A(\mathbb{R}^n) = R(A) = Y_1$

To complete the proof, define

$\phi : A(V) \to Y_2$ by $\phi(A(x)) = \psi(x)$

i.e., $\phi(A(x)) = F(H(x)) - Ax$

i.e., $F(H(x)) = Ax + \phi(Ax)$ for all x

Now the theorem is complete if we prove that ϕ is a C^1-mapping of $A(V)$ into Y_2.

Step 1: ϕ is well-defined

Define $\Phi : V \to Y_1$ by $\Phi(x) = F(H(x))$

Now we prove the following claim

claim ①: To each $x \in V$, if $M =$ range of $\Phi'(x)$ then $P/_m : M \to Y_1$ is an isomorphism

Proof of claim: Let $x \in V$

Now $\Phi'(x) = F'(H(x))H'(x)$

Since $H(x)$ is invertible, rank of $H'(x) = n$

Since rank of $F'(H(x))$ is r [$\because$ rank of $F'(y)$ is r $\forall y \in V$]

($\because x \in V$)

$\Rightarrow$ rank of $\Phi'(x) =$ rank of $(F'(H(x))H'(x))$

$= r$

Let $M =$ range of $\Phi'(x)$

So, $M = \Phi'(x)(\mathbb{R}^n) \subseteq \mathbb{R}^m$ and $\dim M = r$

from ⑤, we have that for any $x \in V$

$PF(H(x)) = Ax$ i.e., $P\Phi(x) = Ax$

and hence $P'(\Phi(x))\Phi'(x) = A'(x)$ [$\because P$ & A are linear]

i.e., $P\phi'(x) = A$ —— ⑦

Thus $P\phi'(x)(\mathbb{R}^n) = A(\mathbb{R}^n)$

i.e. $P(m) = Y_1$

So $P/_m : M \to Y_1$ is linear and onto

Since $\dim M = r = \dim Y_1$, $P/_m : M \to Y_1$ is 1-1 & hence an isomorphism

Hence proof of the claim ① is complete.

Claim ②: $Ah = 0 \to \psi'(x)h = 0 \ \forall x \in V$

Proof of claim ②: Let $Ah=0$

Let $x \in V$

By ⑦, we have $P\Phi'(x) = A$ and hence $P\Phi'(x)h = Ah = 0$

So $\Phi'(x)h \in M$ = range of $\Phi'(x)$ and $P|_M(\Phi'(x)h) = 0$

By claim 1, $P|_M : M \to Y_1$ is 1-1 and hence $\Phi'(x)h = 0$

Now $\varphi'(x)h = \Phi'(x)h - Ah$ $(\because 6)$

$= 0 - 0$

Hence the proof of claim 2.

Claim 3: ϕ is well-defined

Let $x_1, x_2 \in V$ and $Ax_1 = Ax_2$

Put $h = x_2 - x_1$

Define g on $[0,1]$ by $g(t) = \varphi(x_1 + th)$

Since V is convex

$x + th = x_1 + t(x_2 - x_1) = (1-t)x_1 + tx_2 \in V$

So, the definition of g is well

Now $g'(t) = \varphi'(x_1 + th)h = 0$ $\forall t \in [0,1]$

$\Rightarrow g$ is constant

$\Rightarrow g(0) = g(1)$ $[\because h = x_2 - x_1 \Rightarrow x_2 = x_1 + h]$

i.e., $\varphi(x_1) = \varphi(x_2)$

Hence ϕ is well-defined.

Step 2: ϕ is a C'-mapping of $A(V)$:-

Fix $y_0 \in A(V)$

So $y_0 = Ax_0$ for some $x_0 \in V$

Define $f : Y_1 \to \mathbb{R}^n$ by $f(y) = x_0 + S(y - y_0)$

Clearly, f is continuous

Since V is open in $\mathbb{R}^n$, $f^{-1}(V) = W$ is open in $\mathbb{R}^n$,

$f(y_0) = x_0 \in V$ and hence $y_0 \in W$

Also $W \subseteq A(V)$

For this, let $y \in W$

So, $f(y) = x_0 + S(y - y_0) \in V$

Put $x = x_0 + S(y - y_0)$

Then $x \in V$ and $Ax = Ax_0 + AS(y - y_0)$ [$\because$ A is linear]

$= Ax_0 + (y - y_0) = y_0 + y - y_0 = y$ [$\because$ (i)]

and hence $y \in A(V)$

So $W \subseteq A(V)$

Let $y \in W$

So, $y = Ax$ for some $x \in V$

Infact, $x = x_0 + S(y - y_0)$

Now, $\phi(y) = \phi(Ax) = \varphi(x) = \varphi(x_0 + Sy - Sy_0)$

So, $\phi'(y) = \varphi'(x_0 + Sy - Sy_0)\, S'(y)$ [$\because$ chain rule]

$= \varphi'(x_0 + Sy - Sy_0)\, S$ [$\because$ S is linear]

Since φ' and S are continuous

$\Rightarrow \phi'(y)$ is continuous

Thus, to each $y \in W$, $\phi'(y)$ is continuous

Thus, to each $y \in A(V)$ there exists an open set $W_y \subseteq A(V)$ with $y \in W_y$ such that $\phi'(y)$ is continuous on W_y.

But $A(V) = \bigcup_{y \in A(V)} \{y\} \subseteq \bigcup_{y \in A(V)} W_y \subset A(V)$

$\Rightarrow A(V) = \bigcup \{W_y \mid y \in A(V)\}$

So, ϕ' is continuous on $A(V)$

So, ϕ' is a C'-mapping of $A(V)$

Hence the theorem

Determinants:

Definition: If $(j_1, j_2, \ldots, j_n)$ is an ordered, n-tuple of integers define

$$s(j_1, j_2, \ldots, j_n) = \prod_{p<q} \operatorname{sgn}(j_q - j_p)$$

where $\operatorname{sgn} x = 1$ if $x > 0$,

$\operatorname{sgn} x = -1$ if $x < 0$,

$\operatorname{sgn} x = 0$ if $x = 0$

Then $s(j_1, j_2, \ldots, j_n) = 1, -1$ or 0 and it changes sign if any two of the j's are interchanged

Let $[A]$ be the matrix of a linear operator A on $\mathbb{R}^n$, relative to the standard basis $\{e_1, e_2, \dots, e_n\}$ with entries $a(i,j)$ in the i^{th} row and j^{th} column. 13

The determinant of $[A]$ is defined to be the number

$$\det[A] = \sum s(j_1, \dots, j_n)\, a(1,j_1)\, a(2,j_2), \dots, a(n,j_n) \quad \text{—} ①$$

The sum in ① extends over all ordered n-tuples of integers $(j_1, j_2, \dots, j_n)$ with $1 \le j_r \le n$

The column vector x_j of $[A]$ are $x_j = \sum_{i=1}^{n} a(i,j) e_i \quad (1 \le j \le n)$

Note: Let $[A]$ as a function of the column vector of $[A]$. If we write $\det(x_1, x_2, \dots, x_n) = \det[A]$. det is now a real function on the set of all det is now a real function on the set of all ordered n-tuples of vectors in $\mathbb{R}^n$.

Theorem 1

(a) If I is the identity operator on $\mathbb{R}^n$ then $\det[I] = \det(e_1, e_2, \dots e_n) = 1$

(I - 3M)

Proof: Let I be the identity operator on $\mathbb{R}^n$

Put $A = I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$

then $a(i,i) = 1$ and $a(i,j) = 0$ for $i \ne j$

$\therefore$ The column vectors x_j of $[I]$ are

$$x_j = \sum_{i=1}^{n} a(i,j) e_j \quad (1 \le j \le n)$$

$$= a(j,j)\, e_j = 1 \cdot e_j = e_j$$

$\therefore \det[I] = \det(e_1, e_2, \dots, e_n)$

and $\det[I] = \sum s(j_1, j_2, \dots, j_n)\, a(1,j_1)\, a(2,j_2) \dots a(n,j_n)$

$= s(1,2, \dots, n)\, a(1,1)\, a(2,2) \dots a(n,n)$

$= 1 \qquad [\because a(i,j) = 0 \text{ if } i \ne j]$

b) det is a linear function of each of the column vector x_j, if the others are held fixed.

Sol: we know that $s(j_1, j_2, \ldots, j_n) = \prod_{p<q} \operatorname{sgn}(j_q - j_p)$

$= 0$ if any two of the j's are equal.

Each of the remaining n! products in

$\det[A] = \sum s(j_1, j_2, \ldots, j_n)\, a(1,j_1)\, a(2,j_2) \ldots a(n,j_n)$ contains exactly one factor from each column.

If $[A]_1$ is obtained from $[A]$ by interchanging two columns then $\det[A]_1 = -\det[A]$. (II - 3M)

Let $[A]_1$ is obtained from $[A]$ by interchanging two columns

Note that $s(j_1, j_2, \ldots, j_n)$ changes sign if any two of the j's are interchanged.

Now $\det[A] = \sum s(j_1, j_2, \ldots, j_n)\, a(1,j_1), \ldots a(n,j_n)$

Suppose j_1 and j_2 are interchanged

Then $s(j_1, \ldots, j_n) = -s(j_2, j_1, j_3, \ldots, j_n)$

$\Rightarrow \sum s(j_1, \ldots, j_n)\, a(1,j_1)\, a(1,j_2) \ldots a(1,j_n)$

$= -\sum s(j_2, j_1, j_3, \ldots, j_n)\, a(1,j_2)\, a(2,j_1) \ldots a(n,j_n)$

$\Rightarrow \det[A] = -\det[A]$

If $[A]$ has two equal columns, then $\det[A] = 0$

Suppose A has two equal columns

we know that, $s(j_1, j_2, \ldots j_n) = 0$ if any two of the j's are equal

Hence $\det[A] = \sum s(j_1, \ldots, j_n)\, a(1,j_1) \ldots a(n,j_n)$

$= 0$

Theorem: If $[A]$ and $[B]$ are $n \times n$ matrices, then

$\det([B][A]) = \det[B] \det[A]$ (III - 3M)

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Proof If $x_1, x_2, \ldots, x_n$ are the columns of $[A]$,
define $\Delta_B(x_1, x_2, \ldots, x_n) = \Delta_B[A] = \det([B][A])$
The columns of $[B][A]$ are the vectors $Bx_1, \ldots, Bx_n$
Thus $\Delta_B(x_1, x_2, \ldots, x_n) = \det([B][A])$
$= \det(Bx_1, Bx_2, \ldots, Bx_n)$

$\Rightarrow \Delta_B$ satisfies thereom (b), (c) and (d))

since $x_j = \sum_{i=1}^{n} a(i,j) e_i \quad (1 \le j \le n)$

So, $\Delta_B[A] = \Delta_B(x_1, x_2, \ldots, x_n)$
$= \Delta_B\left(\sum_{i=1}^{n} a(i,1) e_i, x_2, \ldots, x_n\right)$
$= \sum_{i=1}^{n} a(i,1)\, \Delta_B(e_{i_1}, x_2, \ldots, x_n)$

Repeating this process with $x_2, \ldots, x_n$
we obtain $\Delta_B[A] = \sum a(i_1,1)\, a(i_2,2), \ldots, a(i_n,n)\, \Delta_B(e_{i_1}, e_{i_2}, \ldots e_{i_n})$ ——— ①

the sum being extended over all ordered n-tuples $(i_1, i_2, \ldots, i_n)$
with $1 \le i_n \le n$.

Also, $\Delta_B(e_{i_1}, e_{i_2}, \ldots, e_{i_n}) = t(i_1, i_2, \ldots, i_n)\, \Delta_B(e_1, e_2, \ldots, e_n)$ ——— ②
where $t = 1, 0$ (or) -1 and since $[B][I] = [B]$,
$\Delta_B(e_1, \ldots, e_n) = \Delta_B[I] = \det[B][I] = \det[B]$ ——— ③

substitute ③ & ② into ①, we obtain
$\Delta_B[A] = \sum a(i_1,1), \ldots, a(i_n,n)\, t(1, i_2, \ldots, i_n)\, \Delta_B(e_1, \ldots, e_n)$
$= \sum a(i_1,1), \ldots, a(i_n,n)\, t(i_1, i_2, \ldots, i_n) \det[B]$

$\Rightarrow \det[B][A] = \left\{\sum a(i_1,1) \ldots a(i_n,n)\, t(i_1, i_2, \ldots, i_n)\right\} \det[B]$ ——— ④

for all n by n matrices $[A]$ and $[B]$

Taking $B = I$ in ④, we get

$\det([I][A]) = \sum a(i_1,1)\cdots a(i_n,n)\, \varepsilon(i_1,i_2,\cdots i_n) \det[I]$

$\Rightarrow \det[A] = \sum a(i,1)\cdots a(i_n,n)\, \varepsilon(i_1,i_2,\cdots,i_n)$

$\{\because [I][A]=[A]$ and $\det[I]=1\}$

from (4),

$\det([B][A]) = \det[B]\cdot\det[A]$

Theorem:-

A linear operator A on $\mathbb{R}^n$ is invertible iff $\det[A] \neq 0$

Proof: Suppose $A \in L(\mathbb{R}^n)$ is invertible

then A^{-1} exists

$[A][A^{-1}] = [I] = [A^{-1}][A]$

Now, $\det[A]\det[A^{-1}] = \det[A][A^{-1}] = \det[I] = 1$

$\Rightarrow \det[A] \neq 0$

Now, we prove the converse in contrary way

Suppose A is not invertible

$\Rightarrow A$ is not a bijection

$\Rightarrow$ the columns $x_1,\cdots,x_n$ of $[A]$ are dependent

$\Rightarrow \exists k \ni x_k + \sum_{j\neq k} c_j x_j = 0$ —— (1) for certain scalars c_j

By theorem, (b) and (d), x_k can be replaced by $x_k + c_j x_j$ without altering the determinant if $j \neq k$

Repeating, we see that x_k can be replaced by the left side of (1)

i.e., by 0, without altering the determinant.

But a matrix which has 0 for one column has determinant 0.

Hence $\det[A] = 0$

Jacobians: If f maps on open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and if f is differentiable at a point $x \in E$, the determinant of the linear operator $f'(x)$ is called the Jacobian of f at x

In symbols $J_f(x) = \det f'(x)$

we shall also use the notation

$$\frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)} \text{ for } J_f(x), \text{ if } (y_1, \dots, y_n) = f(x_1, \dots, x_n)$$

Derivatives of Higher Order :-

Def: Suppose f is a real function defined in an open set $E \subset \mathbb{R}^n$, with Partial derivatives $D_1f_1, \dots, D_nf_n$. If the functions D_jf are themselves differentiable, then the second-order partial derivatives of f are defined by $D_{ij}f = D_iD_jf$ $(i, j = 1, \dots, n)$

If all these functions $D_{ij}f$ are continuous in E we say that f is of class $\mathscr{C}''$ in E, & that $f \in \mathscr{C}''(E)$.

A mapping f of E into $\mathbb{R}^m$ is said to be of class $\mathscr{C}''$ if each component of f is of class $\mathscr{C}''$

Theorem:-

Mean Value theorem:-

Suppose f is defined in an open set $E \subset \mathbb{R}^n$ and D_1f and $D_{2}f$ exists at every point of E. Suppose $Q \subset E$ is a closed rectangle with sides Parallel to the coordinate axes, having (a,b) and $(a+h, b+k)$ as opposite vertices $(h \neq 0, k \neq 0)$ put $\Delta(f,Q) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$ then there is a point (x,y) in the interior of Q such that

$\Delta(f,Q) = hk(D_{21}f)(x,y)$.

Proof: Define $u : [a, a+h] \to \mathbb{R}$ by $u(t) = f(t, b+k) - f(t,b)$

then $u(a+h) - u(a) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$

$= \Delta(f,Q)$

Hence $\Delta(f,Q) = u(a+h) - u(a)$

$= hu'(x)$ for some $x \in (a, a+h)$

$= h[D_1f(x, b+k) - (D_1f)(x,b)]$

$= hk(D_{2}f)(x,y)$ for some $y \in (b, b+k)$

Hence the theorem.

Corollary:-

$$D_{21}f = D_{12}f \text{ if } f \in \mathscr{C}''(E)$$

Proof: Put $A = D_{21}f(a,b)$

choose $\epsilon > 0$

let $Q \in E$ is closed rectangle with sides parallel the coordinate axis, having (a,b) and $(a+h, b+k)$ is opposite vertices $(h \neq 0, k \neq 0)$

If h and k are sufficiently small,

we have $|A - (D_{21}f)(x,y)| < \epsilon \quad \forall (x,y) \in Q$

Thus $\left| \dfrac{\Delta(f,Q)}{hk} - A \right| < \epsilon$

fix h, and let $k \to 0$

Since $D_2 f$ exists in E

$$\left| \frac{D_2 f(a+h,b) - D_2 f(a,b)}{h} - A \right| \leq \epsilon \quad \text{—— (1)}$$

since ϵ was arbitrary and since (1) holds for all sufficiently small $h \neq 0$, it follows that $(D_{12}f)(a,b) = A$

$$\Rightarrow D_{12}f(a,b) = D_{21}f(a,b)$$

$$D_{12}f = D_{21}f$$

*Exercises:

If $f(0,0) = 0$ and $f(x,y) = \dfrac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$ prove that $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ exist at every point of R^2 although f is not continuous at $(0,0)$. (III – 4M)

Proof: $(D_1 f)(x,y) = \lim\limits_{h \to 0} \dfrac{f(x+h,y) - f(x,y)}{h} = \lim\limits_{h \to 0} \dfrac{1}{h}\left[f(x+h,y) - f(x,y)\right]$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(x+h)y}{(x+h)^2+y^2} - \frac{xy}{x^2+y^2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(x^2+y^2)(x+h)y - xy\left((x+h)^2+y^2\right)}{\left((x+h)^2+y^2\right)(x^2+y^2)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{\frac{x^3y + x^2hy + xy^3 + hy^3 - x^3y - 2hx^2y - h^2xy - xy^3}{((x+h)^2+y^2)(x^2+y^2)}\right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^2hy + hy^3 - 2hx^2y - h^2xy}{((x+h)^2+y^2)(x^2+y^2)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{hy^3 - hx^2y - xyh^2}{((x+h)^2+y^2)(x^2+y^2)}\right]$$

$$= \lim_{h\to 0} \left[\frac{y^3 - x^2y - xyh}{((x+h)^2+y^2)(x^2+y^2)}\right]$$

$$= \frac{y^3 - x^2y}{(x^2+y^2)^2}$$

Also $(D_1 f)(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$

$$(D_2 f)(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x(y+h)}{x^2+(y+h)^2} - \frac{xy}{x^2+y^2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x(y+h)(x^2+y^2) - xy(x^2+(y+h)^2)}{(x^2+(y+h)^2)(x^2+y^2)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^3y + xy^3 + x^3h + xhy^2 - x^3y - xy^3 - xyh^2 - 2xy^2h}{(x^2+(y+h)^2)(x^2+y^2)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^3h - xhy^2 - xyh^2}{(x^2+(y+h)^2)(x^2+y^2)}\right]$$

$$= \lim_{h\to 0} \left[\frac{x^3 - xy^2 - xyh}{(x^2+(y+h)^2)(x^2+y^2)}\right]$$

$$= \frac{x^3 - xy^2}{(x^2+y^2)^2}$$

Also $(D_2 f)(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h}$

Hence $(D_1 f)(x,y)$, $(D_2 f)(x,y)$ exists for all $x,y \in \mathbb{R}^2$

since $f(x,y) = \begin{cases} 1/2 & \text{if } x = y,\ (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

f takes the values $1/2$ at each point on the real line $x = y$ except at the origin.

Hence f is not continuous at $(0,0)$

Problem:-

If $f(0,0) = 0$ and $f(x,y) = \frac{x^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$ prove that $D_1 f$ and $D_2 f$ are bounded functions on $\mathbb{R}^2$.

Sol $(D_1 f)(x,y) = \frac{(x^2+y^2)3x^2 - x^3(2x)}{(x^2+y^2)^2}$

$= \frac{3x^4 + 3x^2y^2 - 2x^4}{(x^2+y^2)^2} = \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2} \quad \forall (x,y) \in \mathbb{R}^2$

$(D_2 f)(x,y) = \frac{-2r^4 \cos^3\theta \sin\theta}{r^4}$

$= -2\cos\theta \sin\theta = -\cos^2\theta \sin\theta = -\left(\frac{1+\cos 2\theta}{2}\right) \sin 2\theta$

$= \frac{-1}{2} \sin 2\theta - \frac{1}{2} \sin 2\theta \cos 2\theta$

$= \frac{-1}{2} \sin 2\theta - \frac{2}{4} \sin 2\theta \cos 2\theta$

$= \frac{-1}{2} \sin 2\theta - \frac{1}{4} \sin 4\theta$

$|D_2 f(x,y)| = \left| \frac{-1}{2} \sin 2\theta - \frac{1}{4} \sin 4\theta \right|$

$\leq \frac{1}{2} + \frac{1}{4}$

$= \frac{3}{4}$

Problem:-

Define f in $\mathbb{R}^3$ by $f(x, y_1, y_2) = x^2 y_1 + e^x + y_2$ show that $f(0,1,-1) = 0$, ~~$(D_1 f)(0,1,-1)$~~ $(D_1 f)(0,1,-1) \neq 0$ and that $\exists$ the differential function g in some nbd of $(1,-1)$ in $\mathbb{R}^2 \ni g(1,-1) = 0$ and $f(g(y_1, y_2), y_1, y_2) = 0$ find $(D_1 g)(1,-1)$ $(D_2 g)(1,-1)$

Proof: Given that $f(x, y_1, y_2) = x^2 y_1 + e^x + y_2$

$$f(0,1,-1) = 0+1-1 = 0$$

$$D_1 f(x, y_1, y_2) = 2x y_1 + e^x$$

$$D_1 f(0,1,-1) = 0+1 = 1 \neq 0$$

$$\frac{\partial f}{\partial x} = 2x y_1 + e^x = 0+1 = 1$$

$$\frac{\partial f}{\partial y_1} = x^2 = 0.$$

$$\frac{\partial f}{\partial y_2} = 1$$

$$A = [1\ 0\ 1]$$

$$A_x = [1],\ A_y [0, 1]$$

Since $|A_x| \neq 0$

$$g'(b) = -(A_x)^{-1}(A_y)$$

$$= -[1][0,1]$$

$$= [0,-1]$$

$$= \left[\frac{\partial g}{\partial y_1}, \frac{\partial g}{\partial y_2}\right]$$

$$(D_1 g)(1,-1) = 0$$

$$(D_2 g)(1,-1) = 1$$

Imp Problem:

Let $f = (f_1, f_2)$ be a mapping of $\mathbb{R}^5$ into $\mathbb{R}^2$ given by $f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2 y_1 - 4y_2 + 3$ and $f_2(x_1, x_2, y_1, y_2, y_3) = x_2 \cos x_1 - 6x_1 + 2y_1 - y_3$. Let $a = (0,1)$, $b = (3,2,7)$ then $f(a,b) = a$. Find the matrix of $A = f'(a,b)$ with respect to standard basis.

Sol: Given that, $f = (f_1, f_2)$ be a mapping of $\mathbb{R}^5$ into $\mathbb{R}^2$ given by

$$f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2 y_1 - 4y_2 + 3$$

$$f_2(x_1, x_2, y_1, y_2, y_3) = x_2 \cos x_1 - 6x_1 + 2y_1 - y_3$$

and also $a = (0,1)$, $b = (3,2,7)$ then $f(a,b) = 0$ the matrix of the with respect to the standard basis, the transformation $A = f'(a,b)$ is

$$A = \begin{bmatrix} 2 & 3 & 1 & -4 & 0 \\ -6 & 1 & 2 & 0 & -1 \end{bmatrix}$$

Hence, $[A_x] = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}$, $[A_y] = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}$

we see that the column vectors of $[A_x]$ are independent.

Hence A_x is invertible and the existence of a C'-mapping g, defined in a neighbourhood of $(3,2,7)$ such that

$g(3,2,7) = (0,1)$ and $(f(g(y)), y) = 0$

we can use $g'(b) = -(A_x)^{-1} A_y$ —— (*)

to compute $g'(3,2,7)$

$$[(A_x)^{-1}] = [A_x]^{-1} = \frac{1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix} \qquad \left[\because A^{-1} = \frac{1}{|A|}\operatorname{adj} A\right]$$

by (*), we have

$$[g'(3,2,7)] = \frac{-1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & 1/5 & -3/20 \\ -1/2 & 6/5 & 1/10 \end{bmatrix}$$

In terms of partial derivatives the conclusion is that

$D_1 g_1 = 1/4$; $D_2 g_1 = 1/5$; $D_3 g_1 = -3/20$

$D_1 g_2 = -1/2$; $D_2 g_2 = 6/5$; $D_3 g_3 = 1/10$