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E – CONTENT

PAPER: M 301,

FUNCTIONAL ANALYSIS

M. Sc. II YEAR, SEMESTER - III

UNIT – I : BANACH SPACES

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FUNCTIONAL ANALYSIS
Unit I
Banach Space

The definition and some examples

Definition: Let N be a linear space over field K (where K is either $\mathbb{R}$ the field of real numbers or $\mathbb{C}$ the field of complex numbers). A function $\|\,.\,\| : N \to \mathbb{R}$ is said to be a *norm* on N if it satisfies the following conditions

(i) $\|x\| \geq 0 \ \forall\, x \in N$ (non-negativity)

(ii) $\|x\| = 0$ iff x = 0.

(iii) $\|x+y\| \leq \|x\| + \|y\| \ \forall$ x, y ∈ N (triangle inequality).

(iv) $\|\alpha x\| = |\alpha|\|x\| \ \forall$ x ∈ N, $\alpha \in$ K.

A linear space N over a field K with a norm $\|\,.\,\|$ defined on N is called a *normed linear space* over K.

Result: Every normed linear space N is a metric space with respect to metric d defined by d(x, y) = $\|x-y\|$ ∀ x, y ∈ N.

Proof: Let N be a normed linear space. Let x, y ∈ N.

Then (i) d(x, y) = $\|x-y\| \geq 0$ and d(x, y) = 0 iff $\|x-y\| = 0$ iff x – y = 0 iff x = y.

(ii) d(x, y = $\|x-y\| = \|-(y-x)\| = \|y-x\|$ = d(y, x)

(iii) Let x, y, z ∈ N. Then d(x, y) = $\|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\|$ = d(x, z) + d(z, y). Hence every normed linear space is a metric space.

Definitions: Let (N, $\|\,.\,\|$) be a normed linear space.

(i) A sequence $\{x_n\} \subseteq$ N is said to be convergent to an element x_0 if for each ε > 0 ∃ a positive integer n_0 such that $\|x_n - x_0\| < \varepsilon \ \forall\, n \geq n_0$.

(ii) A sequence $\{x_n\} \subseteq$ N is said to be a Cauchy sequence if for each ε > 0 ∃ a positive integer n_0 such that $\|x_n - x_m\| < \varepsilon \ \forall\, n, m \geq n_0$.

(iii) The space N is said to be complete if every Cauchy sequence in N converges to an element of N.

Theorem 1: Let (N, $\|\,.\,\|$) be a normed linear space.

Then (a) $\big|\|x\| - \|y\|\big| \leq \|x-y\| \ \forall\, x, y \in N.$

(b) $\big|\|x\| - \|y\|\big| \leq \|x+y\| \ \forall\, x, y \in N.$

(c) norm is a real valued continuous function. Ie. $x_n \to x \Rightarrow \|x_n\| \to \|x\|$.

(d)1*: addition and scalar multiplication are joint continuous.

Proof: $\|x\| = \|(x-y)+y\| \leq \|x-y\| + \|y\|$

$\Rightarrow \|x\| - \|y\| \leq \|x-y\|$ … (i)

Again – $(\|x\| - \|y\|) = \|y\| - \|x\| \leq \|y - x\|$ by (i)

$= \| -(x-y)\| = \|x - y\|$

Ie – $(\|x\| - \|y\|) \leq \|x - y\| \ldots$(ii)

Suppose $\|x\| \geq \|y\|$. Then $|\|x\| - \|y\|| = \|x\| - \|y\| \leq \|x - y\|$ by (i)

Suppose $\|x\| < \|y\|$. Then $|\|x\| - \|y\|| = -(\|x\| - \|y\|) \leq \|x - y\|$ by (ii)

Thus $|\|x\| - \|y\|| = \|x\| - \|y\|$ or $-(\|x\| - \|y\|)$

In either case it is $\leq \|x - y\|$. Hence the result.

(b) Replace y by – y in (a) Then $|\|x\| - \|-y\|| \leq \|x - (-y)\|$

$\Rightarrow |\|x\| - \|y\|| \leq \|x + y\|$

__(c)__ Let N be a normed linear space and $\{x_n\}$ be a sequence in N converging to x in N.

Then by the above result $|\|x_n\| - \|x\|| \leq \|x_n - x\|$.

Now since $x_n \to x$, $\|x_n - x\| \to 0 \Rightarrow |\|x_n\| - \|x\|| \to 0. \Rightarrow \|x_n\| \to \|x\|$. Hence the result.

(d) Let $\{x_n\}$ and $\{y_n\}$ be sequences in N ∋ $x_n \to x$ in N and $y_n \to y$ in N.

Now $\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leq \|x_n - x\| + \|y_n - y\|$ …(i)

Since $x_n \to x$ and $y_n \to y$, $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ so that RHS

$\|x_n - x\| + \|y_n - y\| \to 0 + 0 = 0$.

$\therefore$ from (i) $\|(x_n + y_n) - (x + y)\| \to 0$

$\Rightarrow \|x_n + y_n\| \to \|x + y\|$

$\therefore$ addition is jointly continuous

Let $\{\alpha_n\}$ be a sequence in F and $\{x_n\}$ be a sequence in N ∋ $\alpha_n \to \alpha$ in F and $x_n \to x$ in N.

Then $\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\|$

$= \|\alpha_n (x_n - x) + (\alpha_n - \alpha)x\|$

$\leq |\alpha_n| \|x_n - x\| + |\alpha_n - \alpha| \|x\|$

Since $\alpha_n \to \alpha$ and $x_n \to x$, $|\alpha_n - \alpha| \to 0$ and $\|x_n - x\| \to 0$ so that RHS $|\alpha_n| \|x_n - x\| + |\alpha_n - \alpha| \|x\| \to 0$ and hence LHS $\|\alpha_n x_n - \alpha x\| \to 0$ ie $\alpha_n x_n \to \alpha x$.

$\therefore$ Scalar multiplication is jointly continuous.

__**Definition (1*)**__: A *Banach space* is a complete normed linear space.

__**Theorem 2:**__ (8*): Let M be a closed linear subspace of a normed linear space N. If norm of a coset x + M in the quotient space $\frac{N}{M}$ is defined by $\|x + M\| = inf\ \{\|x + m\| : m \in M\}$, then $\frac{N}{M}$ is a normed linear space. Further, if N is a Banach space then so is $\frac{N}{M}$.

Proof: Let x + M $\in \frac{N}{M}$ where x $\in$ N. Then $\frac{N}{M}$ is a linear space.

Define $\|x+M\| = inf\ \{\|x+m\| : m \in M\}$.

(i) Since x + m $\in$N and N is a normed linear space, $\|x+m\| \geq 0\ \forall\ m \in M$.

$\therefore\ inf\ \{\|x+m\| : m \in M\} \geq 0 \Rightarrow \|x+M\| \geq 0\ \forall\ x \in N$.

(ii) Let x + M = M. Then x $\in$ M. $\therefore \|x+M\| = inf\{\|x+m\| : m \in M, x \in M\}$ $= \inf\{\|y\| : y \in M\} = 0$ since M being a subspace contains zero vector whose norm is real number zero.

Thus x + M = M $\Rightarrow \|x+M\| = 0$

Conversely $\|x+M\| = 0 \Rightarrow inf\ \{\|x+m\| : m \in M\} = 0$.

$\Rightarrow \exists$ a subsequence $\{m_k\}$ in M $\ni\ \|x+m_k\| \to 0$.

$\Rightarrow$ x $\in$ M $\Rightarrow$ x + M = M. $\Rightarrow$ x + M is the zero element of $\frac{N}{M}$.

(iii) Let x + M, y + M $\in \frac{N}{M}$ where x, y $\in$ N.

$\therefore \|x+M+y+M\| = \|x+y+M\| = inf\ \{\|x+y+m\| : m \in M\}$.

$= inf\ \{\|x+y+m'+m''\| : m = m'+m'' \in M\}$

$= inf\ \{\|x+m'+y+m''\| : m', m'' \in M\}$

$\leq inf\ \{\|x+m'\| + \|y+m''\| : m', m'' \in M\}$

$= inf\ \{\|x+m'\| : m' \in M\} + inf\ \{\|y+m''\| : m'' \in M\}$

$= \|x+M\| + \|y+M\|$

ie. $\|x+M+y+M\| \leq \|x+M\| + \|y+M\|$

(iv) Let x + M $\in \frac{N}{M}$ where x $\in$ N, α be a scalar.

$\|\alpha x+M\| = inf\ \{\|\alpha x+m\| : m \in M\}$

$= inf\ \{\|\alpha x+\alpha m'\| : m = \alpha m' \in M\}$

$= inf\ \{\|\alpha(x+m')\| : m' \in M\}$

$= inf\ \{|\alpha|\|x+m'\| : m' \in M\}$

$= |\alpha|\ inf\ \{\|x+m'\| : m' \in M\} = |\alpha|\|x+M\|$

$\therefore \frac{N}{M}$ is a normed linear space.

Let N be complete. Let $\{s_n + M\}$ be any Cauchy sequence in $\frac{N}{M}$ where $s_n \in$ N.

For ε = $\frac{1}{2}$. $\exists\ n_1 \in \mathbb{N} \ni$ n, m $\geq n_1 \Rightarrow \|(s_n+M)-(s_m+M)\| < \frac{1}{2}$.

Set $s_{n_1} = x_1$. So $x_1 \in$ N.

Similarly, for ε = $\frac{1}{2^2}$, $\exists\ n_2 \in \mathbb{N}$, $\ni n_2 > n_1 \ni$ n, m $\geq n_2 \Rightarrow \|(s_n+M)-(s_m+M)\| < \frac{1}{2^2}$.

Set $s_{n_2} = x_2$. So $x_2 \in N$.

Having chosen $x_1, x_2, \ldots, x_{k-1}$ and $n_1, n_2, \ldots, n_{k-1}$ now for $½^k$ ∃ a positive integer n_k which we may assume $n_k > n_{k-1}$ ∋ n, m ≥ n_k ⇒ $\|(s_n + M) - (s_m + M)\| < \frac{1}{2^k}$.

Set $s_{n_k} = x_k$. So $x_k \in N$.

And so on,

Thus, we have constructed a subsequence $\{x_k + M\}$ of the sequence $\{s_n + M\}$ such that $\|(x_{k+1} + M) - (x_k + M)\| < \frac{1}{2^k}$ for k = 1, 2, …

Choose $y_1 \in x_1 + M$ where $y_1 = x_1 + m_1$ for $m_1 \in M$.

Now select $y_2 \in x_2 + M$ ∋ $\|y_1 - y_2\| < \frac{1}{2}$.

For $\|(x_1 + M) - (x_2 + M)\| < \frac{1}{2}$.

⇒ $\inf\{\|x_1 - x_2 + m\| : m \;\; M\} < \frac{1}{2}$.

⇒ ∃ $m_0 \in M$ ∋ $\|x_1 - x_2 + m_0\| < \frac{1}{2}$.

⇒ $\|y_1 - y_2\| < \frac{1}{2}$ where $y_2 = x_2 - m_0 + m_1 \in x_2 + M$

Now select y_3 in $x_3 + M$ ∋ $\|y_2 - y_3\| < \frac{1}{2^2}$. …

Continuing, we get a seq $\{y_n\}$ in N ∋ $\|y_n - y_{n+1}\| < \frac{1}{2^n}$,

We claim that $\{y_n\}$ is a Cauchy sequence in N.

Let ε > 0. Select m_0 so large that $\frac{1}{2^{m_0-1}} < \varepsilon$

Then n > m ≥ m_0 ⇒ $\|y_m - y_n\| = \|(y_m - y_{m+1}) + (y_{m+1} - y_{m+2}) + \cdots + (y_{n-1} - y_n)\|$

$\leq \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + \ldots + \|y_{n-1} - y_n\|$

$< \sum_{i=m}^{n-1} \frac{1}{2^i} < \sum_{i=m}^{\infty} \frac{1}{2^i} = \frac{1}{2^{m-1}} \leq \frac{1}{2^{m_0-1}} < \varepsilon.$

Thus $\|y_m - y_n\| \to 0$ as m, n → ∞.

⇒ $\{y_n\}$ is a Cauchy sequence in N.

Since N is complete, $y_n \to y \in N$ for some y.

Now $\|(x_n + M) - (y + M)\| = \|(x_n - y) + M\| = \inf\{\|x_n - y + m\| : m \in M\}$

$\leq \|y_n - y\|$ ∵ $y_n = x_n + m_n$ for some $m_n \in M$,

→ 0 as n → ∞.

⇒ $x_n + M \to y + M \in \frac{N}{M}$ as n → ∞.

∴ A subsequence $\{x_n + M\}$ of Cauchy Sequence $\{s_n + M\}$ converges to y + M.

⇒ $\frac{N}{M}$ is complete.

Hence $\frac{N}{M}$ is a Banach Space.

Example 1: Show that the set of real linear space $\mathbb{R}$ and the complex linear space $\mathbb{C}$ are Banach space under the norm defined by $\|x\| = |x|\ \forall\ x \in \mathbb{R}\ or\ \mathbb{C}$.

Solution: Let $x \in \mathbb{R}\ or\ \mathbb{C}$. Then (i) $\|x\| = x \geq 0$.

(ii) Let $x \in \mathbb{R}$ or $\mathbb{C}$. Then $\|x\| = 0$ iff $|x| = 0$ iff $x = 0$.

(iii) Let $x, y \in \mathbb{R}$. $Then\ \|x + y\| = |x + y| \leq |x| + |y| = \|x\| + \|y\|$

Let $z, \omega \in \mathbb{R}$.

Then $\|z + \omega\|^2 = |z + \omega|^2 = (z + \omega)(\bar{z} + \bar{\omega}) = z\,\bar{z} + z\bar{\omega} + \omega\bar{z} + \omega\bar{\omega}$

$= |z|^2 + 2\mathrm{Rp}(z\bar{\omega}) + |\omega|^2 \leq |z|^2 + 2|z\bar{\omega}| + |\omega|^2 = |z|^2 + 2|z||\bar{\omega}| + |\omega|^2$

$= |z|^2 + 2|z||\omega| + |\omega|^2 = (|z| + |\omega|)^2 = (\|z\| + \|\omega\|)^2$. ie $\|z + \omega\| \leq \|z\| + \|\omega\|$

(iv) Let $\alpha \in K$, $x \in \mathbb{R}$. Or $\mathbb{C}$. Then $\|\alpha x\| = |\alpha x| = |\alpha|\,|x| = |\alpha|\,\|x\|$.

$\therefore$ $\mathbb{R}$ or $\mathbb{C}$ is a normed linear space under the norm defined by $\|x\| = |x|\ \forall\ x \in \mathbb{R}. or\, \mathbb{C}$.

Let $\{x_n\}$ be a Cauchy sequence

$\Rightarrow \{x_n\}$ is a bounded seq

$\Rightarrow \{x_n\}$ has at least one limit point by Bolzano Weierstrass theorem

$\Rightarrow \{x_n\}$ has a convergent sequence converging to that limit point.

$\Rightarrow \{x_n\}$ has a convergent subsequence.

$\Rightarrow$ Cauchy seq $\{x_n\}$ has a convergent subsequence

$\Rightarrow \{x_n\}$ is convergent in $\mathbb{R}$ or $\mathbb{C}$.

$\Rightarrow \mathbb{R}\ or\ \mathbb{C}$ is complete.

Hence $\mathbb{R}$ and $\mathbb{C}$ are Banach spaces under the norm defined by $\|x\| = |x|\ \forall\ x \in \mathbb{R}\ or\, \mathbb{C}$.

Example 2: The set of all n – tuples of real numbers, $\mathbb{R}^n$, is a Banach space under the norm defined by $\|x\| = \left[\sum_{i=1}^{n} |\xi_i|^2\right]^{\frac{1}{2}}$ where $x = (\xi_1, \xi_2, \dots, \xi_n) = (\xi_i)_{i=1}^{n} \in \mathbb{R}^n, \xi_i \in \mathbb{R}\ \forall\ i$

Solution: (i) Let $x = (\xi_i)_{i=1}^{n} \in \mathbb{R}^n for\ \xi_i \in \mathbb{R}\ \forall\ i$.

Then $\|x\| = \left[\sum_{i=1}^{n} |\xi_i|^2\right]^{\frac{1}{2}} \geq 0$

(ii) Let $x = (\xi_i)_{i=1}^{n} \in \mathbb{R}\ for\ \xi_i \in \mathbb{R}\ \forall\ i$.

Then $\|x\| = 0 \Leftrightarrow \left[\sum_{i=1}^{n} |\xi_i|^2\right]^{\frac{1}{2}} = 0 \Leftrightarrow \sum_{i=1}^{n} |\xi_i|^2 = 0$

$\Leftrightarrow |\xi_i|^2 = 0\ \forall\ i. \Leftrightarrow |\xi_i| = 0\ \forall\ i. \Leftrightarrow \xi_i = 0\ \forall\ i. \Leftrightarrow x = 0$.

(iii) Let $x = (\xi_i)_{i=1}^{n}, y = (\eta_i)_{i=1}^{n}, \in \mathbb{R}^n for\ \xi_i, \eta_i \in \mathbb{R}\ \forall\ i$.

$\|x + y\| = \left[\sum_{i=1}^{n} |\xi_i + \eta_i|^2\right]^{\frac{1}{2}}$

$\Rightarrow \|x+y\|^2 = \sum_{i=1}^{n} |\xi_i+\eta_i||\xi_i+\eta_i|$

$\Rightarrow \|x+y\|^2 \le \sum_{i=1}^{n} (|\xi_i|+|\eta_i|)|\xi_i+\eta_i|$

$= \sum_{i=1}^{n} |\xi_i||\xi_i+\eta_i| + \sum_{i=1}^{n} |\eta_i||\xi_i+\eta_i|$

$= \sum_{i=1}^{n} |\xi_i(\xi_i+\eta_i)| + \sum_{i=1}^{n} |\eta_i(\xi_i+\eta_i)|$

$\le \|x\|\|x+y\| + \|y\|\|x+y\|$

ie. $\|x+y\|^2 \le (\|x\|+\|y\|)\|x+y\|$

$\Rightarrow \|x+y\| \le \|x\|+\|y\|$

(iv) Let $x = (\xi_i)_{i=1}^{n} \in \mathbb{R}^n$ for $\xi_i \in \mathbb{R}\ \forall\, i$, $\alpha \in \mathbb{R}$.

$\|\alpha x\| = \left[\sum_{i=1}^{n} |\alpha\xi_i|^2\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{n} |\alpha|^2|\xi_i|^2\right]^{\frac{1}{2}} = |\alpha|\left[\sum_{i=1}^{n} |\xi_i|^2\right]^{\frac{1}{2}} = |\alpha|\|x\|$

Ie. $\|\alpha x\| = |\alpha|\|x\|$ $x = (\xi_i)_{i=1}^{n} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$.

$\therefore \mathbb{R}^n$ is a normed linear space.

Let $\{x_m\}$ be any Cauchy sequence in $\mathbb{R}^n$ where $x_m = \left(\xi_i^{(m)}\right)_{i=1}^{n} \in \mathbb{R}^n$ for $\xi_i^{(m)} \in \mathbb{R}\ \forall\, i$.

$\Rightarrow$ for each $\varepsilon > 0\ \exists\ n_0 \ni \|x_m - x_p\| < \varepsilon\ \forall$ m, p $\ge n_0$.

$\Rightarrow$ for each $\varepsilon > 0\ \exists\ n_0 \ni \left[\sum_{i=1}^{n} \left|\xi_i^{(m)} - \xi_i^{(p)}\right|^2\right]^{\frac{1}{2}} < \varepsilon\ \forall$ m, p $\ge n_0$.

$\Rightarrow$ for each $\varepsilon > 0\ \exists\ n_0 \ni \sum_{i=1}^{n} \left|\xi_i^{(m)} - \xi_i^{(p)}\right|^2 < \varepsilon^2\ \forall$ m, p $\ge n_0$.

$\Rightarrow$ for each $\varepsilon > 0\ \exists\ n_0 \ni \left|\xi_i^{(m)} - \xi_i^{(p)}\right|^2 < \varepsilon^2\ \forall$ m, p $\ge n_0$.

$\Rightarrow$ for each $\varepsilon > 0\ \exists\ n_0 \ni \left|\xi_i^{(m)} - \xi_i^{(p)}\right| < \varepsilon\ \forall$ m, p $\ge n_0$.

$\Rightarrow \{\xi_i^{(m)}\}$ is a Cauchy sequence in $\mathbb{R}$ for each i, $1 \le i \le n$

Since $\mathbb{R}$ is complete $\exists\ \xi_i$ in $\mathbb{R} \ni$ the sequence $\{\xi_i^{(m)}\}$ converges to ξ_i for each i, $1 \le i \le n$ Let $x = (\xi_1, \xi_2, \ldots, \xi_n) = (\xi_i)_{i=1}^{n}$.

Then $x \in \mathbb{R}^n$ and $\|x_m - x\| = \left[\sum_{i=1}^{n} \left|\xi_i^{(m)} - \xi_i\right|^2\right]^{\frac{1}{2}} \to 0$ as $m \to \infty$

$\Rightarrow x_m - x \to 0$ as $m \to \infty \Rightarrow x_m \to x$ as $m \to \infty$

$\therefore$ the sequence $\{x_m\}$ in $\mathbb{R}^n$ converges to x in $\mathbb{R}^n$.

$\therefore\ \mathbb{R}^n$ is complete.

Hence $\mathbb{R}^n$ is a Banach space.

Example 3: The set of all n – tuples of complex numbers, $\mathbb{C}^n$, is a Banach space under the norm defined by $\|z\| = \left[\sum_{i=1}^{n} |\xi_i|^2\right]^{\frac{1}{2}}$ where $z = (\xi_1, \xi_2, \ldots, \xi_n) = (\xi_i)_{i=1}^{n} \in \mathbb{C}^n$, $\xi_i \in \mathbb{C}\ \forall\, i$

Solution: Same as above example

Example 4: The linear space ℓ^∞ of bounded sequences is a Banach space under the norm defined by $\|x\| = |\xi_i|$, where x = $(\xi_i)_{i=1,}^{\infty}\ \xi_i \in \mathbb{R}$ or $\mathbb{C}\ \forall\ i$

(i) Let x $\in \ell^\infty$. $\|x\| = |\xi_i| \geq 0$

(ii) Let x $\in \ell^\infty$. $\|x\| = |\xi_i| = 0$ iff $|\xi_i| = 0\ \forall\ i$ iff $\xi_i = 0\ \forall$ i iff x = 0.

(iii) Let x, y $\in \ell^\infty$. $\|x + y\| = |\xi_i + \eta_i| \leq |\xi_i| + |\eta_i|)$
$= |\xi_i| + |\eta_i| = \|x\| + \|y\|$

(iv) Let x $\in \ell^\infty$, Then $\|\alpha x\| = |\alpha \xi_i| = |\alpha||\xi_i| = |\alpha|\ |\xi_i| = |\alpha|\|x\|$
$\therefore \ell^\infty$ is a normed linear space.

Let $\{x_n\}$ be any Cauchy sequence in ℓ^∞ where $x_n = \left(\xi_i^{(n)}\right)_{i=1}^{\infty} \in \ell^\infty$ for $\xi_i^{(n)} \in \mathbb{R}$ or $\mathbb{C}\ \forall\ i$.

$\Rightarrow$ for each $\varepsilon > 0\ \exists\ n_0 \in \mathbb{N} \ni \|x_m - x_n\| < \varepsilon\ \forall$ m, n $\geq n_0$. where $x_m = \left(\xi_i^{(m)}\right)_{i=1}^{\infty} \in \ell^\infty$ for $\xi_i^{(m)} \in \mathbb{R}$ or $\mathbb{C}\ \forall\ i$.

$\Rightarrow$ for each $\varepsilon > 0\ \exists\ n_0 \in \mathbb{N} \ni Sup_i \left|\xi_i^{(m)} - \xi_i^{(n)}\right| < \varepsilon\ \forall$ m, n $\geq n_0$.

$\Rightarrow$ for each $\varepsilon > 0\ \exists\ n_0 \in \mathbb{N} \ni \left|\xi_i^{(m)} - \xi_i^{(n)}\right| < \varepsilon\ \forall$ m, n $\geq n_0$.

$\Rightarrow \{\xi_i^{(n)}\}$ is a Cauchy sequence of real or complex number.

Since $\mathbb{R}$ and $\mathbb{C}$ are complete $\exists\ \xi_i$ in $\mathbb{R}$ or $\mathbb{C} \ni$ the sequence $\{\xi_i^{(n)}\}$ converges to ξ_i for each i.

Let x $= (\xi_1, \xi_2, \ldots) = (\xi_i)_{i=1}^{\infty}$.

Then $\|x_n - x\| = \sup_i \left|\xi_i^{(n)} - \xi_i\right| \to 0$ as n $\to \infty$

$\Rightarrow x_n \to x$ as n $\to \infty$.

$\therefore$ the sequence $\{x_n\}$ in ℓ^∞ converges to x.

Claim: x $\in \ell^\infty$.

$|\xi_i| = \left|\xi_i - \xi_i^{(n)} + \xi_i^{(n)}\right| \leq \left|\xi_i - \xi_i^{(n)}\right| + \left|\xi_i^{(n)}\right| < \varepsilon + k$ for each i. $\Rightarrow$ x $\in \ell^\infty$.

$\therefore \ell^\infty$ is complete.

Hence ℓ^∞ is a Banach space.

Lemma: Let p > 1, $\frac{1}{p} + \frac{1}{q} = 1, a \geq 0, b \geq 0$. Then ab $\leq \frac{a^p}{p} + \frac{b^q}{q}$.

Proof: If a = 0 or b = 0 then the conclusion is obvious. So let a > 0 and b > 0.
Define f(t) = k(t – 1) – t^k + 1 for t $\geq$ 1, k $\in$ (0, 1).

Note that f(1) = 0 and $f'(t) = k - kt^{k-1} = k\left(1 - \frac{1}{t^{1-k}}\right) \geq 0$.

So, f(t) ≥ 0 ∀ t ∈ [1, ∞). ∴ $t^k \leq kt + 1 - k$

Put $t = a^p b^{-q}$ and replace k by 1/p. We get $\left(a^p b^{-q}\right)^{\frac{1}{p}} \leq 1 - \frac{1}{p} + \frac{1}{p} a^p b^{-q} = \frac{1}{q} + \frac{1}{p} a^p b^{-q}$ Multiplying both sides by bq we get $ab^{q-\frac{q}{p}} \leq \frac{a^p}{p} + \frac{b^q}{q}$ ie. $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

Holder's and Minkowski's inequalities:

<u>Theorem 3</u>: Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ denote n – tuples of scalars (real or complex numbers. Define $\|x\|_p = \left[\sum_{i=1}^n |x_i|^p\right]^{\frac{1}{p}}$ for p ≥ 1

(i) $\sum_{i=1}^n |x_i y_i| \leq \left[\sum_{i=1}^n |x_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^n |y_i|^q\right]^{\frac{1}{q}} = \|x\|_p \|y\|_q$ if p > 1 and $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) $\left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^n |x_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^n |y_i|^p\right]^{\frac{1}{p}}$ ie. $\|x + y\|_p \leq \|x\|_p + \|y\|_p$

<u>Proof</u>: If x = 0 or y = 0 then the conclusion is obvious. So let a ≠ 0 and b ≠ 0.

Then by the lemma for $a_i \geq 0$, $b_i \geq 0$ we have $a_i b_i \leq \frac{a_i^p}{p} + \frac{b_i^q}{q}$.

Put $a_i = \frac{|x_i|}{\|x\|_p}$ and $b_i = \frac{|y_i|}{\|y\|_q}$

Thus, we get $\frac{|x_i|}{\|x\|_p} \frac{|y_i|}{\|y\|_q} \leq \frac{1}{p} \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \frac{|y_i|^q}{\|y\|_q^q}$

Summing from i = 1 to n both sides we get

$$\frac{\sum_{i=1}^n |x_i||y_i|}{\|x\|_p \|y\|_q} \leq \frac{1}{p} \frac{\sum_{i=1}^n |x_i|^p}{\|x\|_p^p} + \frac{1}{q} \frac{\sum_{i=1}^n |y_i|^q}{\|y\|_q^q} = \frac{1}{p} \frac{\|x\|_p^p}{\|x\|_p^p} + \frac{1}{q} \frac{\|y\|_q^q}{\|y\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1.$$

Ie. $\sum_{i=1}^n |x_i y_i| \leq \|x\|_p \|y\|_q$

(ii) This inequality is evident when p = 1. So, assume p > 1.

$\|x + y\|_p^p = \sum_{i=1}^n |x_i + y_i|^p = \sum_{i=1}^n |x_i + y_i||x_i + y_i|^{p-1} \leq \sum_{i=1}^n (|x_i| + |y_i|)|x_i + y_i|^{p-1}$

$= \sum_{i=1}^n (|x_i|)|x_i + y_i|^{p-1} + \sum_{i=1}^n (|y_i|)|x_i + y_i|^{p-1}$

$= \sum_{i=1}^n |x_i (x_i + y_i)^{p-1}| + \sum_{i=1}^n |y_i (x_i + y_i)^{p-1}|$

$\leq \left[\sum_{i=1}^n |x_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^n |(x_i + y_i)^{p-1}|^q\right]^{\frac{1}{q}} + \left[\sum_{i=1}^n |y_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^n \left|(x_i + y_i)^{p-1}\right|^q\right]^{\frac{1}{q}}$

$= \left[\sum_{i=1}^n |x_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^n \left|(x_i + y_i)^{\frac{p}{q}}\right|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^n |y_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^n \left|(x_i + y_i)^{\frac{p}{q}}\right|^p\right]^{\frac{1}{p}}$

$= \|x\|_p \|x + y\|_p^{\frac{p}{q}} + \|y\|_p \|x + y\|_p^{\frac{p}{q}} = (\|x\|_p + \|y\|_p)\|x + y\|_p^{\frac{p}{q}}$

Corollary: Holders and Minkowskie's inequalities for sequences.
Let x = $\{x_n\}$ and y = $\{y_n\}$ be sequences of scalars ∋ $\sum_{i=1}^{\infty}|x_i|^p < \infty$, $\sum_{i=1}^{\infty}|y_i|^q < \infty$.

For p ≥ 1 define $\|x\| = \left[\sum_{i=1}^{\infty}|x_i|^p\right]^{\frac{1}{p}}$.

Then (i) $\sum_{i=1}^{\infty}|x_i y_i| \leq \left[\sum_{i=1}^{\infty}|x_i|^p\right]^{\frac{1}{p}}\left[\sum_{i=1}^{\infty}|y_i|^q\right]^{\frac{1}{q}} = \|x\|_p\|y\|_q$ if p > 1 and $\frac{1}{p}+\frac{1}{q} = 1$.

(ii) $\left[\sum_{i=1}^{\infty}|x_i+y_i|^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{\infty}|x_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{\infty}|y_i|^p\right]^{\frac{1}{p}}$ ie. $\|x+y\|_p \leq \|x\|_p + \|y\|_p$.

Proof: If n is a positive integer, then by above result

$\sum_{i=1}^{n}|x_i y_i| \leq \left[\sum_{i=1}^{n}|x_i|^p\right]^{\frac{1}{p}}\left[\sum_{i=1}^{n}|y_i|^q\right]^{\frac{1}{q}} \leq \left[\sum_{i=1}^{\infty}|x_i|^p\right]^{\frac{1}{p}}\left[\sum_{i=1}^{\infty}|y_i|^q\right]^{\frac{1}{q}} < \varepsilon$ … (1)

Thus the partial sums $\sum_{i=1}^{n}|x_i y_i|$ are bounded and so $\sum_{i=1}^{\infty}|x_i y_i| < \infty$.

If we let n → ∞ in (1) we get $\sum_{i=1}^{\infty}|x_i y_i| \leq \left[\sum_{i=1}^{\infty}|x_i|^p\right]^{\frac{1}{p}}\left[\sum_{i=1}^{\infty}|y_i|^q\right]^{\frac{1}{q}} = \|x\|_p\|y\|_q$.

(ii)

$\sum_{i=1}^{n}|x_i+y_i|^p \leq \left[\sum_{i=1}^{n}|x_i|^p\right]^{\frac{1}{p}}\left[\sum_{i=1}^{n}\left|(x_i+y_i)^{\frac{p}{q}}\right|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n}|y_i|^p\right]^{\frac{1}{p}}\left[\sum_{i=1}^{n}\left|(x_i+y_i)^{\frac{p}{q}}\right|^p\right]^{\frac{1}{p}}$

$= \|x\|_p\|x+y\|_p^{\frac{p}{q}} + \|y\|_p\|x+y\|_p^{\frac{p}{q}}$

Now letting n → ∞

$\|x+y\|_p^{\,p} = \sum_{i=1}^{\infty}|x_i+y_i|^p \leq (\|x\|_p + \|y\|_p)\,\|x+y\|_p^{\frac{p}{q}}$ or

$\|x+y\|_p^{\,p-\frac{p}{q}} \leq \|x\|_p + \|y\|_p$ ie. $\|x+y\|_p \leq \|x\|_p + \|y\|_p$.

Example 5: (1*): The linear space l^p, p ≥ is a Banach space under the norm defined by $\|x\| = \left[\sum_{i=1}^{\infty}|\xi_i|^p\right]^{\frac{1}{p}}$ where x = $(\xi_1, \xi_2, \dots, \xi_n, \dots) = (\xi_i)_{i=1}^{\infty} \in l^p, \xi_i \in \mathbb{R}\ \forall\, i$

Solution: (i) Let x = $(\xi_i)_{i=1}^{\infty} \in l^p$ for $\xi_i \in \mathbb{R}$ or $\mathbb{C}\ \forall\, i$

Then $\|x\| = \left[\sum_{i=1}^{\infty}|\xi_i|^p\right]^{\frac{1}{p}} \geq 0$

(ii) Let x = $(\xi_i)_{i=1}^{\infty} \in l^p$ for $\xi_i \in \mathbb{R}$ or $\mathbb{C}\ \forall\, i$

Then $\|x\| = 0 \Leftrightarrow \left[\sum_{i=1}^{\infty}|\xi_i|^p\right]^{\frac{1}{p}} = 0 \Leftrightarrow \sum_{i=1}^{\infty}|\xi_i|^p = 0$

$\Leftrightarrow |\xi_i|^p = 0\ \forall\, i. \Leftrightarrow |\xi_i| = 0\ \forall\, i. \Leftrightarrow \xi_i = 0\ \forall\, i. \Leftrightarrow$ x = 0.

(iii) Let x = $(\xi_i)_{i=1}^{\infty}, y = (\eta_i)_{i=1}^{\infty}, \in l^p$ for $\xi_i, \eta_i \in \mathbb{R}$ or $\mathbb{C}\ \forall\, i$

$\|x+y\| = \left[\sum_{i=1}^{\infty}|\xi_i+\eta_i|^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{\infty}|\xi_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{\infty}|\eta_i|^p\right]^{\frac{1}{p}}$ by Minkowskie's inequality

$\leq \|x\| + \|y\|$ ie. $\|x+y\| \leq \|x\| + \|y\|$

(iv) Let x = $(\xi_i)_{i=1}^{\infty} \in l^p$ for $\xi_i \in \mathbb{R}$ or $\mathbb{C}$ $\forall$ i $\alpha \in \mathbb{R}$.

$\|\alpha x\| = \left[\sum_{i=1}^{\infty}|\alpha\xi_i|^p\right]^{\frac{1}{p}} = \left[\sum_{i=1}^{\infty}|\alpha|^p|\xi_i|^p\right]^{\frac{1}{p}} = |\alpha|\left[\sum_{i=1}^{n}|\xi_i|^p\right]^{\frac{1}{p}} = |\alpha|\|x\|$

Ie. $\|\alpha x\| = |\alpha|\|x\|$ where x $=\left(\xi_i\right)_{i=1}^{\infty} \in l^p$, $\alpha \in \mathbb{R}$.

$\therefore l^p$ is a normed linear space.

Let $\{x_n\}$ be any Cauchy sequence in l^p where $x_n = \left(\xi_i^{(n)}\right)_{i=1}^{\infty} \in l^p$ for $\xi_i^{(n)} \in \mathbb{R}$ or $\mathbb{C}$ $\forall$ i

$\Rightarrow$ for each $\varepsilon > 0$ $\exists$ n_0 $\ni$ $\|x_m - x_n\| < \varepsilon$ $\forall$ m, n $\geq n_0$, where $x_m = \left(\xi_i^{(m)}\right)_{i=1}^{\infty} \in l^p$ for $\xi_i^{(m)} \in \mathbb{R}$ or $\mathbb{C}$ $\forall$ i

$\Rightarrow \left[\sum_{i=1}^{\infty}\left|\xi_i^{(m)} - \xi_i^{(n)}\right|^p\right]^{\frac{1}{p}} < \varepsilon$ $\forall$ m, n $\geq n_0$

$\Rightarrow \sum_{i=1}^{\infty}\left|\xi_i^{(m)} - \xi_i^{(n)}\right|^p < \varepsilon^p$ $\forall$ m, n $\geq n_0$

$\Rightarrow \left|\xi_i^{(m)} - \xi_i^{(n)}\right|^p < \varepsilon^p$ $\forall$ m, n $\geq n_0$

$\Rightarrow \left|\xi_i^{(m)} - \xi_i^{(n)}\right| < \varepsilon$ $\forall$ m, n $\geq n_0$

$\Rightarrow \{\xi_i^{(m)}\}$ is a Cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$ $\forall$ i

Since $\mathbb{R}$ and $\mathbb{C}$ are complete $\exists$ ξ_i in $\mathbb{R}$ or $\mathbb{C}$ $\ni$ the sequence $\{\xi_i^{(m)}\}$ converges to ξ_i $\forall$ i.

Let x = $\left(\xi_1, \xi_2, \ldots, \xi_n, \ldots\right) = (\xi_i)_{i=1}^{\infty}$.

Then $\|x_n - x\| = \left[\sum_{i=1}^{n}\left|\xi_i^{(n)} - \xi_i\right|^p\right]^{\frac{1}{p}} \to 0$ as n $\to\infty$

$\Rightarrow x_n \to x$ as n $\to\infty$.

$\therefore$ the sequence $\{x_n\}$ in l^p converges to x.

Claim: x $\in l^p$. $x_n - x = \left(\xi_i^{(n)} - \xi_i\right)_{i=1}^{\infty} \to 0$ $\Rightarrow x_n \to x$ as n $\to \infty$

$\Rightarrow$ Given $\varepsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ $\ni$ $\|x_n - x\|$ $\forall$ n $\geq n_0$ $\Rightarrow \sum_{i=1}^{\infty}\left|\xi_i^{(n)} - \xi_i\right|^p < \varepsilon^p = k < \varepsilon$

$\Rightarrow x_n - x \in l^p$.

Now $x = x - x_n + x_n \in l^p$. $\Rightarrow$ x $\in l^p$

$\therefore l^p$ is complete.

Hence l^p is a Banach space.

Example 6: (3*): Let $C(X)$ denote the linear space of all bounded continuous scalar valued functions defined on a topological space X. Show that $C(X)$ is a Banach space under the norm $\|f\| = \sup\{|f(x)|: x \in X\}$, f $\in C(X)$.

Solution: We know that $C(X)$ is a linear space.

Since $|f(x)| \geq 0 \ \forall$ x $\in$ X, we have $\|f\| \geq 0$.

$\|f\| = 0$ iff $\sup\{|f(x)|: x \in X\} = 0$ iff $|f(x)| = 0 \ \forall x \in X$ iff f(x) = 0 $\forall$ x $\in$ X iff f = $\hat{0}$.

$\|f + g\| = \sup\{|(f + g)(x)|: x \in X\} = \sup\{|f(x) + g(x)|: x \in X\}$

$\leq \sup\{|f(x)| + |f(x)|: x \in X\} \leq \sup\{|f(x)|: x \in X\} + \sup\{|g(x)|: x \in X\} = \|f\|$ + $\|g\|$.

$\|\alpha f\| = \sup\{|(\alpha f)(x)|: x \in X\} = \sup\{|\alpha f(x)|: x \in X\} = \sup\{|\alpha||f(x)|: x \in X\}$

$= |\alpha| \sup\{|f(x)|: x \in X\} = |\alpha|\|f\|$

Hence $C(X)$ is a normed linear space.

Let $\{f_n\}$ be any Cauchy sequence in $C(X)$. Then for a given $\varepsilon > 0$, $\exists$ a positive integer $m_0 \ni m, n \geq m_0 \Rightarrow \|f_m - f_n\| < \varepsilon. \Rightarrow \sup\{|(f_m - f_n)(x)|: x \in X\} < \varepsilon$.

$\Rightarrow \sup\{|f_m(x) - f_n(x)|: x \in X\} < \varepsilon. \Rightarrow |f_m(x) - f_n(x)| < \varepsilon \ \forall$ x $\in$ X. But this is the Cauchy's condition for uniform convergence of the sequence of bounded continuous scalar valued functions. Hence the sequence $\{f_n\}$ must converge to a bounded continuous function f on X. $\therefore$ $C(X)$ is complete and hence it is a Banach space.

Example 7: (3*): In the linear space $C[0, 1]$ of real valued continuous functions on [0, 1] define $\|f\| = \max_{0 \leq t \leq 1} |f(t)|$. Prove that $C[0, 1]$ is a Banach space with this norm.

Solution: Since a real valued continuous function on a closed interval is bounded and so $C[0, 1]$ is a Banach space following exactly the same manner as in above example.

CONTINUOUS LINEAR TRANSFORMATION:

Definition: Let N and N′ be normed linear spaces with the same scalars.

(i) A linear transformation T: N $\to$ N′ is said to be ***continuous*** iff for each sequence $\{x_n\}$ in N converging to x in N, the sequence $\{T(x_n)\}$ in N′ converges to T(x) in N′.

(ii) Let T: N $\to$ N′ be a linear transformation.

If $\exists$ a real number k $\geq$ 0 $\ni \|T(x)\| \leq k\|x\| \ \forall$ x $\in$ N, then k is called a ***bound*** for T and T is said to be ***bounded linear transformation.***

Theorem 4: Let T be a linear transformation of a normed linear transformation N into another normed linear space N′. Then the following statements are equivalent.

(i) T is continuous

(ii) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$.

(iii) $\exists$ a real number $k \geq 0 \ni \|T(x)\| \leq k\|x\| \ \forall x \in N$. ie. T is bounded.

(iv) If $S = \{x: \|x\| \leq 1\}$ then its image in N′ is a bounded set.

Proof: Claim: (i) $\Rightarrow$ (ii).

Assume T is continuous in N and $\{x_n\}$ is a sequence in N converging to 0.

Since T is continuous at 0, $\{T(x_n)\}$ converges to T(0). But T(0) = 0.

$\therefore$ Sequence $\{T(x_n)\}$ converges to 0. $\therefore$ T is continuous at the origin.

Claim: (ii) $\Rightarrow$ (iii).

Assume T is continuous at the origin.

If possible, suppose T is not bounded.

Then for each positive integer n, $\exists x_n \in N \ni \|T(x_n)\| > n\|x_n\|$.

$\Rightarrow \frac{1}{n\|x_n\|}\|T(x_n)\| > 1 \Rightarrow \left\|\frac{T(x_n)}{n\|x_n\|}\right\| > 1$ … (1).

Now set $y_n = \frac{x_n}{n\|x_n\|}$ then $\|y_n\| = \frac{1}{n} \to 0$ as $n \to \infty$.

$\therefore \exists$ a seq $\{y_n\}$ in N $\ni y_n \to 0$. But $\|T(y_n)\| > 1$ from (i).

So, $T(y_n) \nrightarrow 0$.

$\Rightarrow$ T is not continuous at origin which is a contradiction.

Hence T must be bounded.

Claim: (iii) $\Rightarrow$ (iv). Assume that T is bounded.

Let $S = \{x \in N: \|x\| \leq 1\}$

Since T is bounded, $\exists$ a real number $k \geq 0 \ni \|T(x)\| \leq k\|x\| \ \forall x \in N$.

$\Rightarrow \|T(x)\| \leq k \ \forall x \in S$. $\therefore$ T(S) is bounded in N′.

Claim: (iv) $\Rightarrow$ (i).

Assume that T(S) is bounded in N′ if $S = \{x: \|x\| \leq 1\}$ is a closed unit sphere in N.

If x = 0 then T(x) = 0 so that $\|T(x)\| \leq k\|x\|$.

If $x \neq 0$, then $\frac{x}{\|x\|} \in S$ and so, $\exists$ a real number $k \geq 0 \ni \left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq k$

$\Rightarrow \frac{1}{\|x\|}\|T(x)\| \leq k \Rightarrow \|T(x)\| \leq k\|x\|$

$\therefore \|T(x)\| \leq k\|x\| \ \forall x \in N$…(1).

Let $x \in N$, and $\{x_n\}$ be a sequence in N $\ni x_n \to x$.

Since $x_n - x \in N$, by (1), $\|T(x_n - x)\| \leq k\|x_n - x\| \to 0$.

$\Rightarrow \|T(x_n) - T(x)\| \to 0$ as $n \to \infty$.

$\Rightarrow T(x_n) - T(x) \to 0$ as $n \to \infty$.

$T(x_n) \to T(x)$ as $n \to \infty$.

$\Rightarrow$ T is continuous. Hence the theorem.

Definition: Let N and N′ be any two normed linear spaces, and T be a bounded linear transformation of N into N′. Define $\|T\| = \text{Sup}\,\{\|T(x)\|: x \in N, \|x\| \leq 1\}$.

Theorem 5: Let N and N′ be any two normed linear spaces, and T be a bounded linear transformation of N into N′. Put a = $Sup\{\|T(x)\|: x \in N, \|x\| = 1\}$,
b = $Sup\left\{\frac{\|T(x)\|}{\|x\|}: x \in N, x \neq 0\right\}$, c = $Inf\{k: k > 0, \|T(x)\| \leq k\|x\|\}$.
Then $\|T\|$ = a = b = c and $\|T(x)\| \leq \|T\|\|x\|$ ∀ x ∈ N.

Proof: Since $\{x \in N, \|x\| = 1\} \subseteq \{x \in N, \|x\| \leq 1\}$, $a = Sup\{\|T(x)\|: x \in N, \|x\| = 1\}$ $\leq \text{Sup}\,\{\|T(x)\|: x \in N, \|x\| \leq 1\} = \|T\|$ ie. a ≤ $\|T\|$ ... (i)
Since T is a linear transformation, b = $Sup\left\{\frac{\|T(x)\|}{\|x\|}: x \in N, x \neq 0\right\}$
= $\text{Sup}\{\|T(y)\|: y = \frac{x}{\|x\|} \in N, \|y\| = 1\}$ = a ie b = a ...(ii)
From definition of b, $b \geq \frac{\|T(x)\|}{\|x\|}$ $\forall\, x \in N, x \neq 0 \Rightarrow \|T(x)\| \leq b\|x\|$
$\Rightarrow$ c = Inf $\{k: k > 0, \|T(x)\| \leq k\|x\|\} \leq$ b. Ie. c ≤ b ...(iii).

From the definition of c, $\|T(x)\| \leq c\|x\|$ ∀ x ∈ N ⇒ $\|T(x)\| \leq c$ ∀ x ∈N with $\|x\| \leq 1$.

⇒ c is an upper bound of $\{\|T(x)\|: x \in N, \|x\| \leq 1\}$.

$\therefore\ \|T\| = \sup\{\|T(x)\|: x \in N, \|x\| \leq 1\} \leq c$. Ie. $\|T\| \leq c \ldots (iv)$

$\therefore\ \|T\|$ ≤ c ≤ b = a ≤ $\|T\|$. Hence $\|T\|$ = a = b = c.

Since $\|T(x)\| \leq b\|x\|$ and b = $\|T\|$, it follows that $\|T(x)\| \leq \|T\|\|x\|$.

Theorem 6: (6*): Let N and N′ be any two normed linear spaces, and $\mathfrak{B}(N, N')$ denote the set of all bounded linear transformation of N into N′. Then $\mathfrak{B}(N, N')$ is itself a normed linear space with respect to pointwise linear operator
(T + U)(x) = T(x) + U(x), (αT)(x) = α{T(x)} and the norm defined by
$\|T\| = \text{Sup}\{\|T(x)\|: x \in N, \|x\| \leq 1\}$. Further if N′ is a Banach space then so is $\mathfrak{B}(N, N')$.

Proof: Claim: $\mathfrak{B}(N, N')$ is a linear space.
Clearly the set S of all linear transformations from a linear space N into another linear space is itself a linear space with respect to pointwise operations.
Let $T_1, T_2 \in \mathfrak{B}(N, N')$. Then T_1, T_2 are bounded and so ∃ real numbers $k_1 \geq 0$, $k_2 \geq 0$ ∋ $\|T_1(x)\| \leq k_1\|x\|$ and $\|T_2(x)\| \leq k_2\|x\|$ ∀x ∈ N.
If α, β are any two scalars then $\|(\alpha T_1+\beta T_2)(x)\| = \|(\alpha T_1)(x)+(\beta T_2)(x)\|$
$= \|\alpha\{T_1(x)\}+\beta\{T_2(x)\}\| \leq \|\alpha\{T_1(x)\}\| + \|\beta\{T_2(x)\}\|$
$= |\alpha|\|T_1(x)\| + |\beta|\|T_2(x)\| \leq (|\alpha|k_1 + |\beta|k_2)\|x\|$.

Thus, $\alpha T_1+\beta T_2$ is bounded and so $\alpha T_1+\beta T_2 \in \mathfrak{B}(N, N')$.
Thus $\mathfrak{B}(N, N')$ is a linear subspace of S.

Claim: $\mathfrak{B}(N, N')$ is a normed linear space. Let $T \in \mathfrak{B}(N, N')$.
$\|T\| \geq 0$ since $\|T\| = Sup\{\|T(x)\|: x \in N, \|x\| \leq 1\}$ and $\|T(x)\| \geq 0 \ \forall \ x \in N$.
$\|T\| = 0$ iff $\text{Sup}\left\{\frac{\|T(x)\|}{\|x\|}: x \in N, x \neq 0\right\} = 0$ iff $\frac{\|T(x)\|}{\|x\|} = 0, x \in N, x \neq 0$
iff $\|T(x)\| = 0, x \in N, x \neq 0$ iff $T(x) = 0 \ \forall \ x \in N$ iff $T = O$ zero transformation.
Let $T, U \in \mathfrak{B}(N, N')$. Then $\|T + U\| = Sup\{\|(T + U)(x)\|: x \in N, \|x\| \leq 1\}$
$= Sup\{\|T(x) + U(x)\|: x \in N, \|x\| \leq 1\}$
$\leq Sup\{\|T(x)\| + \|U(x)\|: x \in N, \|x\| \leq 1\}$
$= Sup\{\|T(x)\|: x \in N, \|x\| \leq 1\} + Sup\{\|U(x)\|: x \in N, \|x\| \leq 1\}$
$= \|T\| + \|U\|$
Let $T \in \mathfrak{B}(N, N')$, $\alpha \in K$.
Then $\|\alpha T\| = Sup\{\|(\alpha T)(x)\|: x \in N, \|x\| \leq 1\}$
$= Sup\ \{\|\alpha T(x)\|: x \in N, \|x\| \leq 1\} = Sup\{|\alpha|\|T(x)\|: x \in N, \|x\| \leq 1\}$
$= |\alpha| Sup\{\|T(x)\|: x \in N, \|x\| \leq 1\} = |\alpha|\|T\|$
Hence $\mathfrak{B}(N, N')$ is a normed linear space.

Claim: $\mathfrak{B}(N, N')$ is complete if N' is complete. Suppose N′ is complete.
Let $\{T_n\}$ be any Cauchy sequence in $\mathfrak{B}(N, N')$. Then $\|T_m - T_n\| \to 0$ as m, n $\to$ 0… (1).
For each $x \in N$ we have $\|T_m(x) - T_n(x)\| = \|(T_m - T_n)(x)\| \leq \|T_m - T_n\|\|x\| \to 0$ by (1).
Hence $\{T_n(x)\}$ is a Cauchy sequence in N′ for each $x \in N$.
Since N′ is complete, $\exists$ a vector in N′, which we denote by T(x) $\ni T_n(x) \to T(x)$.
This defines a mapping T of N into N′. Let $\alpha, \beta \in K$, $x, y \in N$.
Then $T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y) = \lim_{n\to\infty}\{\alpha T_n(x) + \beta T_n(y)\}$
$= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y)\} = \alpha T(x) + \beta T(y)$. $\therefore$ T is linear.
Now $\|T(x)\| = \|\lim_{n\to\infty} T_n(x)\| = \lim_{n\to\infty}\|T_n(x)\| \leq \lim_{n\to\infty}\|T_n\|\|x\| \leq Sup\{\|T_n\|\|x\|\}$
$= (Sup\|T_n\|)\|x\|$ … (2).
Now $|\|T_m\| - \|T_n\|| \leq \|T_m - T_n\| \to 0$ as m, n $\to \infty$.
$\therefore \{\|T_n\|\}$ is a Cauchy sequence of real numbers and hence convergent and bounded. So, $\exists\ k \geq 0 \ni \text{Sup}\|T_n\| \leq k$ … (3).
From (2) and (3) we have $\|T(x)\| \leq k\|x\|$ showing that T is bounded.
$\therefore T \in \mathfrak{B}(N, N')$.
Claim: To show that $T_n \to T$.
Let $\varepsilon > 0$. Then $\exists$ a positive integer $m_0 \ni n, m \geq m_0 \Rightarrow \|T_m - T_n\| < \varepsilon$ … (4).

Let x ∈ N be ∋ $\|x\| \leq 1$.
Then we can choose a +ve integer $m_x > m_0$ ∋ $\|T(x) - T_{m_k}\| \leq \frac{\varepsilon}{2}$ … (5).
Hence ∀ n ≥ m_0 and $\|x\| \leq 1$, $\|T_n(x) - T(x)\| = \|T_n(x) - T_m(x) + T_m(x) - T(x)\|$ $\leq \|T_n(x) - T_m(x)\| + \|T_m(x) - T(x)\| = \|(T_n - T_m)(x)\| + \|T_m(x) - T(x)\|$ $\leq \|T_n - T_m\|\|x\| + \|T_m(x) - T(x)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
Thus $\|T_n(x) - T(x)\| < \varepsilon$ ∀ n ≥ m_0 and x ∈ N ∋ $\|x\| \leq 1$.
Hence $Sup\{\|T_n(x) - T(x)\|: x \in N, \|x\| \leq 1\} < \varepsilon \ \forall n \geq m0$.
⇒ $Sup\{\|(T_n - T)(x)\|: x \in N, \|x\| \leq 1\} < \varepsilon$ ∀ n ≥ m_0. Ie. $\|T_n - T\| < \varepsilon$ ∀ n ≥ m_0.
⇒ $T_n \to T$.

<u>**Example 8**</u>: (3*): If M is a closed linear subspace of a normed linear space N, and if T is a natural mapping of N onto $\frac{N}{M}$ defined by T(x) = x + M show that T is a continuous linear transformation for which $\|T\| \leq 1$.

<u>**Solution**</u>: Let M be a closed linear subspace of a normed linear space N, and T be a natural mapping of N onto N/M defined by T(x) = x + M ∀ x ∈ N.
Clearly $\frac{N}{M}$ is a normed linear space with norm $\|x + M\| = \inf\{\|x + m\|: m \in M\}$. T is linear: Let x, y ∈ N; α, β be scalars. Then T(αx + βy) = αx + βy + M = (αx + M) + (βy + M) = α(x + M) + β(y + M) = αT(x) + βT(y). T is continuous: $\|Tx\| = \|x + M\| = \inf\{\|x + m\|: m \in M\} \leq \|x + m\|$ ∀ m ∈ M.
∴ For $m = \bar{0}$, $\|Tx\| \leq \|x\| = 1.\|x\|$ ∴ T is continuous.
Further $\|T\| = \text{Sup}\{\|Tx\|: x \in N, \|x\| \leq 1\} \leq \text{Sup}\{\|x\|: x \in N, \|x\| \leq 1\} \leq 1$.

Example 9: Let N, and N′ be normed linear spaces and T be a continuous linear transformation of N into N′. If M is the null space of T, show that T induces a natural linear transformation T′ of N/M into N′ and that $\|T'\| = \|T\|$.

<u>**Solution**</u>: Since T is continuous, M is a closed linear subspace of N. So, N/M is a normed linear space with the norm defined by $\|x + M\| = \inf\{\|x + m\|: m \in M\}$. We define T′: N/M → N′ by T′(x + M) = T(x) ∀ x + M ∈ N/M.
Claim: T′ is linear. Let x + M, y + M ∈ N/M and α,β be scalars.
Then T′{α(x + M) + β(y + M)} = T′(αx + βy + M) = T(αx + βy) = αT(x) + βT(y) = αT′(x + M) + βT′(y + M). Claim: $\|T'\| = \|T\|$.
$\|T'\| = \sup\{\|T'(x + M)\|: x \in N, \|x + M\| \leq 1\}$
$= \sup\{\|T(x)\|: x \in N, \inf\{\|x + m\|: m \in M\} \leq 1\}$
$= \sup\{\|T(x) + T(m)\|: x \in N, m \in M, \|x + m\| \leq 1\}$
$= \sup\{\|T(x + m)\|: x + m \in N, \|x + m\| \leq 1\} = \|T\|$.

HAHN BANACH THEOREM

Linear functional: We know that $\mathbb{R}$ and $\mathbb{C}$ are Banach Spaces. If we take $\mathbb{R}$ *or* $\mathbb{C}$ for N′ then $\mathfrak{B}(N, \mathbb{R})$ and $\mathfrak{B}(N, \mathbb{C})$ denote respectively the set of all continuous linear transformations from N into $\mathbb{R}$ *or* $\mathbb{C}$. We denote either of these sets by N* and call N* the conjugate space (or adjoint space or dual space). Members of N* are called ***continuous linear functionals or simply functionals***.
Note: (i) N* is a Banach space. (ii) All the theorems hold good for $\mathfrak{B}(N, N')$ also hold for $\mathfrak{B}(N, \mathbb{R})$ and $\mathfrak{B}(N, \mathbb{C})$. (iii) $\|f\| = \text{Sup}\{|f(x)|: x \in N, \|x\| \leq 1\}$

Lemma: (1*): Let M be a linear subspace of a normed linear space N and let f be a functional defined on M. If $x_0 \notin M$ and if $M_0 = M + \langle x_0 \rangle = \{x + \alpha x_0 : x \in M, \alpha$ real$\}$ is the linear subspace spanned by M and x_0, then f can be extended to a functional f_0 defined on M_0 such that $\|f_0\| = \|f\|$.

Proof: Case (i): Let N be a real normed linear space. Since x_0 is not in M, each vector ω in M_0 is uniquely expressible in the form $\omega = x + \alpha x_0$ with $x \in M$.
Define f_0 by setting $f_0(\omega) = f_0(x + \alpha x_0) = f(x) + \alpha r_0$ where r_0 is any real number.
Claim: For any choice of the real number r_0, f_0 is linear on $M_0 \ni f_0(x) = f(x)\ \forall\ x \in M$. Let β, γ be scalars and $x, y \in M$.

Then $f_0\{\beta(x + \alpha x_0) + \gamma(y + \alpha x_0)\} = f_0\{\beta x + \gamma y + (\beta + \gamma)\alpha x_0\}$
$= f(\beta x + \gamma y) + (\beta + \gamma)\alpha r_0.$
$= \beta f(x) + \gamma f(y) + \beta\alpha r_0 + \gamma\alpha r_0$
$= \beta\{f(x) + \alpha r_0\} + \gamma\{f(y) + \alpha r_0\}$
$= \beta\{f_0(x + \alpha x_0)\} + \gamma\{f_0(y + \alpha x_0)\}$

$\therefore$ f_0 is linear on M_0. Also for $x \in M$, $f_0(x) = f_0(x + 0x_0) = f(x) + 0r_0 = f(x)$.
So, f_0 extends f linearly to M_0.
Claim: $\|f_0\| = \|f\|$.
We have $\|f_0\| = \sup\{|f_0(x)|: x \in M_0, \|x\| \leq 1\}$
$\geq \sup\{|f_0(x)|: x \in M, \|x\| \leq 1\}$ $\because M_0 \supseteq M$.
$= \sup\{|f(x)|: x \in M, \|x\| \leq 1\}$ $\because f_0 = f$ on M.
$= \|f\|$
Thus, $\|f_0\| \geq \|f\|$ … (A)
To choose $r_0 \ni \|f_0\| \leq \|f\|$.
If x_1, x_2 are any two vectors in M_1 then $f(x_2) - f(x_1) = f(x_2 - x_1) \leq |f(x_2 - x_1)|$
$\leq \|f\|\|x_2 - x_1\| = \|f\|\|x_2 + x_0 - (x_1 + x_0)\| \leq \|f\|\{\|x_2 + x_0\| + \|-(x_1 + x_0)\|\}$
$= \|f\|\|x_2 + x_0\| + \|f\|\|x_1 + x_0\|$
Thus, $-f(x_1) - \|f\|\|x_1 + x_0\| \leq -f(x_2) + \|f\|\|x_2 + x_0\|$

Since this inequality holds for arbitrary $x_1, x_2 \in M$, we see that

Sup $\{-f(y) - \|f\|\|y + x_0\|\} \le \inf \{-f(y) + \|f\|\|y + x_0\|\}$.

Choose r_0 to be any real number such that

$\sup\limits_{y\in M} \{-f(y) - \|f\|\|y + x_0\|\} \le r_0 \le \inf\limits_{y\in M} \{-f(y) + \|f\|\|y + x_0\|\}$

$\Rightarrow -f(y) - \|f\|\|y + x_0\| \le r_0 \le -f(y) + \|f\|\|y + x_0\| \ \forall\ y \in M$…. (ii).

With this choice of r_0 we show that $\|f_0\| \le \|f\|$.

Let $\omega = x + \alpha x_0$ be any arbitrary vector in M_0.

Put $\frac{x}{\alpha}$ for y in (ii) to get $-f\left(\frac{x}{\alpha}\right) - \|f\| \left\|\frac{x}{\alpha} + x_0\right\| \le r_0 \le -f\left(\frac{x}{\alpha}\right) + \|f\| \left\|\frac{x}{\alpha} + x_0\right\|$ … (iii).

If $\alpha > 0$, then $r_0 \le -\frac{1}{\alpha} f(x) + \frac{1}{\alpha} \|f\|\|x + \alpha x_0\|$

$\Rightarrow f(x) + \alpha r_0 \le \|f\|\|x + \alpha x_0\|$

$\Rightarrow f_0(x + \alpha x_0) \le \|f\|\|x + \alpha x_0\|$.

If $\alpha < 0$, then $-\frac{1}{\alpha} f(x) + \frac{1}{\alpha} \|f\|\|x + \alpha x_0\| \le r_0$

$\Rightarrow f(x) + \alpha r_0 \le \|f\|\|x + \alpha x_0\|$

$\Rightarrow f_0(\omega) \le \|f\|\|\omega\|$

Thus, when $\alpha \ne 0$. $f_0(\omega) \le \|f\|\|\omega\| \ \forall\ \omega \in M_0$. … (iv).

When $\omega = 0$, $\|f_0\| = \|f\|$.

Replacing ω by $-\omega$, $f_0(-\omega) \le \|f\|\|-\omega\| \Rightarrow -f_0(\omega) \le \|f\|\|\omega\|$ … (v)

From (iv) and (v), $|f0(\omega)| \le \|f\|\|\omega\|$ … (vi).

$\therefore f_0$ is a linear bounded functional on M_0.

Since $\|f_0\| = \sup \{|f_0(\omega)|: \omega \in M_0, \|\omega\| \le 1\}$.

$\|f_0\| \le \|f\|$ … (B)

Hence $\|f_0\| = \|f\|$.

Case (ii) Let N be a complex normed linear space over $\mathbb{C}$.

Let g = R. P. of f, h = I. P. of f so that $f(x) = g(x) + ih(x) \ \forall\ x \in M$.

Let $x, y \in M$. Then $f(x + y) = f(x) + f(y)$ since f is linear.

$\Rightarrow g(x + y) + ih(x + y) = g(x) + ih(x) + g(y) + ih(y)$.

Comparing the real and imaginary parts $g(x + y) = g(x) + g(y)$, $h(x + y) = h(x) + h(y)$.

Let $\alpha \in \mathbb{R}$, $x \in M$.

Then $f(\alpha x) = \alpha f(x) \Rightarrow g(\alpha x) + i\, h(\alpha x) = \alpha\{g(x) + i\, h(x)\}$

Comparing the real and imaginary parts $g(\alpha x) = \alpha\{g(x)$, and $h(\alpha x) = \alpha\ \{h(x)\}$.

Thus, g, and h are linear on M.

Further $|g(x)| \le |f(x)| \le \|f\|\|x\|$, and $|h(x)| \le |f(x)| \le \|f\|\|x\|$.

$\therefore$ g, and h are bounded.

Thus, g, and h are real valued linear bounded functionals on M.

Also, we have $g(ix) + ih(ix) = f(ix) = if(x) = -h(x) + ig(x)$, for all $x \in M$.
Comparing the real and imaginary parts, $g(ix) = -h(x)$, $h(ix) = g(x)$.
Consequently, $f(x) = g(x) - ig(ix) = h(ix) + ih(x)$.
Since g is real functional on M, by case (i) g can be extended to a functional g_0 defined on $M_0 \ni \|g_0\| = \|g\|$.
Now define f_0 for $x \in M_0$ by $f_0(x) = g_0(x) - ig_0(ix)$.
Then f_0 is linear on $M_0 \ni f_0 = f$ on M.
[Let $x, y \in M_0$, $\alpha + i\beta \in K$.
So, $f_0(x + y) = g_0(x + y) - ig_0(ix + iy)$
$= g_0(x) + g_0(y) - ig_0(ix) - ig_0(iy)$
$= f_0(x) + f_0(y)$
and $f_0\{(\alpha + i\beta)x\} = g_0(\alpha x + i\beta x) - ig_0(-\beta x + i\alpha x)$
$= \alpha g_0(x) + \beta g_0(ix) - i\{-\beta g_0(x) + \alpha g_0(ix)\}$
$= \alpha g_0(x) + i\beta g_0(x) + \beta g_0(ix) - i\alpha g_0(ix)$
$= (\alpha + i\beta)\{g_0(x) - ig_0(ix)\}$
$= (\alpha + i\beta)f_0(x)$.
Thus, f_0 is linear on M_0. Also, $g_0 = g$ on $M \Rightarrow f_0 = f$ on M].
Let $x \in M_0$ be arbitrary and write $f_0(x) = r\,e^{i\theta}$ where $r \geq 0$ and θ is real.
Then $|f_0(x)| = r = e^{-i\theta}re^{i\theta} = e^{-i\theta}f_0(x) = f_0(e^{-i\theta}x) = g_0(e^{-i\theta}x) \leq |g_0(e^{-i\theta}x)|$
$\leq \|g_0(e^{-i\theta}x)\| \leq \|g_0\|\|e^{-i\theta}x\| = \|g_0\|\|x\| = \|g\|\|x\| \leq \|f\|\|x\|$.
$\therefore f_0$ is bounded and $\|f_0\| \leq \|f\|$
Also, as in case (i) it is obvious that $\|f_0\| \geq \|f\|$. Hence $\|f_0\| = \|f\|$

Hahn Banach Theorem 8: (5*): Let M be a linear subspace of a normed linear space N and let f be a functional defined on M. Then f can be extended to a functional F defined on the whole space N such that $\|F\| = \|f\|$.
Proof: Let P denote the set of all ordered pairs (f_λ, M_λ) where f_λ is an extension of f to the subspace $M_\lambda \supseteq M$ and $\|f_\lambda\| = \|f\|$.
Relation $\leq$ is defined on P by $(f_\lambda, M_\lambda) \leq (f_\mu, M_\mu)$ iff $M_\lambda \subseteq M_\mu$ and $f_\lambda = f_\mu$ on M_λ.
P is evidently non-empty, for, certainly $(f, M) \in P$.
Clearly $\leq$ is a partially ordering on P.
$\therefore$ $(P, \leq)$ is a poset.
Let $Q = \{(f_i, M_i)\}$ be a chain in P. Then Q has an upper bound $(\varphi, \cup M_i)$ where $\varphi(x) = f_i(x)\ \forall\ x \in M_i$ as detailed below.
Claim: $\cup M_i$ is a subspace of N where $(f_i, M_i) \in Q$.
Let $x, y \in \cup M_i$ and α, β be any scalars.
Let $x \in M_i$, $y \in M_j$ for some i and j.

Since Q is totally ordering either $M_i \subseteq M_j$ or $M_j \subseteq M_i$.
Without loss of generality assume $M_i \subseteq M_j$.
$\therefore x, y \in M_j$.
$\Rightarrow \alpha x + \beta y \in M_j \subseteq \cup M_i$.
$\therefore \cup M_i$ is a subspace of N.
Claim: φ is well defined.
Suppose $x \in M_i$ is such that $x \in M_i$ and $x \in M_j$.
Then by definition, $\varphi(x) = f_i(x)$ and $\varphi(x) = f_j(x)$.
By total ordering of Q, either f_i extends f_j or vice versa.
In either case $f_i(x) = f_j(x)$.
Thus, φ is well defined.
$(\varphi, \cup M_i)$ is an upper bound of Q.
By Zorn's lemma $\exists$ maximal element (F, H) in P.
Claim: H = N.
Suppose, if possible, N contains H properly. Then $\exists\, x_0 \in N\text{-}H$ and so, by the above lemma, f can be extended to a functional F_0 on $H_0 = H + \langle x_0 \rangle$ which contains H properly. But this contradicts the maximality of (F, H)
$\therefore$ H = N.

<u>Theorem 9</u>: (3*): Let N be a normed linear space and x_0 be a non-zero vector in N. Then there exists a functional F in N* ∍ $F(x_0) = \|x_0\|$ and $\|F\| = 1$. In particular, if $x, y \in N$ and $x \neq y$, then there exists a functional $f \in N^*$ ∍ $f(x) \neq f(y)$.

<u>Proof</u>: Let $M = \langle x_0 \rangle$ be the linear subspace of N spanned by x_0.
Define f_0 on M by $f_0(\alpha x_0) = \alpha\|x_0\|$.
Claim: f_0 is a functional on M ∍ $\|f_0\| = 1$.
Let $y_1, y_2 \in M$ so that $y_1 = \alpha x_0$, $y_2 = \beta x_0$ for some scalars α, β.
If γ, δ are any two scalars then $f_0(\gamma y_1 + \delta y_2) = f_0(\gamma\alpha x_0 + \delta\beta x_0) = f_0\{(\gamma\alpha + \delta\beta)x_0\}$
$= (\gamma\alpha + \delta\beta)\,\|x_0\| = \gamma\alpha\|x_0\| + \delta\beta\|x_0\| = \gamma f_0(\alpha x_0) + \delta f_0(\beta x_0) = \gamma f_0(y_1) + \delta f_0(y_2)$.
$\therefore f_0$ is linear.
Let $y = \alpha x_0 \in M$ so that $\|y\| = \|\alpha x_0\| = |\alpha|\|x_0\|$.
Now $|f_0(y)| = |f_0(\alpha x_0)| = |\alpha\|x_0\|| = |\alpha|\|x_0\| = \|y\|$.
$\therefore f_0$ is bounded.
Hence f_0 is a functional on M.
Further $\|f_0\| = \sup\{|f_0(y)| : y \epsilon M, \|y\| \leq 1\} = \sup\{\|y\| : y \epsilon M, \|y\| \leq 1\} = 1$.
Also, $f_0(x_0) = \|x_0\|$ by definition of f_0.
Hence by Hahn Banach theorem, f_0 can be extended to norm preserving functional $F \in N^*$ so that $F(x_0) = f_0(x_0) = \|x_0\|$ and $\|F\| = \|f_0\| = 1$.
In the particular case, since $x \neq y$, $x - y \neq \bar{0}$ and by the above part of this theorem,

$\exists$ f $\in$ N* such that f(x – y) = $\|x-y\| \neq 0$
$\Rightarrow$ f(x) – f(y) $\neq$ 0
$\Rightarrow$ f(x) $\neq$ f(y).

Theorem 10: (1*): Let M be a closed linear subspace of a normed linear space N and x_0 a vector not in M. Then there exists a functional F in N* such that F(M) = {0} and F(x_0) $\neq$ 0.

Proof: Consider the natural map φ: N $\to \frac{N}{M}$ such that φ(x) = x + M.
Then φ is a continuous linear transformation and if m $\in$ M, then φ(m) = m + M = 0 (Here 0 denotes zero element in $\frac{N}{M}$ which is M.)
In other words, φ(M) = {0}… (i).
Also, since $x_0 \notin$ M, we have φ(x_0) = x_0 + M $\neq$ 0 ($\neq$ zero element in M/N which is M). Hence, by the previous theorem, $\exists$ a functional f $\in \left(\frac{N}{M}\right)^*$ $\ni$ f(x_0 + M) = $\|x_0+M\| \neq 0$ $\because$ x_0 + M $\neq$ 0 (zero element in N/M ie. M) and $\|f\| = 1$… (ii).
We now define F by F(x) = f{φ(x)}.
Then F is a linear functional on N with the desired properties as shown below.
F is linear: F(αx + βy) = f{φ(αx + βy)}
= f{αx + βy + M}
= f{α(x + M) + β(y + M)}
= αf(x + M) + βf(y + M) $\because$ f is linear on $\frac{N}{M}$
= α[f{φ(x)}] + β[f{φ(y)}]
= αF(x) + βF(y).
F is bounded: $|F(x)| = |f\{\varphi(x)\}|$
$\leq \|f\|\|\varphi(x)\|$
$\leq \|f\|\|\varphi\|\|x\|$
$\leq \|f\|\|x\|$ $\because \|\varphi\| \leq 1$ by example 8.
$\therefore$ F is bounded. Thus, F is a functional on N ie. F $\in$ N*.
Further, if m $\in$ M, then F(m) = f{φ(m)}= f(**0**) = 0 so that F(M) = {0}.
and F(x_0) = f{φ(x_0)}.
= f(x_0 + M) $\neq$ 0 by (ii).

Example 11: Let M be a closed linear subspace of a normed linear space N and let x_0 be a point not in M. If d is the distance of x_0 from M, show that $\exists$ a functional F in N* such that F(M) = {0}, F(x_0) = 1 and $\|F\| = \frac{1}{d}$.

Solution: By definition, d = inf $\{\|x_0 - x\|$: x $\in$M$\}$ … (i).
Since M is closed and $x_0 \notin$ M, d > 0.
Now consider the subspace $M_0 = \{x + \alpha x_0 : x \in M, \alpha \in \mathbb{R}\}$ spanned by M and x_0.
Since $x_0 \notin M$, the representation of each vector y in M_0 in the form $y = x + \alpha x_0$ is unique.
Define the map f_0 on M_0 by $f_0(x + \alpha x_0) = \alpha$.
The map f_0 is well defined and linear on M_0.
Also, $f_0(x_0) = f_0(0 + 1\, x_0) = 1$
and if m $\in$M, then $f_0(m) = f_0(m + 0\, x_0) = 0$, ∍ $f_0(M) = \{0\}$.
Now $\|f_0\| = Sup\left\{\frac{|f_0(y)|}{\|y\|} : y \in M_0, y \neq 0\right\} = Sup\left\{\frac{|f_0(x+\alpha x_0)|}{\|x+\alpha x_0\|} : x \in M, \alpha \in \mathbb{R}, \alpha \neq 0\right\}$ $= Sup\left\{\frac{|\alpha|}{\|x+\alpha x_0\|} : x \in M, \alpha \in \mathbb{R}, \alpha \neq 0\right\} = Sup\left\{\frac{1}{\left\|\frac{x}{\alpha}+x_0\right\|} : x \in M, \alpha \in \mathbb{R}, \alpha \neq 0\right\}$ $= Sup\left\{\frac{1}{\|x_0 - z\|} : z = -\frac{x}{\alpha} \in M, \alpha \in \mathbb{R}, \alpha \neq 0\right\} = \frac{1}{inf\{\|x_0 - z\| : z \in M\}} = \frac{1}{d}$
Thus, f_0 is linear functional on M_0 such that $f_0(M) = \{0\}$, $f_0(x_0) = 1$ and $\|f_0\| = \frac{1}{d}$
∴ By Hahn Banach Theorem ∃ F ∈ N* such that $F(y) = f_0(y)$ ∀ y ∈ M_0 and $\|F\| = \|f_0\|$. Hence it follows that $F(M) = \{0\}$, $F(x_0) = 1$ and $\|F\| = \frac{1}{d}$.

Example 12: Let M be a closed linear subspace of a normed linear space N and let x_0 be a point not in M. If d is the distance of x_0 from M, show that ∃ a functional F in N* such that $F(M) = \{0\}$, $F(x_0) = d$ and $\|F\| = 1$.

Solution: By definition, d = inf $\{\|x_0 - x\|$: x $\in$M$\}$ … (i).
Since M is closed and $x_0 \notin$ M, d > 0.
Now consider the subspace $M_0 = \{x + \alpha x_0 : x \in M, \alpha \in \mathbb{R}\}$ spanned by M and x_0.
Since $x_0 \notin M$, the representation of each vector y in M_0 in the form $y = x + \alpha x_0$ is unique.
Define the map f_0 on M_0 by $f_0(x + \alpha x_0) = \alpha d$.
By the uniqueness of y, the map f_0 is well defined and also linear on M_0.
Also, $f_0(x_0) = f_0(0 + 1\, x_0) = 1.d$ and if m ∈ M, then $f_0(m) = f_0(m + 0\, x_0) = 0$, ∍ $f_0(M) = \{0\}$.
Now $\|f_0\| = Sup\left\{\frac{|f_0(y)|}{\|y\|} : y \in M_0, y \neq 0\right\}$
$= Sup\left\{\frac{|f_0(x+\alpha x_0)|}{\|x+\alpha x_0\|} : x \in M, \alpha \in \mathbb{R}, x \neq 0, \alpha \neq 0\right\}$

$= Sup\left\{\frac{|\alpha d|}{\|x+\alpha x_0\|} : x \in M, \alpha \in \mathbb{R}, \alpha \neq 0\right\} = Sup\left\{\frac{d}{\left\|\frac{x}{\alpha}+x_0\right\|} : x \in M, \alpha \in \mathbb{R}, \alpha \neq 0\right\}$

$= \text{d } Sup\left\{\frac{1}{\|x_0-z\|} : z = -\frac{x}{\alpha} \in M, \alpha \in \mathbb{R}, \alpha \neq 0\right\} = \frac{d}{inf\{\|x_0-z\|: z\in M\}} = 1$

Thus, f_0 is linear functional on M_0 such that $f_0(M) = \{0\}$, $f_0(x_0) = d$ and $\|f_0\| = 1$

∴ by Hahn Banach Theorem ∃ F ∈ N* such that F(y) = $f_0(y)$ ∀ y ∈ M_0 and $\|F\| = \|f_0\|$. Hence it follows that F(M) = {0}, $F(x_0)$ = d and $\|F\| = 1$.

Example 13: (3*): Prove that a normed linear space is separable if its conjugate space is separable.

Solution: Let N be a normed linear space whose conjugate space N* is separable.

Consider S = {f: f ∈ N*, $\|f\|$ = 1}.

Since every subspace of a metric space is separable, S must be separable.

Hence S contains countable dense subset, say, A = {f_1, f_2, …, f_n, …}.

Since each f_n ∈ S we have $\|f_n\|$ = 1 ∀ n.

Since $\|f_n\|$ = sup {$|f_n(x)|$: $\|x\|$ = 1}, for each n there must exist some vector x_n with $\|x_n\|$ = 1 ∋ $|f_n(x_n)| > \frac{1}{2}$. [If such x_n does not exist, this would contradict the fact that $\|f_n\|$ = 1].

Let M be the closed linear subspace in N generated by the sequence {x_n}.

We assert that M = N. Suppose, if possible, that M ≠ N and let x_0 ∈ N – M.

Then ∃ a functional F ∈ N* ∋ $\|F\|$ = 1, $F(x_0) \neq 0$ and F(x) = 0 if x ∈ M.

Since $\|F\|$ = 1, F ∈ S and since each x_n ∈ M, we have $F(x_n)$ = 0 for n = 1, 2, ….

Now $\frac{1}{2} < |f_n(x_n)| = |f_n(x_n) - F(x_n) + F(x_n)| \leq |f_n(x_n) - F(x_n)| + |F(x_n)|$

$= |(f_n - F)(x_n)|$ ∵ $F(x_n) = 0$.

$\leq \|f_n - F\|\|x_n\| = \|f_n - F\|$ ∵ $\|x_n\| = 1$.

Thus, $\|f_n - F\| > \frac{1}{2}$ ∀ n. Now since A is dense in S, every point of S is an adherent point of A so that each sphere centered at arbitrary f ∈ S must contain a point of A. But the open sphere {f: $\|f - F\| < \frac{1}{2}$ } centered at F ∈ S contains no point of A by (i).

We thus arrive at a contradiction and so we must have M = N.

It then follows that the set of all linear combinations of the x_n's whose coefficients are rational or if N is complex have rational real and imaginary parts, contribute a countable set everywhere dense in N and consequently N is separable.

THE NATURAL IMBEDDING OF N IN N**

Since N* is a normed linear space, whenever N is, (N*)* is called a second conjugate of N and is denoted by N**.

Definition: A normed linear space is said to be reflexive if N = N**.

Definition: Week topology on a normed linear space N:

Definition: Let N and N′ be normed linear spaces. An isometric isomorphism of N into N′ is a one – to – one linear transformation T of N into N′ such that $\|T(x)\| = \|x\|$ for every x in N; and N is said to be isometrically isomorphic to N′ if there exists an isometric isomorphism of N onto N′.

Theorem 11: (4*): Let N be an arbitrary normed linear space. Then, each vector x in N induces a functional F_x on N* defined by $F_x(f) = f(x)\ \forall\ f \in N^*$ ∋ $\|F_x\| = \|x\|$. Further the mapping J: N → N** ∋ $J(x) = F_x\ \forall\ x \in N$ defines an isometric isomorphism of N into N**.

Proof: Let x ∈ N and f∈N*. Define a function F_x: N*→ K by $F_x(f) = f(x)\ \forall\ f \in N^*$. Claim: F_x is linear and bounded.

Let f, g ∈ N* and α, β be scalars (∈ K).

Now $F_x(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha F_x(f) + \beta F_x(g)$ and

$|F_x(f)| = |f(x)| \le \|f\|\|x\|$…(1) where the constant $\|x\|$ is a bound for F_x.

Thus F_x (∈N**) is a functional on N*.

Claim: $\|F_x\| = \|x\|$.

$\|F_x\| = \text{Sup}\ \{|F_x(f)|: f \in N^*, \|f\| \le 1\}$

$= \text{Sup}\ \{\|f\|\|x\|: f \in N^*, \|f\| \le 1\} \le \|x\|$ ... (2).

Again, when x = 0, by (2) $\|F_0\| \le \|0\| = 0$. But for any x, $\|F_x\| \ge 0$.

Thus $\|F_0\| = 0$ and so $\|F_x\| = \|x\|$ when x = 0.

Let x be any non – zero vector in N.

By a theorem ∃ a function F ∈ N* ∋ $F(x) = \|x\|$ and $\|F\| = 1$.

But $\|F_x\| = \text{Sup}\ \{|F_x(f)|: f \in N^*, \|f\| = 1\} = \text{Sup}\ \{|f(x)|: f \in N^*, \|f\| = 1\}$ and since $\|x\| = F(x) = |F(x)| \le \text{Sup}\ \{|F(x)|: x \in N, F \in N^*, \|F\| = 1\}$ we have $\|F_x\| \ge \|x\|$… (3).

From (2) and (3) $\|F_x\| = \|x\|$ … (4).

Claim: The mapping J: N → N** ∋ $J(x) = F_x\ \forall\ x \in N$ is linear.

For any x, y ∈ N, f ∈ N* and α∈K, $F_{x+y}(f) = f(x + y) = f(x) + f(y) = F_x(f) + F_y(f)$ $= (F_x + F_y)(f)$. and $F_{\alpha x}(f) = f(\alpha x) = \alpha f(x) = \alpha F_x(f) = (\alpha F_x)(f)\ \forall\ f \in N^*$.

Thus, $F_{x+y} = F_x + F_y$ and $F_{\alpha x} = \alpha F_x$. ∀ x, y ∈ N, and α∈K.

Now $J(x + y) = F_{x+y} = F_x + F_y = J(x) + J(y)$ and $J(\alpha x) = F_{\alpha x} = \alpha F_x = \alpha J(x)$.

Thus, J is linear.

Claim: J is an isometry

Let x, y ∈N. Then $\|J(x) - J(y)\| = \|F_x - F_y\| = \|F_{x-y}\| = \|x - y\|$ … (5) by (4)
Thus J preserves norm and hence it is an isometry.
Also, from (5), J(x) – J(y) = 0 ⇒ x – y = 0. Ie. J(x) = J(y) ⇒ x = y so that J is one-one. Also $\|J(x)\| = \|F_x\| = \|x\|$
Hence J defines isometric isomorphism of N into N**.

OPEN MAPPING THEOREM:

Lemma: (3*): If B and B′ are Banach Spaces, and if T is a continuous linear transformation of B onto B′, then the image of each open sphere centered on the origin in B contains an open sphere centered on the origin in B′.

Proof: We denote by S_r and S_r' the open spheres with radius r centered at origin in B and B′ respectively. Then $S_r = rS_1$.
Then clearly $T\{S_r\} = T\{rS_1\} = rT\{S_1\}$ so that it suffices to show that $T\{S_1\}$ contains some S_s'.
We begin by proving that $\overline{T\{S_1\}}$ contains some $S_{s'}$.
For each positive integer n, consider the open sphere S_n in B.
Then clearly $B = \bigcup_{n=1}^{\infty} S_n$. Since T is onto, we see that $B' = T[B] = T[\bigcup_{n=1}^{\infty} S_n] = \bigcup_{n=1}^{\infty} T\{S_n\}$.
Since B′ is complete, Baire's theorem implies that for some n_0, $\overline{T\{S_{n_0}\}}$ has an interior point y_0, which may be assumed to lie in $T\{S_{n_0}\}$.
The mapping $y \to y - y_0$ is a homeomorphism of B′ onto itself, so $\overline{T\{S_{n_0}\}} - y_0$ has the origin as an interior point.
Since y_0 is in $T\{S_{n_0}\}$, we have $T\{S_{n_0}\} - y_0 \subseteq T\{S_{2n_0}\}$; and from this we obtain $\overline{T\{S_{n_0}\}} - y_0 = \overline{T\{S_{n_0}\} - y_0} \subseteq \overline{T\{S_{2n_0}\}}$, which shows that the origin is an interior point of $\overline{T\{S_{2n_0}\}}$.
Multiplication by any non-zero scalar is a homeomorphism of B′ onto itself, so $\overline{T\{S_{2n_0}\}} = \overline{2n_0 T\{S_1\}} = 2n_0\, \overline{T\{S_1\}}$; and it follows from this that the origin is an interior point of $\overline{T\{S_1\}}$.
So for some ε > 0, $S_\varepsilon' \subseteq \overline{T\{S_1\}}$.
We conclude the proof by showing that $S_\varepsilon' \subseteq T\{S_3\}$, which is clearly equivalent to $S_{\frac{\varepsilon}{3}}' \subseteq T\{S_1\}$.
Let y be a vector in B′ so that $\|y\| < \varepsilon$.
Since y is in $\overline{T\{S_1\}}$, ∃ a vector x_1 in B ∋ $\|x_1\| < 1$ and $\|y - y_1\| < \frac{\varepsilon}{2}$, where $y_1 = T(x_1)$. We next observe that $S_{\frac{\varepsilon}{3}}' \subseteq T\left\{S_{\frac{1}{2}}\right\}$, so ∃ a

vector x_2 in B ∋ $\|x_2\| < \frac{1}{2}$ and $\|(y - y_1) - y_2\| < \frac{\varepsilon}{4}$, where $y_2 = T(x_2)$.
Continuing in this way, we obtain a sequence $\{x_n\}$ in B such that $\|x_n\| < \frac{1}{2^{n-1}}$, and $\|y - (y_1 + y_2 + \cdots + y_n)\| < \frac{\varepsilon}{2^n}$, where $y_n = T(x_n)$.
If we put $s_n = x_1 + x_2 + \ldots + x_n$, then it follows from $\|x_n\| < \frac{1}{2^{n-1}}$, that $\{s_n\}$ is a Cauchy sequence in B for which $\|s_n\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_n\| < 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 2$.
Since B is complete, so there exists a vector x in B such that $s_n \to x$; and $\|x\| = \|\lim s_n\| = \lim \|s_n\| \leq 2 < 3$ shows that x is in S_3.
All that remains is to notice that the continuity of T yields $T(x) = T(\lim s_n) = \lim T(s_n) = \lim (y_1 + y_2 + \ldots + y_n) = y$, from which we see that y is in $T(S_3)$.
Hence the lemma.

Open Mapping theorem: 1*: Let B and B′ be Banach Spaces and T be a continuous linear transformation of B onto B′. Then T is an open mapping.
Proof: Let G be an open set in B.
Let y ∈ T(G) be an arbitrary point.
Since T is onto ∃ x ∈ G ∋ T(x) = y.
⇒ ∃ an open sphere $x + S_r(0) = S_r(x) \subseteq G$ for some r > 0…(1).
Now by the above lemma, ∃ an open sphere S_ε' in B′ ∋ $S_\varepsilon' \subseteq T\{S_r\}$ for some ε > 0.
Now $y + S_\varepsilon' \subseteq y + T\{S_r\} = T(x) + T\{S_r\} = T\{x + S_r\} \subseteq T(G)$ by (1).
Thus, to each y ∈ T(G) ∃ an open sphere in B′ centered at y and contained in T(G) and consequently T(G) is an open set.
So, T(G) is open in B′ whenever G is open in B.
Hence T is an open map.

Theorem: (2*): (Banach's Theorem) Let B and B′ be Banach Spaces and T be a continuous one – one linear transformation of B onto B′. Then T is a homeomorphism.
Proof: Let G be an open set in B. Let y ∈ T(G) be an arbitrary point.
Since T is onto ∃ x ∈ G ∋ T(x) = y.
⇒ ∃ an open sphere $S_r(x) = x + S_r(0) \subseteq G$ for some r > 0…(1).
Now by the above lemma, ∃ an open sphere $S_\varepsilon'(0)$ in B′ ∋ $S_\varepsilon' \subseteq T\{S_r(0)\}$ for some ε > 0. Now $y + S_\varepsilon' \subseteq y + T\{S_r\} = T(x) + T\{S_r\} = T\{x + S_r(0)\} \subseteq T(G)$ by (1).
Thus, to each y ∈ T(G) ∃ an open sphere in B′ centered at y and contained in T(G) and consequently T(G) is an open set.

So, T(G) is open in B′ whenever G is open in B.
Hence T is an open map.
Since T is also one to one, onto and continuous T is a homeomorphism.

Projections:

Definition: A projection E on a linear space L is simply an idempotent ($E^2 = E$) linear transformation of L into itself.

Note: Projection on L can be described geometrically as follows.

(1) a projection E determines a pair of linear subspaces M and N such that $L = M \oplus N$ where $M = \{E(x): x \in L\}$ and $N = \{x \in L: E(x) = 0\}$ are the range and null spaces of E respectively.

(2) A pair of linear subspaces M and N such that $L = M \oplus N$ determines a projection E whose range and null space are M and N (If $z = x + y$ is a unique representation of a vector in L as a sum of vectors in $x \in M$ and $y \in N$, then E is defined by $E(z) = x$.

Definition: Projection on a Banach space is an idempotent operator E on B. ie.
(i) It is a projection on B if $E^2 = E$ ie. E is a projection in the algebraic sense
(ii) E is continuous.

Theorem: If P is a projection on a Banach space B, and M and N are it's range and null space, then M and N are closed linear subspaces of B such that $B = M \oplus N$.

Proof: Let P is a projection on a Banach space B, and M and N be it's range and null spaces. So, $P^2 = P$, P is continuous, $M = \{P(z): z \in B\}$ and $N = \{z \in B: P(z) = 0\}$.

Let $z \in B$. Then for identity operator I, $z = I(z) + P(z) – P(z) = P(z) + (I – P)(z)$.
Now $P(z) \in M$ and since $P\{(I – P)(z)\} = P(z) – P^2(z) = P(z) – P(z) = 0$, $(I – P)(z) \in N$. So, that $B = M + N$.
Let $z = x + y$ where $x \in M$, and $y \in N$.
Then $P(z) = P(x + y) = P(x) + P(y) = P(x) + 0 = P(x)$… (i).
Again since, $P(x) = P^2(x)$, $P(I – P)(x) = 0$ so that $(I – P)(x) = x – P(x) \in N$.
But $(I – P)(x) = x – P(x) \in M$.
$\therefore\ x – P(x) \in M \cap N = \{0\} \Rightarrow P(x) = x \in M$ … (ii).
$(I – P)(z) = z – Pz = x + y – P(x) = x + y – x = y$ … (iii).
From (i), (ii) and (iii) $z = x + y = P(z) + (I – P)(z)$ where $P(z) \in M$, $(I – P)(z) \in N$.
$\therefore\ B = M \oplus N$. Moreover $P(z) = P(x + y) = x$.
Since Null space of any continuous linear transformation is closed N is closed.

[Let $z \in \overline{N}$. $\Rightarrow \exists\{z_n\}$ in N $\ni \{z_n\} \to z$.
Now $P(z) = P\{\lim (z_n)\} = \lim P(z_n) = 0$ since $z_n \in N \Rightarrow z \in N$.
$\therefore \overline{N} \subseteq N \Rightarrow \overline{N} = N$].
Now $M = \{P(z): z \in B\} = \{z \in B: z = P(z)\} = \{z: (I - P)(z) = 0\}$.
$\therefore$ M is null space of linear operator I – P. Hence M is closed. Hence the theorem.

Theorem: Let B be a Banach space and M, N be closed linear subspaces of B such that $B = M \oplus N$. If $z = x + y$ is the unique representation of a vector in B as a sum of vectors in M and N, then the mapping P defined by $P(z) = x$ is a projection on B whose range and null space are M and N.

Proof: Since $B = M \oplus N$, every element z of B can be uniquely expressed as $z = x + y$ where $x \in M$, $y \in N$.
$P: B \to B$ is defined by $P(z) = P(x + y) = x \ \forall z \in B$.
Clearly $P(x) = x \ \forall x \in M$ and $P(y) = 0 \ \forall y \in N$.
P is linear, for, $P(\alpha z_1 + \beta z_2) = P\{\alpha(x_1 + y_1) + \beta(x_2 + y_2)\}$.
$= P(\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2 = \alpha P(z_1) + \beta P(z_2)$.
Range of $P = \{P(z) : z \in B\} = \{P(x + y) : x \in M, y \in N\} = \{x : x \in M\} = M$.
Null space of $P = \{z \in B : Pz = 0\} = \{ x + y \in M \oplus N : P(x + y) = 0\}$
$= \{ x + y \in M \oplus N : x = 0\} = \{y : y \in N\} = N$.
Let $z = x + y \in M \oplus N$. Then $P^2 z = P\{P(z)\} = P(x + 0) = x = P(z)$
ie. $P^2(z) = P(z) \ \forall z \in B$. So, $P^2 = P$. Thus, P is idempotent.
If B' denotes the linear space B equipped with the norm defined by $\|z\|' = \|x\| + \|y\|$
Then, B' is a Banach space and since for $z = x + y \in M \oplus N$, $\|P(z)\|' = \|x\| \leq \|x\| + \|y\| = 1\ \|z\|' \ \forall z \in B'$, P is continuous as a mapping of B' into B. If I denotes the identity mapping of B' onto B, then $\|I(z)\| = \|z\| = \|x + y\| \leq \|x\| + \|y\| = 1\ \|z\|' \ \forall z \in B'$ shows that I is continuous as a one to one, linear transformation of B' onto B. $\therefore$ I is a homeomorphism and so B' and B have the same topology. Hence the theorem.

Graph of a mapping:

Definition: Let X, Y be any two non – empty sets and $f: X \to Y$ be a mapping. Then the graph of f, denoted by f_G, is defined as $\{(x, f(x)): x \in X\}$.

Remark: Let N, N′ be normed linear spaces. Then $N \times N'$ is a normed linear space with coordinate wise linear operations and the norm $\|(x, y)\| = (\|x\|^p + \|y\|^p)^{\frac{1}{p}}$ where $x \in N$, $y \in N'$ and $1 \leq p \leq \infty$.

Moreover, this norm induces the product topology on $N \times N'$ and $N \times N'$ is complete iff both N and N′ are complete. In future we mostly use norm when p = 1.

Definition: Let B, B′ be Banach spaces and T: B → B′ be a linear transformation. $T_G = \{(x, T(x)): x \in B\}$ is called ***graph*** of T.
Note: T_G is a subspace of $B \times B'$.

Definition: Let N, N′ be normed linear spaces and D be a subspace of N. Then a linear transformation T: D → N′ is said to be ***closed linear transformation*** if $x_n \in D$, $x_n \to x$ and $T(x_n) \to y$ implies $x \in D$ and $y = T(x)$.

Theorem: Let N, N′ be normed linear spaces and D be a subspace of N. Then a linear transformation T: D → N′ is closed if and only if the graph of T_G is closed.

Proof: Given that N and N′ are normed linear spaces and D is a subspace of N.
Suppose the linear transformation T: D → N′ is a closed.
Required to prove that T_G is closed.
Let (x, y) be a limit point of T_G. We prove that $(x, y) \in T_G$.
By definition of limit point, ∃ a sequence $(x_n, T(x_n))$ of points in T_G, where $x_n \in D$, converging to (x, y).
Now $(x_n, T(x_n)) \to (x, y) \Rightarrow \|(x_n, T(x_n)) - (x, y)\| \to 0$.
$\Rightarrow \|(x_n - x, T(x_n) - y)\| \to 0$.
$\Rightarrow \|x_n - x\| + \|(T(x_n) - y)\| \to 0 \quad \because \|(x, y)\| = \|x\| + \|y\|$
$\Rightarrow \|x_n - x\| \to 0, \ \|(T(x_n) - y)\| \to 0$
$\Rightarrow x_n \to x, T(x_n) \to y$.
$\Rightarrow x \in D$ and $T(x) = y$ since T is closed linear transformation.
$\Rightarrow (x, y) = (x, T(x)) \in T_G$. ∴ T_G is closed.

Conversely suppose T_G is closed.
Let $\{x_n\}$ be a sequence in $D \ni x_n \to x$ and $T(x_n) \to y$.
To prove T is closed we have to show that $x \in D$ and $y = T(x)$.
Now (x, y) is an adherent point of T_G so that $(x, y) \in \overline{T_G}$.
But $T_G = \overline{T_G}$ since T_G is closed. $\Rightarrow (x, y) \in T_G$
Then, by the definition of T_G, $x \in D$ and $T(x) = y$.
∴ T is a closed linear transformation.

The Closed Graph Theorem: 4*: Let B, B′ be Banach spaces and T: B → B′ be a linear transformation. Then T is continuous mapping if and only if it's graph is closed.

Proof: Suppose T: B $\rightarrow$ B′ be a continuous linear transformation and T_G be it's graph. Ie. $x_n \rightarrow x \Rightarrow T(x_n) \rightarrow T(x)$.

Claim: $T_G = \{(x, T(x)) : x \in B\}$ is closed.

Let (x, y) be a limit point of T_G.

$\Rightarrow \exists$ a sequence $(x_n, T(x_n)) \in T_G \ni (x_n, T(x_n)) \rightarrow (x, y)$.

$\Rightarrow (x_n, T(x_n)) - (x, y) \rightarrow 0$

$\Rightarrow \|(x_n, T(x_n)) - (x, y)\| \rightarrow 0$.

$\Rightarrow \|(x_n - x, T(x_n) - y)\| \rightarrow 0$.

$\Rightarrow \|x_n - x\| + \|(T(x_n) - y)\| \rightarrow 0$ since $\|(x, y)\| = \|x\| + \|y\|$

$\Rightarrow \|x_n - x\| \rightarrow 0$, $\|(T(x_n) - y)\| \rightarrow 0$

$\Rightarrow x_n \rightarrow x$, $T(x_n) \rightarrow y$.

But $x_n \rightarrow x \Rightarrow T(x_n) \rightarrow T(x)$ since T is continuous linear transformation.

$\Rightarrow y = T(x)$. $\therefore$ limit point $(x, y) = (x, T(x)) \in T_G$.

$\Rightarrow T_G$ contains all its limit points. $\therefore$ T_G is closed.

Conversely suppose T_G is closed.

Then T_G is a subspace of B $\times$ B′.

For, $T_G = \{(x, T(x)): x \in B\} \subseteq B \times B'$

and $(x_1, T(x_1)), (x_2, T(x_2)) \in T_G$ and $\alpha, \beta \in K$ implies that

$$\alpha(x_1, T(x_1)) + \beta (x_2, T(x_2)) = (\alpha x_1, \alpha T(x_1)) + (\beta x_2, \beta T(x_2))$$
$$= (\alpha x_1+\beta x_2, \alpha T(x_1) + \beta T(x_2))$$
$$= (\alpha x_1+\beta x_2, T(\alpha x_1+\beta x_2)) \in T_G.$$

Now T_G is complete, and so Banach space.

Define a map $\varphi: T_G \rightarrow B$ by $\phi (x, T(x)) = x \ \forall \ (x, T(x)) \in T_G$.

φ is one- one, for, let $\varphi (x_1, T(x_1)) = \varphi (x_2, T(x_2))$

$\Rightarrow x_1 = x_2$

$\Rightarrow (x_1, T(x_1)) = (x_2, T(x_2))$.

φ is onto, for, let $x \in B$. Then $\exists$ (x, Tx) $\in T_G$ and then φ (x, Tx) = x.

φ is linear. For, $\varphi \{\alpha(x_1, Tx_1) + \beta(x_2, Tx_2)\} = \varphi\{\alpha x_1+ \beta x_2, \alpha Tx_1 + \beta Tx_2)\} = \alpha x_1+ \beta x_2$ $= \alpha\varphi(x_1, Tx_1) + \beta\varphi(x_2, Tx_2)$.

Also $\varphi$is continuous. For, $\|\varphi(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|(x, Tx)\|$

$\Rightarrow \|\varphi(x, Tx)\| \leq 1 \ \|(x, Tx)\| \ \forall$ (x, Tx) $\in T_G$.

Thus, $\varphi: T_G \rightarrow B$ is one-one, onto and continuous linear transformation.

Then by a theorem following open mapping theorem φ is a homeomorphism.

So, φ^{-1}: B $\rightarrow T_G$ is bounded (continuous).

Now $\|Tx\| \leq \|x\| + \|Tx\| = \|(x, Tx)\| = \|\varphi^{-1}(x)\| \leq k \ \|x\|$.

Ie. $\|Tx\| \leq k \ \|x\| \ \forall x \in B$. $\therefore$ T is continuous. Hence the theorem.

Lemma: (3*) (In detail) If B and B′ are Banach Spaces, and if T is a continuous linear transformation of B onto B′, then the image of each open sphere centered on the origin in B contains an open sphere centered on the origin in B′.

Proof: We denote by S_r and S_r' the open spheres with radius r centered at origin in B and B′ respectively. Then $S_r = rS_1$.

Then clearly $T[S_r] = T[rS_1] = rT[S_1]$ so that it suffices to show that $T[S_1]$ contains some S_s'.

We begin by proving that $\overline{T[S_1]}$ contains some $S_{s'}$.

For each positive integer n, consider the open sphere S_n in B.

Then clearly $B = \bigcup_{n=1}^{\infty} S_n$.

Since T is onto, we see that $B' = T[B] = T[\bigcup_{n=1}^{\infty} S_n] = \bigcup_{n=1}^{\infty} T[S_n]$.

Since B′ is complete, by Baire's theorem, for some n_0, $\overline{T[S_{n_0}]}$ has an interior point y_0, which may be assumed to lie in $T[S_{n_0}]$.

[y is an interior point of $\overline{T[S_{n_0}]}$ ⇒ ∃ open set G ∋ y ∈ G ⊆ $\overline{T[S_{n_0}]}$.

y ∈ $\overline{T[S_{n_0}]}$ ⇒ y is a limit point of $T[S_{n_0}]$. ⇒ the nbd G of y must contain a point y_0 of $T[S_{n_0}]$. Thus $y_0 \in T[S_{n_0}]$ ∋ $y_0 \in G \subseteq \overline{T[S_{n_0}]}$.

⇒ y_0 is an interior point of $\overline{T[S_{n_0}]}$.]

The mapping f: B′ → B′ ∋ f(y) = y – y_0 is a homeomorphism.

For, f is clearly one – one, onto and if y_n ∈B′ is ∋ $y_n \to y$ then $f(y_n) = y_n - y_0 \to y - y_0 = f(y)$ and $f^{-1}(y_n) = y_n + y_0 \to y + y_0 = f^{-1}(y)$ so that f and f^{-1} are both continuous. Claim: $\overline{T[S_{n_0}]} - y_0$ has the origin as an interior point. y_0 is an interior point of $\overline{T[S_{n_0}]}$ ⇒ ∃ open set G ∋ y_0 ∈ G ⊆ $\overline{T[S_{n_0}]}$ ⇒ f(y_0) ∈ f(G) ⊆ f[$\overline{T\{S_{n_0}\}}$].

⇒ 0 = $y_0 - y_0$ ∈ f(G) ⊆ $\overline{T[S_{n_0}]} - y_0$. ⇒ 0 is an interior point of $\overline{T[S_{n_0}]} - y_0$. … (2). Claim:
$T[S_{n_0}] - y_0 \subseteq T[S_{2n_0}]$.

Let y ∈ $T[S_{n_0}] - y_0$. Then ∃ x ∈ S_{n_0} ∋ y = T(x) – y_0.

But $y_0 \in T[S_{n_0}]$ ⇒ $y_0 = T(x_0)$ for some $x_0 \in S_{n_0}$.

Thus y = T(x) – T(x_0) = T(x – x_0) … (3) where x, $x_0 \in S_{n_0}$.

Also x, $x_0 \in S_{n_0}$ ⇒ $\|x\| < n_0$ and $\|x_0\| < n_0$.

⇒ $\|x - x_0\| \le \|x\| + \|x_0\| < 2n_0$.

⇒ x – $x_0 \in S_{2n_0}$ ⇒ T(x – x_0) ∈ $T[S_{2n_0}]$ ⇒ y ∈ $T[S_{2n_0}]$ by (3).

Thus we have $T[S_{n_0}] - y_0 \subseteq T[S_{2n_0}] = 2n_0\, T[S_1]$

Since f is a homeomorphism, $f[\overline{T[S_{n_0}]}] = \overline{f[T[S_{n_0}]]}$

$\Rightarrow \overline{T[S_{n_0}]} - y_0 = \overline{T[S_{n_0}] - y_0} \subseteq \overline{T[S_{2n_0}]} = 2n_0\, \overline{T[S_1]}$ and it follows from (2) that the origin is an interior point of $\overline{T[S_1]}$.

$\Rightarrow$ for some $\varepsilon > 0$, $S_\varepsilon' \subseteq \overline{T[S_1]}$ … (4)

We conclude the proof by showing that $S_\varepsilon' \subseteq T[S_3]$, which is clearly equivalent to $S_{\frac{\varepsilon}{3}}' \subseteq T[S_1]$

Let $y \in S_\varepsilon'$. So $\|y\| < \varepsilon$.

$\therefore$ y is in $\overline{T[S_1]} \Rightarrow$ y is a limit point of $T[S_1] \Rightarrow \exists\, y_1 \in T[S_1] \ni \|y - y_1\| < \frac{\varepsilon}{2}$.

But $y_1 \in T[S_1] \Rightarrow \exists$ a vector x_1 in $S_1 \ni \|x_1\| < 1$ and $\|y - y_1\| < \frac{\varepsilon}{2}$, where $y_1 = T(x_1)$. From (4), $S'_{\varepsilon/2} \subseteq \overline{T[S_{1/2}]}$ and since $\|y - y_1\| < \frac{\varepsilon}{2}$, $y - y_1 \in S'_{\varepsilon/2} \subseteq \overline{T\left[S_{\frac{1}{2}}\right]}$. $\therefore$ as above

$\exists\, y_2 \in T\left[S_{1/2}\right] \ni \|(y - y_1) - y_2\| < \varepsilon/4$ where $y_2 = T(x_2)$ and $\|x_2\| < 1/2$

$T\left[S_{1/2}\right] \ni \|(y - y_1) - y_2\| < \frac{\varepsilon}{4}$ *where* $y2 = T(x2), x2 \in S_{1/2}$ & $\|x_2\| < \frac{1}{2}$.

Continuing in this way, we obtain a sequence $\{x_n\}$ in B such that $\|x_n\| < \frac{1}{2^{n-1}}$, and $\|y - (y_1 + y_2 + \cdots + y_n)\| < \frac{\varepsilon}{2^n}$, … (5) where $y_n = T(x_n)$.

If we put $s_n = x_1 + x_2 + \ldots + x_n$, then $\|s_n\| \le \|x_1\| + \|x_2\| + \cdots + \|x_n\|$

$< 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 2$… (6)

Also for $n > m$, $\|s_n - s_m\| = \|x_{m+1} + x_{m+2} + \cdots + x_n\|$

$\le \|x_{m+1}\| + \|x_{m+2}\| + \cdots + \|x_n\|$

$< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{n-1}} = \frac{\frac{1}{2^m}\left(1 - \frac{1}{2^{n-m}}\right)}{1 - \frac{1}{2}} = \frac{1}{2^{m-1}} - \frac{1}{2^{n-m-1}} \to 0$ as $m, n \to \infty$.

$\therefore \{s_n\}$ is a Cauchy sequence in B. Since B is complete, so there exists a vector x in B such that $s_n \to x$; and so $\|x\| = \|\lim s_n\| = \lim \|s_n\| \le 2 < 3. \Rightarrow x \in S_3$.

Now $y_1 + y_2 + \ldots + y_n = T(x_1) + T(x_2) + \ldots + T(x_n)$

$= T(x_1 + x_2 + \ldots + x_n) = T(s_n)$.

Since T is continuous, $x = \lim s_n$

$\Rightarrow T(x) = T(\lim s_n) = \lim T(s_n) = \lim (y_1 + y_2 + \ldots + y_n) = y$, by (5)

Thus $y = T(x)$ where $\|x\| < 3$, so that $y \in T[S_3]$. $\therefore S_\varepsilon' \subseteq T[S_3]$.

Hence the lemma.

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E – CONTENT

PAPER: M 301,

FUNCTIONAL ANALYSIS

M. Sc. II YEAR, SEMESTER - III

UNIT – II: HILBERT SPACES

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301, FUNCTIONAL ANALYSIS
UNIT – II

THE CONJUGATE OF AN OPERATOR

Theorem: 5 *: **(The uniform bounded theorm or Banach Steinhaus theorem)**.
Let B be a Banach space and N be a normed linear space. If $\{T_i\}$ is a non-empty set of bounded (i.e, continuous) linear transformation of B into N having the property that $\{T_i(x)\}$ is a bounded subset of N for each vector x in B, then $\{\|T_i\|\}$ is a bounded set of numbers ie. $\{T_i\}$ is bounded as a subset of $\mathfrak{B}$ (B, N).

Proof: For each positive integer n, define $F_n = \{x \in B: \|T_i(x)\| \leq n \ \forall \ i\}$ ... (1).
Then F_n is a closed subset of B as shown below.

$x \in F_n$ iff $\|T_i(x)\| \leq n \ \forall \ i$

$\Leftrightarrow T_i(x) \in S_n[0] \ \forall \ i$

$\Leftrightarrow x \in T_i^{-1}\{S_n[0]\} \ \forall \ i$

$\Leftrightarrow x \in \bigcap_i T_i^{-1}\{s_n[0]\}$

so that $F_n = \bigcap_i T_i^{-1}\{s_n[0]\}$ which is closed being intersection of closed sets.
[Note that $\because$ each T_i is continuous and $S_n[0]$ is closed in N each $T_i^{-1}\{S_n[0]$ is closed in B]

Further, $B = \bigcup_{n=1}^{\infty} F_n$

For, if $B \neq \bigcup_{n=1}^{\infty} F_n$ then $\exists$ some $x \in B$ such that $x \notin F_n$ for any n.

$\Rightarrow \|T_i(x)\| > n \ \forall \ n$ by (1)

$\Rightarrow$ The set $\{T_i(x)\}$ is not bounded, which contradicts hypothesis.

Hence, we must have $B = \bigcup_{n=1}^{\infty} F_n$, so that this complete space B is the union of a sequence of its subsets.

$\therefore$ By Baire's category theorem, $\exists$ an integer $n_0 \ni \overline{F_{n_0}}$ has non empty interior.

Since F_{n_0} is closed $\overline{F_{n_0}} = F_{n_0}$ and so F_{n_0} must have non empty interior, i.e., $\exists$ some $x_0 \in {F_{n_0}}^o$ so that F_{n_0} is a nbd of x_0.

Since F_{n_0} is closed, $\exists$ a closed sphere $S = \{x \in$ **B:** $\|x - x_0\| \leq r_0\} \subseteq F_{n_0} \ldots$ (2)

Now if $\|y\| \leq 1$, and $z = r_0 y$ then, $\|z + x_0 - x_0\| = \|r_0 y\| = r_0\|y\| \leq r_0$

so that $z + x_0 \in S \subseteq F_{n_0}$, and $x_0 \in S \subseteq F_{n_0}$

Now for arbitrary but fixed i,

$\|T_i(y)\| = \left\|T_i\left(\frac{z}{r_0}\right)\right\| = \frac{1}{r_0}\|T_i(z)\|$

$= \frac{1}{r_0}\|T_i(z + x_0 - x_0)\|$

$= \frac{1}{r_0}\|T_i(z + x_0) - T_i(x_0)\|$

$\leq \frac{1}{r_0}[\|T_i(z + x_0)\| + \|T_i(x_0)\|$

$\leq \frac{1}{r_0}(n_0 + n_0) = \frac{2n_0}{r_0}$ since z + x_0 and $x_0 \in F_{n_0}$.

Thus $\|T_i(y)\| \leq \frac{2n_0}{r_0}$ if $\|y\| \leq 1$

$\therefore \|T_i\| = \sup\{\|T_i(y)\|: \|y\| \leq 1\} \leq \frac{2n_0}{r_0}$

It follows that $\{\|T_i(y)\|\}$ is a bounded set of numbers.

Theorem: 2*: A nonempty subset S of a normed linear space N is bounded if and only if f(S) is a bounded set for each f ∈ N*.

Proof: Let S be a bounded subset of N. ∴ ∃ real k > 0 ∍ $\|x\| \leq k$ ∀ x ∈ S.
Now for each f ∈ N*, since f is bounded linear functional ∃ real $k_1 > 0$ ∍
for all x ∈ S, $|f(x)| \leq \|f\|\|x\| \leq k\, k_1$. ∴ f(S) is bounded for each f ∈ N*.
Conversely let f(S) be bounded set for each f ∈ N*.
Claim: S is bounded.
For convenience we exhibit vectors in S by S = $\{x_i\}$.
By assumption f(S) = $\{f(x_i): x_i \in S\}$ is a bounded set for each f ∈ N*.
Now by a theorem, each vector x_i in N induces a functional F_{x_i} on N* defined by
$F_{x_i}(f) = f(x_i)$ ∀ f ∈ N* ∍ $\|F_{x_i}\| = \|x_i\|$.
∴ $\{F_{x_i}(f)\}$ is a bounded set of numbers for each f ∈ N*.
⇒ By uniform bounded theorem, $\{\|F_{x_i}\|\}$ is a bounded set of numbers ∀ f ∈ N*.
⇒ $\{\|x_i\|\}$ is a bounded set of numbers ∀ f ∈ N*. ⇒ S is a bounded subset of N.

Notation: Let N be a normed linear space and denote by Ns, the linear space of all scalar valued functions defined on N.

Definition: Let N be a normed linear space and T be an operator on N i.e, T be a continuous linear transformation of N into itself. Define a linear transformation T* of N* into itself as follows.

If f ∈ N* then T*(f) is given by [T*(f)](x) = f{T(x)} ∀ x ∈ N…(1)
We call T* the conjugate (or adjoint) of T.

Theorem: 2*: Let T be an operator on a normed linear space N, then its conjugate T*, defined by T*: N* → N* such that T*(f) = f oT and [T*(f)](x) = f{T(x)} ∀ f ∈ N* and ∀ x ∈ N is an operator on N* and the mapping φ: $\mathcal{B}$(N) → $\mathcal{B}$(N*) ∋ φ(T) = T* ∀ T ∈ $\mathcal{B}$(N) is an isometric isomorphism of $\mathcal{B}$(N) into $\mathcal{B}$(N*) which reverses products and preserves the identity transformation.

Proof: Claim: T* is linear on N*.
Let α, β ∈ K, f, g ∈ N*.
Then for each x ∈ N, T*(αf + βg)](x) = (αf + βg){T(x)} = (αf){T(x)} + (βg){T(x)} = α[f{T(x)}] + β[g{T(x)}] = α[T*(f)](x) + β[T*(g)](x) = [α{T*(f)} + β{T*(g)}](x)
Thus, T*(αf + βg) = αT*(f) + βT*(g) ∴ T* is linear on N*.
Claim: T* is continuous operator on N*
$\|T^*\|$ = sup {$\|T^*(f)\|$: $\|f\| \le 1$}
= sup {$|[T^*(\text{f})](\text{x})|$: $\|f\| \le 1$, $\|x\| \le 1$}
= sup {|f[T(x)]|: $\|f\| \le 1$, $\|x\| \le 1$} by (1)
≤ sup {$\|f\|\|T\|\|x\|$: $\|f\| \le 1$, $\|x\| \le 1$} ≤ $\|T\|$… (2).
Since T is bounded, T* is bounded
∴ T* is an operator on N*
Claim: $\|T^*\| = \|T\|$
For each x ∈ N, x ≠ $\bar{0}$ ∃ a functional f ∈ N* ∋ $\|f\| = 1$ and f(T(x)) = $\|T(x)\|$ …(3) by a theorem.
Therefore $\|T\|$ = sup $\left\{\frac{\|T(x)\|}{\|x\|} : x \neq \bar{0}\right\}$.
= sup $\left\{\frac{|f[T(x)]|}{\|x\|} : \|f\| = 1, x \neq \bar{0}\right\}$ by (3)
= sup $\left\{\frac{|[T^*(f)](x)]|}{\|x\|} : \|f\| = 1, x \neq \bar{0}\right\}$ by (1)
≤ sup $\left\{\frac{\|T^*(f)\|\|x\|}{\|x\|} : \|f\| = 1, x \neq \bar{0}\right\}$.
= sup {$\|T^*(f)\|$: $\|f\| = 1$} = $\|T^*\|$ … (4).
From (2) and (4) $\|T^*\| = \|T\|$ …(5)
Claim: φ is linear:
Let T, U be arbitrary elements of $\mathcal{B}$(N) and α, β be scalar.
Then φ (αT + βU) = (αT + βU)* by def
But for any f ∈N*, x ∈ N, [(αT+ βU)*(f)](x) = f[(αT+ βU)(x)] by (1)
= f[αT(x) + βU(x)]

= αf[T(x)] + βf[U(x)] ∵ f is linear
= α[T*(f)](x) + β[U*f](x) from (1)
= {α[T*(f)] + β[U*f]}(x) = [(αT* + βU*)(f)](x)

∴ (αT+ βU)*(f) = (αT* + βU*)(f) ∀ f ∈ N* so that (αT + βU)* = αT* + βU*… (6)

∴ φ (αT + βU) = (αT + βU)* = αT* +βU* = α φ(T) + β φ(U)

∴ φ is linear.

Claim: φ is one-one: Let φ(T)= φ(U) for T, U ∈ $\mathcal{B}$(N)

⇒ T* = U*

⇒ $\|T^* - U^*\| = 0$

⇒ $\|(T-U)^*\| = 0 \Rightarrow \|T-U\| = 0$

⇒ T = U

∴ φ is one-one.

Claim: $\|\varphi(T)\| = \|T\|$:

Now $\|\varphi(T)\| = \|T^*\|$ by def

= $\|T\|$ by (5)

∴ φ is isometric isomorphism.

Claim: φ reverses products.

[(TU)*(f)](x) = f[(TU)(x)] by (1)
= f[T{U(x)}]
= [T*(f)]{U(x)} by (1)
= [U*{T*(f)}](x) by (1)
= [U*T*(f)](x).

So, [(TU)*(f)] = (U*T*)(f) ∀ f ∈ N*.

Hence (TU)* = U*T*… (7).

Now φ(TU) = (TU)* = U*T* = φ(U) φ(T)

Thus, φ reverses products.

Claim: φ preserves identity:

Let f ∈ N*, x ∈ N. Then [I*(f)](x) = f[I(x)] by (1)
= f(x) = (If)(x) so that I*(f) = I(f) ∀ f ∈ N*

Hence, I* = I so that φ(I) = I* = I

Thus, φ preserves Identity.

HILBERT SPACES

Definition: Let H be a complex Banach Space. Then H is said to be a Hilbert Space if a complex number (x, y), called the inner product of x and y, is associated to each of the two vectors x and y in such a way that (i) $\overline{(x,y)}$ = (y, x), (ii) (αx + βy, z) = α(x, z) + β(y, z) and (iii) (x, x) = $\|x\|^2$.

Example 1: Consider the Banach Space l_2^n consisting of all n – tuples of complex numbers with the norm of a vector x = ($x_1, x_2, \dots, x_n$) defined by $\|x\| = \left(\sum_{i=1}^{n}|x_i|^2\right)^{\frac{1}{2}}$. If the inner product of two vectors x = ($x_1, x_2, \dots, x_n$), y = ($y_1, y_2, \dots, y_n$) is defined by (x, y) = $\sum_{i=1}^{n} x_i\overline{y_i}$ then l_2^n is a Hilbert Space.

Example 2: Consider the Banach Space ℓ_2 consisting of all infinite sequences of complex numbers x = < x_n > = ($x_1, x_2, \dots, x_n, \dots$) ∋ $\sum_{i=1}^{n}|x_i|^2 < \infty$ with the norm of a vector defined by $\|x\| = \left(\sum_{i=1}^{\infty}|x_i|^2\right)^{\frac{1}{2}}$. If the inner product of two vectors x = ($x_1, x_2, \dots, x_n, \dots$), y = ($y_1, y_2, \dots, y_n, \dots$) is defined by (x, y) = $\sum_{i=1}^{\infty} x_i\overline{y_i}$ then ℓ_2 is a Hilbert Space.

Theorem: In a Hilbert Space H prove that

(i) (αx – βy, z) = α(x, z) – β(y, z).

(ii) (x, αy + βz) = $\bar{\alpha}$(x, y) + $\bar{\beta}$ (x, z).

(iii) (x, αy – βz) = $\bar{\alpha}$(x, y) – $\bar{\beta}$ (x, z).

(iv) (x, $\bar{0}$) = 0 ∀ x ∈ H and $(\bar{0}, x)$ = 0 ∀ x ∈ H.

Proof: In what follows let α, β ∈ K and x, y, and z ∈ H.

(i) (αx – βy, z) = (αx + {–β}y, z) = α(x, z) + (–β)(y, z) = α(x, z) – β(y, z).

(ii) (x, αy + βz) = $\overline{(\alpha y + \beta z, x)} = \overline{\alpha(y,x) + \beta(z,x)} = \overline{\alpha(y,x)} + \overline{\beta(z,x)}$ $= \bar{\alpha}\overline{(y,x)} + \bar{\beta}\,\overline{(z,x)} = \bar{\alpha}$(x, y) + $\bar{\beta}$ (x, z).

(iii) (x, αy – βz) = (x, αy +{–β}z) = $\bar{\alpha}$(x, y) + $(-\bar{\beta})$(x, z) = $\bar{\alpha}$(x, y) – $\bar{\beta}$ (x, z).

(iv) $(\bar{0}, x) = (0\bar{0}, x) = 0(\bar{0}, x) = 0$ ∀ x ∈ H and (x, $\bar{0}$) = $\overline{(\bar{0}, x)} = \bar{0}$ = 0 ∀ x ∈ H.

Schwartz Inequality: 4*: If x and y are any two vectors in a Hilbert Space H then $|(x, y)| \le \|x\|\|y\|$

Proof: If y = $\bar{0}$, then $\|y\|$ = 0 and (x, y) = (x, $\bar{0}$) = 0 so that both sides vanish and the equality holds.

Now let y ≠ $\bar{0}$.

For any scalar λ, (x + λy, x + λy) ≥ 0

⇒ (x, x + λy) + λ(y, x + λy) ≥ 0

⇒ (x, x) + $\bar{\lambda}$(x, y) + λ(y, x) + $\lambda\bar{\lambda}$(y, y) ≥ 0.

⇒ $\|x\|^2$ + $\bar{\lambda}$(x, y) + λ(y, x) + $\lambda\bar{\lambda}$ $\|y\|^2$ ≥ 0… (1)

Since y ≠ $\bar{0}$, $\|y\| \neq 0$. ∴ Put $\lambda = \frac{-(x,y)}{\|y\|^2}$.

$\Rightarrow \|x\|^2 - \frac{\overline{(x,y)}}{\|y\|^2}(x, y) - \frac{(x,y)}{\|y\|^2}(y, x) + \frac{(x,y)}{\|y\|^2}\frac{\overline{(x,y)}}{\|y\|^2}\|y\|^2 \geq 0$

$\Rightarrow \|x\|^2 - \frac{\overline{(x,y)}}{\|y\|^2}(x, y) - \frac{(x,y)}{\|y\|^2}\overline{(x,y)} + \frac{(x,y)}{\|y\|^2}\frac{\overline{(x,y)}}{\|y\|^2}\|y\|^2 \geq 0$

$\Rightarrow \|x\|^2 - \frac{|(x,y)|^2}{\|y\|^2} - \frac{|(x,y)|^2}{\|y\|^2} + \frac{|(x,y)|^2}{\|y\|^2} \geq 0$

$\Rightarrow \|x\|^2 - \frac{|(x,y)|^2}{\|y\|^2} \geq 0. \Rightarrow \|x\|^2\|y\|^2 \geq |(x,y)|^2$

$\Rightarrow |(x,y)| \leq \|x\|\|y\|$

Theorem: In a Hilbert space the inner product is jointly continuous
i.e., $x_n \to x$, $y_n \to y \Rightarrow (x_n, y_n) \to (x, y)$.

Proof: Let $x_n \to x$, $y_n \to y$.

$|(x_n, y_n) - (x, y)| = |(x_n, y_n) - (x_n, y) + (x_n, y) - (x, y)|$

$= |(x_n, y_n - y) + (x_n - x, y)| \leq |(x_n, y_n - y)| + |(x_n - x, y)|$

$\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\|$ by Schwartz inequality,

But $\|y_n - y\| \to 0$ and $\|x_n - x\| \to 0$ as $n \to \infty$ (∵ $x_n \to x$, $y_n \to y$)

∴ $|(x_n, y_n) - (x, y)| \to 0$ as $n \to \infty$.

Hence $(x_n, y_n) \to (x, y)$.

Theorem: 2*: If x and y are any two vectors in a Hilbert space then

(1) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (parallelogram law)

(2) $4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$ (Polarization identity)

Proof (i) $\|x + y\|^2 = (x + y, x + y) = (x, x + y) + (y, x + y)$

$= (x, x) + (x, y) + (y, x) + (y, y)$

$= \|x\|^2 + (x, y) + (y, x) + \|y\|^2$ … (1)

$\|x - y\|^2 = (x - y, x - y) = (x, x - y) + (y, x - y)$

$= (x, x) - (x, y) - (y, x) + (y, y)$

$= \|x\|^2 - (x, y) - (y, x) + \|y\|^2$ … (2)

Adding (1) and (2), $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$

(ii) Subtracting (2) from (1) $\|x + y\|^2 - \|x - y\|^2 = 2(x, y) + 2(y, x)$ … (3)

Replacing y by iy in (3)

$\|x + iy\|^2 - \|x - iy\|^2 = 2(x, iy) + 2(iy, x)$

$= 2\bar{i}(x, y) + 2i(y, x)$

$= -2i(x, y) + 2i(y, x)$ … (4)

Multiplying both sides of (4) by i we get

$i\|x + iy\|^2 - i\|x - iy\|^2 = 2(x, y) - 2(y, x)$ … (5)

Adding (3) and (5)

$\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = 4(x, y)$

Theorem: 4*: If B is a complex Banach space whose norm obeys the parallelogram law and if the inner product is defined on B by

$4(x, y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$ then B is a Hilbert space.

Proof: Given that parallelogram law

$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \ldots (1)$

Also, $4(x, y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \ldots (2)$

Claim: $(x, x) = \|x\|^2$

Replace, y by x in (2)

$4(x, x) = \|x+x\|^2 - \|\bar{0}\|^2 + i\|(1+i)x\|^2 - i\|(1-i)x\|^2$

$= 4\|x\|^2 + i|1+i|^2\|x\|^2 - i|1-i|^2\|x\|^2$

$= 4\|x\|^2 + 2i\|x\|^2 - 2i\|x\|^2 \because |1+i|^2 = |1-i|^2 = 1 + 1 = 2$

$= 4\|x\|^2$

Thus, $(x, x) = \|x\|^2$

Claim: $\overline{(x, y)} = (y, x)$

Taking the complex conjugate on both sides of (2)

$4\overline{(x,y)} = \|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2$

$= \|y+x\|^2 - \|-(y-x)\|^2 - i\|i(y-ix)\|^2 + i\|-i(y+ix)\|^2$

$= \|y+x\|^2 - \|y-x\|^2 - i|i|^2\|y-ix\|^2 + i|-i|^2\|y+ix\|^2$

$= \|y+x\|^2 - \|y-x\|^2 - i\|y-ix\|^2 + i\|y+ix\|^2$

$= 4(y, x)$

$\therefore \overline{(x,y)} = (y, x)$

Claim: $(x + y, z) = (x, z) + (y, z)$

Replacing x by x + y and y by z in (2)

$4(x+y, z) = \|x+y+z\|^2 - \|x+y-z\|^2 + i\|x+y+iz\|^2 - i\|x+y-iz\|^2 \ldots (3)$

Replacing x by x + z in (1)

$\|x+z+y\|^2 + \|x+z-y\|^2 = 2\|x+z\|^2 + 2\|y\|^2$

(or) $\|x+y+z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - \|x+z-y\|^2 \ldots (4)$

Again $\|x+z-y\|^2 = \|z-y+x\|^2 = 2\|z-y\|^2 + 2\|x\|^2 - \|z-y-x\|^2$ by (1)

$= 2\|y-z\|^2 + 2\|x\|^2 - \|x+y-z\|^2 \ldots (5)$

Substituting the value of $\|x+z-y\|^2$ from (5) in (4)

$\|x+y+z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - \{2\|y-z\|^2 + 2\|x\|^2 - \|x+y-z\|^2\}$

(or) $\|x+y+z\|^2 - \|x+y-z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - 2\|y-z\|^2 - 2\|x\|^2 \ldots (6)$

Interchanging x and y in (6) we get

$\|x+y+z\|^2 - \|x+y-z\|^2 = 2\|y+z\|^2 + 2\|x\|^2 - 2\|x-z\|^2 - 2\|y\|^2 \ldots (7)$

Adding (6) and (7) we get

$\|x + y + z\|^2 - \|x + y - z\|^2 = \|x + z\|^2 - \|x - z\|^2 + \|y + z\|^2 - \|y - z\|^2$… (8)

Replacing z by iz in (8) and multiplying both sides by i

$i\,\|x + y + iz\|^2 - i\|x + y - iz\|^2 = i\|x + iz\|^2 - i\|x - iz\|^2 + i\|y + iz\|^2 - i\|y - iz\|^2$… (9)

Adding (8) and (9)

$\|x + y + z\|^2 - \|x + y - z\|^2 + i\,\|x + y + iz\|^2 - i\,\|x + y - iz\|^2 = \|x + z\|^2 - \|x - z\|^2 +$
$i\|x + iz\|^2 - i\|x - iz\|^2 + \{\|y + z\|^2 - \|y - z\|^2 + i\|y + iz\|^2 - i\|y - iz\|^2\}$…(10)

By (3) and (10) we get $4(x + y, z) = 4(x, z) + 4(y, z)$

i.e., $(x + y, z) = (x, z) + (y, z)$…(11)

Claim: $(\alpha x, y) = \alpha(x, y)$

(a) let α be a positive integer

Then $(x + x, y) = (x, y) + (x, y)$

$\Rightarrow (2x, y) = 2(x, y)$ so that result is true for n = 2

Assume the result is true for n i.e., $(nx, y) = n(x, y)$…(12)

Then $(\{n+1\}x, y) = (nx + x, y) = (nx, y) + (x, y)$
$= n(x, y) + (x, y) = (n + 1)(x, y)$.

∴ Result is true for n + 1 if it were true for n

∴ By induction result is true for all positive integral values of α

Replacing x by – x in (2) we get,

$4(-x, y) = \|-x + y\|^2 - \|-x - y\|^2 + i\,\|-x + iy\|^2 - i\|-x - iy\|^2$
$= \|x - y\|^2 - \|x + y\|^2 + i\,\|x - iy\|^2 - i\|x + iy\|^2$ ($\because \|-x\| = \|x\|$)
$= -4(x, y)$

∴ $(-x, y) = -(x, y)$ … (13)

Let α be negative integer. Then ∃ a positive integer β ∋ $\alpha = -\beta$.

∴ $(\alpha x, y) = (-\beta x, y) = (-(\beta x), y) = -(\beta x, y) = -\beta(x, y) = \alpha(x, y)$

(c) Let α be rational say $\alpha = \frac{p}{q}$ where p, q are integers and $q \neq 0$

∴ $(\alpha x, y) = \left(\frac{p}{q}x, y\right) = (pz, y)$ where $z = \frac{x}{q}$
$= p(z, y)$ … (14)

Now $(qz, y) = q(z, y)$

∴ $(z, y) = \frac{1}{q}(qz, y)$ … (15)

Substituting the value of (z, y) from (15) in (14) we get $(\alpha x, y) = \frac{p}{q}(qz, y) = \alpha(x, y)$

Similarly, we can prove the result if α is any real number.

(d) Let α be a complex number.

Replacing x by ix in (2),

$4(ix, y) = \|ix + y\|^2 - \|ix - y\|^2 + i\|ix + iy\|^2 - i\|ix - iy\|^2$
$= |i|^2\|x - iy\|^2 - |i|^2\|x + iy\|^2 + i|i|^2\,\|x + y\|^2 - i|i|^2\|x - y\|^2$
$= \|x - iy\|^2 - \|x + iy\|^2 + i\|x + y\|^2 - i\|x - y\|^2$
$= -i^2\|x - iy\|^2 + i^2\|x + iy\|^2 + i\|x + y\|^2 - i\|x - y\|^2$

= i{$\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$} = i4(x, y) so that (ix, y) = i(x, y)

Now let α = a + ib where a, b are real numbers.

(αx, y) = ({a + ib}x, y) = (ax, y) + i(bx, y)

= a(x, y) + ib(x, y) = (a + ib)(x, y) = α(x, y)

Ie. (αx, y) = α(x, y) when α is a complex number

Hence B is a Hilbert space.

Convex Set: Let L be a real or complex linear Space. A non-empty subset S of L is said to be convex if x, y ∈ S ⇒ (1 – α)x + αy ∈ S where α is any real number ∍ 0 ≤ α ≤ 1.

Note: Taking $\alpha = \frac{1}{2}$ we see that if S is a convex subset of a linear space L, then x, y ∈ S ⇒ $\frac{x+y}{2}$ ∈ S.

Theorem: 8*: A closed convex subset C of a Hilbert Space H contains a unique vector of smallest norm.

Proof: Let d = inf {$\|x\|$: x ∈ C}.

Then ∃ a sequence {x_n} of vectors in C ∍ $\|x_n\| \to d$

Consider two vectors x_m and x_n belonging to sequence {x_n}

Since C is a convex subset of H and $x_m, x_n \in C$, $\frac{x_m+x_n}{2} \in C$.

∴ By the definition of d, $\left\|\frac{x_m+x_n}{2}\right\| \geq d$ so that $\|x_m + x_n\| \geq 2d$ … (1).

Appling parallelogram law for the vectors x_m and x_n we get

$\|x_m + x_n\|^2 + \|x_m - x_n\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2$

$\Rightarrow \|x_m - x_n\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2$

$\leq 2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2$ … (2).

Since $\|x_m\| \to d$ and $\|x_n\| \to d$, we have $2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2 \to 2d^2 + 2d^2 - 4d^2 = 0$

∴ {x_n} is a Cauchy sequence in C.

Since H is complete and C is closed, C is also complete.

Hence Cauchy sequence {x_n} converges in C. ∴ ∃ x in C ∍ $x_n \to x$.

Now $\|x\| = \|lt\ x_n\|$ = lt $\|x_n\|$ since norm is a continuous mapping

= d.

∴ x is vector in C with smallest norm.

Uniqueness of x:

If possible, suppose y is another vector in C ∍ $\|y\| = d$.

Then $\frac{x+y}{2}$ ∈ C and again by parallelogram law,

$\left\|\frac{x+y}{2}\right\|^2 = 2\left\|\frac{x}{2}\right\|^2 + 2\left\|\frac{y}{2}\right\|^2 - \left\|\frac{x-y}{2}\right\|^2 < \frac{\|x\|^2}{2} + \frac{\|y\|^2}{2} = \frac{d^2}{2} + \frac{d^2}{2} = d^2$
which contradicts the definition of d.
∴ A closed convex subset C of a Hilbert Space H contains a unique vector of smallest norm.

Theorem: 3*: Let M be a closed linear subspace of a Hilbert space H, x be a vector not in M and d be the distance of M from x. Then ∃ a unique vector y_0 in M ∋ $\|x - y_0\| = d$.

Proof: Let d(x, M) = d = inf $\{\|x - z\|: z \in M\}$ by definition.
∴ ∃ a sequence $\{y_n\}$ of vectors in M ∋ $\|x - y_n\| \to d$
Consider two vectors y_m and y_n belonging to sequence $\{y_n\}$.
Since M is a linear subspace of H, $\frac{y_m+y_n}{2} \in M$.
$\therefore \left\|x - \frac{y_m+y_n}{2}\right\| \geq d \Rightarrow \|2x - (y_m + y_n)\| \geq 2d \ldots (1)$.
Appling parallelogram for the vectors $x - y_m$ and $x - y_n$ we get
$\|x - y_m - (x - y_n)\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - \|x - y_m + x - y_n\|^2$
$\Rightarrow \|y_n - y_m\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - \|2x - (y_m + y_n)\|^2$
$\leq 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - 4d^2$
Since $\|x - y_m\| \to d$ and $\|x - y_n\| \to d$ we have
$\|y_n - y_m\|^2 \leq 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - 4d^2 \to 2d^2 + 2d^2 - 4d^2 = 0$
∴ $\{y_n\}$ is a Cauchy sequence in M.
∴ ∃ $y_0 \in M$ ∋ $y_n \to y_0$ ∵ M is complete being closed subspace of complete space.
Now $\|x - y_0\| = \|x - lt\, y_n\| = \|lt(x - y_n)\| = lt\, \|x - y_n\| = d$
∴ y_0 is vector in M ∋ $\|x - y_0\| = d$
Uniqueness of y_0:
If possible, suppose y is another vector in M ∋ $\|x - y\| = d$
Then $\frac{y_0+y}{2} \in M$ and again by parallelogram law
$\left\|\frac{x-y_0+x-y}{2}\right\|^2 = 2\left\|\frac{x-y_0}{2}\right\|^2 + 2\left\|\frac{x-y}{2}\right\|^2 - \left\|\frac{x-y_0-(x-y)}{2}\right\|^2 < \frac{d^2}{2} + \frac{d^2}{2} = d^2 \Rightarrow$
$\|2x - (y_0 + y)\| < 2d$ which is a contradiction to (1). Hence y_0 is unique.

Example: For the special Hilbert space l_2^n use Cauchy's inequality to prove Schwartz inequality.
Solution: Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ be any two members of the Hilbert space l_2^n.
By Cauchy's inequality, $\sum_{i=1}^n |x_i y_i| \leq (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}} (\sum_{i=1}^n |y_i|^2)^{\frac{1}{2}}$

But (x, y) = $\sum_{i=1}^{n} x_i \overline{y_i}$
$\therefore |(x,y)| = |\sum_{i=1}^{n} x_i \overline{y_i}| \le \sum_{i=1}^{n} |x_i \overline{y_i}| = \sum_{i=1}^{n} |x_i y_i|$
$\le (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}} (\sum_{i=1}^{n} |y_i|^2)^{\frac{1}{2}} = \sqrt{(x,x)}\sqrt{(y,y)} = \|x\|\|y\|$

ORTHOGONAL COMPLEMENT:

Orthogonality: Definition

Let x and y be vectors in a Hilbert space H. Then x is said to be *orthogonal* to y if (x, y) = 0 (written as x ⊥ y).

Note: 1. If x, y ∈ H and x ⊥ y then y ⊥ x.

Solution: Let x, y ∈ H and x ⊥ y
⇒ (x, y) = 0 ⇒ (y, x) = $\overline{(x,y)}$ = 0 so that y ⊥ x.

2. If x is orthogonal to y, then every scalar multiple of x is orthogonal to y.

Solution: Let x, y ∈ H and x ⊥ y. ∴ (x, y) = 0. Let α be any scalar
Then (αx, y) = α(x, y) = α0 = 0 so that αx ⊥ y

3. The zero vector is orthogonal to every vector.

Solution: Let x ∈ H. Then ($\bar{0}$, x) = 0. ∴ $\bar{0}$ ⊥ x. Thus, $\bar{0}$ ⊥ x ∀ x ∈ H.

4. The zero vector is the only vector which is orthogonal to itself.

Ie. x ⊥ x iff x = $\bar{0}$.

Solution: Let x ∈ H. Then x ⊥ x iff (x, x) = 0 iff $\|x\|^2 = \bar{0}$ iff $\|x\| = 0$ iff x = $\bar{0}$.

5. ⊥ is not transitive. ie. x, y, z ∈ H, x ⊥ y and y ⊥ z ⇏ x ⊥ z.

Solution: Consider x = (1, 0, 0), y = (0, 1, 0), z = (1, 0, 1) ∈ $\mathbb{C}^3$.
Then (x, y) = 1(0) + 0(1) + 0(0) = 0 and (y, z) = 0(1) + 1(0) + 0(1) = 0 so that x ⊥ y and y ⊥ z. But 1(1) + 0(0) + 0(1) = 1 ≠ 0. So, x is not orthogonal to z.

The Pythagorean theorem: If x and y are any two orthogonal vectors in a Hilbert Space H, then $\|x+y\|^2 = \|x-y\|^2 = \|x\|^2 + \|y\|^2$

Proof: Let x ⊥ y ie. (x, y) = 0 then (y, x) = 0.
Now $\|x+y\|^2$ = (x + y, x + y) = (x, x) + (x, y) + (y, x) + (y, y).
= $\|x\|^2$+ 0 + 0 + $\|y\|^2$ = $\|x\|^2 + \|y\|^2$ and
$\|x-y\|^2$= (x – y, x – y) = (x, x) – (x, y) – (y, x) + (y, y) = $\|x\|^2 + \|y\|^2$.

Definition: A vector x is said to be orthogonal to a non – empty subset S of a Hilbert Space H (written x ⊥ S) if x ⊥ y ∀ y ∈ S.
Two non – empty subsets S_1 and S_2 of a Hilbert space H are said to be orthogonal (written $S_1 \perp S_2$) if x ⊥ y ∀ x ∈ S_1 and ∀ y ∈ S_2.

Definition: Let S be a non-empty subset of a Hilbert Space H. The orthogonal complement of S (written as $S^{\perp}$) is defined by $S^{\perp} = \{x \in H: x \perp y \ \forall\ y \in S\}$.

Theorem: 4*: Let S be a non – empty subset of a Hilbert Space H then $S^{\perp}$ is a closed linear subspace of H.

Proof: Claim: $S^{\perp}$ is linear subspace of H.

$S^{\perp}$ is non-empty, since $(\bar{0}, x) = 0 \ \forall\ x \in S$.

Let $x_1, x_2 \in S^{\perp}$ and α, β be any scalars. Let $y \in S$ then, $(x_1, y) = 0 = (x_2, y)$.

$\therefore\ (\alpha x_1 + \beta x_2, y) = \alpha(x_1, y) + \beta(x_2, y) = \alpha 0 + \beta 0 = 0$.

$\therefore (\alpha x_1 + \beta x_2, y) = 0 \ \forall\ y \in S$ so that $\alpha x_1 + \beta x_2 \perp S^{\perp}$.

$\therefore\ S^{\perp}$ is a subspace of H.

Claim: $S^{\perp}$ is closed.

Let x be a limit point of $S^{\perp}$.

$\therefore\ \exists$ a sequence $\{x_n\}$ of points of $S^{\perp} \ni x_n \to x$.

Then for every n, $(x_n, y) = 0 \ \forall\ y \in S$.

Now let $y \in S$. Then $(x, y) = (\text{lt } x_n, y) = \text{lt } (x_n, y) = \text{lt } 0 = 0$. $\therefore\ x \in S^{\perp}$.

Hence $S^{\perp}$ is closed subspace of H.

Note: $S^{\perp}$ is complete. $\therefore\ S^{\perp}$ is a Hilbert Space.

Orthogonal complement of an orthogonal complement.

Definition: Let S be any non-empty subset of a Hilbert Space H.

We define $(S^{\perp})^{\perp} = S^{\perp\perp} = \{x \in H: (x, y) = 0 \ \forall\ y \in S^{\perp}\}$.

Theorem: 1*: If S, S_1, S_2 are non-empty subsets of a Hilbert Space H, then (i) $\{0\}^{\perp} = H$, (ii) $H^{\perp} = \{0\}$, (iii) $S \cap S^{\perp} \subseteq \{\bar{0}\}$ (iv) $S_1 \subseteq S_2 \Rightarrow {S_2}^{\perp} \subseteq {S_1}^{\perp}$ and (v) $S \subseteq S^{\perp\perp}$.

Proof: (i) Clearly $\{0\}^{\perp} \subseteq H$, Let $x \in H$. Then $(x, \bar{0}) = 0$. $\therefore\ x \in \{0\}^{\perp}$ so that $H \subseteq \{0\}^{\perp}$. Hence $\{0\}^{\perp} = H$.

(ii) Let $x \in H^{\perp}$. $\therefore (x, y) = 0 \ \forall\ y \in H$. $\therefore$ In particular $(x, x) = 0$. $\Rightarrow \|x\|^2 = 0 \Rightarrow x = \bar{0}$. $\therefore\ H^{\perp} \subseteq \{\bar{0}\}$. Since $(\bar{0}, x) = 0 \ \forall x \in H, \{\bar{0}\} \subseteq H^{\perp}$. Hence $H^{\perp} = \{\bar{0}\}$.

(iii) Let $x \in S \cap S^{\perp}$. Then $x \in S$ and $x \in S^{\perp}$. $\Rightarrow x \in S$ and $x \perp y \ \forall\ y \in S$. $\Rightarrow$ In particular $x \perp x$. $\Rightarrow (x, x) = 0 \ \Rightarrow\ \|x\|^2 = 0 \Rightarrow x = \bar{0}$. $\therefore\ S \cap S^{\perp} \subseteq \{\bar{0}\}$.

(iv) Let $S_1 \subseteq S_2$ and $x \in {S_2}^{\perp}$

$\therefore x \perp y \ \forall\ y \in S_2 \Rightarrow x \perp y \ \forall\ y \in S_1$ since $S_1 \subseteq S_2$.

$\Rightarrow x \in {S_1}^{\perp}$. $\therefore\ {S_2}^{\perp} \subseteq {S_1}^{\perp}$

(v) Let $x \in S$. Let $y \in S^{\perp}$. Then $x \perp y$. $\therefore\ x \perp y \ \forall\ y \in S^{\perp} \Rightarrow x \in (S^{\perp})^{\perp} = S^{\perp\perp}$.

$\therefore\ S \subseteq S^{\perp\perp}$.

Theorem: 7*: If M is a proper closed linear subspace of a Hilbert Space H, then ∃ a non – zero vector z_0 in H ∋ $z_0 \perp M$.

Proof: Since M is a proper closed linear subspace of a Hilbert Space H, ∃ a vector $x \in H$ which is not in M.

Let d be the distance of M from x. Then d = inf $\{\|x - y\| : y \in M\}$.

Since $x \neq y \ \forall \ y \in M$, $d > 0$.

Since M is a closed linear subspace of H, ∃ a vector y_0 in M ∋ $\|x - y_0\| = d$.

Now we set $z_0 = x - y_0$.

∴ We have $\|z_0\| = \|x - y_0\| = d > 0$ so that z_0 is non-zero vector.

Let y be any arbitrary vector in M. For any scalar α, we have $z_0 - \alpha y = x - (y_0 + \alpha y)$. Since M is a subspace of H and $y_0, y \in M$, $y_0 + \alpha y \in M$.

By definition of d, $\|x - (y_0 + \alpha y)\| \geq d.$

Now $\|(z_0 - \alpha y\| = \|x - (y_0 + \alpha y\| \geq d = \|z_0\|$

∴ $\|z_0 - \alpha y\|^2 \geq \|z_0\|^2$

$\Rightarrow (z_0 - \alpha y, z_0 - \alpha y) - (z_0, z_0) \geq 0$

$\Rightarrow -\bar{\alpha}(z_0, y) - \alpha\overline{(z_0, y)} + \alpha\bar{\alpha}(y, y) \geq 0$ … (1).

The relation (1) is true ∀ scalars α.

Let us take $\alpha = \beta(z_0, y)$ where β is real number.

Then $\bar{\alpha} = \beta\overline{(z_0, y)}$.

Putting the values of α, $\bar{\alpha}$ in (1),

$-\beta\overline{(z_0, y)}(z_0, y) - \beta(z_0, y)\overline{(z_0, y)} + \beta^2(z_0, y)\overline{(z_0, y)}\|y\|^2 \geq 0$

$\Rightarrow -2\beta|(z_0, y)|^2 + \beta^2|(z_0, y)|^2\|y\|^2 \geq 0$

$\Rightarrow \beta|(z_0, y)|^2(\beta\|y\|^2 - 2) \geq 0$ … (2).

The relation (2) is true ∀ real β.

Suppose that $(z_0, y) \neq 0$. Then choose β positive and so small that $\beta\|y\|^2 < 2$.

∴ $\beta|(z_0, y)|^2(\beta\|y\|^2 - 2) < 0$ which contradicts (2)

∴ $(z_0, y) = 0$

$\Rightarrow z_0 \perp y$. ie $y \in M \Rightarrow z_0 \perp y$

∴ $z_0 \perp y \ \forall \ y \in M$.

Hence $z_0 \perp M$.

Theorem: 2*: If M is a linear subspace of Hilbert Space H, show that M is closed if and only if $M = M^{\perp\perp}$.

Proof: Let M be a subspace of a Hilbert Space H and $M = M^{\perp\perp}$.

We know that $(M^{\perp})^{\perp}$ is a closed subspace of H.

∴ M is a closed subspace of H.

Conversely suppose that M is closed subspace of H.
As proved earlier M $\subseteq M^{\perp\perp}$.
If possible, suppose that M is a proper subset of $M^{\perp\perp}$.
Now $M^{\perp\perp}$ is a Hilbert Space and M is a proper closed subspace of $M^{\perp\perp}$.
$\therefore$ By previous theorem, $\exists$ non-zero vector z_0 in $M^{\perp\perp}$ $\ni$ $z_0 \perp$ M.
$\Rightarrow z_0 \in M^{\perp}$.
$\therefore z_0 \in M^{\perp}$ and $z_0 \in M^{\perp\perp}$
$\Rightarrow z_0 \in M^{\perp} \cap M^{\perp\perp} \subseteq \{\bar{0}\}$.
$\therefore z_0 = \bar{0}$ which is a contradiction. Hence M = $M^{\perp\perp}$.

Theorem: 2*: If M and N are closed linear subspaces of a Hilbert Space H such that M $\perp$ N, then the linear subspace M + N is also closed.
Proof: Let z be a limit point of M + N.
$\therefore$ $\exists$ a sequence $\{z_n\}$ of points of M + N $\ni z_n \to z$.
Since M $\perp$ N, M $\cap$ N = $\{\bar{0}\}$ and so the subspace M + N is the direct sum of the subspaces M and N.
$\therefore$ each z_n can be uniquely written as $z_n = x_n + y_n$ where $x_n \in M$, $y_n \in N$.
Consider two vectors $z_m = x_m + y_m$ and $z_n = x_n + y_n$ belonging to $\{z_n\}$.
Since $x_m - x_n \in M$ and $y_m - y_n \in N$ and M $\perp$ N, $(x_m - x_n) \perp (y_m - y_n)$.
By Pythogorian theorem, $\|x_m - x_n + y_m - y_n\|^2 = \|x_m - x_n\|^2 + \|y_m - y_n\|^2$.
$\Rightarrow \|z_m - z_n\|^2 = \|x_m - x_n\|^2 + \|y_m - y_n\|^2$. … (i)
Now $\{z_n\}$ is a convergent sequence in the Hilbert Space H.
$\therefore$ $\{z_n\}$ is a Cauchy sequence in the Hilbert Space H.
$\therefore$ as m, n $\to \infty$, we have $\|z_m - z_n\|^2 \to 0 \Rightarrow \|x_m - x_n\|^2 + \|y_m - y_n\|^2 \to 0$.
$\Rightarrow \|x_m - x_n\|^2 \to 0, \|y_m - y_n\|^2 \to 0$.
$\Rightarrow$ $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in M and N respectively.
But M and N are complete being closed subspaces of a complete space.
$\therefore$ $\{x_n\}$ and $\{y_n\}$ in M and N are convergent sequences in M and N.
$\therefore$ $\exists$ x $\in$ M and y $\in$ N $\ni x_n \to x$ and $y_n \to y$.
Now z = lim z_n = lim $(x_n + y_n)$ = lim x_n + lim y_n = x + y $\in$ M + N.
Thus, if z is a limit point of M + N then z $\in$ M + N.
$\therefore$ M + N is closed.

Projection Theorem: 6*: If M is a closed linear subspace of a Hilbert Space H, then H = M $\oplus$ $M^{\perp}$.
Proof: Let M be a closed linear subspace of a Hilbert Space H.
$\therefore$ M $\cap$ $M^{\perp}$ = $\{\bar{0}\}$ since M is a subspace of H.
Now $M^{\perp}$ is a closed subspace of H.

M is given to be a closed subspace of H.
$\therefore$ By the preceding theorem M + $M^{\perp}$ is closed subspace of H.
Put N = M + $M^{\perp}$ … (i)
Then, By (i), M $\subseteq$ N and $M^{\perp} \subseteq$ N.
$\therefore N^{\perp} \subseteq M^{\perp}$ and $N^{\perp} \subseteq M^{\perp\perp}$.
$\Rightarrow N^{\perp} \subseteq M^{\perp} \cap M^{\perp\perp} = \{\bar{0}\}$.
$\therefore N^{\perp} = \{\bar{0}\}$
$\Rightarrow N^{\perp\perp} = \{\bar{0}\}^{\perp}$ = H
$\Rightarrow$ N = H since N = M + $M^{\perp}$ is a closed subspace of H $\Rightarrow N^{\perp\perp}$ = N
Thus, N = M + $M^{\perp}$ = H.
Finally, H = M + $M^{\perp}$ and M $\cap$ $M^{\perp} = \{\bar{0}\}$.
$\Rightarrow$ H = M $\oplus$ $M^{\perp}$.

Example 1: 2*: If S is a non-empty subset of a Hilbert Space H, show that $S^{\perp} = S^{\perp\perp\perp}$.
Solution: We know that if M is a closed subspace of a Hilbert Space H then M = $M^{\perp\perp}$. Since $S^{\perp}$ is a closed subspace of Hilbert Space H, $S^{\perp} = S^{\perp\perp\perp}$.

Example 2: 1*: If S is a non-empty subset of a Hilbert Space H, show that the set of all linear combinations of vector in S is dense in H if and only if $S^{\perp} = \{\bar{0}\}$.
Solution: Put M = [S]. Suppose M is dense in H. ie. $\bar{M}$ = H.
Let z be a limit point of M.
Then $\exists$ a sequence $\{z_n\}$ of points of M $\ni z_n \to z$.
Let x $\perp$ M. Then x $\perp$ z_n $\forall$ n since $z_n \in$ M $\forall$ n.
$\Rightarrow (x, z_n) = 0\ \forall$ n.
$\therefore$ 0 = lt (x, z_n) = (x, lt z_n) = (x, z). ie. (x, z) = 0. So, x $\perp$ z.
Thus, x $\perp$ M $\Rightarrow$ x $\perp$ z where z is a limit point of M. $\therefore$ x $\perp$ $\bar{M}$ ($\Rightarrow$ x $\in \bar{M}^{\perp}$.)
Now let x $\in S^{\perp}$. Then x $\perp$ S $\Rightarrow$ x is orthogonal to every vector in [S] = M.
$\Rightarrow$ x $\perp$ $\bar{M}$ $\Rightarrow$ x $\perp$ H. In particular x $\perp$ x which $\Rightarrow$ x = $\bar{0}$. $\therefore S^{\perp} = \{\bar{0}\}$.
Conversely suppose $S^{\perp} = \{\bar{0}\}$.
Claim: $\bar{M}$ = H. Clearly, $\bar{M} \subseteq H$.
If possible, suppose H $\not\subseteq \bar{M}$.
Then $\exists$ a vector x in H $\ni$ x $\notin \bar{M}$.
Since $\bar{M}$ is a closed subspace of H, H = $\bar{M} \oplus \bar{M}^{\perp}$.
$\therefore$ we can write x = y + z where y $\in \bar{M}$ and z $\in \bar{M}^{\perp}$.
Now z cannot be zero vector. If z = $\bar{0}$, then x = y $\in \bar{M}$ which is a contradiction.
Then $\exists$ non-zero vector z $\ni$ z $\in \bar{M}^{\perp} \Rightarrow$ z $\perp$ $M^{\perp}$ $\because$ M $\subseteq \bar{M}$.
Thus, z $\in \bar{M}^{\perp}$ $\Rightarrow$ z $\in \bar{M}$ which is a contradiction. Hence H = $\bar{M}$.

Example 3: 1*: If S is a non-empty subset of a Hilbert Space H, show that $S^{\perp\perp} = \overline{[S]}$.

Solution: We know that S $\subseteq S^{\perp\perp}$. Also $S^{\perp\perp}$ is a closed subspace of H. But $\overline{[S]}$ is the smallest closed subspace of H containing S. $\therefore$ $\overline{[S]} \subseteq S^{\perp\perp}$ … (1)
Now S $\subseteq$ [S] and [S] $\subseteq \overline{[S]}$ $\therefore$ S $\subseteq \overline{[S]}$. $\therefore$ $\overline{[S]}^{\perp} \subseteq S^{\perp}$ by a theorem $\Rightarrow S^{\perp\perp} \subseteq \overline{[S]}^{\perp\perp}$… (2) Since $\overline{[S]}$ is the closed subspace of H, $\overline{[S]} = \overline{[S]}^{\perp\perp}$.
From (2) we get $S^{\perp\perp} \subseteq \overline{[S]}$ … (3)
From (1) and (3) $S^{\perp\perp} = \overline{[S]}$.

ORTHONORMAL SETS

Definition: A non-empty set $\{e_i\}$ of a Hilbert Space H is said to be an orthonormal set if (i) $\|e_i\| = 1$ for every i (ii) i ≠ j ⇒ $e_i \perp e_j$.
Note: (i) Orthonormal set cannot contain $\bar{0}$ vector.
(ii) Every Hilbert Space which is not equal to zero space possesses an orthonormal set. For, $\bar{0} \neq x \in H \ni \frac{x}{\|x\|}$ is a unit vector and so $\left\{\frac{x}{\|x\|}\right\}$ is an orthonormal set of H.
(iii) If $\{x_i\}$ is a non-empty set of mutually orthogonal non-zero vectors in H, then $\{e_i\}$ where $e_i = \frac{x_i}{\|x_i\|}$ is an orthonormal set in H.

Example: In the Hilbert Space l_2^n, $\{e_1, e_2, \ldots, e_n\}$ of l_2^n where $e_i = (x_1, x_2, \ldots, x_n)$ such that $x_i = 1$ and $x_j = 0$ if j ≠ i is an orthonormal set, exactly it is an orthonormal basis of l_2^n.

Bessel's inequality for finite sets
Theorem: 4*: Let $\{e_1, e_2, \ldots, e_n\}$ be a finite orthonormal set in a Hilbert Space H. If x is any vector in H, then $\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$. Further, $x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j$ for each j.
Proof: Given that $\{e_1, e_2, \ldots, e_n\}$ be a finite orthonormal set in a Hilbert Space H and x be any vector in H. Consider the vector $y = x - \sum_{i=1}^{n} (x, e_i) e_i$.
$\therefore 0 \leq \|y\|^2 = (y, y) = \left(x - \sum_{i=1}^{n} (x, e_i) e_i,\ x - \sum_{j=1}^{n} (x, e_j) e_j\right)$
$= (x, x) - \sum_{i=1}^{n} (x, e_i)(e_i, x) - \sum_{j=1}^{n} \overline{(x, e_j)}(x, e_j) + \sum_{i=1}^{n} \sum_{j=1}^{n} (x, e_i) \overline{(x, e_j)} (e_i, e_j)$
$= \|x\|^2 - \sum_{i=1}^{n} (x, e_i) \overline{(x, e_i)} - \sum_{j=1}^{n} \overline{(x, e_j)}(x, e_j) + \sum_{i=1}^{n} (x, e_i) \overline{(x, e_i)}$
$= \|x\|^2 - \sum_{i=1}^{n} |(x, e_i)|^2 = \|x\|^2 - \sum_{i=1}^{n} |(x, e_i)|^2$

$\therefore \sum_{i=1}^{n}|(x, e_i)|^2 \leq \|x\|^2 \ldots$ (i)
Further, for each $1 \leq j \leq n$, $\left(x - \sum_{i=1}^{n}(x, e_i)e_i , e_j\right) = (x, e_j) - \sum_{i=1}^{n}(x, e_i)(e_i, e_j)$
$= (x, e_j) - (x, e_j) = 0$.
$x - \sum_{i=1}^{n}(x, e_i)e_i \perp e_j$ for each $j \ni 1 \leq j \leq n$.

Theorem: If $\{e_i\}$ is an orthonormal set in a Hilbert Space H, and if x is any vector in H, then the set $S = \{e_i: (x, e_i) \neq 0\}$ is either empty or countable.
Proof: For each positive integer n, consider $S_n = \left\{e_i: |(x, e_i)|^2 > \frac{\|x\|^2}{n}\right\}$.
If the set S_n contains n or more than n vectors, then $\sum_{e_i \in S_n}|(x, e_i)|^2 > n\frac{\|x\|^2}{n} = \|x\|^2$
… (1) which is a contradiction since by a theorem $\sum_{e_i \in S_n}|(x, e_i)|^2 \leq \|x\|^2$.
$\therefore$ S_n contains at most $n - 1$ vectors.
Thus, for each + ve integer n, the set S_n is finite.
Now suppose $e_i \in S$. Then $(x, e_i) \neq 0$.
However small may be the value of $|(x, e_i)|^2$, we can take n so large that
$|(x, e_i)|^2 > \frac{\|x\|^2}{n}$.
$\therefore$ If $e_i \in S$, then e_i must belong to some S_n.
$\therefore$ $S = \bigcup_{n=1}^{\infty} S_n$. $\therefore$ S is a countable being countable union of finite sets.
If $(x, e_i) = 0$ for every i, then S is empty. Otherwise, S is a finite or a countable infinite set.

Theorem: 2*: Bessel's inequality.
If $\{e_i\}$ is an orthonormal set in a Hilbert Space H, then $\sum|(x, e_i)|^2 \leq \|x\|^2$ for each vector $x \in H$.
Proof: Let $x \in H$ and $S = \{e_i: (x, e_i) \neq 0\}$. Then S is either empty or countable.
If S is empty then $(x, e_i) = 0 \ \forall$ i.
In this case define $\sum|(x, e_i)|^2$ to be 0 and we have $0 \leq \|x\|^2$.
Thus, $\sum|(x, e_i)|^2 \leq \|x\|^2$.
Now suppose $S \neq \phi$.
$\therefore$ either S is finite or countably infinite.
If S is finite $S = \{e_1, e_2, \ldots, e_n\}$ for some + ve integer n.
In this case we can define $\sum|(x, e_i)|^2$ to be $\sum_{i=1}^{n}|(x, e_i)|^2$ which is $\leq \|x\|^2$ by Bessel's inequality for finite cases.
Finally suppose that S be arranged in a definite order say $S = \{e_1, e_2, \ldots, e_n, \ldots\}$.
For each n, the set $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal set and by Bessel's inequality for finite cases $\sum_{i=1}^{n}|(x, e_i)|^2 \leq \|x\|^2$.

This says that the infinite series $\sum_{i=1}^{\infty}|(x,e_i)|^2$ is absolutely convergent since all terms of this series are + ve, and so $\sum_{i=1}^{\infty}|(x,e_i)|^2 \le \|x\|^2$ … (i).
Also, it's sum will not change by any rearrangement of it's terms. In this case we can define $\sum|(x,e_i)|^2$ to be $\sum_{i=1}^{\infty}|(x,e_i)|^2$ and from (i) this is $\le \|x\|^2$.

Theorem: 2*: If $\{e_i\}$ is an orthonormal set in a Hilbert Space H, and if x is an arbitrary vector in H, then $x - \sum(x,e_i)e_i \perp e_j$ for each j.

Proof: Let S = $\{e_i: (x, e_i) \neq 0\}$. Then S is either empty or countable.
If S is empty then $(x, e_i) = 0 \ \forall$ i.
In this case define $\sum(x,e_i)e_i$ to be vector $\bar{0}$ and then $x - \sum(x,e_i)e_i = x - \bar{0} = x$.
Since S = φ, $(x, e_j) = 0 \ \forall$ j so that $x \perp e_j \ \forall$ j and hence the result.
Now suppose S ≠ ϕ.
∴ either S is finite or countably infinite.
If S is finite S = $\{e_1, e_2, \ldots, e_n\}$ for some + ve integer n.
In this case define $\sum(x,e_i)e_i = \sum_{i=1}^{n}(x,e_i)e_i$ and we have already proved that $x - \sum_{i=1}^{n}(x,e_i)e_i \perp e_j$ for each j.
Finally suppose S be countably infinite and be arranged in a definite order say, S = $\{e_1, e_2, \ldots, e_n, \ldots\}$.
Put $s_n = \sum_{i=1}^{n}(x,e_i)e_i$.
For m > n, $\|s_n - s_m\|^2 = \|\sum_{i=n+1}^{m}(x,e_i)e_i\|^2 = \sum_{i=n+1}^{m}|(x,e_i)|^2$.
But by Bessel's inequality the series $\sum_{i=1}^{\infty}|(x,e_i)|^2$ is convergent.
∴ m, n → ∞, we have $\sum_{i=n+1}^{\infty}|(x,e_i)|^2 \to 0 \Rightarrow \|s_n - s_m\|^2 \to 0$.
∴ the sequence $\{s_n\}$ is a Cauchy sequence in H.
Since H is complete, $\{s_n\}$ is convergent in H.
∴∃ a vector s ∈ H ∋ $s_n \to s$ which we write $s = \sum_{n=1}^{\infty}(x,e_n)e_n$.
Now define $\sum(x,e_i)e_i = \sum_{n=1}^{\infty}(x,e_n)e_n$.
$\therefore \left(x - \sum(x,e_i)e_i\,, e_j\right) = \left(x - s, e_j\right) = \left(x,e_j\right) - \left(s,e_j\right) = \left(x,e_j\right) - \left(lt\, s_n, e_j\right)$
$= \left(x,e_j\right) - lt\left(s_n, e_j\right)$.
If $e_j \notin$ S then $(s_n, e_j) = \left(\sum_{i=1}^{n}(x,e_i)e_i\,, e_j\right) = 0$.
∴ $lt\left(s_n, e_j\right) = 0$ in this case.
$\therefore \left(x - \sum(x,e_i)e_i\,, e_j\right) = (x, e_j) = 0$ since $e_j \notin$ S.
If $e_j \in$ S, then $(s_n, e_j) = \left(\sum_{i=1}^{n}(x,e_i)e_i\,, e_j\right) = (x, e_j)$ for n > j.
∴ $lt\,(s_n, e_j) = (x, e_j)$ in this case.
So that $\left(x - \sum(x,e_i)e_i\,, e_j\right) = (x, e_j) - (x, e_j) = 0$.
Thus, we have $\left(x - \sum(x,e_i)e_i\,, e_j\right) = 0$ for each j.
ie. $x - \sum(x,e_i)e_i \perp e_j$ for each j.

Claim: Definition of $\sum (x, e_i)e_i$ is valid.
Let the vectors in S be arranged in any manner say S = {f_1, f_2, … , f_n, …}.
Put $s_n' = \sum_{i=1}^{n} (x, f_i)f_i$.
As shown above {s_n'} will converge to vector say s′ in H.
We write $s' = \sum_{n=1}^{\infty} (x, f_n)f_n$.
For any ε > 0, let n_0 be a + ve integer so large that if $n \geq n_0$, then $\|s_n - s\| < \varepsilon$,
$\|s_n' - s'\| < \varepsilon$ and $\sum_{i=n_0+1}^{\infty} |(x, e_i)|^2 < \varepsilon^2$.
For some + ve integer $m_0 > n_0$, all terms of s_{n_0} occur among those of s_{m_0}'.
$\therefore s_{m_0}' - s_{n_0}$ is a finite sum consisting of the type $(x, e_i)e_i$ for $i = n_0 + 1, n_0 + 2, \ldots$
This gives $\|s_{m_0}' - s_{n_0}\|^2 \leq \sum_{i=n_0+1}^{\infty} |(x, e_i)|^2 < \varepsilon^2$.
Consequently, $\|s_{m_0}' - s_{n_0}\| < \varepsilon$.
Now, $\|s' - s\| = \|s' - s_{m_0}' + s_{m_0}' - s_{n_0} + s_{n_0} - s\|$

$$\leq \|s' - s_{m_0}'\| + \|s_{m_0}' - s_{n_0}\| + \|s_{n_0} - s\| < 3\varepsilon.$$

Since ε is arbitrary, $\|s' - s\| < 3\varepsilon$ gives s′ = s.

COMPLETE ORTHONORMAL SET

Definition: An orthonormal set is said to be complete if it is not contained in any larger orthonormal set.

Theorem: 2*: Every orthonormal set in a Hilbert Space is contained in some complete orthonormal set. Further every non – zero Hilbert Space contains a complete orthonormal set.
Proof: Let S be an orthonormal set in a Hilbert Space H.
Let $\mathcal{P}$ be the class of orthonormal sets containing S.
Then $\mathcal{P}$ is non-empty since $S \in \mathcal{P}$.
Now $\mathcal{P}$ is partially ordered set w. r. t. set inclusion.
Let T be any totally ordered subset of $\mathcal{P}$ and let $T = \{A_\lambda : \lambda \in \Lambda\}$.
Obviously for every λ, $A_\lambda \subseteq \cup\{A_\lambda, \lambda \in \Lambda\}$.
Since each A_λ contains S, $\cup A_\lambda$ contains S.
Let x, y be any two distinct vectors belonging to $\cup A_\lambda$, $\lambda \in \Lambda$.
Then $\exists\, A_\alpha$ and $A_\beta \in T \ni x \in A_\alpha$ and $y \in A_\beta$.
But T is totally ordered.
$\therefore$ either $A_\alpha \subseteq A_\beta$ or $A_\beta \subseteq A_\alpha$.
Without loss of generality let us take $A_\alpha \subseteq A_\beta$.
Then $x, y \in A_\beta$. But A_β is an orthonormal set.
$\therefore$ $x \perp y$ and $\|x\| = 1$, $\|y\| = 1$.

∴ ∪A_λ, λ∈Λ is an orthonormal set.
Thus, ∪A_λ, λ∈Λ is an orthonormal set containing S and each $A_\lambda \subseteq \cup A_\lambda$, λ∈Λ.
∴ ∪A_λ, λ∈Λ is an upper bound for T in $\mathcal{P}$.
Thus, $\mathcal{P}$ satisfies all the conditions of Zorn's lemma.
∴ there must exist a maximal element in $\mathcal{P}$. Let it be M.
Then M is a complete orthonormal set containing S.
For, if it is not so, then ∃ an orthonormal set containing S and also containing M properly. This will contradict the maximality of M.
Further, let H be a non-zero Hilbert Space.
Let x be a non-zero vector in H.
Then S = $\left\{\frac{x}{\|x\|}\right\}$ is an orthonormal set in H.
∴ By the above part of this theorem ∃ a complete orthonormal set in H containing S.

Theorem: 5*: Let H be a Hilbert Space, and let $\{e_i\}$ be an orthonormal set in H. Then the following are equivalent.
(i) $\{e_i\}$ is complete.
(ii) x ⊥ $\{e_i\}$ ⇒ x = $\bar{0}$.
(iii) If x is an arbitrary vector in H, then x = $\sum (x, e_i)e_i$.
(iv) If x is an arbitrary vector in H, then $\|x\|^2 = \sum |(x, e_i)|^2$.
Proof: Claim: (i) ⇒ (ii).
Let $\{e_i\}$ be an orthonormal set. Suppose x ⊥ $\{e_i\}$ and x ≠ $\bar{0}$.
Then e = $\frac{x}{\|x\|}$ is a unit vector ∍ e ⊥ $\{e_i\}$. ie. $(e, e_i) = 0$ for each i. Then $\{e, e_i\}$ is an orthonormal set containing $\{e_i\}$ which contradicts the fact that $\{e_i\}$ is complete.
∴ x ⊥ $\{e_i\}$ ⇒ x = $\bar{0}$.
Claim: (ii) ⇒ (iii).
Let x ⊥ $\{e_i\}$ ⇒ x = $\bar{0}$ and x ∈ H.
Then by a theorem $x - \sum (x, e_i)e_i$ is orthogonal to every vector in the set $\{e_i\}$.
Ie. $x - \sum (x, e_i)e_i \perp \{e_i\}$.
By (ii) $x - \sum (x, e_i)e_i = \bar{0}$. ⇒ x = $\sum (x, e_i)e_i$.
Claim: (iii) ⇒ (iv). Assume (iii).
Let x ∈ H. Then x = $\sum (x, e_i)e_i$ by (iii).
∴ $\|x\|^2 = (x, x) = \left(\sum_i (x, e_i)e_i , \sum_j (x, e_j)e_j \right) = \sum_i \sum_j (x, e_i)\overline{(x, e_j)}(e_i, e_j)$
$= \sum_i (x, e_i)\overline{(x, e_i)} = \sum |(x, e_i)|^2$
Claim: (iv) ⇒ (i). Assume (iv). Ie $\|x\|^2 = \sum |(x, e_i)|^2$ ∀ x ∈ H.
If possible, suppose $\{e_i\}$ is not complete.

Then $\{e_i\}$ is a proper subset of an orthonormal set $\{e, e_i\}$.
$\therefore \|e\|^2 = \sum|(e, e_i)|^2 = 0$ which is a contradiction to the fact that e is a unit vector.

Standard Terminology. Let $\{e_i\}$ be a complete orthonormal set in a Hilbert Space H and x be any vector in H. Then w.r.t this complete orthonormal set the scalars (x, e_i) are called the Fourier coefficients of x, the expression $x = \sum(x, e_i)e_i$ is called the Fourier expansion of x, and the equation $\|x\|^2 = \sum|(x, e_i)|^2$ is called Parseval's equation or Parseval's identity.

Gram – Schmidth Orthogonalisation

<u>Theorem</u>: 1*: Let $S = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a linearly independent set of vectors in a Hilbert Space H. Then $\exists$ an orthonormal set of vectors $S_o = \{e_1, e_2, \ldots, e_n, \ldots\}$ such that for each n, $[\{x_1, x_2, \ldots, x_n\}] = [\{e_1, e_2, \ldots, e_n\}]$ ie, for any n, the linear subspace spanned by $\{x_1, x_2, \ldots, x_n\}$ is same as that spanned by $\{e_1, e_2, \ldots, e_n\}$.

<u>Proof</u>: We prove it by induction on n.

Let n = 1. Then $x_1 \neq \bar{0}$ since S is linearly independent. Put $e_1 = \frac{x_1}{\|x_1\|}$.

Then e_1 is a unit vector and so $\{e_1\}$ is an orthonormal set.

Since e_1 and x_1 are non-zero vectors and they are linearly dependent, the subspace spanned by $\{x_1\}$ is the same as that spanned by $\{e_1\}$.

Thus, the theorem is true for n = 1.

Now assume that we have constructed an orthonormal set $\{e_1, e_2, \ldots, e_{n-1}\}$ such that $[\{x_1, x_2, \ldots, x_i\}] = [\{e_1, e_2, \ldots, e_i\}]$ for any integer i ∍ $1 \leq i \leq n - 1$.

Now consider $y = x_n - \sum_{i=1}^{n-1}(x_n, e_i)e_i$.

Then y is orthogonal to each of vectors $e_1, e_2, \ldots, e_{n-1}$ by a known theorem.

Further, $y \neq \bar{0}$. For, if $y = \bar{0}$, then $x_n = \sum_{i=1}^{n-1}(x_n, e_i)e_i$.

ie. x_n is a linear combination of $e_1, e_2, \ldots, e_{n-1}$.

$\Rightarrow x_n$ is a linear combination of $x_1, x_2, \ldots, x_{n-1}$ by induction hypothesis which is contrary to the fact that $\{x_1, x_2, \ldots, x_n\}$ is linearly independent.

Now take $e_n = \frac{y}{\|y\|}$.

Then e_n is a unit vector orthogonal to each of $e_1, e_2, \ldots, e_{n-1}$ so that $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal set. Now $e_n = \frac{y}{\|y\|} = \frac{x_n - \sum_{i=1}^{n-1}(x_n, e_i)e_i}{\|y\|}$ … (i).

So, e_n is a linear combination of $e_1, e_2, \ldots, e_{n-1}, x_n$

$\Rightarrow e_n$ is a linear combination of $x_1, x_2, \ldots, x_{n-1}, x_n$ by induction.

Also from (i), $x_n = \|y\|e_n + \sum_{i=1}^{n-1}(x_n, e_i)e_i$

Thus, x_n is a linear combination of $e_1, e_2, \ldots, e_n$. $\therefore$ the linear subspace of H spanned by $\{x_1, x_2, \ldots, x_n\}$ is same as that spanned by the set $\{e_1, e_2, \ldots, e_n\}$. The proof of the theorem is complete by induction.

Example 1: Show that in the Hilbert space $l_2{}^n$ the set $\{e_1, e_2, \dots, e_n\}$ where e_i is the n-tuple with 1 in the i^{th} place and 0 elsewhere is a complete orthonormal set.

Solution: Let $S = \{e_1, e_2, \dots, e_n\}$ where $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, …, $e_n = (0, 0, \dots, 1)$. If $x = (x_1, x_2, \dots, x_n) \in l_2{}^n$ then by definition of norm in $l_2{}^n$, $\|x\| = \{\sum_{i=1}^{n} |x_i|^2\}^{\frac{1}{2}}$.

Also if $y = (y_1, y_2, \dots, y_n) \in l_2{}^n$ then by definition of inner product in $l_2{}^n$ we have $(x, y) = \sum_{i=1}^{n} x_i \bar{y_i}$.

Now $\|e_i\| = 1$ for each i. Also, if $i \neq j$ then, $(e_i, e_j) = 0$.

$\therefore$ S is an orthonormal set.

$x \perp ei$ for each $i = 1, 2, \dots, n \Rightarrow x_1 0 + x_2 0 + \dots + x_{i-1} 0 + x_i 1 + x_{i+1} 0 + \dots + x_n 0 = 0$.

$\Rightarrow x_i = 0$ for each $i = 1, 2, \dots, n$.

$\therefore x \perp S \Rightarrow x = \bar{0}$. $\therefore$ S is complete.

Example 2: Show that in the Hilbert space l_2 the set $\{e_1, e_2, \dots, e_n, \dots\}$ where e_i is the n-tuple with 1 in the i^{th} place and 0 elsewhere is a complete orthonormal set.

Solution: Let $S = \{e_1, e_2, \dots, e_n, \dots\}$. If $x = (x_1, x_2, \dots, x_n, \dots) \in l_2$ then by definition of norm in l_2, $\|x\| = \{\sum_{i=1}^{\infty} |x_i|^2\}^{\frac{1}{2}}$.

Also if $y = (y_1, y_2, \dots, y_n, \dots) \in l_2$ then by definition of inner product in l_2 we have $(x, y) = \sum_{i=1}^{\infty} x_i \bar{y_i}$.

Now $\|e_i\| = 1$ for each i. Also, if $i \neq j$ then, $(e_i, e_j) = 0$.

$\therefore$ S is an orthonormal set.

$x \perp e_n$ for each $n = 1, 2, \dots, \Rightarrow x_1 0 + x_2 0 + \dots + x_{n-1} 0 + x_n 1 + x_{n+1} 0 + \dots = 0$.

$\Rightarrow x_n = 0$ for each $n = 1, 2, \dots$

$\therefore x \perp S \Rightarrow x = \bar{0}$. $\therefore$ S is complete.

Example 3: Prove that an orthonormal set in a Hilbert Space is linearly independent.

Solution: Let S be any orthonormal set of vectors in a Hilbert Space H.

Let $S_n = \{e_1, e_2, \dots, e_n\}$ be a finite subset of S.

Let $\sum_{i=1}^{n} \alpha_i e_i = \bar{0}$ for scalars α_i, $1 \leq i \leq n$. …(i).

Now for each j, $1 \leq j \leq n$, $0 = (\bar{0}, ej) = \left(\sum_{i=1}^{n} \alpha_i e_i, e_j\right) = \sum_{i=1}^{n} \alpha_i (e_i, e_j) = \alpha_j$.

ie. $\alpha_j = 0$ for each j, $1 \leq j \leq n$.

$\therefore$ the set S_n is linearly independent. Thus, every finite subset of S is linearly independent. $\therefore$ S is linearly independent.

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E – CONTENT

PAPER: M 301,

FUNCTIONAL ANALYSIS

M. Sc. II YEAR, SEMESTER - III

UNIT – III : HILBERT SPACES - 2

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FUNCTIONAL ANALYSIS
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THE CONJUGATE SPACE H*

Let H be a Hilbert Space. A continuous linear transformation from H into $\mathbb{C}$ is called a functional on H. The set $\mathfrak{B}(H, \mathbb{C})$ of all functionals on H is denoted by H* and is called conjugate space of H. If we define addition and scalar multiplication in H* pointwise and if the norm of a functional f is defined by $\|f\|$ = Sup $\{|f(x)|: \|x\| \leq 1\}$, then H* is a Banach Space. To give H* the structure of a Hilbert Space we can define a suitable inner product on H*. Consequently by the same process (H*)* will also become a Hilbert Space.

Theorem 1: 3*: Let y be a fixed vector in a Hilbert Space H and let f_y be a scalar valued function on H defined by $f_y(x) = (x, y)\ \forall\ x \in H$. Then f_y is a functional in H*. Further show that $\|y\| = \|f_y\|$.

Proof: By definition $f_y(x) = (x, y)\ \forall\ x \in H$. Since (x, y) is a scalar, f_y is a mapping from H into $\mathbb{C}$.

f_y is linear: For, let $x_1, x_2 \in H$ and $\alpha, \beta \in K$.

Then $f_y(\alpha x_1 + \beta x_2) = (\alpha x_1 + \beta x_2, y) = \alpha(x_1, y) + \beta(x_2, y) = \alpha f_y(x_1) + \beta f_y(x_2)$.

f_y is continuous: For, $x \in H \Rightarrow |f_y(x)| = |(x, y)| \leq \|x\|\|y\|$ … (1) by Schwartz inequality. $= k\|x\|$ where $\|y\| = k \geq 0$. $\therefore |f_y(x)| \leq k\|x\|\ \forall\ x \in H$ so that f_y is bounded and hence continuous. Thus, f_y is a functional on H. ie. $f_y \in H^*$.

Now $\|f_y\| = \sup\{|f_y(x)|: \|x\| \leq 1\} \leq \|y\|$ from (1). Ie. $\|f_y\| \leq \|y\|$…(2).

If $y = \bar{0}$, then $\|y\| = 0$ so that $f_y(x) = (x, \bar{0}) = 0\ \forall\ x \in H$. ie. zero functional

Also $\|f_y\| = 0$.

Let $y \neq \bar{0}$. Then H is not a zero space.

$\therefore \|f_y\| = \sup\{|f_y(x)|: \|x\| = 1\}$…(3).

Since $y \neq \bar{0}$, $\frac{y}{\|y\|}$ is a unit vector.

Taking $x = \frac{y}{\|y\|}$, $\|f_y\| \geq \left|f_y\left(\frac{y}{\|y\|}\right)\right| = \left(\frac{y}{\|y\|}, y\right) = \frac{1}{\|y\|}(y, y) = \frac{1}{\|y\|}\|y\|^2 = \|y\|$.

Thus $\|f_y\| \geq \|y\|$ … (4). From (2) and (4) $\|f_y\| = \|y\|$.

Note: $\psi : H \to H^*$ defined by $\psi(y) = f_y\ \forall\ y \in H$ is a norm preserving map.

Theorem 2: 7*: (Riesz Representation theorem) Let H be a Hilbert space and f be an arbitrary functional in H*. Then $\exists$ a unique vector y in H $\ni f = f_y$ ie. $f(x) = (x, y)\ \forall\ x \in H$.

Proof: Case (i):

If f is zero functional, so $f(x) = 0\ \forall\ x \in H$. Also, if $y = \bar{0}$ then $(x, y) = (x, \bar{0}) = 0\ \forall\ x \in H$.

Thus, $\exists\ y = \bar{0} \ni f(x) = (x, y)\ \forall\ x \in H$.

Case (ii): Now suppose f is not a zero functional ie. $f(x) \neq 0$ for some $x \in H$.

Then the null space of f is $N = \{x \in H: f(x) = 0\}$. Then N is a proper subspace of H.

Since f is also continuous N is proper closed subspace of H.

$\therefore\ \exists$ a non-zero vector $y_0 \in H \ni y_0 \perp N$ ie. $y_0 \in N^{\perp}$.

Now define $v = f(x)y_0 - f(y_0)x\ \forall\ x \in H$

Then $f(v) = f(x)f(y_0) - f(y_0)f(x) = 0\ \forall\ x \in H$.

$\Rightarrow v \in N. \Rightarrow (v, y_0) = 0 \Rightarrow (f(x)y_0 - f(y_0)x, y_0) = 0$.

$\Rightarrow (f(x)y_0, y_0) - (f(y_0)x, y_0) = 0. \Rightarrow f(x)(y_0, y_0) = f(y_0)(x, y_0)$.

$\Rightarrow f(x) = \frac{f(y_0)}{\|y_0\|^2}(x, y_0) = \left(x, \frac{\overline{f(y_0)}}{\|y_0\|^2} y_0\right)$

Let $y = \frac{\overline{f(y_0)}}{\|y_0\|^2} y_0$. Then $f(x) = (x, y)\ \forall\ x \in H$

[Let α be any scalar $\ni y = \alpha y_0$.

If $x \in N$, then $f(x) = 0$ and $(x, y) = (x, \alpha y_0) = \bar{\alpha}(x, y_0) = 0\ \forall\ y_0 \perp N$.

$\therefore$ If $x \in N$ and $y = \alpha y_0$ then, $f(x) = (x, y)$.

Now choose α in such a way that $y = \alpha y_0$ satisfies $f(x) = (x, y)$ for $x = y_0$.

Ie. $f(y_0) = (y_0, \alpha y_0) = \bar{\alpha}(y_0, y_0) = \bar{\alpha}\ \|y_0\|^2$

If we take $\alpha = \frac{\overline{f(y_0)}}{\|y_0\|^2}$ then, $y = \alpha y_0$ satisfies $f(x) = (x, y)$ for every x in N and for $x = y_0$. Now let $x \in H$. Since $N \cap N^{\perp} = \{\bar{0}\}$ and y_0 is a non-zero vector belonging to $N^{\perp}$, $y_0 \notin N$. $\therefore$ $f(y_0) \neq 0$. Now $f(x) = \frac{f(x)}{f(y_0)} f(y_0) = \beta f(y_0)$ where $\beta = \frac{f(x)}{f(y_0)}$. Then $f(x) = \beta f(y_0)$. $\Rightarrow f(x) - \beta f(y_0) = 0 \Rightarrow f(x - \beta y_0) = 0 \Rightarrow x - \beta y_0 \in N \Rightarrow x - \beta y_0 = n$ for some $n \in N$. Thus, $x \in H \Rightarrow x = n + \beta y_0$ where β is some scalar and $n \in N$. Now $f(x) = f(n + \beta y_0) = f(n) + \beta f(y_0) = (n, y) + \beta(y_0, y) = (n, y) + (\beta y_0, y) = (n + \beta y_0, y) = (x, y)$. $\therefore\ \exists\ y = \alpha y_0$ where $\alpha = \frac{\overline{f(y_0)}}{\|y_0\|^2} \ni f(x) = (x, y)\ \forall\ x \in H$]

Uniqueness of y:

If possible, suppose y and z are two vectors in H ∋ $f(x) = (x, y)\ \forall\ x \in H$ and $f(x) = (x, z)\ \forall\ x \in H$.

Then $(x, y) = (x, z)\ \forall\ x \in H$.

$\Rightarrow (x, y - z) = 0\ \forall\ x \in H$.

$\Rightarrow (y - z, y - z) = 0$.

$\Rightarrow y - z = \bar{0} \Rightarrow y = z$.

Hence the Theorem.

Theorem 3: Show that the mapping $\psi : H \to H^*$ defined by $\psi(y) = f_y$ where $f_y(x) = (x, y)\ \forall\ x \in H$ is one to one, onto additive but not linear but an isometry.

Proof: (i) ψ is one to one. For let $y, z \in H$ and $\psi(y) = \psi(z)$.

$\Rightarrow f_y = f_z$

$\Rightarrow f_y(x) = f_z(x)\ \forall\ x \in H$.

$\Rightarrow (x, y) = (x, z)\ \forall\ x \in H$.

$\Rightarrow (x, y) - (x, z) = 0\ \forall\ x \in H$.

$\Rightarrow (x, y - z) = 0\ \forall\ x \in H$.

$\Rightarrow (y - z, y - z) = 0$.

$\Rightarrow y - z = \bar{0} \Rightarrow y = z$.

(ii) ψ is onto: For, let f be an arbitrary functional in H*.

∃ a unique vector y ∋ $f = f_y$.

Then $\psi(y) = f_y = f$.

(iii) ψ is additive: For, let $y, z \in H$.

$$\begin{aligned} f_{y+z}(x) &= (x, y + z)\ \forall\ x \in H \\ &= (x, y) + (x, z)\ \forall\ x \in H \\ &= f_y(x) + f_z(x)\ \forall\ x \in H \\ &= (f_y + f_z)(x)\ \forall\ x \in H. \end{aligned}$$

$\therefore\ f_{y+z} = f_y + f_z \ldots (1)$.

Now $\psi(y + z) = f_{y+z} = f_y + f_z = \psi(y) + \psi(z) \ldots (2)$

(iv) ψ is not linear: For, let $y \in H$, and $\alpha \in K$.

$$\begin{aligned} \text{Then } f_{\alpha y}(x) &= (x, \alpha y)\ \forall\ x \in H \\ &= \bar{\alpha}(x, y)\ \forall\ x \in H \\ &= \bar{\alpha} f_y(x)\ \forall\ x \in H. \end{aligned}$$

$\therefore f_{\alpha y} = \bar{\alpha} f_y \ldots (3)$.

Now $\psi(\alpha y) = f_{\alpha y} = \bar{\alpha} f_y = \bar{\alpha}\psi(y) \ldots (4)$.

Thus, ψ is not linear. Such a mapping is called a conjugate linear map.

(v) ψ is an isometry: For, let y, z $\in$ H.
Then $\|\psi(y) - \psi(z)\| = \|f_y - f_z\| = \|f_y + f_{-z}\| = \|f_{y-z}\| = \|y - z\|$ by theorem (1). $\therefore \psi$ is an isometry.

<u>Theorem 4</u>: If H is a Hilbert Space, then show that H* is also a Hilbert Space with respect to the inner product defined by $(f_x, f_y) = (y, x)$.

<u>Proof</u>: We know that H* is a Banach Space with suitable definitions of addition and scalar multiplication in H* and norm of a functional in H*.

(i) Claim: $(\alpha f_x + \beta f_y, f_z) = \alpha(f_x, f_z) + \beta(f_y, f_z)$.
We know that $f_{\alpha y} = \bar{\alpha} f_y$.
$(\alpha f_x + \beta f_y, f_z) = (f_{\bar{\alpha}x} + f_{\bar{\beta}y}, f_z) = (f_{\bar{\alpha}x+\bar{\beta}y}, f_z) = (z, \bar{\alpha}x + \bar{\beta}y)$
$= \bar{\bar{\alpha}}(z, x) + \bar{\bar{\beta}}(z, y) = \alpha(f_x, f_z) + \beta(f_y, f_z)$.

(ii) Claim: $\overline{(f_x, f_y)} = (f_y, f_x)$.
$\overline{(f_x, f_y)} = \overline{(y, x)} = (x, y) = (f_y, f_x)$

(iii) Claim: $(f_x, f_x) = \|f_x\|^2$.
$(f_x, f_x) = (x, x) = \|x\|^2 = \|f_x\|^2$

Hence H* is a Hilbert Space with inner product $(f_x, f_y) = (y, x)$.

Corollary: If we denote the elements of H** by F_f, F_g etc. where f, g are their corresponding elements in H*, then by theorem 4, H** is also a Hilbert Space with respect to inner product defined by $(F_f, F_g) = (g, f)$.

<u>Theorem 5</u>: 1*: If H is a Hilbert Space, then H is reflexive.

<u>Proof</u>: Let H be a Hilbert Space.
We prove that there exists a natural isometric isomorphism from H onto H**.
For this we will define two natural mappings from H to H** which are equal.
Let x be any fixed vector in H.
Let F_x be a scalar valued function defined on H* by $F_x(f) = f(x)$ $\forall$ f $\in$ H*.
Then F_x is a functional on H*.
F_x is called functional on H* induced by x.
Now define T: H $\rightarrow$ H** by $T(x) = F_x$ $\forall$ x $\in$ H.
Since T is linear and therefore T is isometric isomorphism from H onto H**.
Let T_1 be a mapping from H into H* defined by $T_1(x) = f_x$ where $f_x(y) = (y, x)$ $\forall$ y $\in$ H.
Let T_2 be a mapping from H* into H** defined by $T_2(f_x) = F_{f_x}$ where $F_{f_x}(f) = (f, f_x)$ $\forall$ f $\in$ H*.
Then T_2T_1 is a mapping of H into H**.

Then T_2T_1 is also one-one and onto, since T_1, T_2 are one-one and onto.
Claim: $T = T_2T_1$.
Both T and T_2T_1 are mappings from H into H**.
By definition of T, $T(x) = F_x$.
Also, $T_2T_1(x) = T_2\{T_1(x)\} = T_2(f_x) = F_{f_x}$.
Now we have to prove that $F_x = F_{f_x}$
Clearly both F_x and F_{f_x} are scalar valued functions defined on H*.
Let f be an arbitrary element of H*.
Then ∃ unique y in H such that $f = f_y$.
Now $F_{f_x}(f) = (f, f_x) = (f_y, f_x)$ ∵ $f = f_y$
$= (x, y) = f_y(x) = f(x)$ ∵ $f = f_y$.
$= F_x(f)$.
Thus, $F_{f_x}(f) = F_x(f)\ \forall\ f \in H^*$.
∴ $F_{f_x} = F_x$
∴ $T = T_2T_1$
∴ T is a mapping of H onto H**.
Hence H is reflexive.

THE ADJOINT OF AN OPERATOR:

<u>Theorem 1</u>: 5*: Let T be an operator on a Hilbert Space H. Then ∃ a unique operator T* on H ∍ $(Tx, y) = (x, T^*y)\ \forall\ x, y \in H$. The operator T* is called adjoint of the operator T.
<u>Proof</u>: ***Existence***: Let y be a fixed vector in H.
Define a scalar valued function $f_y : H \to K$ such that $f_y(x) = (Tx, y)\ \forall\ x \in H$.
f_y is linear: For, If $x_1, x_2 \in H$, $\alpha, \beta \in K$; then $f_y(\alpha x_1 + \beta x_2) = (T(\alpha x_1 + \beta x_2), y)$
$= (\alpha Tx_1 + \beta Tx_2, y)$ ∵ T is linear.
$= \alpha\, (Tx_1, y) + \beta\, (Tx_2, y)$
$= \alpha f_y(x_1) + \beta f_y(x_2)$
f_y is continuous: For, If $x \in H$, then $|f_y(x)| = |(Tx, y)|$
$\leq \|Tx\|\ \|y\|$ by Schwartz inequality.
$\leq \|T\|\|y\|\|x\|$ … (i) as T is bounded.
Hence $f_y \in H^*$.
Now by Riesz representation theorem ∃ a unique vector $z \in H$ ∍ $f_y(x) = (x, z)\ \forall\ x \in H$.
Ie. ∃ a unique vector $z \in H$ ∍ $(Tx, y) = (x, z)\ \forall\ x \in H$ … (ii).

And $\|f_y\| = \|z\|$ … (iii).
Thus, for each y in H ∃ a unique vector z in H ∋ (Tx, y) = (x, z) ∀ x ∈ H.
Hence, we get a mapping (say) T^*: H → H ∋ T^*y = z ∀ y ∈ H.
So, from (ii), we have (Tx, y) = (x, T^*y) ∀ x, y ∈ H…(iv).
Thus, existence of T^*: H → H ∋ (Tx, y) = (x, T^*y) ∀ x, y ∈ H is established.
We call this new mapping T^*: H → H, the adjoint of T.
Claim: T^* is linear. Let y_1, y_2 be any two vectors in H and α, β be any scalars.
For any vector x ∈ H, $(x, T^*(\alpha y_1 + \beta y_2)) = (Tx, \alpha y_1 + \beta y_2)$ by (1)
$= \bar{\alpha}\,(Tx, y_1) + \bar{\beta}(Tx, y_2)$
$= \bar{\alpha}(x, T^*y_1) + \bar{\beta}(x, T^*y_2)$ by (1)
$= (x, \alpha T^*y_1 + \beta T^*y_2)$
Thus, $(x, T^*(\alpha y_1 + \beta y_2)) = (x, \alpha T^*y_1 + \beta T^*y_2)$ ∀ x ∈ H.
∴ $T^*(\alpha y_1 + \beta y_2) = \alpha T^*y_1 + \beta T^*y_2$. ∴ T^* is linear
Claim: T^* is continuous.
For any vector y ∈ H, $\|T^*y\|^2 = (T^*y, T^*y) = (TT^*y, y)$ by (1)
$= |(TT^*y, y)|$ ∵ $\|T^*y\|^2$ is real ≥ 0
$\leq \|TT^*y\|\,\|y\|$ by Schwartz inequality.
$\leq \|T\|\|T^*y\|\,\|y\|$
Thus, $\|T^*y\|^2 \leq \|T\|\|T^*y\|\,\|y\|$ ∀ y ∈ H … (2).
If $\|T^*y\| = 0$ then $\|T^*y\| \leq \|T\|\|y\|$ ∵ $\|T\|\,\|y\| \geq 0$.
If $\|T^*y\| \neq 0$ then from (2), $\|T^*y\| \leq \|T\|\|y\|$ ∀ y ∈ H.
Let $\|T\|$ = k then, k ≥ 0 and $\|T^*y\| \leq k\|y\|$ ∀ y ∈ H.
∴ T^* is bounded and hence continuous. ∴ T^* is an operator on H.
Uniqueness: Suppose ∃ another mapping T′: H → H ∋ (Tx, y) = (x, T′y) ∀ x, y ∈ H.
Thus, (Tx, y) = (x, T′y) and (Tx, y) = (x, T^*y) ∀ x, y ∈ H.
⇒ (x, T′y) = (x, T^*y) ∀ x, y ∈ H.
⇒ T′y = T^*y ∀ y ∈ H. ⇒ T′ = T^*.

Theorem 2: The adjoint operator T → T^* on 𝔅(H) has the following properties.

(1) $(T_1 + T_2)^* = T_1^* + T_2^*$ 4*
(2) $(\alpha T)^* = \bar{\alpha}\, T^*$ 1*
(3) $(T_1T_2)^* = T_2^*\, T_1^*$ 4*
(4) $T^{**} = T$ 3*
(5) $\|T^*\| = \|T\|$ 3*
(6) $\|T^*T\| = \|T\|^2$ 5*

Proof: (1) For every x, y ∈ H, $(x, (T_1 + T_2)^*y) = (\{T_1 + T_2\}x, y)$
$= (T_1x + T_2x, y)$

$= (T_1x, y) + (T_2x, y)$
$= (x, T_1^*y) + (x, T_2^*y)$
$= (x, \{T_1^* + T_2^*\}y)$

$\therefore$ From Uniqueness of T*, $(T_1 + T_2)^* = T_1^* + T_2^*$

(2) For every x, y $\in$ H, $(x, \{\alpha T^*\}y) = (\{\alpha T\}x, y)$
$= (\alpha\{Tx\}, y)$
$= \alpha\,(Tx, y)$
$= \alpha(x, T^*y)$
$= (x, \bar{\alpha}\{T^*y\})$
$= (x, \{\bar{\alpha}T^*\}y)$

$\therefore$ From uniqueness $(\alpha T)^* = \bar{\alpha}T^*$.

(3) For every x, y $\in$ H, $(x, \{T_1T_2\}^*y) = (\{T_1T_2\}x, y)$
$= (T_1\{T_2x\}, y)$
$= (T_2x, T_1^*y)$
$= (x, T_2^*\{T_1^*y\})$
$= (x, \{T_2^*T_1^*\}y)$

$\therefore$ From uniqueness $(T_1T_2)^* = T_2^*\, T_1^*$

(4) For every x, y $\in$ H, $(x, T^{**}y) = (x, \{T^*\}^*y)$
$= (T^*x, y)$
$= \overline{(y, T^*x)}$
$= \overline{(Ty, x)}$
$= (x, Ty)$

$\therefore$ From uniqueness $T^{**} = T$.

(5) For every y $\in$ H,

$\|T^*y\|^2 = (T^*y, T^*y)$
$= (TT^*y, y)$
$= |(TT^*y, y)| \because (TT^*y, y) = \|T^*y\|^2 \geq 0.$
$\leq \|TT^*y\|\|y\|$ by Schwartz inequality.
$\leq \|T\|\|T^*y\|\|y\|$

Thus, $\|T^*y\|^2 \leq \|T\|\|T^*y\|\|y\|\ \forall$ y $\in$ H.

$\Rightarrow \|T^*y\| \leq \|T\|\|y\|\ \forall$ y $\in$ H…(A)

Now $\|T^*\| = \sup\{\|T^*y\|: \|y\| \leq 1\} \leq \|T\| \Rightarrow \|T^*\| \leq \|T\|$ … (B)

Now apply result (B) for T* in p[lace of T. $\|(T^*)^*\| \leq \|T^*\|$
$\Rightarrow \|T^{**}\| \leq \|T^*\|$
$\Rightarrow \|T\| \leq \|T^*\|$ … (C).

From (B) and (C), $\|T^*\| = \|T\|$

(6)We have $\|T^*T\| \leq \|T^*\|\|T\| \because \|ST\| \leq \|S\|\|T\|$
$= \|T\|\|T\| = \|T\|^2$.

Thus, $\|T^*T\| \leq \|T\|^2$… (D).
Further for every x ∈ H, $\|Tx\|^2$ = (Tx, Tx)

= (T*Tx, x)

= $|(T^*Tx, x)|$

≤ $\|T^*Tx\|\|x\|$ by Schwartz inequality

≤ $\|T^*T\|\|x\|\|x\| = \|T^*T\|\|x\|^2$

Thus, $\|Tx\|^2 \leq \|T^*T\|\|x\|^2$ ∀ x ∈ H… (E).
Now $\|T\| = \sup\{\|Tx\|: \|x\| \leq 1\}$
∴ $\|T\|^2 = [\sup\{\|Tx\|: \|x\| \leq 1\}]^2 = \sup\{\|Tx\|^2: \|x\| \leq 1\} \leq \|T^*T\|$
⇒ $\|T\|^2 \leq \|T^*T\|$… (F). From (D) and (F), $\|T^*T\| = \|T\|^2$.

Theorem 3: If O and I be zero and identity operators on a Hilbert Space H, then O* = O and I* = I. Hence show that if T is a non-singular operator on H then T* is also non-singular and in this case $(T^*)^{-1} = (T^{-1})^*$.
Proof: For every x, y ∈ H, (x, O*y) = (Ox, y) = ($\bar{0}$, y) = 0 = (x, $\bar{0}$) = (x, Oy).
∴ O* = O (∵ adjoint operator is unique).
Again (x, I*y) = (Ix, y) = (x, y) = (x, Iy). ∴ I* = I.
Now suppose that T is non-singular operator on H.
Let T^{-1} be the inverse of T.
Then T^{-1} is also an operator on H and $TT^{-1} = I = T^{-1}T$ …(1).
Taking adjoint of (1), $(TT^{-1})^* = I^* = (T^{-1}T)^*$
or $(T^{-1})^*T^* = I = T^*(T^{-1})^*$.
∴ T* is invertible and hence non-singular.
Inverse of T* is $(T^{-1})^*$. ie. $(T^*)^{-1} = (T^{-1})^*$.

Example 1: Show that the adjoint operator is one-to-one, onto as a mapping of $\mathfrak{B}(H)$ into itself.
Solution: Let ψ: $\mathfrak{B}(H) \to \mathfrak{B}(H)$ defined by ψ(T) = T* ∀ T ∈ $\mathfrak{B}(H)$.
Claim: ψ is one-one.
Let $T_1, T_2 \in \mathfrak{B}(H)$ and $\psi(T_1) = \psi(T_2) \Rightarrow T_1^* = T_2^*$.
⇒ $(T_1^*)^* = (T_2^*)^* \Rightarrow T_1^{**} = T_2^{**}$.
⇒ $T_1 = T_2$. ∴ ψ is one-one.
Claim: ψ is onto.
Let T be any arbitrary member of $\mathfrak{B}(H)$.
Then T* ∈ $\mathfrak{B}(H)$ and ψ(T*) = $(T^*)^*$ = T** = T.
∴ ψ is onto.

Example 2: Show that $\|TT^*\| = \|T\|^2$.

Solution: We have $\|T^*T\| = \|T\|^2 \ldots$ (i).
Take the operator T* in place of the operator T.
$\therefore \|T^{**}T^*\| = \|T^*\|^2 = \|T\|^2 \because \|T^*\| = \|T\|$.
$\Rightarrow \|TT^*\| = \|T\|^2$.

SELF - ADJOINT OPERATORS

Definition: An operator T on a Hilbert Space H is said to be ***self-adjoint*** if T* = T ie. (Tx, y) = (x, Ty) $\forall$ x, y $\in$ H.
Note: O and I are self-adjoint operators.

Theorem 1: 3*: The self-adjoint operators in $\mathfrak{B}(H)$ form a closed real linear subspace of $\mathfrak{B}(H)$ and therefore a real Banach Space which contains the identity transformation.
Proof: Let S be the collection of all self-adjoint operators on a Hilbert Space H.
Claim: S is real linear subspace
Clearly S is a non-empty subset of $\mathfrak{B}(H)$.
Let $A_1, A_2 \in$ S and α, β be any two real numbers.$\mathbb{R}$
Then $A_1^* = A_1$ and $A_2^* = A_2$.
Then $(\alpha A_1 + \beta A_2)^* = (\alpha A_1)^* + (\beta A_2)^* = \bar{\alpha} A_1^* + \bar{\alpha} A_2^* = \alpha A_1 + \beta A_2$.
$\therefore$ $\alpha A_1 + \beta A_2$ is also self-adjoint operator on H.
$\therefore$ S is a real linear subspace of $\mathfrak{B}(H)$.
S is closed: Let A be any limit point of S.
$\exists$ a sequence $\{A_n\}$ of distinct points of S $\ni A_n \to A$.

$$\begin{aligned}\|A - A^*\| &= \|A - A_n + A_n - A_n{}^* + A_n{}^* - A^*\| \\ &\le \|A - A_n\| + \|A_n - A_n{}^*\| + \|A_n{}^* - A^*\| \\ &= \|(A_n - A)\| + \|A_n - A_n\| + \|(A_n - A)^*\| \\ &= \|(A_n - A)\| + \|(A_n - A)\| \\ &= 2\|(A_n - A)\| \to 0 \text{ as } A_n \to A.\end{aligned}$$

$\therefore$ $\|A - A^*\| = 0 \quad \Rightarrow A - A^* = O \quad \Rightarrow A = A^*$
$\Rightarrow$ A is self-adjoint. $\therefore$ A $\in$ S.
Thus, S is closed.
$\therefore$ S is complete $\because \mathfrak{B}(H)$ is complete.
$\therefore$ S is real Banach Space.
Since I* = I, S contains Identity transformation.
Theorem 2: If A_1 and A_2 are self-adjoint operators on H, then their product A_1A_2 is self- adjoint if and only if $A_1A_2 = A_2A_1$.
Proof: Let A_1, A_2 be self-adjoint operators on a Hilbert Space H.
Then $A_1^* = A_1$ and $A_2^* = A_2$.

Suppose $A_1A_2 = A_2A_1$
Then $(A_1A_2)^* = A_2^*A_1^* = A_2A_1 = A_1A_2$.
$\therefore$ A_1A_2 is self-adjoint.
Conversely suppose A_1A_2 is self-adjoint. Then $A_1A_2 = (A_1A_2)^* = A_2^*A_1^* = A_2A_1$.
Ie. $A_1A_2 = A_2A_1$.

Theorem 3: If T is an arbitrary operator on a Hilbert Space H, then T = O if and only if (Tx, y) = 0 $\forall$ x, y.
Proof: Suppose T = O. Then $\forall$ x, y; (Tx, y) = (Ox, y) = $(\bar{0}, y)$ = 0.
Ie. (Tx, y) = 0 $\forall$ x, y.
Conversely suppose (Tx, y) = 0 $\forall$ x, y $\in$ H
$\Rightarrow$ (Tx, Tx) = 0 $\forall$ x $\in$ H
$\Rightarrow \|Tx\|^2 = 0 \quad \forall x \in H$
$\Rightarrow$ Tx = 0 $\forall$ x $\in$ H.
$\Rightarrow$ T = O.

Theorem 4: 4*: If T is an operator on a Hilbert Space H, then (Tx, x) = 0 $\forall$ x $\in$ H if and only if T = O.
Proof: Suppose T = O. Then $\forall$ x $\in$ H, (Tx, x) = (Ox, x) = $(\bar{0}, x)$ = 0.
Conversely suppose that (Tx, x) = 0 $\forall$ x $\in$ H.
Let α, β be any scalars and x, y $\in$ H.
Then 0 = (T{αx + βy}, αx + βy) = (αTx + βTy, αx + βy)
= $\alpha\bar{\alpha}$(Tx, x) + $\alpha\bar{\beta}$ (Tx, y) + $\beta\bar{\alpha}$ (Ty, x) + $\beta\bar{\beta}$(Ty, y).
$\Rightarrow$ $\alpha\bar{\beta}$ (Tx, y) + $\beta\bar{\alpha}$ (Ty, x) = 0 … (1) $\forall$ scalars α, β and x, y $\in$ H.
Put α = 1, β = 1 in (1). Then (Tx, y) + (Ty, x) = 0 …(2).
Put α = i, β = 1 in (1). Then i(Tx, y) – i(Ty, x) = 0 ie. i(Tx, y) – i(Ty, x) = 0 … (3).
(2) + (3) gives 2(Tx, y) = 0 $\forall$ x, y $\in$ H. $\Rightarrow$ (Tx, y) = 0 $\forall$ x, y $\in$ H.
$\Rightarrow$ (Tx, Tx) = 0 $\forall$ x $\in$ H. $\Rightarrow$ Tx = 0 $\forall$ x $\in$ H. $\Rightarrow$ T = O.

Theorem 5: 3*: An operator T on a Hilbert Space H is self-adjoint if and only if (Tx, x) is real $\forall$ x $\in$ H.
Proof: Suppose T is a self-adjoint operator on a Hilbert Space H.
Let x $\in$ H. Then (Tx, x) = (x, T*x) = (x, Tx) = $\overline{(Tx, x)}$ $\therefore$ (Tx, x) is real $\forall$ x $\in$ H.
Conversely suppose that (Tx, x) is real $\forall$ x $\in$ H.
Then (Tx, x) = $\overline{(Tx, x)} = \overline{(x, T^*x)} = (T^*x, x)$
$\Rightarrow$ (Tx, x) – (T*x, x) = 0 $\forall$ x $\in$ H.
$\Rightarrow$ (Tx – T*x, x) = 0 $\forall$ x $\in$ H

$\Rightarrow (\{T - T^*\}x, x) = 0 \ \forall \ x \in H$
$\Rightarrow T - T^* = 0$ by Theorem (4)
$\Rightarrow T = T^*$.

Definition: Let S be the set of all self-adjoint operators on a Hilbert Space H. We define $\leq$ on S as follows. We write $A_1 \leq A_2$ for $A_1, A_2 \in S$, if $(A_1x, x) \leq (A_2x, x) \ \forall \ x \in H$.

Theorem 6: 1*: The real Banach Space of all self-adjoint operators on a Hilbert Space H is a partially ordered set whose linear structure and order structure are related by the following properties.
(a) If $A_1 \leq A_2$ then $A_1 + A \leq A_2 + A$ for every A.
(b) If $A_1 \leq A_2$ and $\alpha \geq 0$ then $\alpha A_1 \leq \alpha A_2$.
Proof: Let S denote the set of all self-adjoint operators on H. For $A_1, A_2 \in S$, define $\leq$ on S by $A_1 \leq A_2$ if $(A_1x, x) \leq (A_2x, x) \ \forall \ x \in H$.
$\leq$ *is reflexive*: For, let $A \in S$. Observe that $(Ax, x) = (Ax, x) \ \forall \ x \in H$.
$\therefore$ we may say $(Ax, x) \leq (Ax, x) \ \forall \ x \in H. \ \Rightarrow A \leq A$.
Thus, $A \leq A \ \forall \ A \in S$.
$\leq$ *is antisymmetric:* For, let $A_1, A_2 \in S \ni A_1 \leq A_2$ and $A_2 \leq A_1$.
$\therefore (A_1x, x) \leq (A_2x, x)$ and $(A_2x, x) \leq (A_1x, x) \ \forall \ x \in H$.
$\Rightarrow (A_1x, x) = (A_2x, x) \ \forall \ x \in H$.
$\Rightarrow (A_1x - A_2x, x) = 0 \ \forall \ x \in H$.
$\Rightarrow (\{A_1 - A_2\}x, x) = 0 \ \forall \ x \in H$.
$\Rightarrow A_1 - A_2 = 0. \Rightarrow A_1 = A_2$.
$\leq$ *is transitive:* For, let $A_1, A_2, A_3 \in S \ni A_1 \leq A_2$ and $A_2 \leq A_3$.
$\therefore (A_1x, x) \leq (A_2x, x)$ and $(A_2x, x) \leq (A_3x, x) \ \forall \ x \in H$.
$\Rightarrow (A_1x, x) \leq (A_3x, x) \ \forall \ x \in H$.
$\Rightarrow A_1 \leq A_3$.
Thus, $\leq$ is a partial order relation on S.

(a) Let $A, A_1, A_2 \in S \ni A_1 \leq A_2$.
Then $(A_1x, x) \leq (A_2x, x) \ \forall \ x \in H$.
$\therefore (A_1x, x) + (Ax, x) \leq (A_2x, x) + (Ax, x) \ \forall \ x \in H$.
$\Rightarrow (A_1x + Ax, x) \leq (A_2x + Ax, x) \ \forall \ x \in H$.
$\Rightarrow (\{A_1 + A\}x, x) \leq (\{A_2 + A\}x, x) \ \forall \ x \in H$.
$\Rightarrow A_1 + A \leq A_2 + A$.

(b) Let $A_1, A_2 \in S$ and a scalar $\alpha \geq 0 \ni A_1 \leq A_2$.
Then $(A_1x, x) \leq (A_2x, x) \ \forall \ x \in H$.

$\therefore\ \alpha(A_1x, x) \le \alpha(A_2x, x)\ \forall\ x \in H.$
$\Rightarrow (\alpha A_1x, x) \le (\alpha A_2x, x)\ \forall\ x \in H.$
$\Rightarrow (\{\alpha A_1\}x, x) \le (\{\alpha A_2\}x, x)\ \forall\ x \in H.$
$\Rightarrow \alpha A_1 \le \alpha A_2.$

POSITIVE OPERATORS

Definition: A self-adjoint operator A on a Hilbert Space H is said to be *positive* if A $\ge$ O. ie. if $(Ax, x) \ge 0\ \forall\ x \in H$.
Note: O, I are positive operators.
Note: Let T be any arbitrary operator on H. Then both TT* and T*T are positive operators.

For, (TT*)* = (T*)*T* = TT* so that TT* is self-adjoint. Again (T*T)* = (T*)(T*)* = T*T so that T*T is self-adjoint.
Now (TT*x, x) = (T*x, T*X) = $\|T^*x\|^2 \ge 0$
and (T*Tx, x) = (Tx, {T*}*X) = (Tx, Tx) = $\|Tx\|^2 \ge 0$.

Theorem 7: 2*: If T is a positive operator on a Hilbert Space H, then I + T is non-singular.
Proof: Claim: I + T is one – one.
Let $x \in$ Ker (I + T) ie. (I + T)x = 0
$\Rightarrow Ix + Tx = 0 \Rightarrow x + Tx = 0 \Rightarrow Tx = -x \Rightarrow (Tx, x) = (-x, x) = -\|x\|^2 \ge 0 \because$ T is positive.

$\Rightarrow \|x\|^2 \le 0.$
$\Rightarrow \|x\|^2 = 0 \because \|x\|^2 \ge 0.$
$\Rightarrow x = \bar{0}.$

$\therefore$ Ker (I + T) = $\{\bar{0}\}$.
Hence, I + T is one-one.
Claim: I + T is onto.
Let M be the range of I + T.
First, we prove that M is closed.
For any vector $x \in H$, $\|(I+T)x\|^2 = \|x + Tx\|^2 = (x + Tx, x + Tx)$
$= (x, x) + (x, Tx) + (Tx, x) + (Tx, Tx) = \|x\|^2 + \overline{(Tx, x)} + (Tx, x) + \|Tx\|^2$
$= \|x\|^2 + 2(Tx, x) + \|Tx\|^2$ [$\because$ T is +ve $\Rightarrow$ T is self-adjoint $\Rightarrow$ (Tx, x) is real.]
$\ge \|x\|^2 \because$ T is positive.
Thus, $\|x\| \le \|(I+T)x\|\ \forall\ x \in H$.
Now let $\{(I + T)x_n\}$ be a Cauchy sequence in M.
For any 2 positive integers m, n; $\|x_m - x_n\| \le \|(I+T)(x_m - x_n)\|$

$= \|(I+T)x_m - (I+T)x_n\| \to 0$ ∵ {(I + T)x_n} be a Cauchy sequence.
$\therefore \|x_m - x_n\| \to 0$
⇒ {x_n} is a Cauchy sequence in H.
⇒ {x_n} converges to say x ∈ H ∵ H is complete.
∴ lt {(I + T)x_n} = (I + T)(lt x_n) ∵ I + T is continuous.
= (I + T)x ∈ M.
Thus, the Cauchy sequence {(I + T)x_n} in M converges to (I + T)x in M.
Thus, M is complete.
Hence M is closed. ∵ complete subspace of a complete space is closed.
To prove I + T is onto it suffices if we prove that M = H.
If possible, suppose M ≠ H.
Then ∃ a non-zero vector x_0 in H ∋ $x_0 \perp$ M.
⇒ ({I + T}x_0, x_0) = 0 ∵ $x_0 \perp$ M
⇒ (x_0 + Tx_0, x_0) = 0.
⇒ (x_0, x_0) + (Tx_0, x_0) = 0
⇒ $\|x_0\|^2$ + (Tx_0, x_0) = 0.
⇒ $-\|x_0\|^2$ = (Tx_0, x_0).
⇒ $-\|x_0\|^2 \geq 0$ ∵ T is positive.
⇒ $\|x_0\|^2 \leq 0 \Rightarrow \|x_0\|^2 = 0$.
⇒ $x_0 = \bar{0}$ which contradicts the fact that x_0 is a non-zero vector.
∴ M = H and so I + T is onto.
Claim: I + T is non-singular.
Since I + T is a bijection, I + T is invertible.
Hence, I + T is non-singular.

Corollary: If T is an arbitrary operator on H, then the operators I + TT* and I +T*T are non-singular.
Proof: For an arbitrary operator T on H, T*T and TT* are both positive operators. Hence by the above theorem both the operators I + TT* and I + T*T are non-singular.

NORMAL AND UNITARY OPERATORS

Normal Operator: Definition: An operator T on a Hilbert Space H is said to be *normal* if it commutes with it's adjoint. Ie. TT* = T*T.
Note: Obviously every self-adjoint operator is normal. For if T is a self-adjoint operator ie. T* = T. Then T*T = TT = TT*

Theorem 1: 2*: The set of all normal operators on a Hilbert Space H is a closed subset of $\mathfrak{B}(H)$ which contains the set of all self – adjoint operators and is closed under scalar multiplication.

Proof: Let M be the set of all normal operators on a Hilbert Space H.

Let T be a limit point of M. ∃ a sequence {Tn} of distinct points of M ∍ $T_n \to T$.

$\therefore \|T_n{}^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\| \to 0$.

$\therefore \|T_n{}^* - T^*\| \to 0. \Rightarrow T_n{}^* \to T^*$

Now $\|TT^* - T^*T\| = \|TT^* - T_nT_n{}^* + T_nT_n{}^* - T_n{}^*T_n + T_n{}^*T_n - T^*T\|$

$\leq \|TT^* - T_nT_n{}^*\| + \|T_nT_n{}^* - T_n{}^*T_n\| + \|T_n{}^*T_n - T^*T\|$

$= \|TT^* - T_nT_n{}^*\| + \|T_n{}^*T_n - T^*T\| \to 0 \because T_n{}^* \to T^*$ and $T_n \to T$.

Thus, $\|TT^* - T^*T\| \to 0. \Rightarrow TT^* - T^*T = O$

$\Rightarrow TT^* = T^*T \Rightarrow$ T is normal operator on H.

∴ T ∈ M and so M is closed.

Since every self-adjoint operator is normal, M contains the set of all self – adjoint operators on H.

Let T ∈ M and α be any scalar.

Now $(\alpha T)(\alpha T)^* = (\alpha T)(\bar{\alpha}T^*) = \alpha\bar{\alpha}(TT^*) = \alpha\bar{\alpha}(T^*T) = (\bar{\alpha}T^*)(\alpha T) = (\alpha T)^*(\alpha T)$.

∴ αT is normal ie. αT ∈ M.

∴ M is closed under scalar multiplication.

Theorem 2: 3*: If N_1 and N_2 are normal operators on a Hilbert Space H with the property that either commutes with the adjoint of the other, then $N_1 + N_2$ and N_1N_2 are also normal operators.

Proof: Let N_1 and N_2 be normal operators so that $N_1N_1^* = N_1^*N_1$ and $N_2N_2^* = N_2^*N_2$

Also given $N_1N_2^* = N_2^*N_1$ and $N_2N_1^* = N_1^*N_2$.

Now $(N_1 + N_2)(N_1 + N_2)^* = (N_1 + N_2)(N_1^* + N_2^*)$

$= N_1N_1^* + N_1N_2^* + N_2N_1^* + N_2N_2^*$

$= N_1^*N_1 + N_2^*N_1 + N_1^*N_2 + N_2^*N_2$.

$= N_1^*(N_1 + N_2) + N_2^*(N_1 + N_2)$

$= (N_1^* + N_2^*)(N_1 + N_2)$.

$= (N_1 + N_2)^*(N_1 + N_2)$

∴ $N_1 + N_2$ is normal.

Again $(N_1N_2)(N_1N_2)^* = (N_1N_2)(N_2^*N_1^*)$

$= N_1(N_2N_2^*)N_1^*$

$= N_1(N_2^*N_2)N_1^*$

$= (N_1N_2^*)(N_2N_1^*)$

$= (N_2^*N_1)(N_1^*N_2)$

$= N_2^*(N_1N_1^*)N_2$
$= N_2^*(N_1^*N_1)N_2$
$= (N_2^*N_1^*)(N_1N_2)$
$= (N_1N_2)^*(N_1N_2)$.

$\therefore$ N_1N_2 is normal.

Theorem 3: 3*: An operator T on a Hilbert Space H is normal if and only if $\|T^*x\| = \|Tx\|$ $\forall$ x $\in$ H.

Proof: T is normal iff $TT^* = T^*T$ iff $TT^* - T^*T = O$
iff $((TT^* - T^*T)x, x) = 0$ $\forall$ x $\in$ H.
iff $((TT^*)x, x) = ((T^*T)x, x)$ $\forall$ x $\in$ H
iff $(T^*x, T^*x) = (Tx, T^{**}x)$ $\forall$ x $\in$ H.
iff $(T^*x, T^*x) = (Tx, Tx)$ $\forall$ x $\in$ H.
iff $\|T^*x\|^2 = \|Tx\|^2$ $\forall$ x $\in$ H
iff $\|T^*x\| = \|Tx\|$ $\forall$ x $\in$ H.

Theorem 4: 2*: If N is a normal operator on Hilbert Space H, then $\|N^2\| = \|N\|^2$

Proof: Let N be a normal operator on H. $\therefore \|Nx\| = \|N^*x\|$ $\forall$ x $\in$ H … (i)
Replace x by Nx, we have, $\|NNx\| = \|N^*Nx\|$ $\forall$ x $\in$ H.
$\Rightarrow \|N^2x\| = \|N^*Nx\|$ $\forall$ x $\in$ H… (ii).
Now $\|N^2\| = \text{Sup}\{\|N^2x\| : \|x\| \le 1\} = \text{Sup}\{\|N^*Nx\|: \|x\| \le 1\} = \|N^*N\| = \|N\|^2$.

Theorem 5: 1*: Any arbitrary operator T on a Hilbert Space H can be uniquely expressed as $T = T_1 + iT_2$ where T_1 and T_2 are self – adjoint operators on H.

Proof: Let $T_1 = \frac{T+T^*}{2}$ and $T_2 = \frac{T-T^*}{2i}$.
Then $T = T_1 + iT_2$
Now $T_1^* = \left(\frac{T+T^*}{2}\right)^* = \frac{T^*+T^{**}}{2} = \frac{T+T^*}{2} = T_1$ so that T_1 is self - adjoint.
Again $T_2^* = \left(\frac{T-T^*}{2i}\right)^* = \frac{T^*-T^{**}}{2\bar{\imath}} = -\frac{T^*-T}{2i} = \frac{T-T^*}{2i} = T_2$ so that T_2 is self - adjoint.
If possible, let $T = U_1 + iU_2$ where U_1 and U_2 are self – adjoint.
Then $T^* = (U_1 + iU_2)^* = U_1^* + (iU_2)^* = U_1^* + \bar{\imath}\, U_2^* = U_1^* - i\, U_2^* = U_1 - iU_2$.
Now $T + T^* = U_1 + iU_2 + U_1 - iU_2 = 2\, U_1$.
$\therefore$ $U_1 = \frac{T+T^*}{2} = T_1$
Again $T - T^* = U_1 + iU_2 - U_1 + iU_2 = 2i\, U_2$.
$\therefore$ $U_2 = \frac{T-T^*}{2i} = T_2$
Hence the expression $T = T_1 + iT_2$ is unique where T_1 and T_2 are self – adjoint.

Theorem 6: 2*: If T is an operator on a Hilbert Space H, then T is normal if and only if it's real and imaginary parts commute.

Proof: Let $T = T_1 + iT_2$ where T_1 and T_2 are the real and imaginary parts of T.
Then T_1 and T_2 are self – adjoint operators and
$T^* = (T_1 + iT_2)^* = T_1^* + (iT_2)^* = T_1^* + \bar{\iota}\, T_2^* = T_1^* - iT_2^* = T_1 - iT_2$.
Now $TT^* = (T_1 + iT_2)(T_1 - iT_2) = T_1^2 + T_2^2 + i(T_2T_1 - T_1T_2)$ … (i)
Again $T^*T = (T_1 - iT_2)(T_1 + iT_2) = T_1^2 + T_2^2 + i(T_1T_2 - T_2T_1)$ … (ii)
Suppose T is normal then $TT^* = T^*T$
$\Rightarrow T_1^2 + T_2^2 + i(T_2T_1 - T_1T_2) = T_1^2 + T_2^2 + i(T_1T_2 - T_2T_1)$
$\Rightarrow T_2T_1 - T_1T_2 = T_1T_2 - T_2T_1$
$\Rightarrow T_1T_2 = T_2T_1$
$\Rightarrow T_1$ and T_2 commute.
Conversely suppose T_1 and T_2 commute. Ie. $T_1T_2 = T_2T_1$.
Then from (i) and (ii), $TT^* = T_1^2 + T_2^2 = T^*T$
$\therefore$ T is normal.

Unitary Operator: Definition: An operator U on a Hilbert Space H is said to be *unitary* if $UU^* = U^*U = I$.

Note: (i) Obviously every unitary operator is normal.
(ii) U is unitary iff U is invertible and $U^{-1} = U^*$.

Theorem 7: If T is an operator on a Hilbert Space H, then the following are equivalent:

(i) $T^*T = I$
(ii) $(Tx, Ty) = (x, y)\ \forall\ x, y \in H$
(iii) $\|Tx\| = \|x\|\ \forall\ x \in H$.

Proof: Assume (i). ie. $T^*T = I$.
Then for $x, y \in H$, $(Tx, Ty) = (x, T^*Ty) = (x, Iy) = (x, y)$. $\therefore$ (i) $\Rightarrow$ (ii)
Assume (ii). Ie $(Tx, Ty) = (x, y)\ \forall\ x, y \in H$.
$\Rightarrow (Tx, Tx) = (x, x)\ \forall\ x \in H$.
$\Rightarrow \|Tx\|^2 = \|x\|^2\ \forall\ x \in H$.
$\Rightarrow \|Tx\| = \|x\|\ \forall\ x \in H$.
$\therefore$ (ii) $\Rightarrow$ (iii).
Assume (iii). Ie. $\|Tx\| = \|x\|\ \forall\ x \in H$.
$\Rightarrow \|Tx\|^2 = \|x\|^2\ \forall\ x \in H$.
$\Rightarrow (Tx, Tx) = (x, x)\ \forall\ x \in H$.
$\Rightarrow (T^*Tx, x) = (Ix, x)\ \forall\ x \in H$.
$\Rightarrow ((T^*T - I)\, x, x) = 0\ \forall\ x \in H$.

$\Rightarrow T^*T - I = O$.

$\Rightarrow T^*T = I$

$\therefore$ (iii) $\Rightarrow$ (i). Hence the theorem.

Theorem 8: 2*: An operator T on a Hilbert Space H is unitary if and only if it is an isometric isomorphism of H onto itself.

Proof: Suppose T is unitary operator on H.

T is invertible and so T is onto.

Also, $TT^* = I$.

$\therefore$ By the above theorem, $\|Tx\| = \|x\|\ \forall\ x \in H$.

Thus, T preserves norm and so T is an isometric isomorphism of H onto itself.

Conversely suppose T is an isometric isomorphism of H onto itself.

Then T is one-one and onto.

$\therefore$ T^{-1} exists.

Also, T is an isometric isomorphism.

$\Rightarrow \|Tx\| = \|x\|\ \forall\ x \in H$.

$\Rightarrow T^*T = I$ by the above theorem

$\Rightarrow (T^*T)T^{-1} = I\,T^{-1}$

$\Rightarrow T^*I = T^{-1}$

$\Rightarrow T^* = T^{-1}$

$\therefore TT^* = I = T^*T$

Hence T is unitary.

Example 1: 3*: If T is any arbitrary operator on a Hilbert Space H, and if α, β are scalars ∍ $|\alpha| = |\beta|$, then $\alpha T + \beta T^*$ is normal.

Solution: $(\alpha T + \beta T^*)^* = (\alpha T)^* + (\beta T^*)^* = \bar{\alpha}T^* + \bar{\beta}T^{**} = \bar{\alpha}T^* + \bar{\beta}\,T$

Now $(\alpha T + \beta T^*)(\alpha T + \beta T^*)^* = (\alpha T + \beta T^*)(\bar{\alpha}T^* + \bar{\beta}\,T)$

$= \alpha\bar{\alpha}TT^* + \alpha\bar{\beta}T^2 + \beta\bar{\alpha}(T^*)^2 + \beta\bar{\beta}\,T^*T.$

$= |\alpha|^2TT^* + \alpha\bar{\beta}T^2 + \beta\bar{\alpha}(T^*)^2 + |\beta|^2T^*T$… (i)

Also $(\alpha T + \beta T^*)^*(\alpha T + \beta T^*) = (\bar{\alpha}T^* + \bar{\beta}T)(\alpha T + \beta T^*)$

$= \bar{\alpha}\alpha T^*T + \bar{\beta}\alpha T^2 + \bar{\alpha}\beta(T^*)^2 + \bar{\beta}\,\beta TT^*.$

$= |\alpha|^2T^*T + \alpha\bar{\beta}T^2 + \beta\bar{\alpha}(T^*)^2 + |\beta|^2TT^*$… (ii)

Since $|\alpha| = |\beta|$, RHS's of (i) and (ii) are same.

$\therefore (\alpha T + \beta T^*)(\alpha T + \beta T^*)^* = (\alpha T + \beta T^*)^*(\alpha T + \beta T^*)$.

Hence $\alpha T + \beta T^*$ is normal.

Example 2: If T is a normal operator on a Hilbert Space H and λ is any scalar, then $T - \lambda I$ is also normal.

Solution: Let T be a normal operator.

∴ TT* = T*T.

Now $(T - \lambda I)^* = T^* - (\lambda I)^* = T^* - \bar{\lambda} I^* = T^* - \bar{\lambda} I$.

∴ $(T - \lambda I)(T - \lambda I)^* = (T - \lambda I)(T^* - \bar{\lambda} I) = TT^* - \bar{\lambda}T - \lambda T^* + |\lambda|^2 I$…(i)

Also $(T - \lambda I)^* (T - \lambda I) = (T^* - \bar{\lambda} I)(T - \lambda I) = T^*T - \lambda T^* - \bar{\lambda}T + |\lambda|^2 I$…(ii).

Since TT* = T*T, RHS of (i) and (ii) are equal. ∴ T – λI is normal.

Example 3: If H is a finite dimensional Hilbert Space, show that every isometric isomorphism of H into itself is unitary.

Solution: Let T be an isometric isomorphism of a finite dimensional Hilbert Space H into itself.

Since H is a finite dimensional linear space and T is an isomorphism of H into itself. ∴ T must be onto.

∴ T is unitary.

Example 4: 3*: Show that the unitary operators on a Hilbert space H form a group.

Solution: Closure: Let T_1, T_2 be two unitary operators.

Then T_1 and T_2 are invertible and ${T_1}^{-1} = T_1^*$ and ${T_2}^{-1} = T_2^*$.

Since the mappings T_1, T_2 are continuous, T_1T_2 is also continuous.

∴ T_1T_2 is an operator on H.

Also, T_1, T_2 are invertible ⇒ T_1T_2 is also invertible.

∴ $(T_1T_2)^{-1} = {T_2}^{-1}{T_1}^{-1} = T_2^* T_1^* = (T_1T_2)^*$

∴ T_1T_2 is also unitary.

Associativity: We know that product of mappings is associative.

Existence of Identity: The identity operator I on H, is one – one, and onto so that I is invertible.

Also $I^{-1} = I = I^*$. ∴ I is unitary.

Existence of inverse: Let T be unitary on H. Then T is invertible and $T^{-1} = T^*$.

The mapping T^{-1} is continuous. ∴ T* is an operator on H.

Also, T^{-1} is invertible and $(T^{-1})^{-1} = (T^*)^{-1} = (T^{-1})^*$

∴ T^{-1} is unitary.

Hence unitary operators on a Hilbert Space form a group.

Example 5: 5*: Show that an operator T on a Hilbert Space H is unitary if and only if $T(\{e_i\})$ is a complete orthonormal set whenever $\{e_i\}$ is.

Solution: Suppose T is unitary operator on H and $\{e_i\}$ is a complete orthonormal set in H.

∴ TT* = I ∵ T is unitary.

$\therefore$ By a theorem $(Te_i, Te_j) = (e_i, e_j) = \begin{cases} 0, & if\ i \neq j \\ 1, & if\ i = j \end{cases}$.

$\therefore$ $\{T(e_i)\}$ is orthonormal set in H. To show that $\{T(e_i)\}$ is complete let $x \perp \{T(e_i)\}$

$\Rightarrow (x, Te_j) = 0\ \forall\ e_i$

$\Rightarrow (T^*x, e_i) = 0\ \forall\ e_i$

$\Rightarrow T^*x \perp \{e_i\}$

$\Rightarrow T^*x = \bar{0}$ by theorem 6.

$\Rightarrow TT^*x = T\bar{0}$

$\Rightarrow Ix = \bar{0} \Rightarrow x = \bar{0}$.

Thus, $x \perp T(\{e_i\}) \Rightarrow x = \bar{0}$.

$\therefore$ The orthonormal set $\{T(e_i)\}$ is complete.

Conversely suppose that $\{T(e_i)\}$ is a complete orthonormal set whenever $\{e_i\}$ is.

Claim: T is isometry.

If $x = \bar{0}$ then obviously $\|Tx\| = \|x\|$.

Let $x \neq \bar{0}$. Obviously $\left\{\frac{x}{\|x\|}\right\}$ is an orthonormal set.

$\therefore$ $\exists$ a complete orthonormal set in H containing singleton set $\left\{\frac{x}{\|x\|}\right\}$.

By hypothesis T maps this complete orthonormal set onto a complete orthonormal set. $\therefore T\left(\frac{x}{\|x\|}\right)$ is a unit vector. Ie. $\left\|T\left(\frac{x}{\|x\|}\right)\right\| = 1$.

$\Rightarrow \frac{1}{\|x\|}\|Tx\| = 1$ ie. $\|Tx\| = \|x\|$

$\therefore$ T preserves norms and so T is also one-one.

To show $T: H \to H$ is onto.

Let $T(H) = M$.

We show that M is closed subspace of H.

Let y be a limit point of M.

$\therefore$ $\exists$ a sequence $\{T(x_n)\}$ of distinct points of M $\ni Tx_n \to y$.

Now $\|x_m - x_n\|^2 = \|T(x_m - x_n)\|^2 = \|Tx_m - Tx_n\|^2$ $\because$ T is linear.

$\to 0$ $\because$ $\{T(x_n)\}$ is a convergent sequence in H.

$\Rightarrow \{x_n\}$ is a Cauchy sequence in H.

$\therefore$ $\{x_n\}$ is convergent $\because$ H is complete.

$\therefore \exists\ x \in H \ni x_n \to x$.

Now $y = lt\ Tx_n = T(lt\ x_n)$ $\because$ T is continuous.

$= Tx$.

But $y = Tx$

$\Rightarrow$ y is in the range of T which is M.

$\therefore$ M is closed.

Let $M \neq H$.

Then M is a proper closed subspace of H.
$\therefore$ $\exists$ non-zero $y_0 \in M \ni y_0 \perp M$.
M itself is a Hilbert Space because M is a closed subspace of H.
If M is zero space, then T is one-one $\Rightarrow$ H itself is a zero space.
$\therefore$ In this case everything is trivial.
So let $M \neq \{\bar{0}\}$.
Then M must contain a complete orthonormal set.
Since $y_0 \perp M$, y_0 is also orthonormal to this complete orthonormal set.
Then $y_0 = 0$ by theorem 6 which is a contradiction.
$\therefore$ M = H.
$\therefore$ T is onto.
$\therefore$ T is unitary.

PROJECTIONS

Definition: A projection P on a Hilbert Space H is said to be a *perpendicular projection* on H if the range M and the null space N of P are orthogonal.

Theorem 1: 2*: If P is a projection on a Hilbert Space H with range M and null space N then $M \perp N$ if and only if P is self-adjoint; and in this case, $N = M^{\perp}$.
Proof: Suppose P is a projection on a Hilbert Space H with range M and null Space N. Then $H = M \oplus N$.
Assume $M \perp N$. Let $z \in H$. Then z can be uniquely written as $z = x + y$ where $x \in M$ and $y \in N$.
$\therefore$ $(Pz, z) = (x, z) = (x, x + y) = (x, x) + (x, y) = (x, x)$ $\because$ $M \perp N$
Also $(P^*z, z) = (z, Pz) = (z, x) = (x + y, x) = (x, x) + (y, x) = (x, x)$.
$\therefore$ $(Pz, z) = (P^*z, z)$ $\forall$ $z \in H$.
$\Rightarrow$ $((P - P^*)z, z) = 0$ $\forall$ $z \in H$.
$\Rightarrow$ $P - P^* = O \Rightarrow P = P^* \Rightarrow$ P is self- adjoint.
Conversely suppose, P is self- adjoint.
Let $x \in M$, $y \in N$. Then $(x, y) = (Px, y) = (x, P^*y) = (x, Py) = (x, 0) = 0$.
$\therefore$ $M \perp N$.
Finally let P be a projection on a H with range M and null Space N.
Then $M \perp N$ by above part. Suppose $y \in N$. then $N \perp M \Rightarrow y \in M^{\perp}$.
$\therefore$ $N \subseteq M^{\perp}$.
Suppose N is a proper subset of $M^{\perp}$.
$\therefore$ N is a proper closed linear subspace of the Hilbert Space $M^{\perp}$.
$\therefore$ $\exists$ a non-zero vector $z_0 \in M^{\perp} \ni z_0 \perp N$.

But $z_0 \in M^{\perp} \Rightarrow z_0 \perp M$.
$\therefore z_0 = \bar{0}$ which contradicts the fact that $z_0 \neq \bar{0}$.
$\therefore\ N = M^{\perp}$.

Note: From now onwards by a projection P on H we mean a perpendicular projection on H
$\therefore$ An operator P on a Hilbert Space is a projection on H iff P is linear, continuous, $P^2 = P$ and $P^* = P$.

Note: The zero operator O and identity operator I are projections on H.

Note: If M is a closed linear subspace of H then $H = M \oplus M^{\perp}$.
$\therefore\ \exists$ a projection P on H with range M defined by $P(x + y) = x$ where $x \in M$, $y \in M^{\perp}$.

Remark: If P is a projection on a Hilbert Space H with range M, then the null space of P is uniquely determined and is always $M^{\perp}$.

Theorem 2: P is a projection on a closed linear subspace M of H if and only if $I - P$ is the projection on $M^{\perp}$.
Proof: Suppose P is a projection on M. $\therefore\ P^2 = P$ and $P^* = P$, P is linear & continuous.
Clearly $I - P$ is linear and continuous.
Now $(I - P)^* = I^* - P^* = I - P$.
Also $(I - P)^2 = (I - P)(I - P) = I - P - P + P^2 = I - P - P + P = I - P$.
$\therefore\ I - P$ is a projection.
Let M be the range of P, N be the range of $I - P$.
Now $x \in N$, $(I - P)x = x \Rightarrow x - Px = x$
$\Rightarrow Px = 0$.
$\Rightarrow$ x is in the null space of $P \Rightarrow x \in M^{\perp}$.
$\therefore\ N \subseteq M^{\perp}$.
Again $x \in M^{\perp}$.
$\Rightarrow Px = 0$
$\Rightarrow x - Px = x$ ie. $(I - P)x = x$.
$\Rightarrow$ x is in the range of $I - P$.
$\Rightarrow x \in N$.
$\therefore\ M^{\perp} \subseteq N$
Hence $M^{\perp} = N$.

∴ If P is the projection on the closed linear subspace M of H, then I – P is the projection on $M^{\perp}$.
Conversely suppose that I – P is the projection on $M^{\perp}$.
⇒ I – (I – P) ie. P is the projection on $(M^{\perp})^{\perp}$ ie. $M^{\perp\perp}$.
Since M is closed $M^{\perp\perp} = M$.
∴ P is a projection on M.

Theorem 3: If P is a projection on the closed linear subspace M of H, then x ∈ M if and only if Px = x if and only if $\|Px\| = \|x\|$.
Proof: Let P be a projection on a closed linear subspace M of H.
Claim: x ∈ M iff Px = x
Suppose Px = x. Then x is in the range of P ∵ Px is in the range of P.
∴ x ∈ M.
Conversely suppose that x ∈ M.
Let Px = y. ⇒ P(Px) = Py ⇒ $P^2x = Py$ ⇒ Px = Py ∵ $P^2 = P$.
⇒ P(x – y) = 0 ⇒ x – y is in the null space of P.
⇒ x – y = z ∈ $M^{\perp}$.
⇒ x = y + z
Now y = Px ⇒ y is in the range of P ie. y ∈ M.
Thus, x = y + z where y ∈ M, z ∈ $M^{\perp}$.
But x is in M. So, x = x + $\bar{0}$ where x ∈ M, $\bar{0}$ ∈ $M^{\perp}$.
But H = M ⊕ $M^{\perp}$.
∴ y = x, z = $\bar{0}$.
Claim: Px = x iff $\|Px\| = \|x\|$.
If Px = x then obviously $\|Px\| = \|x\|$.
Conversely suppose $\|Px\| = \|x\|$.
∴ $\|x\|^2 = \|Px + (I - P)x\|^2$ … (i)
Now Px ∈ M.
Also, P is the projection on M
⇒ I – P is the projection.
∴ Px and (I – P)x are orthogonal vectors.
Then, by Pythagorean theorem, $\|Px + (I - P)x\|^2 = \|Px\|^2 + \|(I - P)x\|^2$ … (ii)
From (i) and (ii) $\|x\|^2 = \|Px\|^2 + \|(I - P)x\|^2$
⇒ $\|(I - P)x\|^2 = 0$ ∵ $\|Px\| = \|x\|$.
⇒ $\|(I - P)x\| = 0$
⇒ Px = x.

Theorem 4: If P is a projection on a Hilbert Space H, then

(i) P is a positive vector ie. $P \geq O$.
(ii) $O \leq P \leq I$
(iii) $\|Px\| \leq \|x\|$ $\forall$ x $\in$ H.
(iv) $\|P\| \leq 1$.

Proof: Let P be a projection on a Hilbert Space H.
Then, $P^2 = P$, $P^* = P$.
Let M be the range of P. (i) Let x be any vector in H.
Then $(Px, x) = (PPx, x) = (Px, P^*x) = (Px, Px) = \|Px\|^2 \geq 0$.
Thus, $(Px, x) \geq 0$ $\forall$ x $\in$ H.
$\therefore$ P is a positive operator. Ie. $P \geq O$.
Note: If P is a projection on a Hilbert Space H and x $\in$ H, then $(Px, x) = \|Px\|^2$.
(ii) Since P is a projection on H, I – P is also a projection on H.
Thus, by part (i) we have $I - P \geq O$ ie. $P \leq I$.
But $P \geq O$.
$\therefore 0 \leq P \leq I$
(iii)Let x $\in$ H. If M is the range of P, then $M^\perp$ is the range of I – P.
Now Px $\in$ M and $(I - P)x \in M^\perp$.
$\therefore$ Px and (I – P)x are orthogonal vectors.
$\therefore$ By Pythagorean theorem, $\|Px + (I-P)x\|^2 = \|Px\|^2 + \|(I-P)x\|^2$
$\Rightarrow \|x\|^2 = \|Px\|^2 + \|(I-P)x\|^2$
$\Rightarrow \|x\|^2 \geq \|Px\|^2 \Rightarrow \|Px\| \leq \|x\|$.
(iv) $\|P\| = Sup\{\|Px\|: \|x\| \leq 1\} \leq 1 \because \|Px\| \leq \|x\|$ $\forall$ x $\in$ H.
$\therefore \|P\| \leq 1$.

INVARIANCE AND REDUCIBILITY.

Definition: Let T be an operator on a Hilbert Space H. If M is a closed linear subspace of H, then M is said to be *invariant* under T if x $\in$ M $\Rightarrow$ Tx $\in$ M. ie. if $T(M) \subseteq M$.
Since M is closed linear subspace of H, M itself is a Hilbert Space.
T may be regarded as an operator on M.
Thus, the operator T on H induces an operator T_M on M defined by TM(x) = Tx $\forall$ x $\in$ M.
The operator T_M is called the restriction of T on M.

Reducibility: Definition: Let T be an operator on a Hilbert Space H. If M is a closed linear subspace of H, then T is said to be reducible by M if both M and $M^\perp$ are invariant under T.

Theorem 5: 2*: A closed linear subspace M of a Hilbert Space H is invariant under an operator T if and only if $M^{\perp}$ is invariant under T*.

Proof: Suppose M is invariant under T.
Let $y \in M^{\perp}$ and $x \in M$.
Then $Tx \in M$ ∵ M is invariant under T.
Also, $y \in M^{\perp}$.
⇒ y is orthogonal to every vector in M.
∴ $y \perp Tx$. Ie. $(Tx, y) = 0$.
⇒ $(x, T^*y) = 0$.
∴ $T^*y \perp x \ \forall \ x \in M$.
∴ $T^*y \in M^{\perp}$.
∴ $M \perp$ is invariant under T*.
Conversely suppose that $M\perp$ is invariant under T*.
Since $M^{\perp}$ is a closed linear subspace of H and invariant under T*, by first case $(M^{\perp})^{\perp}$ is invariant under $(T^*)^*$. But $(M^{\perp})^{\perp} = M^{\perp\perp} = M$ and $(T^*)^* = T^{**} = T$.
∴ M is invariant under T.

Theorem 6: A closed linear subspace M of a Hilbert Space H reduces an operator T if and only if M is invariant under both T and T*.

Proof: Suppose M reduces T.
∴ both M and $M^{\perp}$ are invariant under T.
But by theorem 5, $M^{\perp}$ is invariant under both T and T*.
Conversely suppose that M is invariant under both T and T*.
Since M is invariant under T*, by theorem 5, $M^{\perp}$ is invariant under $(T^*)^*$ ie. T.
Thus, both M and $M^{\perp}$ are invariant under T.
∴ M reduces T.

Theorem 7: If P is a projection on a closed linear subspace M of a Hilbert Space H, then M is invariant under an operator T if and only if TP = PTP.

Proof: Suppose M is invariant under T.
Let $x \in H$. Then Px is in the range of T ie. $Px \in M$.
⇒ $TPx \in M$ ∵ M is invariant under T.
(⇒ TPx will remain unchanged under P ∵ P is projection, M is the range of P.)
∴ PTPx = TPx   Hint: Px = x
Ie. $PTPx = TPx \ \forall \ x \in H$.
∴ PTP = TP.
Conversely suppose that PTP = TP.

Let $x \in M$.

$\therefore$ $Px = x$ $\because$ P is projection with range M.

$\Rightarrow TPx = Tx$

$\Rightarrow PTPx = Tx$ $\because$ $PTP = TP$.

$\Rightarrow PTPx = TPx$ $\because$ $TPx = Tx$

$\Rightarrow TPx \in M$ $\because$ P is the projection with range M. Hint: $Tx = x \Rightarrow x \in M$.

$\Rightarrow Tx \in M$ $\because$ $TPx = Tx$.

Thus, $x \in M \Rightarrow Tx \in M$.

$\therefore$ M is invariant under T.

Theorem 8: If P is a projection on a closed linear subspace M of a Hilbert Space H, then M reduces an operator T if and only if $PT = TP$.

Proof: Let P be the projection on a closed linear subspace M.

Then M reduces T iff M is invariant under both T and T*. Hint: By theorem 6

iff $TP = PTP$ and $T^*P = PT^*P$ Hint By theorem 7

iff $TP = PTP$ and $(T^*P)^* = (PT^*P)^*$

iff $TP = PTP$ and $P^*T^{**} = P^*T^{**}P^*$

iff $TP = PTP$ and $PT = PTP$ $\because$ P is projection, $P^* = P$

ie. M reduces T iff $TP = PTP$ and $PT = PTP$ … (i)

Now suppose M reduces T.

Then from (i), $TP = PTP$ and $PT = PTP$.

$\therefore$ $TP = PT$.

Conversely suppose that $TP = PT$.

$\Rightarrow PTP = P^2T$

$\Rightarrow PTP = PT$ $\because$ $P^2 = P$.

Similarly, $TP^2 = PTP \Rightarrow TP = PTP$.

Thus, $TP = PT \Rightarrow TP = PTP$ and $PT = PTP$

$\therefore$ from (i), M reduces T.

Theorem 9: 2*: If M and N are closed linear subspaces of a Hilbert Space H and P and Q are the projections on M and N respectively, then $M \perp N$ if and only if $PQ = O$ if and only if $QP = O$.

Proof: Let M and N be closed linear subspaces of a Hilbert Space H and P and Q be the projections on M and N respectively.

$\therefore$ $P^* = P$ and $Q^* = Q$

Claim: $PQ = O$ iff $QP = O$.

Now $PQ = O$ iff $(PQ)^* = O^*$ iff $Q^*P^* = O^*$ iff $QP = O$. ie. $PQ = O$ iff $QP = O$

Claim: $M \perp N$ iff $PQ = O$.

Now suppose $M \perp N$. Let $y \in N$. Then $y \perp M$ ie. $y \in M^{\perp}$
Thus, $y \in N \Rightarrow y \in M^{\perp} \therefore N \subseteq M^{\perp}$….(i)
Now let $z \in H$. Then Qz is in the range of Q ie. $Qz \in N$.
From (i), $Qz \in M^{\perp}$ which is the null space of P.
$\therefore P(Qz) = \bar{0}$.
Thus, $PQz = \bar{0}\ \forall\ z \in H$. $\therefore PQz = Oz\ \forall\ z \in H$.
Hence PQ = O.
Conversely suppose, that PQ = O and $x \in M$ and $y \in N$.
$\therefore$ Px = x $\because$ M is the range of P.
And Qy = y $\because$ N is the range of Q.
$\therefore (x, y) = (Px, Qy)$
$= (x, P^*Qy)$
$= (x, PQy) \because P^* = P$.
$= (x, Oy) \because PQ = O$
$= (x, \bar{0})$
$= 0$.
$\therefore M \perp N$

ORTHOGONAL PROJECTIONS

Definition: Two projections P and Q on a Hilbert Space H are said to be orthogonal if PQ = O.
By theorem 9, P and Q are orthogonal iff their ranges M and N are orthogonal.

Theorem 10: 1*: If $P_1, P_2, \ldots, P_n$ are the projections on closed linear subspaces $M_1, M_2, \ldots, M_n$ of a Hilbert Space H, then $P = P_1 + P_2 + \ldots + P_n$ is a projection if and only if the P_i's are pairwise orthogonal. Also, then P is the projection on $M = M_1 + M_2 + \ldots + M_n$.
Proof: Let $P_1, P_2, \ldots, P_n$ be pairwise orthogonal projections on H.
$\therefore P_i$'s linear, continuous, $P_i^2 = P_i = P_i^*$ for each i = 1, 2, …, n. and $P_iP_j = O$ if $i \neq j$.
Let $P = P_1 + P_2 + \ldots + P_n$. Then clearly P is linear and continuous.
Also $P^* = (P_1 + P_2 + \ldots + P_n)^* = P_1^* + P_2^* + \ldots + P_n^* = P_1 + P_2 + \ldots + P_n = P$.
And $P^2 = (P_1 + P_2 + \ldots + P_n)^2 = \sum_{i=1}^{n} P_i^2 + \sum_{1 \le i \neq j \le n} P_iP_j = \sum_{i=1}^{n} P_i = P$.
Thus, P is linear, continuous, $P^2 = P = P^*$.
$\therefore$ P is a projection on H.
Conversely suppose P is a projection on H. ie. let P is linear, continuous $P^2 = P = P^*$.

To prove $P_iP_j = O \ \forall \ i \neq j$ it suffices to prove that $M_i \perp M_j \ \forall \ i \neq j$ in view of theorem 9.
Let T be any projection on H and $z \in H$.
Then $(Tz, z) = (TTz, z) = (Tz, T^*z) = (Tz, Tz) = \|Tz\|^2$ … (i).
Let $x \in M_i$, and $y \in M_j$. Since M_i is range of P_i, $P_ix = x \ \forall \ i = 1, 2, \ldots, n$.
Then $\|x\|^2 = \|P_ix\|^2 \leq \sum_{j=1}^{n} \|P_jx\|^2$
$= \sum_{j=1}^{n} (P_jx, x)$ by (i).
$= (P_1x, x) + (P_2x, x) + \ldots + (P_nx, x)$.
$= ((P_1 + P_2 + \ldots + P_n)x, x)$
$= (Px, x) = \|Px\|^2$ by (i).
$\leq \|x\|^2$ by using projection theorem (or theorem 4) … (ii).
Ie. $\|x\|^2 = \|P_ix\|^2 \leq \sum_{j=1}^{n} \|P_jx\|^2 \leq \|x\|^2$.
$\Rightarrow \|P_ix\|^2 = \sum_{j=1}^{n} \|P_jx\|^2$.
$\Rightarrow \|P_jx\|^2 = 0 \ \forall \ j \neq i$.
$\Rightarrow P_jx = 0 \ \forall \ j \neq i$.
$\Rightarrow$ x is in the null space of $P_j \ \forall \ j \neq i$.
$\Rightarrow x \in M_j^{\perp} \ \forall \ j \neq i$.
$\Rightarrow M_i \subseteq M_j^{\perp} \ \forall \ j \neq i$.
$\Rightarrow M_i \perp M_j$.
Hence $P_iP_j = O$ whenever $i \neq j$.
Claim: P is a projection on $M = M_1 + M_2 + \ldots + M_n$. ie. Range of P = P.
Let $x \in M$. Then $x = x_1 + x_2 + \ldots + x_n$ where $x_i \in M_i$, $1 \leq i \leq n$.
Now $Px = P(x_1 + x_2 + \ldots + x_n) = Px_1 + Px_2 + \ldots + Px_n$
$= (P_1 + P_2 + \ldots + P_n)x_1 + (P_1 + P_2 + \ldots + P_n)x_2 + \ldots + (P_1 + P_2 + \ldots + P_n)x_n$
$= P_1x_1 + P_2x_2 + \ldots + P_nx_n$
$= x_1 + x_2 + \ldots + x_n = x$
Ie. $Px = x$
So, $x \in$ Range of P.
$\therefore$ $M \subseteq$ Range of P.
Now suppose $x \in$ Range of P. Then $Px = x$
$\Rightarrow (P_1 + P_2 + \ldots + P_n)x = x$.
$\Rightarrow P_1x + P_2x + \ldots + P_nx = x$.
But $P_1x \in M_1$, $P_2x \in M_2$, …, $P_nx \in M_n$.
$\therefore$ $x \in M_1 + M_2 + \ldots + M_n = M$
$\therefore$ Range of $P \subseteq M$.
$\therefore$ $M =$ Range of P.
Hence P is a projection on M.

Example 1: 3*: If P and Q are the projections on a closed linear subspaces M and N of H, then prove that PQ is projection if and only if $PQ = QP$. In this case, show that PQ is the projection on $M \cap N$.

Solution: Let P and Q be the projections on closed linear subspaces M and N.

$\therefore$ P, Q are linear, continuous, $P^2 = P = P^*$ and $Q^2 = Q = Q^*$.

Suppose PQ is a projection on H.

$\therefore (PQ)^* = PQ \Rightarrow Q^*P^* = PQ \Rightarrow QP = PQ$.

Conversely suppose $PQ = QP$.

Since P, Q are Projections, they are linear and continuous so that PQ is linear and continuous.

$\therefore (PQ)^* = Q^*P^* = QP = PQ$

Also $(PQ)^2 = (PQ)(PQ) = P(QP)Q = P(PQ)Q = (PP)(QQ) = P^2Q^2 = PQ$. Ie. $(PQ)^2 = PQ$

$\therefore$ PQ is a projection.

Claim: Range of PQ, denoted by R(PQ), is $M \cap N$.

Let $x \in M \cap N \Rightarrow x \in M$ and $x \in N$.

Then $(PQ)x = P(Qx) = Px$ $\because N = R(Q)$ and $x \in N \Rightarrow Qx = x$.

$= x$ $\because M = R(P)$ and $x \in M \Rightarrow Px = x$

Thus, $(PQ)x = x$

$\therefore x \in R(PQ)$.

$\therefore M \cap N \subseteq R(PQ)$…(a)

Now suppose that $x \in R(PQ)$.

Then $(PQ)x = x$ … (i)

$\Rightarrow P[(PQ)x] = Px$

$\Rightarrow (P^2Q)x = Px$.

$\Rightarrow (PQ)x = Px$…(ii)

$\therefore$ from (i) and (ii), $Px = x \Rightarrow x \in R(P) = M$

Ie, $x \in M$…(iii)

As, $PQ = QP$, from (i), $(QP)x = x$ … (iv)

$\Rightarrow Q[(QP)x = Qx$

$\Rightarrow (Q^2P)x = Qx$

$\Rightarrow (QP)x = Qx$. … (v)

From (iv) and (v) $Qx = x$

$\Rightarrow x \in N$.

From (iii) and (v) $x \in M \cap N$.

$\therefore R(PQ) \subseteq M \cap N$… (b)

Hence, from (a) and (b) $R(PQ) = M \cap N$.

Example 2: 2*: If P and Q are the projections on closed linear subspaces M and N of H, prove the following statements are all equivalent to one another.

(i) $P \leq Q$
(ii) $\|Px\| \leq \|Qx\|$ for every $x \in H$.
(iii) $M \subseteq N$.
(iv) $QP = P$.
(v) $PQ = P$.

Solution: Remember if P is any projection on H, then $(Px, x) = \|Px\|^2 \ \forall\, x \in H$.

Claim: (i) $\Rightarrow$ (ii).
Let $P \leq Q$. $\Rightarrow (Px, x) \leq (Qx, x) \ \forall\, x \in H$.
$\Rightarrow \|Px\|^2 \leq \|Qx\|^2 \ \forall\, x \in H$.
$\Rightarrow \|Px\| \leq \|Qx\|$ for every x.

Claim: (ii) $\Rightarrow$ (iii)
Assume $\|Px\| \leq \|Qx\|$ for every x in H.
Let $x \in M$.
$\Rightarrow Px = x$
$\Rightarrow \|Px\| = \|x\|$.
$\Rightarrow \|x\| \leq \|Qx\|$
$\Rightarrow \|x\| = \|Qx\| \because \|Qx\| \leq \|x\| \ \forall\, x \in H$ by theorem 4.
$\Rightarrow Qx = x$ by theorem 3.
$\Rightarrow x \in N$. $\therefore M \subseteq N$.

Claim: (iii) $\Rightarrow$ (iv).
Assume $M \subseteq N$.
Let $x \in H$. Then $(QP)x = Q(Px)$.
Since $Px \in M$, $M \subseteq N \Rightarrow Px \in N$.
$\therefore (QP)x = Px \ \forall\, x \in H$.
$\therefore QP = P$.

Claim: (iv) $\Rightarrow$ (v).
Let $QP = P$
$\Rightarrow (QP)^* = P^* \Rightarrow P^*Q^* = P^*$
$\Rightarrow PQ = P$.

Claim: (v) $\Rightarrow$ (i).
Let $PQ = P$.
Let $x \in H$. Then $(Px, x) = \|Px\|^2 = \|PQx\|^2 \because PQ = P$.
$= \|P(Qx)\|^2$
$\leq \|Qx\|^2 \because \|Px\| \leq \|x\| \ \forall\, x \in H$.
$= (Qx, x)$

$\therefore P \leq Q$

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E – CONTENT

PAPER: M 301,

FUNCTIONAL ANALYSIS

M. Sc. II YEAR, SEMESTER - III

UNIT – IV: FIN DIM SPECTRAL THEORY

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FUNCTIONAL ANALYSIS

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UNIT IV

FINITE DIMENSIONAL SPECTRAL THEORY

EIGEN VALUES AND EIGEN VECTORS

Definition:- Let T be an operator on a Hilbert Space H. Then a scalar λ is called an ***eigen value or Characteristic value, or proper value or latent value*** of T if $\exists$ non-zero vector x in H $\ni$ Tx = λx.

Also, if λ is an eigen value of T, then any non-zero vector x in H $\ni$ Tx = λx is called an ***eigen vector or Characteristic vector, or proper vector or latent vector*** of T corresponding to the eigen value λ.

The set of all eigen values of T is called the ***spectrum*** of T and is denoted by $\sigma(T)$.

Note: Eigen vector is always a non-zero vector.

If H = $\{ \bar{0} \}$, then H has no eigen vector and hence no eigen value.

So, here after we assume that H $\neq \{ \bar{0} \}$.

Therom-1: If x is an eigen vector of T corresponding to eigen value λ, then αx is also an eigen vector of T corresponding to the same eigen value λ where α is any non-zero scalar.

Proof: Let x be an eigen vector of T corresponding to the eigen value λ.

Then $x \neq \bar{0}$ and Tx = λx

If α is any non - zero scalar, then $\alpha x \neq \bar{0}$ and $T(\alpha x) = \alpha Tx = \alpha(\lambda x) = \lambda(\alpha x)$

$\therefore$ αx is an eigen vector of T corresponding to the eigen value λ.

Theorem-2: If x is an eigen vector of T, then x cannot correspond to more than one eigen value of T.

Proof: If possible, suppose x is an eigen vector of T corresponding to two distinct eigen values λ_1 and λ_2 of T.

Then Tx = $\lambda_1 x$ and also Tx = $\lambda_2 x$

$$\Rightarrow \lambda_1 x = \lambda_2 x$$

$\Rightarrow (\lambda_1 - \lambda_2)x = \bar{0}$

$\Rightarrow \lambda_1 - \lambda_2 = 0$ (since, $x \neq \bar{0}$).

$\Rightarrow \lambda_1 = \lambda_2$ which is a contradiction.

$\therefore$ x cannot correspond to more than one eigen value of T.

Theorem-3: Let λ be an eigen value of an operator T on a Hilbert space H. If M_λ is the set consisting of all eigen vectors of T which corresponds to the eigen value of λ together with the null vector $\bar{o}$, then M_λ is a non-zero closed linear subspace of H invariant under T. M_λ is called the ***eigen space*** of T corresponding to the eigen value λ.

Proof: Since, by definition, an eigen vector is a non-zero vector, M_λ necessarily contains some non-zero vector. Also given that the vector $\bar{o} \in M_\lambda$.

$\therefore$ $x \in M_\lambda$ if and only if $Tx = \lambda x$.

$\therefore$ $M_\lambda = \{x \in H: Tx = \lambda x\} = \{x \in H: (T - \lambda I)x = \bar{0}\}$.

Thus, M_λ is the null space of the linear transformation $T - \lambda I$ on H.

$\therefore$ M_λ is a linear sub space of H.

Recall that the null space of a continuous linear transformation is closed.

Since the linear transformation $T - \lambda I$ is a continuous mapping, M_λ is closed

Claim: M_λ is invariant under T.

Let $x \in M_\lambda$. Then $Tx = \lambda x$

But $\lambda x \in M_\lambda$ (since M_λ is linear subspace of H)

$\therefore$ $Tx \in M_\lambda$

Thus, $T(M_\lambda) \subseteq M_\lambda$

$\therefore M_\lambda$ is invariant under T.

We assume that H is a finite dimensional Hilbert space with dimension n throughout the remaining part of this chapter.

Note: Every linear transformation on H is continuous and so is an operator on H. $\mathfrak{B}$(H) is the collection of all linear transformations.

Total Matrix Algebra of degree n:

Let A_n be the set of all $n \times n$ matrices over the field $\mathbb{C}$. Then A_n is a Complex Algebra with identity with respect to matrix addition, scalar multiplication and matrix multiplication. It is called the total matrix algebra of degree n.

MATRIX OF LINEAR TRANSFORMATION:

Definition: Let H be an n-dimensional Hilbert space and let B = $\{e_1, e_2, . . . e_n\}$ be an ordered basis for H. Let T be an operator on H. Since each T(e_j) ∈ H and B is a basis, ∃ scalars α_{ij} , i = 1, 2, 3, ..., n ∍ T(e_j) = $\alpha_{1j}e_1$ + $\alpha_{2j}e_2$ + ...+ $\alpha_{nj}e_n$ = $\sum_{i=1}^{n} \alpha_{ij}e_i$

Then n × n matrix whose j^{th} column (j = 1, 2, 3, ..., n) consists of the scalars α_{1j},α_{2j},....,α_{nj} is called the matrix of the operator T relative to the ordered basis B.

∴ $[T]_B$ = Matrix of T relative to the ordered basis B is $[\alpha_{ij}]_{n\times n}$ where T(e_j) = $\sum_{i=1}^{n} \alpha_{ij}e_i$ for each j = 1, 2, ..., n.

Matrices of identity and zero operator :

Theorem 1:- Let H be an n-dimensional Hilbert space and B be an ordered basis for H. If I is an identity operator and O be the zero operator on H then

(i) $[\ I\ ]_B = I = [\delta_{ij}]_{n\times n}$, unit matrix of order n.

(ii) $[O]_B = O$, Null matrix of the type n × n.

Proof: Let B = $\{e_1, e_2, . . ., e_n\}$ be an ordered basis for H.

(i) I(e_j) = e_j = $\sum_{i=1}^{n} \delta_{ij}e_i$ where $\delta_{ij} = \begin{cases} 1 \ if\ i = j \\ 0 \ if\ i \neq j \end{cases}$

∴ $[\ I\ ]_B = [\delta_{ij}]_{n\times n}$= I, unit matrix of order n.

We have O(e_j) = $\bar{0}$ (for j = 1, 2, ..., n)

$= 0e_1 + 0e_2 + \cdots + 0e_n$

$= \sum_{i=1}^{n} \alpha_{ij}e_i$ where α_{ij} = 0 ∀ i, j.

∴ $[O]_B = [\alpha_{ij}]_{n\times n}$ = O, null matrix of the type n × n.

Theorem 2: Let H be a finite dimensional Hilbert space of dimension n and let B = $\{e_1, e_2, . . ., e_n\}$ be an ordered basis for H. If $f_1, f_2, . . ., f_n$ are any n vectors in H then ∃ unique operator T on H ∍ T(e_i) = f_i, i = 1, 2, ..., n.

Proof: Existence of T:

Let x ∈ H. Since B = $\{e_1, e_2, . . . e_n\}$ is a basis for H ∃ unique scalars $\alpha_1, \alpha_2, . . . \alpha_n$ ∍ x = α_1e_1+α_2e_2+...+α_ne_n

Now define T: H → H by T(x) = T($\alpha_1 e_1$+....+$\alpha_n e_n$) = $\alpha_1 f_1$+....+$\alpha_n f_n$
Clearly T is well defined.
Let $e_i \in B$. Then $0e_1$+....+ $0e_{i-1}$ + $1e_i$+ $0e_{i+1}$+...+ $0e_n$ for i = 1, 2, ..., n.
∴ T(e_i) = $0f_1$+....+ $0f_{i-1}$ + $1f_i$ + $0f_{i+1}$+ ... + $0f_n$ = f_i for i = 1, 2, ..., n.
Let α, β be any scalars and x, y ∈ H.
Then ∃ scalars $\alpha_1, \alpha_2, . . . \alpha_n, \beta_1, \beta_2, . . . \beta_n$ ∍
x = $\alpha_1 e_1 + \alpha_2 e_2$ + ... + $\alpha_n e_n$ and y = $\beta_1 e_1 + \beta_2 e_2$ + ... + $\beta_n e_n$
Then T(αx + βy) = T{$\alpha(\alpha_1 e_1$+...+$\alpha_n e_n$) + β($\beta_1 e_1$+....+$\beta_n e_n$)}

$$= T\{(\alpha\alpha_1+\beta\beta_1)e_1 +...+ (\alpha\alpha_n+\beta\beta_n)e_n\}$$
$$= (\alpha\alpha_1+\beta\beta_1)f_1 +...+ (\alpha\alpha_n+\beta\beta_n)f_n$$
$$= \alpha\alpha_1 f_1+\beta\beta_1 f_1 + ... + \alpha\alpha_n f_n+ \beta\beta_n f_n$$
$$= \alpha(\alpha_1 f_1 + \ldots + \alpha_n f_n) + \beta(\beta_1 f_1+ \ldots + \beta_n f_n).$$
$$= \alpha T(x) + \beta T(y)$$

∴ T is a linear transformation
Thus, ∃ an operator T on H ∍ T(e_i) = f_i for i = 1, 2, ..., n.
Uniqueness of T:
Let T′ be an operator on H ∍ $T'(e_i) = f_i$ for i = 1, 2, ..., n.
Now for the vector x = $\alpha_1 e_1$+ ... + $\alpha_n e_n$ ∈ H, T′ (x) = T′($\alpha_1 e_1$+ ... + $\alpha_n e_n$)

$$= \alpha_1 T'(e_1) + ... + \alpha_n T'(e_n)$$
$$= \alpha_1 f_1 + ... + \alpha_n f_n$$
$$= T(x)$$

Thus, T′(x) = T(x) ∀ x ∈ H
∴ T′ = T.
Note: Two operators on H are equal if they agree on a basis of H.

Theorem 3: 3*: If B is an ordered basis for a finite dimensional Hilbert space H of dimension n then the mapping T → [T] which assigns to each operator T it's matrix relative to B is an isomorphism of the algebra 𝔅 (H) onto the total matrix algebra A_n.
Proof: Let B = {$e_1, e_2, . . ., e_n$}.
Define ψ : 𝔅 (H) → A_n by ψ(T) = $[T]_B$ ∀ T ∈ B(H).
Let $T_1, T_2 \in \mathfrak{B}(H)$ and let $[T_1]_B = [\alpha_{ij}]_{n\times n}$ and $[T_2]_B = [\beta_{ij}]_{n\times n}$ where

$T_1(e_j) = \sum_{i=1}^{n} \alpha_{ij} e_i$, j = 1, 2, ..., n. ... (1).
and $T_2(e_j) = \sum_{i=1}^{n} \beta_{ij} e_i$, j =1, 2, ..., n. ... (2).

Claim: ψ is one-one:

Let $\psi(T_1) = \psi(T_2)$

$\Rightarrow [T_1]_B = [T_2]_B$

$\Rightarrow [\alpha_{ij}]_{n\times n} = [\beta_{ij}]_{n\times n}$

$\Rightarrow \alpha_{ij} = \beta_{ij}$ for i = 1, 2, ..., n, j = 1, 2, ..., n

$\Rightarrow \sum_{i=1}^{n} \alpha_{ij} e_i = \sum_{i=1}^{n} \beta_{ij} e_i$ for j = 1, 2, ..., n.

$\Rightarrow T_1(e_j) = T_2(e_j)$ for j = 1, 2, ..., n.

$\Rightarrow T_1 = T_2$

ψ <u>is onto:</u>

Let $[\gamma_{ij}]_{n\times n}$ be any matrix in A_n

Then for each j = 1, 2, ..., n, $\sum_{i=1}^{n} \gamma_{ij} e_i \in H$

By theorem 2, ∃ a unique operator T on H ∋ T(e_j) $= \sum_{i=1}^{n} \gamma_{ij} e_i$ for j = 1, 2, ..., n.

$\therefore [T]_B = [\gamma_{ij}]_{n\times n} \Rightarrow \psi(T) = [\gamma_{ij}]_{n\times n}$

ψ <u>preserves addition:</u>

Let $T_1, T_2 \in \mathfrak{B}$ (H).

From (1) and (2), $(T_1 + T_2)(e_j) = T_1(e_j) + T_2(e_j)$ for j = 1, 2, ..., n.

$= \sum_{i=1}^{n} \alpha_{ij} e_i + \sum_{i=1}^{n} \beta_{ij} e_i$

$= \sum_{i=1}^{n} (\alpha_{ij} + \beta_{ij}) e_i$

$\therefore [T_1 + T_2] = [\alpha_{ij} + \beta_{ij}]_{n\times n}$

$= [\alpha_{ij}]_{n\times n} + [\beta_{ij}]_{n\times n}$

$= [T_1] + [T_2]$

$\therefore \psi(T_1 + T_2) = [T_1 + T_2] = [T_1] + [T_2] = \psi[T_1] + \psi[T_2]$

ψ <u>preserves scalar multiplication:</u>

Let α be any scalar then $(\alpha T_1)(e_j) = \alpha T_1(e_j)$ for j = 1, 2, ..., n.

$= \alpha \sum_{i=1}^{n} \alpha_{ij} e_i$

$= \sum_{i=1}^{n} \alpha \alpha_{ij} e_i$

$\therefore [\alpha T_1] = [\alpha \alpha_{ij}]_{n\times n} = \alpha[\alpha_{ij}]_{n\times n} = \alpha[T_1]$

$\therefore \psi(\alpha T_1) = [\alpha T_1] = \alpha[T_1] = \alpha \psi[T_1]$

ψ <u>preserves multiplication:</u>

We have $(T_1 T_2)(e_j) = T_1(T_2(e_j))$, j = 1, 2, ..., n.

$= T_1(\sum_{k=1}^{n} \beta_{kj} e_k)$

$= \sum_{k=1}^{n} \beta_{kj} T_1(e_k)$

$= \sum_{k=1}^{n} \beta_{kj} (\sum_{i=1}^{n} \alpha_{ik} e_i)$

$= \sum_{i=1}^{n} (\sum_{j=1}^{n} \alpha_{ik} \beta_{kj}) e_i$

$\therefore [T_1 T_2] = [\sum_{j=1}^{n} \alpha_{ik} \beta_{kj})]_{n \times n}$

$= [\alpha_{ij}]_{n \times n} [\beta_{ij}]_{n \times n}$

$= [T_1][T_2]$

$\therefore \psi(T_1 T_2) = [T_1 T_2]$

$= [T_1][T_2]$

$= \psi[T_1]\, \psi[T_2]$.

$\therefore \psi$ is an isomorphism of the algebra 𝔅 (H) onto the Matrix algebra A_n.

Theorem 4: 2*: Let B be an ordered basis for a finite dimensional Hilbert space H of dimension n and T an operator on H whose matrix relative to B is $[\alpha_{ij}]$. Then T is non-singular if and only if $[\alpha_{ij}]$ is non-singular and in this case $[\alpha_{ij}]^{-1} = [T^{-1}]$.

Proof: T is non-singular iff ∃ an operator T^{-1} on H ∍ $T^{-1}T = I = TT^{-1}$

iff $[T^{-1}T] = [I] = [TT^{-1}]$

iff $[T^{-1}][T] = [\delta_{ij}] = [T][T^{-1}]$

iff $[T^{-1}][\alpha_{ij}] = [\delta_{ij}] = [\alpha_{ij}][T^{-1}]$

iff the matrix $[\alpha_{ij}]$ is non-singular and $[\alpha_{ij}]^{-1} = [T^{-1}]$

SIMILARITY OF MATRICES:

Definition: Let A and B be square matrices of order n over the field of complex numbers. Then B is said to be similar to A if there exists an n×n non-singular matrix C over the field of complex numbers ∍ B = C^{-1}A C.

Note: The relation of similarity on the set of all n×n matrices over the field of complex numbers is an equivalence relation.

Theorem 5: Similar matrices have the same determinant.

Proof: Suppose A and B are similar matrices.

Then there exists a non-singular matrix C such that B = $C^{-1}AC$

Then det B = det $(C^{-1}AC)$

$= (\det C^{-1})(\det A)(\det C)$

$= (\det C^{-1})(\det C)(\det A)$
$= (\det C^{-1}C)(\det A)$
$= (\det [\delta_{ij}])(\det A)$
= 1. det A
= det A

Thus, det B = det A
Hence the result.

Similarity of operators:

Definition: Let A and B be operator on a Hilbert space H. Then B is said to be similar to A if there exists a non-singular operator C on H ∋ B = $C^{-1}AC$.
Note: The relation of similarity on B(H) is an equivalence relation.

Theorem 6: 2*: Two matrices in A_n are similar if and only if they are the matrices of a single operator on H relative to (possibly) different bases.
Proof: [First we prove that if T is an operator on an n – dimensional Hilbert Space H and if B and B′ are two ordered bases for H, then the matrix of T relative to B is similar to the matrix of T relative to B′]
Suppose T is an operator on an n-dimensional Hilbert space H.
Let B = $\{e_1, e_2, \ldots, e_n\}$ and B′ = $\{f_1, f_2, \ldots f_n\}$ be two ordered bases for H.
Let $[T]_B = [\alpha_{ij}]_{n\times n}$ and $[T]_{B'} = [\beta_{ij}]_{n\times n}$ so that $T(e_j) = \sum_{i=1}^{n} \alpha_{ij} e_i$, j = 1, 2, ..., n. ... (1)
$T(f_j) = \sum_{i=1}^{n} \beta_{ij} f_i$, j = 1, 2, ..., n. … (2).
Let S be an operator in H defined by $S(e_j) = f_j$, j = 1, 2, ..., n. ... (3).
Then S is non-singular since S maps a basis B onto a basis B′.
Let $[\gamma_{ij}]_{n\times n}$ be the matrix of S relative to B.
Then $[\gamma_{ij}]_{n\times n}$ is also non-singular, (by theorem 4.)
Also $S(e_j) = \sum_{i=1}^{n} \gamma_{ij} e_i$, j = 1, 2, ..., n. ... (4).
We have $T(f_j) = T\{S(e_j)\}$ [from (3)]
$= T(\sum_{k=1}^{n} \gamma_{kj} e_k)$ [from (4) on replacing i by k]
$= \sum_{k=1}^{n} \gamma_{kj} T(e_k)$
$= \sum_{k=1}^{n} \gamma_{kj} \sum_{i=1}^{n} \alpha_{ik} e_i$ [from (1) on replacing j by k]
$= \sum_{i=1}^{n} (\sum_{k=1}^{n} \alpha_{ik}\gamma_{kj}) e_i$ ... (5)

Again $T(f_j) = \sum_{k=1}^{n} \beta_{kj} f_k$ [from (2) on replacing i by k]

$= \sum_{k=1}^{n} \beta_{kj} S(e_k)$ [from (3)]

$= \sum_{k=1}^{n} \beta_{kj} \sum_{i=1}^{n} \gamma_{ik} e_i$ [from (4), on replacing j by k]

$= \sum_{i=1}^{n} (\sum_{k=1}^{n} \gamma_{ik} \beta_{kj}) e_i$ ... (6)

From (5) and (6), $\sum_{i=1}^{n} (\sum_{k=1}^{n} \alpha_{ik} \gamma_{kj}) e_i = \sum_{i=1}^{n} (\sum_{k=1}^{n} \gamma_{ik} \beta_{kj}) e_i$

$\sum_{k=1}^{n} \alpha_{ik} \gamma_{kj} = \sum_{k=1}^{n} \gamma_{ik} \beta_{kj}$ since $e_1, e_2, \ldots e_n$ are linearly independent

$\Rightarrow [\alpha_{ij}]_{n\times n} [\gamma_{ij}]_{n\times n} = [\gamma_{ij}]_{n\times n} [\beta_{ij}]_{n\times n}$

$\Rightarrow [\gamma_{ij}]^{-1} [\alpha_{ij}][\gamma_{ij}] = [\gamma_{ij}]^{-1} [\gamma_{ij}][\beta_{ij}]$ since $[\gamma_{ij}]$ is non-singular.

$\Rightarrow [\gamma_{ij}]^{-1} [\alpha_{ij}][\gamma_{ij}] = [\beta_{ij}]$ … (7)

$\Rightarrow [\alpha_{ij}]$ *and* $[\beta_{ij}]$ are similar matrices

$\Rightarrow [T]_B$ *is simialr to* $[T]_{\boldsymbol{B'}}$

From (7) we note that $[\beta_{ij}] = [\gamma_{ij}]^{-1} [\alpha_{ij}][\gamma_{ij}]$

$[T]_{\boldsymbol{B'}} = [\gamma_{ij}]^{-1} [T]_B [\gamma_{ij}]$ … (8)

where $[\gamma_{ij}]$ is the matrix of the operators S relative to the basis B.

[The relation (8) gives us a formula which enables us to write the matrix of T relative to basis B´ when we already know the matrix of T relative to the basis B.]

Converse: Suppose that $[\alpha_{ij}]$ and $[\beta_{ij}]$ are two $n \times n$ similar matrices.

Then ∃ a non-singular matrix $[\gamma_{ij}]_{n\times n}$ such that $[\gamma_{ij}]^{-1} [\alpha_{ij}][\gamma_{ij}] = [\beta_{ij}]$ … (9)

Let B = $\{e_1, e_2, \ldots e_n\}$ be any ordered basis for H and let T be the operator on H whose matrix relative to B is $[\alpha_{ij}]$. *i.e.*, $[T]_B = [\alpha_{ij}]$.

Let S be the operator on H whose matrix relative to B is $[\gamma_{ij}]$.

Then S is also non-singular since $[\gamma_{ij}]$ is non–singular.

Let B′ = $\{Se_1, Se_2, \ldots Se_n\}$.

Then B′ is also a basis since non-singular S carries basis onto a basis

We have $[S]_B = [\gamma_{ij}]$

By the result (8), proved in this theorem,

$[T]_{\boldsymbol{B'}} = [\gamma_{ij}]^{-1} [T]_B [\gamma_{ij}]$

$= [\gamma_{ij}]^{-1} [\alpha_{ij}][\gamma_{ij}]$

$= [\beta_{ij}]$ by (9)

Thus, $[\alpha_{ij}]$ and $[\beta_{ij}]$ are the matrices of T relative to the basis B and B′ respectively.

Definition: Let T be an operator on an n-dimensional Hilbert space H. Then the determinant of the operator T is the determinant of the matrix of T relative to any ordered basis for H.

Theorem – 7: 1*: Let S and T be operators on a finite dimensional Hilbert space H of dimension n. Then (i) det (I) = 1 where I is the identity operator

(ii) det (ST) = (det S)(det T)

(iii) det T $\neq$ 0 iff T is non-singular.

Proof: Let B be any ordered basis for H. We have det T = det $[T]_B$.

(i) det (I) = det $([I]_B)$ = det $([\delta_{ij}])$ = 1

(ii) det (ST) = det $([ST]_B)$

= det $([S]_B [T]_B)$

= (det $[S]_B$)(det $[T]_B$)

= (det S)(det T)

(iii) T is non-singular iff $[T]_B$ is non- singular iff det $[T]_B \neq 0$ iff det (T) $\neq$ 0.

Theorem – 8: An operator T on a finite dimensional Hilbert space H is singular if and only if there exists a non-zero vector x in H ∍ Tx = $\bar{0}$.

Proof: Suppose ∃ a non-vector x in H ∍ Tx = $\bar{0}$

But T$\bar{0}$ = $\bar{0}$). Ie. Two distinct elements in H have the same image.

∴ T is not one – one.

∴ T is not non-singular.

i.e., T is singular.

Conversely suppose that T is singular.

If possible, suppose there exists no non-zero vector x ∍ Tx = $\bar{0}$.

i.e., Tx = $\bar{0}$ ⇒ x = $\bar{0}$.

Let y, z ∈ H ∍ Ty = Tz ⇒ T(y – z) = $\bar{0}$.

⇒ y – z = $\bar{0}$.

⇒ y = z

∴ T is one – one.

Since H is finite dimensional and T is one – one, ⇒ T is onto and so, T is non-singular which is a contradiction. Hence there must exist a non-zero vector x ∋ Tx = $\bar{0}$.

Theorem -9: 1*: If T is an arbitrary operator on a finite dimensional Hilbert space H, then the eigen values of T constitute a non-empty finite subset of the complex plane. Furthermore, the number of points in this does not exceed the dimension n of the space H.

Proof: Let T be an operator on a finite dimensional Hilbert space H of dimension n.

A scalar λ is an eigen value of T iff ∃ a non-zero vector x in H ∋ Tx = λx
iff ∃ a non-zero vector x ∋ (T – λI)x = $\bar{0}$.
iff the operator T – λI is singular [by theorem 8]
iff det (T – λI) = 0 [by theorem 7]

Thus, λ is an eigen value of T iff λ satisfies the equation det (T – λI) = 0.
Let B be any ordered basis for H.

Then det (T – λI) = det ($[T - \lambda I]_B$)
= det ($[T]_B - \lambda\ [I]_B$)
= det ($[T]_B - \lambda \left[\delta_{ij}\right]_{n\times n}$)

Let $[T]_B = \left[\alpha_{ij}\right]_{n\times n}$

Then det (T – λI) = 0 takes the form

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & .. & .. & a_{1n} \\ a_{21} & a_{22}-\lambda & .. & .. & a_{2n} \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & .. & .. & a_{nn}-\lambda \end{vmatrix} = 0 \ldots (1)$$

The equation (1) is called the characteristic equation of the operator T.
If we expand the determinant on the left hand side of (1), then (1) is a polynomial equation with complex coefficients of degree n in the complex variable λ.
By fundamental theorem of Algebra, the equation (1) has a root in the field of complex coefficients of degree n in the complex variable λ. ∴ equation (1) has a root in the field of complex numbers. Hence every operator T on H has an eigen value.
Also, the equation (1) has exactly n roots in the complex field.
Some of these roots may be repeated.
Hence T has an eigen value and the number of distinct eigen values of T ≤ n.

Theorem 1: 1*: If T is a normal operator on a Hilbert Space H, then x is an eigen vector of T with eigen value λ if and only if x is an eigen vector of T* with eigen value $\bar{\lambda}$.

Proof: Let T be a normal operator on H.

Now for any scalar λ, $(T - \lambda I)(T - \lambda I)^* = (T - \lambda I)(T^* - \bar{\lambda} I^*)$

$= (T - \lambda I)(T^* - \bar{\lambda} I)$

$= T T^* - \bar{\lambda} T I - \lambda T^* I + \lambda\bar{\lambda} I^2.$

$= T^* T - \lambda T^* I - \bar{\lambda} T I + |\lambda|^2 I^2$

$= T^* (T - \lambda I) - \bar{\lambda} I (T - \lambda I)$

$= (T^* - \bar{\lambda} I)(T - \lambda I)$

$= (T - \bar{\lambda} I)^*(T - \lambda I)$

$\therefore T - \lambda I$ is also a normal operator on H where λ is any scalar.

Recall that T is normal iff $\|Tx\| = \|T^*x\|$.

Since $T - \lambda I$ is normal, $\|(T - \lambda I)x\| = \|(T - \lambda I)^*x\| \ \forall\ x \in H$

iff $\|(T - \lambda I)x\| = \|(T^* - \bar{\lambda} I)x\| \ \forall\ x \in H$.

iff $\|Tx - \lambda x\| = \|T^*x - \bar{\lambda}x\| \ \forall\ x \in H$ … (1).

$\therefore Tx - \lambda x = \bar{0}$ iff $T^*x - \bar{\lambda}x = \bar{0}$.

$\therefore$ x is an eigen vector of T with eigen value λ iff it is the eigen vector of T* with eigen value $\bar{\lambda}$.

Theorem 2: If T is a normal operator on a Hilbert Space H, then eigen spaces of T are pairwise orthogonal.

Proof: Let M_i, M_j be eigen Spaces of a normal operator T on H corresponding to the distinct eigen values λ_i, λ_j

Let x_i be any vector in M_i and x_j be any vector in M_j.

Then $Tx_i = \lambda_i x_i$ and $Tx_j = \lambda_j x_j$.

$\therefore \lambda_i(x_i, x_j) = (\lambda_i x_i, x_j)$

$= (Tx_i, x_j)$

$= (x_i, T^*x_j)$

$= (x_i, \bar{\lambda}_j x_j)$

$= \lambda_j(x_i, x_j)$.

$\therefore (\lambda_i - \lambda_j)(x_i, x_j) = 0$

$\Rightarrow (x_i, x_j) = 0 \because \lambda_i \neq \lambda_j$.

$\Rightarrow x_i \perp x_j$.

Thus, $x_i \perp x_j \ \forall\ x_i \in M_i$ and $x_j \in M_j$.

$\therefore M_i \perp M_j$.

Theorem 3: If T is a normal operator on a Hilbert Space H, then each eigen space of T reduces T.

Proof: Let T be a normal operator on a Hilbert Space H and M be an eigen space of T corresponding to the eigen value λ.

Claim: M is invariant under T.

Let $x \in M$. Then $Tx = \lambda x$.

But $\lambda x \in M$ $\because$ M is a linear subspace.

$\Rightarrow Tx \in M$. ie. $T(M) \subseteq M$.

$\therefore$ M is invariant under T.

Claim: M is invariant under T*.

Let $x \in M$. Then $Tx = \lambda x$.

$\therefore T^*x = \bar{\lambda}x$ by Theorem 1.

But $\bar{\lambda}x \in M$ $\because$ M is a linear subspace.

$\Rightarrow T^*x \in M$. ie. $T^*(M) \subseteq M$.

$\therefore$ M is also invariant under T*. Hence M reduces T.

THE SPECTRAL THEOREM

Theorem: 9*: Let T be an operator on a finite dimensional Hilbert Space H. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the distinct eigen values of T and $M_1, M_2, \ldots, M_m$ be their corresponding eigen spaces, and $P_1, P_2, \ldots, P_m$ be the projections on these spaces. Then the following statements are all equivalent to one another.

(i) The M_i's are pairwise orthogonal and span H.

(ii) The P_i's are pairwise orthogonal, $\sum_{i=1}^{m} P_i = I$ and $T = \sum_{i=1}^{m} \lambda_i P_i$.

(iii) T is normal.

Proof: Claim: (i) $\Rightarrow$ (ii). Assume (i).

Let $x \in H$. Then x can be uniquely expressed as $x = x_1 + x_2 + \ldots + x_m$ … (1) where $x_i \in M_i$ for each $i = 1, 2, \ldots, m$, since M_i's are pairwise orthogonal and span H.

$P_iP_j = O$ if $i \neq j$, since P_i's are projections on M_i's which are pairwise orthogonal.

Then from (1), for each i, $P_ix = P_i(x_1 + x_2 + \ldots + x_m) = P_ix_1 + P_ix_2 + \ldots + P_ix_m$ … (2).

Now $P_ix_i = x_i$ $\because$ $x_i \in M_i$ which is the range of P_i.

Further, $P_ix_j = 0$ if $j \neq i$ $\because$ $x_j \in M_i^{\perp}$ which is null space of P_i. $M_j \perp M_i$.

$\therefore$ From (2), $P_ix = x_i$ … (3).

Now $\forall\, x \in H$, $Ix = x = x_1 + x_2 + \ldots + x_m = P_1x + P_2x + \ldots + P_mx$ from (3)

$= (P_1 + P_2 + \ldots + P_m)x$.

$\therefore P_1 + P_2 + \ldots + P_m = I$.

Again, $\forall\, x \in H$, $Tx = T(x_1 + x_2 + \ldots + x_m) = Tx_1 + Tx_2 + \ldots + Tx_m$

$= \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m$. $\because x_i \in M_i \Rightarrow Tx_i = \lambda_i x_i$.

$= \lambda_1 P_1 x + \lambda_2 P_2 x + \ldots + \lambda_m P_m x$ from (3)

$= (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m)x$.

$\therefore\ T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m$.

Claim: (ii) $\Rightarrow$ (iii)

Assume (ii).

Since each P_i is a projection, $P_i^* = P_i = P_i^2$. Also, $P_iP_j = 0$ if $i \neq j$.

Now $T^* = (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m)^*$

$= \overline{\lambda_1} P_1^* + \overline{\lambda_2} P_2^* + \ldots + \overline{\lambda_m} P_m^*$

$= \overline{\lambda_1} P_1 + \overline{\lambda_2} P_2 + \ldots + \overline{\lambda_m} P_m$.

$\therefore\ T T^* = (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m)(\overline{\lambda_1} P_1 + \overline{\lambda_2} P_2 + \ldots + \overline{\lambda_m} P_m)$

$= |\lambda_1|^2 P_1^2 + |\lambda_2|^2 P_2^2 + \cdots + |\lambda_m|^2 P_m^2$ $\because P_iP_j = 0$ for $i \neq j$.

$= |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_m|^2 P_m$ $\because P_i = P_i^2$.

Similarly, $T^*T = |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_m|^2 P_m$.

Hence $TT^* = T^*T$ so that T is normal.

Claim: (iii) $\Rightarrow$ (i).

Assume (iii). Since T is a normal, M_i's are pairwise orthogonal. By Theorem 2

$\therefore$(By theorem 9,) P_i's are pairwise orthogonal $\because$ P_i's are projections on M_i's and M_i's are pairwise orthogonal.

Let $M = M_1 + M_2 + \ldots + M_m$.

Then M is a closed linear subspace of H and its associated projection is $P = P_1 + P_2 + \ldots + P_m$ (by theorem 10).

Since T is normal, each eigen space M_i of T reduces T (by theorem 3).

Also, P_i is the projection on the closed linear subspace M_i of H.

$\therefore$ M_i reduces $T \Rightarrow P_iT = TP_i$ (by theorem 8).

Thus, $P_iT = TP_i$ for each P_i.

$\therefore\ TP = T(P_1 + P_2 + \ldots + P_m)$

$= TP_1 + TP_2 + \ldots + TP_m$

$= P_1T + P_2T + \ldots + P_mT$.

$= (P_1 + P_2 + \ldots + P_m)T = PT$.

Now $TP = PT$ and P is the projection on M.

$\therefore$ (By theorem 8,) M reduces T and so $M^\perp$ is invariant under T.

Let U be the restriction of T to $M^\perp$.

Then U is an operator on a finite dimensional Hilbert Space $M^\perp$ and $Ux = Tx\ \forall\ x \in M^\perp$.

If x is an eigen vector for U corresponding to the eigen value λ, then $x \in M^\perp$ and $Ux = \lambda x$.

$\therefore\ Tx = \lambda x$ and so x is also an eigen vector for T.

∴ each eigen vector for U is also an eigen vector for T.
But T has no eigen vector in $M^\perp$ since all the eigen vectors for T are on M and $M \cap M^\perp = \{ \bar{0} \}$.
So U is an operator on a finite dimensional Hilbert Space $M^\perp$ and U has no eigen vector and so no eigen value.
∴ $M^\perp = \{ \bar{0} \}$ because if $M^\perp \neq \{ \bar{0} \}$ then every operator on a nonzero finite dimensional Hilbert Space must have an eigen value.
Now $M^\perp = \{ \bar{0} \} \Rightarrow M = H$.
Thus, $M_1 + M_2 + \ldots + M_m = H$ and so M_i's span H.

SPECTRAL RESOLUTION.

Definition: Let T be an operator on a Hilbert Space H. If there exist distinct complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ and pairwise orthogonal projections $P_1, P_2, \ldots, P_m$ such that $T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m$ … (i) and $P_1 + P_2 + \ldots + P_m = I$, then the expression (i) for T is called ***Spectral Resolution*** for T.

Note: Every normal operator T on a non-zero finite dimensional Hilbert Space H has a spectral resolution.

Theorem 5: The spectral resolution of a normal operator on a finite dimensional non – zero Hilbert Space is unique.
Proof: Let T be a normal operator on a finite dimensional non – zero Hilbert Space H.
Let $T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m$ … (i) be a spectral resolution of T. Then $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct complex numbers and P_i's are non-zero pairwise orthogonal projections such that $P_1 + P_2 + \ldots + P_m = I$… (ii)
Claim: $\lambda_1, \lambda_2, \ldots, \lambda_m$ are precisely the distinct eigen values of T.
Since $P_i \neq O$, ∃ a non-zero vector x in the range of P_i.
But P_i is a projection. ∴ $P_i x = x$.
Now $Tx = (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m)x$
$= (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m)P_i x$
$= \lambda_1 P_1 P_i x + \lambda_2 P_2 P_i x + \ldots + \lambda_m P_m P_i x$
$= \lambda_i P_i^2 x = \lambda_i P_i x = \lambda_i x$.
Thus, x is a non-zero vector ∍ $Tx = \lambda_i x$.
∴ λ_i is an eigen value of T.
Since T is an operator on a finite dimensional Hilbert Space, T must possess an eigen value.

Let λ be an eigen value of T.

Then $\exists$ a non – zero vector x such that $Tx = \lambda x$.

$\Rightarrow Tx = \lambda Ix \;\Rightarrow (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m)x = \lambda(P_1 + P_2 + \ldots + P_m)x$.

$\Rightarrow (\lambda_1 - \lambda)P_1x + (\lambda_2 - \lambda)P_2x + \ldots + (\lambda_m - \lambda)P_mx = \bar{0}$.

Operating on this with P_i and remembering that $P_i^2 = P_i$ and $P_iP_j = O$ if $i \neq j$ we get $(\lambda_i - \lambda)P_ix = \bar{0}$ for $i = 1, 2, \ldots, m$.

If $\lambda_i \neq \lambda$ for each i, then we have $P_ix = \bar{0}$ for each i.

$\therefore\; P_1x + P_2x + \ldots + P_mx = \bar{0} \Rightarrow (P_1 + P_2 + \ldots + P_m)x = \bar{0} \Rightarrow Ix = \bar{0}$

$\Rightarrow x = \bar{0}$ which is a contradicts that $x \neq \bar{0}$.

Hence λ must be equal to λ_i for each i.

Thus, we have proved that in the spectral resolution (i) of T the scalars λ_i's are precisely the distinct eigen values of T.

$\therefore$ If $T = \alpha_1Q_1 + \alpha_2Q_2 + \ldots + \alpha_mQ_m \ldots$ (iii) is another spectral resolution of T, then scalars α_i's are precisely distinct eigen value of T.

$\therefore$ Renaming the projections Q_i's, if necessary, we can write (iii) in the form

$T = \lambda_1Q_1 + \lambda_2Q_2 + \ldots + \lambda_mQ_m$.

We have $T^0 = I = P_1 + P_2 + \ldots + P_m$

$T = \lambda_1P_1 + \lambda_2P_2 + \ldots + \lambda_mP_m$.

$T^2 = (\lambda_1P_1 + \lambda_2P_2 + \ldots + \lambda_mP_m)(\lambda_1P_1 + \lambda_2P_2 + \ldots + \lambda_mP_m) = {\lambda_1}^2P_1 + {\lambda_2}^2P_2 + \cdots + {\lambda_m}^2P_m$

Similarly, $T^n = {\lambda_1}^nP_1 + {\lambda_2}^nP_2 + \cdots + {\lambda_m}^nP_m$ where n is a non-negative integer.

$\therefore$ If g(t) is any polynomial with complex coefficients in the complex variable t, then taking linear combinations of the above relations, we get $g(T) = g(\lambda_1)P_1 + g(\lambda_2)P_2 + \ldots + g(\lambda_m)P_m = \sum_{j=1}^{m} g(\lambda_j)P_j$.

Now suppose that p_i is a polynomial such that $p_i(\lambda_j) = \delta_{ij}$. Ie. $p_i(\lambda_i) = 1$ and $p_i(\lambda_j) = 0$ if $j \neq i$.

Taking p_i in the place of g, $p_i(T) = \sum_{j=1}^{m} p_i(\lambda_j)P_j = \sum_{j=1}^{m} \delta_{ij}P_j = P_i$.

Thus, for each i, $P_i = p_i(T)$ which is a polynomial in T. But we must show the existence of such a polynomial p_i over the field of complex numbers.

Obviously, $p_i(t) = \frac{(t-\lambda_1)\ldots(t-\lambda_{i-1})(t-\lambda_{i+1})\ldots(t-\lambda_m)}{(\lambda_i-\lambda_1)\ldots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\ldots(\lambda_i-\lambda_m)}$ serves the purpose ie. $p_i(\lambda_i) = 1$ and $p_i(\lambda_j) = 0$ if $j \neq i$.

If we apply the above discussion for Q_i's then we shall get $Q_i = p_i(T)$ for each i.

$\therefore P_i = Q_i$ for each i.

Hence the two spectral resolutions of T are the same.

Theorem 6: 2*: If T is a normal operator on a finite dimensional Hilbert Space H, then prove that there exists an orthonormal basis for H relative to which the matrix of T is diagonal.

Proof: Let $T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m$ be the spectral resolution of normal operator T.

Then $\lambda_1, \lambda_2, \ldots, \lambda_m$, are precisely the distinct eigen values of T and $P_1, P_2, \ldots, P_m$, are the projections on $M_1, M_2, \ldots, M_m$ which are the eigen spaces of the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_m$ respectively.

Also, $P_1 + P_2 + \ldots + P_m = I$.

Now $M_i \perp M_j$ if $i \neq j$; since M_i's are eigen spaces of a normal operator.

Now each M_i is a finite dimensional non-zero Hilbert Space.

$\therefore$ each M_i contains a complete orthonormal set which will be a basis for it.

Let $B_1, B_2, \ldots, B_m$ be orthonormal basis for the spaces $M_1, M_2, \ldots, M_m$ respectively.

Claim: $B = \cup B_i$ is an orthonormal basis for H

Obviously, B is an orthonormal set since each B_i is an orthonormal set and any vector in B_i is orthonormal to any vector in B_j, if $i \neq j$.

Note that the vectors in B_i are some elements of M_i and the vectors in B_j are some elements of M_j.

The eigen spaces M_i and M_j are orthogonal if $i \neq j$.

Since B is an orthonormal set, B is linearly independent.

Now B will be a basis for H if we prove that B generates H.

Let $x \in H$.

Then $x = Ix = (P_1 + P_2 + \ldots + P_m)x = P_1x + P_2x + \ldots + P_mx$.

$= x_1 + x_2 + \ldots + x_m$ where $x_i = P_ix$.

Since P_ix is in the range of P_i, x_i is in M_i. So for each i, the vector x_i can be expressed as a linear combination of vectors in B_i which is a basis for M_i.

$\therefore$ x can be expressed as a linear combination of the vectors in B.

Hence H is generated by B.

$\therefore$ B is an orthonormal basis for H.

Since each non-zero vector in M_i is an eigen vector of T, each vector in B_i is an eigen vector for T.

Consequently, each vector in B is an eigen vector of T.

Then B is an orthonormal basis for H and each vector in B is an eigen vector for T.

Let us find the matrix of T relative to the basis B.

Let $B = \{e_1, e_2, \ldots, e_n\}$.

Since each vector in B is an eigen vector of T, $Te_1 = \alpha_1 e_1, Te_2 = \alpha_2 e_2, \ldots, Te_n = \alpha_n e_n$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are some scalars.

Now $Te_1 = \alpha_1 e_1 = \alpha_1 e_1 + 0e_2 + \ldots + 0e_n$.

$Te_2 = \alpha_2 e_2 = 0e_1 + \alpha_2 e_2 + 0e_3 + \ldots + 0e_n$.

$\vdots$

$Te_n = \alpha_n e_n = 0e_1 + 0e_2 + \ldots + 0e_{n-1} + \alpha_n e_n$.

$$[T]_B = \begin{bmatrix} \alpha_1 & 0 & \ldots & 0 \\ 0 & \alpha_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \alpha_n \end{bmatrix}$$ which is a diagonal matrix.

Example 1: 2*: Let T is an operator on a finite dimensional Hilbert Space H. Prove that

(a) T is singular if and only if $0 \in \sigma(T)$ and

(b) if T is non – singular then $\lambda \in \sigma(T)$ if and only if $\lambda^{-1} \in \sigma(T^{-1})$.

(c) If A is non – singular, then $\sigma(ATA^{-1}) = \sigma(T)$

(d) If $\lambda \in \sigma(T)$ and if p is any polynomial, then $p(\lambda) \in \sigma\{p(T)\}$.

Here $\sigma(T)$ denotes the spectrum of T i.e. the set of all eigen values of T.

Solution: (a) T is singular iff $\exists$ a non – zero vector x $\ni$ $Tx = \bar{0}$.

iff $\exists$ a non – zero vector x $\ni$ $Tx = 0x$

iff 0 is an eigen value of T

iff $0 \in \sigma(T)$.

(b) Suppose T is non – singular and $\lambda \in \sigma(T)$.

$\Rightarrow \lambda \neq 0$ by part (a)

So, λ^{-1} exists.

$\exists$ a non – zero vector $x \in H \ni Tx = \lambda x$ $\because \lambda \in \sigma(T)$.

$\Rightarrow \exists$ a non – zero vector $x \in H \ni T^{-1}(Tx) = T^{-1}(\lambda x)$

$\Rightarrow \exists$ a non – zero vector $x \in H \ni (T^{-1}T)x = \lambda T^{-1}(x)$

$\Rightarrow \exists$ a non – zero vector $x \in H \ni I(x) = \lambda T^{-1}(x)$

$\Rightarrow \exists$ a non – zero vector $x \in H \ni x = \lambda T^{-1}(x)$

$\Rightarrow \exists$ a non – zero vector $x \in H \ni \lambda^{-1}x = T^{-1}(x)$

$\Rightarrow \lambda^{-1}$ is the eigen value of T^{-1} ie. $\lambda^{-1} \in \sigma(T^{-1})$.

Conversely suppose λ^{-1} is the eigen value of T^{-1}.

$\Rightarrow (\lambda^{-1})^{-1}$ is the eigen value of $(T^{-1})^{-1}$.

$\Rightarrow \lambda$ is an eigen value of T. ie $\lambda \in \sigma(T)$.

(c) Let $ATA^{-1} = S$.

Then $S - \lambda I = ATA^{-1} - \lambda I = AT A^{-1} - A(\lambda I) A^{-1} = A(T - \lambda I) A^{-1}$.

$\therefore$ $\det(S - \lambda I) = \det\{A(T - \lambda I) A^{-1}\} = \det A \det(T - \lambda I) \det A^{-1} = \det(AA^{-1}) \det (T - \lambda I)$.

$= \det(T - \lambda I)$.

$\therefore$ $\det(S - \lambda I) = 0$ iff $\det(T - \lambda I) = 0$.

But λ is an eigen value of T iff $\det(T - \lambda I) = 0$.

$\therefore$ S and T have the same eigen values.

Ie. $\sigma(T) = \sigma(S) = \sigma(ATA^{-1})$

(d) Let $\lambda \in \sigma(T)$.

$\therefore$ $\exists$ a non–zero vector $x \in H \ni Tx = \lambda x$.

$\Rightarrow$ $\exists$ a non–zero vector $x \in H \ni T(Tx) = T(\lambda x)$.

$\Rightarrow$ $\exists$ a non–zero vector $x \in H \ni T^2x = \lambda Tx$

$\Rightarrow$ $\exists$ a non–zero vector $x \in H \ni T^2x = \lambda(\lambda x)$

$\Rightarrow$ $\exists$ a non–zero vector $x \in H \ni T^2x = \lambda^2 x$.

$\therefore$ $\lambda^2 \in \sigma(T^2)$.

Repeating k times we get $T^k x = \lambda^k x$.

$\therefore \lambda^k \in \sigma(T^k)$ where k is any + ve integer.

Let $p(t) = \alpha_0 + \alpha_1 t + \ldots + \alpha_m t^m$ where α's are scalars.

Then $p(T) = \alpha_0 + \alpha_1 T + \ldots + \alpha_m T^m$.

We have $[p(T)]x = \alpha_0 Ix + \alpha_1 Tx + \ldots + \alpha_m T^m x$.

$= \alpha_0 x + \alpha_1(\lambda x) + \ldots + \alpha_m(\lambda^m x)$

$= (\alpha_0 + \alpha_1\lambda + \ldots + \alpha_m\lambda^m)x$.

$\therefore$ $p(\lambda) = \alpha_0 + \alpha_1\lambda + \ldots + \alpha_m\lambda^m$ is an eigen value of p(T). ie. $p(\lambda) \in \sigma\{p(T)\}$.

Example 2: 2*: If T is any arbitrary operator on a finite dimensional Hilbert Space H, and N, a normal operator on H. Show that if T commutes with N, then T also commutes with N*.

Solution: Let T be any arbitrary operator on a finite dimensional Hilbert Space H, and N, a normal operator on H such that T commutes with N.

Ie. $TN = NT$.

Claim: $TN^k = N^kT$ $\forall$ $k \in \mathbb{N}$.

Obviously, the result is true for $k = 1$.

Suppose $TN^{k-1} = N^{k-1}T$.

Then $TN^k = (TN^{k-1})N = N^{k-1}T)N = N^{k-1}(TN) = N^{k-1}(NT) = N^kT$.

$\therefore$ By induction $TN^k = N^kT$ $\forall$ + ve integral values of k.

Claim: T commutes with every polynomial in N

Now let $p(t) = \alpha_0 + \alpha_1 t + \ldots + \alpha_s t^s$ be any polynomial with complex coefficients.

Then $p(N) = \alpha_0 I + \alpha_1 N + \ldots + \alpha_s N^s$.

$\therefore Tp(N) = T(\alpha_0 I + \alpha_1 N + \ldots + \alpha_s N^s)$.

$= \alpha_0\, TI + \alpha_1 TN + \ldots + \alpha_s TN^s$.

$= \alpha_0 IT + \alpha_1 NT + \ldots + \alpha_s N^sT$.

$= (\alpha_0 I + \alpha_1 N + \ldots + \alpha_s N^s)T$

$= p(N)T.$

Thus, T commutes with every polynomial in N.

Now let $N = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m$ be the spectral resolution of the normal operator N.

Then $N^* = (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m)^*$

$= \overline{\lambda_1} P_1^* + \overline{\lambda_2} P_2^* + \ldots + \overline{\lambda_m} P_m^*$

$= \overline{\lambda_1} P_1 + \overline{\lambda_2} P_2 + \ldots + \overline{\lambda_m} P_m.$

But, for each i, the operator P_i is a polynomial in N.

$\therefore$ N* is also a polynomial in N.

$\therefore$ T also commutes with N* $\because$ T commutes with every polynomial in N.

Example 3: 3*: Show that an operator T on a finite dimensional Hilbert Space H is normal if and only if its adjoint T* is a polynomial in T.

Solution: Suppose T* is a polynomial in T.

Let $T^* = \alpha_0 I + \alpha_1 T + \ldots + \alpha_k T^k$.

Then $T^*T = (\alpha_0 I + \alpha_1 T + \ldots + \alpha_k T^k)\, T = \alpha_0 IT + \alpha_1 T^2 + \ldots + \alpha_k T^{k+1}$

$= T(\alpha_0 I + \alpha_1 T + \ldots + \alpha_k T^k) = TT^*$

$\therefore$ T is normal.

Conversely suppose that T is normal.

Let $T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m$ be a spectral resolution of T.

Then $T^* = \overline{\lambda_1} P_1^* + \overline{\lambda_2} P_2^* + \ldots + \overline{\lambda_m} P_m^* = \overline{\lambda_1} P_1 + \overline{\lambda_2} P_2 + \ldots + \overline{\lambda_m} P_m.$

But, for each i, the operator P_i is a polynomial in T.

$\therefore$ T* is also a polynomial in T.

Example 4: 1*: Let T be a normal operator on a finite dimensional Hilbert Space H with spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$. Then prove that

(a)* T is a self – adjoint if and only if each λ_i is real

(b) T is positive if and only if each eigen value λ_i of T is ≥ 0.

(c) T is unitary if and only if $|\lambda_i| = 1$ for each i.

Solution: Let $T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m$ be the spectral resolution of P.

Then $\lambda_1, \lambda_2, \ldots + \lambda_m$ are preciously the distinct eigen values of T, $P_i \neq O$ and $P_i P_j = O$ if $i \neq j$. Also, $P_1 + P_2 + \ldots + P_m = I$

(a) $T^* = (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m)^* = \overline{\lambda_1} P_1^* + \overline{\lambda_2} P_2^* + \ldots + \overline{\lambda_m} P_m^*$

$= \overline{\lambda_1} P_1 + \overline{\lambda_2} P_2 + \ldots + \overline{\lambda_m} P_m$ ... (1)

Suppose each λ_i is real. Then $\overline{\lambda_i} = \lambda_i$ for each i.

From (1), $T^* = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m = T$.

Hence T is self – adjoint.

Conversely suppose T is self – adjoint. Then T* = T.

∴ from (i) $\overline{\lambda_1}P_1 + \overline{\lambda_2}P_2 + \ldots + \overline{\lambda_m}P_m = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m$.

$\Rightarrow (\overline{\lambda_1} - \lambda_1)P_1 + (\overline{\lambda_2} - \lambda_2)P_2 + \ldots + (\overline{\lambda_m} - \lambda_m)P_m = O$.

$\Rightarrow (\overline{\lambda_1} - \lambda_1)P_iP_1 + (\overline{\lambda_2} - \lambda_2)P_iP_2 + \ldots + (\overline{\lambda_m} - \lambda_m)P_iP_m = P_iO = O$ for each i.

$\Rightarrow (\overline{\lambda_i} - \lambda_i)P_i = O$ for each i.

$\Rightarrow \overline{\lambda_i} = \lambda_i$ for each i.

$\Rightarrow \lambda_i$ is real.

(b)For each x in H, $(Tx, x) = (Tx, Ix) = (\sum_{i=1}^{m} \lambda_i P_i x, \sum_{j=1}^{m} P_j x)$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i (P_i x, P_j x)$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i (x, P_i^{*} P_j x)$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i (x, P_i P_j x)$$
$$= \sum_{i=1}^{m} \lambda_i (x, P_i P_i x)$$
$$= \sum_{i=1}^{m} \lambda_i (P_i^{*} x, P_i x)$$
$$= \sum_{i=1}^{m} \lambda_i (P_i x, P_i x)$$
$$= \sum_{i=1}^{m} \lambda_i \|P_i x\|^2 \ \ldots (2).$$

Now suppose that each eigen value λ_i of T is ≥ 0. Then each λ_i is real.

∴ T is self-adjoint by part (a). Also $\|P_i x\|^2 \geq 0$ for each i.

∴ if $\lambda_i \geq 0$ for each i, $(Tx, x) \geq 0 \ \forall \ x \in H$. ∴ T is positive.

Conversely suppose that T is positive.

∴ $\sum_{i=1}^{m} \lambda_i \|P_i x\|^2 \geq 0 \ \forall \ x \in H \ \ldots (3)$

Now for any fixed i, suppose x is in the range of P_i.

Then $P_i x = x$ and $P_j x = \overline{0}$ for $j \neq i$.

∴ from (3), $\lambda_i \|x\|^2 \geq 0 \Rightarrow \lambda_i \geq 0$ for each i.

(c) We have $TT^* = (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m)(\overline{\lambda_1}P_1 + \overline{\lambda_2}P_2 + \ldots + \overline{\lambda_m}P_m)$

$= |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_m|^2 P_m \ \ldots (4)$.

Suppose each eigen value λ_i of T is of unit modules ie. $|\lambda_i| = 1$ for each i.

Then, from (4), $TT^* = P_1 + P_2 + \ldots + P_m = I$.

Similarly, T*T = I.

Hence T is unitary.

Conversely suppose T is unitary.

Then TT* = I.

From (4), $|\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_m|^2 P_m = I$.

$\Rightarrow P_i\{|\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_m|^2 P_m\} = P_i$ for each i.

$\Rightarrow |\lambda_i|^2 {P_i}^2 = P_i$ for each i.

$\Rightarrow |\lambda_i|^2 P_i = P_i$ for each i.

$\Rightarrow (|\lambda_i|^2 - 1)P_i = 0$ for each i.

$\Rightarrow |\lambda_i|^2 = 1$ or $|\lambda_i| = 1$ for each i.