Fax: 08816-227318 off: 08816-224072, 224119,228342 Mobile: 9491334119
DANTULURI NARAYANA RAJU COLLEGE
(Autonomous)
BHIMAVARAM, W.G.DIST, ANDHRA PRADESH, INDIA, PIN- 534202.
(Accredited at 'B++' level by NAAC)
(Affiliated to Adikavi Nannaya University, Rajamahendravaram)

E – CONTENT

PAPER: M 104, TOPOLOGY

M. Sc. I YEAR, SEMESTER - I

UNIT – I: METRIC SPACES

PREPARED BY

K, C. TAMMI RAJU, M. Sc.

HEAD OF THE DEPARTMENT

DEPARTMENT OF MATHEMATICS,

PG COURSES

DNR COLLEGE (A),

BHIMAVARAM – 534202

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METRIC SPACES

Definition: Let X be a nonempty set and d: $X \times X \to \mathbb{R}$ be a function. d is said to be a ***metric*** on X if

(i) $d(x, y) \geq 0 \ \forall\ x, y \in X$ and $d(x, y) = 0$ iff $x = y$. (Non negativity)

(ii) $d(x, y) = d(y, x) \ \forall\ x, y \in X$. (symmetry)

(iii) $d(x, y) \leq d(x, z) + d(z, y) \ \forall\ x, y, z \in X$ (Triangle in equality).

If d is a metric on X then (X, d) is called a ***metric space***. d(x, y) is called the distance between x and y.

Example: Define d: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x, y) = |x - y|$ where $\mathbb{R}$ is the set of all real numbers. Then d is a metric called ***usual metric*** on $\mathbb{R}$.

Solution: (i) $d(x, y) = |x - y| \geq 0$. $d(x, y) = 0$ iff $|x - y| = 0$ iff $x = y$.

(ii) $d(x, y) = |x - y| = |y - x| = d(y, x)$

(iii) $d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$.

Hence d is a metric on $\mathbb{R}$.

Example: Define d: $\mathbb{C} \times \mathbb{C} \to \mathbb{R}$ by $d(z_1, z_2) = |z_1 - z_2|$ where $\mathbb{C}$ is the set of all complex numbers. Then d is a metric on $\mathbb{C}$.

Solution: Let $z_1, z_2, z_3 \in \mathbb{C}$.

(i) $d(z_1, z_2) = |z_1 - z_2| \geq 0$ and $d(z_1, z_2) = 0$ iff $|z_1 - z_2| = 0$ iff $z_1 = z_2$.

(ii) $d(z_1, z_2) = |z_1 - z_2| = |-(z_1 - z_2)| = |z_2 - z_1| = d(z_2, z_1)$.

(iii) $d(z_1, z_2) = |z_1 - z_2| = |z_1 - z_3 + z_3 - z_2| \leq |z_1 - z_3| + |z_3 - z_2|$
$= d(z_1, z_3) + d(z_3, z_2)$. $\therefore$ d is a metric called usual metric on $\mathbb{C}$.

Problem: Let X be a nonempty set and d: $X \times X \to \mathbb{R}$ be a function satisfying the following two conditions.

(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) \leq d(x, z) + d(y, z) \ \forall\ x, y, z \in X$.

Then d is a metric on X.

Solution: (i) Put $y = x$ in (ii). Then $d(x, x) \leq d(x, z) + d(x, z) \Rightarrow 0 \leq 2\, d(x, z)$
$\Rightarrow d(x, z) \geq 0$.

(ii) Put $x = z$ in (ii). $d(z, y) \leq d(z, z) + d(y, z) \Rightarrow d(z, y) \leq 0 + d(y, z)$.
$\Rightarrow d(z, y) \leq d(y, z)$ and this is true $\forall\ y, z \in X$.
$\therefore d(y, z) \leq d(z, y)$ is also true. Hence $d(y, z) = d(z, y) \ \forall\ y, z \in X$.

(iii) By (ii) d(x, y) ≤ d(x, z) + d(y, z) = d(x, z) + d(z, y) since d(y, z) = d(z, y),
∴ d(x, y) ≤ d(x, z) + d(z, y) ∀ x, y, z ∈ X. Hence d is a metric on X.

Example: Let X ≠ ϕ. Define d: X × X → ℝ by d(x, y) = 0 if x = y and d(x, y) = 1 if x ≠ y. Then d is a metric on X called ***discrete metric*** and so (X, d) is a metric space called ***discrete metric space***.

Solution: (i) Clearly d(x, y) ≥ 0 and d(x, y) = 0 iff x = y.
(ii) If x = y then d(x, y) = 0 = d(y, x). If x ≠ y, then d(x, y) = 1 = d(y, x).
Thus, d(x, y) = d(y, x).
(iii) Suppose x = y = z, then d(x, y) = 0 = 0 + 0 = d(x, z) + d(z, y).
Suppose x = y ≠ z. Then d(x, y) = 0 ≤ 1 + 1 = d(x, z) + d(z, y).
Suppose x ≠ y. If x = z, y ≠ z then d(x, y) = 1 = 0 + 1 = d(x, z) + d(z, y).
Similar is the case when x ≠ y, x ≠ z, y = z.
Suppose no two are equal. Then d(x, y) = 1 ≤ 1 + 1 = d(x, z) + d(z, y).
Thus, in all the cases d(x, y) ≤ d(x, z) + d(z, y). Hence d is a metric on X.

Problem: Let (X, d) be a metric space. Show that d_1 defined by $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$ is a metric on X. Show that X is a bounded set in (X, d_1).

Solution: Let x, y, z ∈ X. Since d(x, y) ≥ 0, $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} \geq 0$.

$d_1(x,y) = 0$ iff $\frac{d(x,y)}{1+d(x,y)} = 0$ iff d(x, y) = 0 iff x = y.

Also $d_1(x,y) = \frac{d(x,\ y)}{1+d(x,\ y)} = \frac{d(y,\ x)}{1+d(y,\ x)} = d_1(y,x)$. ∴ d_1 is symmetric.

Again $d_1(x,y) = \frac{d(x,\ y)}{1+d(x,\ y)} \leq \frac{d(x,\ z)}{1+d(x,\ z)} + \frac{d(z,\ y)}{1+d(z,\ y)} = d_1(x,z) + d_1(z,y)$.

Hence d_1 is also a metric on X.

(ii) For any x, y ∈ X, 0 ≤ d(x, y) ≤ 1 + d(x, y)

$\Rightarrow 0 \leq \frac{d(x,\ y)}{1+d(x,\ y)} \leq 1$

$\Rightarrow 0 \leq d_1(x,y) \leq 1$

∴ d(X) = sup {d_1 (x, y): x, y ∈ X} ≤ 1.
This shows that X is bounded in the metric space (X, d_1).

Definition: Let X be a nonempty set and d: X × X → ℝ be a function such that

(i) d(x, y) ≥ 0 ∀ x, y ∈ X and x = y ⇒ d(x, y) = 0.

(ii) d(x, y) = d(y, x) ∀ x, y ∈ X.

(iii) $d(x, y) \le d(x, z) + d(z, y) \ \forall\ x, y, z \in X$.
Then d is said to be a ***pseudo – metric*** on X.

Note: Every metric is a pseudo – metric. But converse is not true.

Example: Let X be a set with $|X| \ge 2$. Define $d(a, b) = 0 \ \forall\ a, b \in X$. Then d is a pseudo metric but not a metric.
Solution: Clearly d is a pseudo metric. Let $a \ne b$. Then also $d(a, b) = 0$.
$\therefore$ d is not a metric.

Example: Let $X = \{1, 2, 3\}$. Define $d : X \times X \to \mathbb{R}$ by $d(1, 1) = d(2, 2)$ $= d(3, 3) = d(1, 2) = d(2, 1) = 0$; $d(2, 3) = d(3, 2) = d(3, 1) = d(1, 3) = 1$.
Then d is a pseudo metric but not a metric.
Solution: Clearly d is a pseudo metric. $1 \ne 2$ but $d(1, 2) = 0$. So, d is not a metric.

Example: Give two examples of pseudo – metric which are not metrics.

Problem: Let X be a Pseudo metric on X and define '~' on X by
$x \sim y \Leftrightarrow d(x, y) = 0$. (i) Show that '~' is an equivalence relation (ii) Define a metric on the set of all equivalence classes.
Solution: (i) ~ is reflexive: $x \sim x \ \forall\ x \in X$ since $d(x, x) = 0$.
~ is symmetric: Suppose $x \sim y$.
$\Rightarrow d(x, y) = 0 \Rightarrow d(y, x) = 0. \Rightarrow y \sim x$.
~ is transitive: Suppose $x \sim y$, $y \sim z$
$\Rightarrow d(x, y) = 0$ and $d(y, z) = 0$.
Now $d(x, z) \le d(x, y) + d(y, z) = 0 + 0 = 0$.
$\Rightarrow d(x, z) = 0 \Rightarrow x \sim z$.
Hence ~ is an equivalence relation.
Define $d^*([x], [y]) = d(x, y)$.
Then $d^*([x], [y]) = d(x, y) \ge 0$.
$d^*([x], [y]) = 0$ iff $d(x, y) = 0$ iff $x \sim y$ iff $[x] = [y]$.
$d^*([x], [y]) = d(x, y) = d(y, x) = d^*([y], [x])$.
$d^*([x], [y]) = d(x, y) \le d(x, z) + d(z, y) = d^*([x], [z]) + d^*([z], [y])$.
Hence d^* is a metric on the set of all equivalence classes $\{[x] : x \in X\}$.

Definition: Let X be a nonempty set. If for each x $\in$ X, there corresponds a real number $\|x\|$, and it satisfies the conditions

(i) $\|x\| \geq 0$ and $\|x\| = 0$ iff x = 0.

(ii) $\|-x\| = \|x\|$ $\forall$ x $\in$ X.

(iii) $\|x+y\| \leq \|x\| + \|y\|$ $\forall$ x, y $\in$ X

then $\|x\|$ is called ***norm*** of x $\in$ X**.**

Example : Let $\|x\|$ be norm of x $\in$ X as defined as above. If we define d(x, y) = $\|x-y\|$ then (X, d) is a metric space and 'd' is called the metric ***induced by the norm*.**

Proof: Let x, y $\in$ X. (i) Then d(x, y) = $\|x-y\| \geq 0$.

Now d(x, y) = 0 iff $\|x-y\| = 0$ iff x – y = 0 iff x = y.

(ii)d(x, y) = $\|x-y\| = \|-(y-x)\| = \|y-x\|$ = d(y, x)

(iii)Let x, y, z $\in$ X. Then d(x, y) = $\|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\|$ = d(x, z) + d(z, y).

$\therefore$(X, d) is a metric space.

Define: Let f : [0, 1] $\to \mathbb{R}$. F is said to be ***bounded*** if there exists k $\in \mathbb{R}$ such that $|f(x)| \leq$ k for every x $\in$[0, 1].

Example: Let X = {f / f: [0, 1] $\to \mathbb{R}$, f is bounded and continuous}. Define $\|f\|$ by $\|f\| = \int_0^1 |f(x)|dx$ (here the integral involved is the Riemann integral) Then d defined by d(f, g) = $\|f-g\| = \int_0^1 |f(x)-g(x)|dx$ is induced metric.

Solution: $\|f\| = \int_0^1 |f(x)|dx \geq 0 \because |f(x)| \geq 0$.

$\|f\| = 0$ iff $\int_0^1 |f(x)|dx$ = iff $|f(x)| = 0$ $\forall$ x iff f = 0 (zero function.

$\|-f\| = \int_0^1 |-f(x)|dx = \int_0^1 |f(x)|dx = \|f\|$

Let f, g $\in$ X. Then $\|f+g\| = \int_0^1 |(f+g)(x)|dx \leq \int_0^1 \{|f(x)| + |g(x)|\}dx$

$= \int_0^1 |f(x)|dx + \int_0^1 |g(x)|dx = \|f\| + \|g\|$. $\therefore \|f\| = \int_0^1 |f(x)|dx$ defines norm on X.

Hence d defined by d(f, g) = $\|f-g\| = \int_0^1 |f(x)-g(x)|dx$ is induced metric.

Example : Let X = {f / f: [0, 1] $\to \mathbb{R}$, f is bounded and continuous}. Define $\|f\|$ by $\|f\|$ = sup $\{|f(x)|: x \in [0,1]\}$. Then d defined by d(f, g) = $\|f-g\|$ = sup

$\{|f(x) - g(x)|: x \in [0,1]\}$ is a metric and this metric space is denoted by C[0, 1]
.
Solution: Let f ∈ X. Then $\|f\|$ = sup $\{|f(x)|: x \in [0,1]\} \geq 0 \because |f(x)| \geq 0$.
$\|f\| = 0$ iff sup $\{|f(x)|: x \in [0,1]\} = 0$ iff $|f(x)|$ $\forall\, x \in [0,1]$ iff f = 0 (zero function.
$\|-f\|$ = sup $\{|-f(x)|: x \in [0,1]\}$ = sup $\{|f(x)|: x \in [0,1]\} = \|f\|$
Let f, g ∈ X. Then $\|f + g\|$ = sup $\{|(f + g)(x)|: x \in [0,1]\}$
= sup $\{|f(x) + g(x)|\} \leq$ sup $\{|f(x)| + |g(x)|\} \leq$ sup $\{|f(x)|\}$ + sup $\{|g(x)|\}$ = $\|f\| + \|g\|$. $\therefore \|f\|$ = sup $\{|f(x)|: x \in [0,1]\}$ defines norm on X. $\therefore$ d defined on X by d(f, g) = $\|f - g\|$ = sup $\{|f(x) - g(x)|: x \in [0,1]\}$ is a metric on X.

SUBSPACE

Definition: Let (X, d) be a metric space and Y ⊆ X. Then the restrictions of 'd' to Y, then (Y, d) is a metric space and (Y, d) is called *subspace* of (X, d).
Definition: Let (X, d) be a metric space and A ⊆ X.

(i) If x ∈ X then the distance from x to A, d(x, A) = inf {d(x, a) / a ∈ A}.
(ii) The diameter of the set A, d(A) = sup {d(x, y) / x, y ∈ A}.
(iii) If d(A) = ±∞ then A is said to have infinite diameter, otherwise, it is said to have finite diameter. Note that if A = ϕ then d(ϕ) = sup {d(x, y) / x, y ∈ ϕ} = sup ϕ = - ∞ and so ϕ has infinite diameter.
(iv) A is said to be bounded if d(A) is finite. A mapping f:Y → X where Y ≠ ϕ and (X, d) is a metric space is said to be bounded if the set f(Y) is bounded in (X, d).

Example: Let $\mathbb{R}^k$ be the Euclidean space.
Define d(x, y) = $|x - y|$ ∀ x, y ∈ $\mathbb{R}^k$. Then d is a metric on $\mathbb{R}^k$.

OPEN SETS

Let (X, d) be a metric space. Let x_0 ∈ X and r be a positive real number. Then $N_r(x_0) = S_r(x_0) = \{x \in X / d(x, x_0) < r\}$ is called the open sphere with centre x_0 and radius r. It is also called neighbourhood of x_0 with radius r.

Note: $S_r(x_0) \neq \phi$.

Example: (i) If (X d) is a metric space where $X \neq \phi$ and 'd' is a metric on X, defined by d(x. y) = 0 if x = y and 1 if $x \neq y$. Then for every $x_0 \in X$, $S_l(x_o) = \{x_0\}$.

(ii) Consider $(\mathbb{R}, d)$ where $\mathbb{R}$ is the set of all real numbers, d is a usual metric on $\mathbb{R}$. Then for any $x_0 \in \mathbb{R}$, $S_r(x_0) = (x_0 - r, x_o + r)$.

Definition : Let X be a metric space. All points and sets mentioned here are elements and subsets of X.

(i) A point p is a limit point of the set E if every neighbourhood of p contains a point q such that $p \neq q$ and $q \in E$; The set of all limit points of E is denoted by D(E).

(ii) If $p \in E$ and p is not a limit point of E, then p is called an isolated point of E;

(iii) A set E is said to be closed if every limit point of E is a point of E;

(iv) A point p of E is said to be an interior point of E if there exists a neighbourhood N of p such that $p \in N \subseteq E$. The set of all interior points of A, is called the interior of A. It is denoted by Int (A);

(v) A set E is open if every point of E is an interior point. Equivalently, a subset G of the metric space X is called an open set if given $x \in G$ there exists a positive real number r such that $S_r(x) \subseteq G$;

(vi) A set E is said to be perfect if E is closed and every point of E is a limit point of E;

(vii) E is bounded if there exists a real number M and a point $q \in X$ such that $d(p, q) < M$, for all $p \in E$.

Definition: A subset E of a metric space X is said to be dense in X if every point of X is a limit point of E or a point of E, or both.

Note: Consider the set $\mathbb{R}$ of real numbers with usual metric d. The set [0, 1) is not open as a subset of $\mathbb{R}$, since $0 \in [0, 1)$ is not an interior point. If we consider [0, l) as a metric space X in its own right, as a subspace of the real line, then [0, 1) is open as a subset of X, since from this point of view it is the full space.

Theorem: In any metric space X the empty set and the full space X are open sets.

Proof: To show that ϕ is open, we must show that each point in ϕ is the centre of an open sphere contained in ϕ; but since there are no points in ϕ, the requirement is automatically satisfied. Hence ϕ is open.
Since every open sphere centred on each of the points in X, is contained in X, we have X is open.

Lemma: In a metric space X, $x \in S_r(x_o) \Rightarrow$ there exists $s > 0 \ni S_s(x) \subseteq S_r(x_0)$.
In other words, every open sphere (or neighbourhood) is an open set.
Proof: Let $S_r(x_0)$ be an open sphere in X. Let $x \in S_r(x_0)$.
Then $d(x_0, x) < r \Rightarrow r - d(x_0, x) > 0$.
Put $s = r - d(x_0, x)$. Then $s > 0$.
Consider the sphere $S_s(x)$.
Let $y \in S_s(x)$
$\Rightarrow d(y, x) < s$.
Now $d(y, x_0) \leq d(y, x) + d(x, x_0)$ (by triangle inequality)
$< s + d(x, x_0) = [r - d(xo, x)] + d(x, x_0) = r$
Therefore $d(y, x_0) < r$
$\Rightarrow y \in S_r(x_0)$. Hence $S_s(x) \subseteq S_r(x_0)$. $\therefore$ $S_r(x_0)$ is an open set.

Theorem: Let X be a metric space. A subset G of X is open if and only if it is a union of open spheres
Proof: Suppose G is Open. If $G = \phi$, then it is the union of the empty class of open spheres. If $G \neq \phi$, then for any $x \in G\ \exists\ r_x > 0$ such that $S_{r_x}(x) \subseteq G$.
Then $G = \bigcup_{x \in G} S_{r_x}(x)$
Conversely suppose $G = \bigcup_{x \in I} S_{r_x}(x)$, where $\{S_{r_x}(x)) / x \in I\}$ is a collection of open spheres.
If $I = \phi$, then $G = \phi$ which is an open set.
Suppose $I \neq \phi$. Let $y \in G$.
Since $G = \bigcup_{x \in I} S_{r_x}(x)$, we have $y \in S_{r_x}(x)$ for some $x \in I$.
By above lemma, $\exists\ r > 0 \ni S_r(y) \subseteq S_{r_x}(x)$.
Hence $S_r(y) \subseteq S_{r_x}(x) \subseteq G$. This shows that G is open.

Theorem: Let X be a metric space. Then (i) union of open sets in X is open; and (ii) finite intersection of open sets in X is open.
Proof: (i) Let $\{G_i\}_{i \in I}$ be a collection of open sets. Write $G = \bigcup_{i \in I} G_i$. We have to show that G is open.
If $I = \phi$ then the union of the empty class of open sets G_i is $G = \phi$ which is open If I

≠ ϕ, then by above theorem, each G_i is a union of open spheres. Again by above Theorem G is open.

(ii) Let $\{G_i\}_{1 \le i \le n}$ be a finite collection of open sets in X.

Claim: $G = \bigcap_{i=1}^{n} G_i$ is open.

If $I = \phi$ then the class of $\{G_i\}_{1 \le i \le n}$ is ϕ and hence $\bigcap_{i=1}^{n} G_i = X$ which is open.

Let $I \neq \phi$. If $G = \phi$ then G is open. Suppose $G \neq \phi$. Let $x \in G = \bigcap_{i=1}^{n} G_i$. Since each G_i is open $\exists\, r_i > 0$ such that $S_{r_i}(x) \subseteq G_i$. Write $r = \min\{r_1, r_2, \ldots, r_n\}$. Then $S_r(x) \subseteq S_{r_i}(x) \subseteq G_i$ for all $1 \le i \le n$, which shows that $S_r(x) \subseteq \bigcap_{i=1}^{n} G_i = G$. Hence G is open.

Remark: Intersection of infinite collection of open sets need not be open.

For, consider $\mathbb{R}$ with usual metric. Write $G_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$. Then $G = \bigcap_{i=1}^{\infty} G_i = \{0\}$ which is not open.

Problem: Let G be an open set in $\mathbb{R}$. Define ~ on G as $x, y \in G$, $x \sim y$ if and only if $\exists$ open interval (a, b) such that $x, y \in (a, b) \subseteq G$. Then

(i) ~ is an equivalence relation

(ii) For any $x \in G$, if $I_x = \cup \{(a, b) / x \in (a, b) \subseteq G\}$, then I_x is an open interval such that $x \in I_x \subseteq G$.

(iii) $[x] = I_x$ and

(iv) $G = \cup I_x$, $x \in G$.

Solution: (i) Let $x \in G$. Since G is open $\exists\, r > 0 \ni x \in S_r(x) = (x - r, x + r) \subseteq G$. $\therefore$ $x \sim x \ \forall\, x \in G$. Viz. ~ is reflexive.

Let $x, y \in G \ni x \sim y$. Then $\exists$ open interval (a, b) such that $x, y \in (a, b) \subseteq G$.

$\Rightarrow \exists$ open interval (a, b) such that $y, x \in (a, b) \subseteq G$.

$\Rightarrow y \sim x$. Viz. ~ is symmetric.

Let $x, y, z \in G \ni x \sim y$ and $y \sim z$.

Then $\exists$ open intervals (a, b), (c, d) $\ni x, y \in (a, b) \subseteq G$ and $y, z \in (c, d) \subseteq G$. Since $y \in (a, b) \cap (c, d)$ and $(a, b) \cup (c, d)$ is an interval we get $x, z \in (a, b) \cup (c, d) \subseteq G$. $\therefore$ $x \sim z$. Viz. ~ is transitive.

Hence ~ is an equivalence relation.

(ii) Let $x \in G$ and $I_x = \cup \{(a, b) / x \in (a, b) \subseteq G\}$.
Then I_x is an open set. Since the intersection of all the intervals involved in this union contains x, we have that I_x is nonempty.
Thus, I_x is an interval such that $x \in I_x \subseteq G$.

(iii) Let $u \in [x]$.
Then $u \sim x \Rightarrow \exists$ open interval (a, b) such that $u, x \in (a, b) \subseteq G$

$\Rightarrow u \in (a, b) \subseteq I_x$. $\therefore$ $[x] \subseteq I_x$.

Let $y \in I_x$. Then $y \in (a, b)$ for some (a, b) with $x \in (a, b) \subseteq G$

$\Rightarrow y, x \in (a, b) \subseteq G \Rightarrow y \sim x \Rightarrow y \in [x]$. Hence $[x] = I_x$.

(iv) Since the set of equivalence classes $[x] = I_x$, $x \in G$ for some partition for G, we have $G = \cup I_x$, $x \in G$.

Theorem: Every non-empty open set on the real line is the union of a countable disjoint class of open intervals.

Proof: Let G be a non-empty open subset of the real line. Let x be a point of G. Since G is open, x is the centre of a bounded open interval contained in G. Define $I_x = \cup \{(a, b) / x \in (a, b) \subseteq G\}$.

Next we observe that if x and y are two distinct points of G then I_x and I_y are either disjoint or identical.

For, suppose $z \in I_x \cap I_y . \Rightarrow z \in I_x$ and $z \in I_y$.

Then $I_z = I_x$ and $I_y = I_z$ (by above problem). Therefore $I_x = I_y$.

Consider the class I of all distinct sets of the form I_x for some point x in G.

This is a disjoint class of open intervals, and G is its union. It remains to prove that I is countable.

Let G_r be the set of rational points in G. Clearly G_r is non-empty.

Define f: $G_r \to I$ as $f(r) = [r] = I_r$. If $I_x \in I$ then I_x contains at least one rational number u. Now $u \in I_x \subseteq G \Rightarrow u \in G_r$. Also $f(u) = [u] = I_u = I_x$. Hence f is onto.

Since G_r is countable and f: $G_r \to I$ is onto, we have that I is countable.

Definition: Let (X, d) be a metric space, $A \subseteq X$ and $x \in A$. Then x is said to be an interior point of A if there exists $r > 0$ such that $S_r(x) \subseteq A$.

The set of all interior points of A is called the interior of A. It is denoted by Int (A). So Int (A) = $\{x \in A$ and $Sr(x) \subseteq A$ for some r$\}$.

Proposition: Write $X = \mathbb{R}$, the set of real numbers with usual metric. Find Int (Q), where Q is the set of all rational numbers.

Solution: Let $x \in$ 1nt (Q) $\Rightarrow$ there exists a real number $r > 0$ such that $S_r(x) \subseteq Q$

$\Rightarrow (x - r, x + r) \subseteq Q$. Since $r > 0$, we have that $x - r \neq x + r$.

We know that between any two real numbers there is an irrational number.

$\therefore \exists$ an irrational number q such that $x - r < q < x + r \Rightarrow q \in (x - r, x + r) \subseteq Q$. $\therefore$ Q contains an irrational number q, a contradiction. Hence Int Q = ϕ.

Result: (i) Int (A) is an open subset of A; (ii) lnt (A) contains every open subset of A; (iii) Int (A) is the largest open subset of A.

Proof: (i) Clearly Int (A) $\subseteq$ A. Let x $\in$ Int (A). Then $\exists$ r > 0 such that $S_r(x) \subseteq A$.
Let y $\in$ $S_r(x)$. Then $\exists$ s > 0 such that $S_s(y) \subseteq S_r(x) \subseteq A$. $\Rightarrow$ y $\in$ Int (A).
$\Rightarrow S_r(x) \subseteq$ Int (A) for all x $\in$ Int (A). Hence Int (A) is an open set.
(ii) Let G be an open set of A. Let x $\in$ G. Since G is open $\exists$ r > 0 such that $S_r(x) \subseteq$ G. Now $S_r(x) \subseteq G \subseteq A \Rightarrow S_r(x) \subseteq A \Rightarrow$ x $\in$ Int (A).
Therefore G $\subseteq$ Int (A).
(iii) From (i), Int (A) is an open set. If Int (A) is not the largest open set contained in A, then there exists an open set G in A such that Int (A) $\subset$ G. But form (ii), we get G $\subseteq$ Int (A).
Therefore G $\subseteq$ Int (A) $\subset$ G $\Rightarrow$ G $\subset$ G, a contradiction.
Hence Int (A) is the largest open subset of A.

Result: A is open if and only if A = Int (A).
Proof: Suppose A is open.
Then by a result Int (A) is the largest open subset of A.
Hence A = Int (A).
Conversely A = Int (A) implies that A is open since Int (A) is open.

Result : Int (A) is the union of all open subsets of A.
Proof: Let $\{G_i / i \in I\}$ be the collection of all open subsets contained in A. Since each G_i is open and $G_i \subseteq$ Int (A).
$\Rightarrow \bigcup_{i \in I} G_i \subseteq$ Int (A). Let x $\in$ Int (A)
$\Rightarrow \exists$ r > 0 $\ni$ $S_r(x) \subseteq A$.
Since $S_r(x)$ is open, we have that $S_r(x)$ = G for some j $\in$ I.
So x $\in S_r(x) = G_j \subseteq \bigcup_{i \in I} G_i$.
Hence Int (A) $\subseteq \bigcup_{i \in I} G_i$ Thus Int(A) = $\bigcup_{i \in I} G_i$.

CLOSED SETS

Definition: A subset F of a metric Space X is called a closed set if it contains each of its limit points.
Theorem: In any metric space X, the empty set ϕ and the full space X are closed sets.

Proof: Since ϕ contains no limit points, we have that ϕ is closed. Since X contains all points of the metric space, we have that X is closed.

Theorem: A set E is open if and only if E^c (the complement of E) is closed.

Proof: Suppose E is open. Let x be a limit point of E^c. we have to show that $x \in E^c$. If $x \notin E^c$ then $x \in (E^c)^c = E$. Since E is open and $x \in E$, there exists $r > 0$ such that $S_r(x) \subseteq E \Rightarrow S_r(x) \cap E^c = \phi$. $\Rightarrow$ x is not a limit point of E^c, a contradiction. $\therefore$ $x \in E^c$. Hence E^c is closed.
Converse: Suppose E^c is closed. Now we show that E is open. Let $y \in E$. Then $y \notin E^c \Rightarrow$ y is not a limit point of $E^c \Rightarrow \exists$ a neighbourhood N of y such that $N \cap E^c = \phi \Rightarrow y \in N \subseteq E$. $\therefore$ y is an interior point of E. Since y is an arbitrary point in E, we have that every point of E is interior point of E. Hence E is open.

Corollary: A set F is closed if and only if F^c is open.
Proof: Follows from the above theorem.

Definition: Let X be a metric space. $x_0 \in X$, r be a non negative real number. Then $S_r[x_0] = \{x / x \in X, d(x, x_0) \le r\}$ is called the *closed sphere* with centre x_0 and radius r.

Theorem: In a metric space X, each closed sphere $S_r[x_0]$ is a closed set.
Proof: First we show that Y = the complement of $S_r[x_0]$ is open.
If $Y = \phi$, then it is open. Suppose $Y \neq \phi$. Let $x \in Y$ then $d(x, x_0) > r$.
Let $s = d(x, x_0) - r > 0$. Consider $S_s(x)$. Let $z \in S_s(x)$. Then $d(x, z) < s$.
So $d(x_0, x) \le d(x_0, z) + d(z, x)$.
$\Rightarrow d(x_0, z) \ge d(x_0, x) - d(x, z) > d(x_0, x) - s = r$
$\Rightarrow d(x_0, z) > r \Rightarrow z \notin S_r[x_0]$
$\Rightarrow z \in Y$. Hence $S_s(x) \subseteq Y$.
$\therefore$ for any $x \in Y, \exists\, s > 0 \ni x \in S_s(x) \subseteq Y$.
$\therefore$ Y is open. Hence $S_r[x_0]$ is closed.

Theorem: (i) Let X be a metric space. Then (i) any intersection of closed sets in X is closed; ie. If $\{F_\alpha / \alpha \in I\}$ is a collection of closed sets then $\cap F_\alpha$ is closed.
(ii) any finite union of closed sets in X is closed. Ie. For any finite collection $F_1, F_2, .., F_n$ of closed sets, $F_1 \cup F_2 \cup ... \cup F_n$ is closed.

Proof: (i) Let $\{F_\alpha / \alpha \in I\}$ be a collection of closed sets
Since each F_α is closed, we have that F_α^c is open. $\therefore$
$\{F_\alpha^c : \alpha \in I\}$ is a collection of open sets.
By a theorem, $\cup F_\alpha^c$ is open. $\Rightarrow (\cap F_\alpha)^c = \cup F_\alpha^c$ is open
$\Rightarrow \cap F_\alpha$ is closed.
(ii) Let F_i, $1 \le i \le n$, be closed sets $\Rightarrow F_i^c$ $1 \le i \le n$, are open sets.

Now $(F_1 \cup F_2 \cup \ldots \cup F_n)^c = F_1^c \cap F_2^c \cap \ldots \cap F_n^c$ is open
$\Rightarrow F_1 \cup F_2 \cup \ldots \cup F_n$ is closed.

Example: Consider the following sub sets of $\mathbb{R}^2$
(i) $\{z \in \mathbb{C} / |z| < 1\}$ is open, not closed, not perfect, bounded.
(ii) $\{z \in \mathbb{C} / |z| \leq 1\}$ is closed, not open, perfect and bounded.
(iii) A finite set is closed, not open, not perfect, bounded.
(iv) The set of all integers is closed, not open, not perfect and not bounded.
(v) $E = \{1/n : n \in \mathbb{N}\}$ is not closed, not open, not perfect but bounded.
Here note that this set has only limit point 0, and $0 \notin E$.
(vi) $\mathbb{C}$ (set of complex numbers) is closed, open, perfect but not bounded.
(vii) (a, b) as a subset of $\mathbb{R}^2$, is not closed, open, not perfect but bounded.

Note: (i) If $\{F_\alpha\}$ is a collection of sets then $\{(\cup F_\alpha)^c = \cap F_\alpha^c$.
(ii) An arbitrary union of closed sets need not be closed.
For, Consider $A_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$ for $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} A_n = (0, 1)$ which is not closed, because 0 and 1 are limit points of (0, 1) and these are not in (0, 1).

Theorem: Let E be a nonempty set of real numbers which is bounded above. Let y = sup E. Then (i) $y \in \bar{E}$ and (ii) $y \in E$ if E is closed.

Proof: (i) If $y \in E$, then clearly $y \in E \subseteq \bar{E}$.
Suppose $y \notin E$.
Now $y = \sup E \Rightarrow$ for any $\varepsilon > 0$, $y - \varepsilon$ is not an upper bound
$\Rightarrow \exists\, x \in E$ such that $y - \varepsilon < x < y$
$\Rightarrow x \in (y - \varepsilon, y + \varepsilon) = S_\varepsilon(y)$ and $x \in E \Rightarrow x \in \{E \cap S_\varepsilon(y)\} - \{y\}$
$\Rightarrow$ y is a limit point of $E \Rightarrow y \in D(E) \subseteq E \cup D(E) = \bar{E}$.
(ii) If E is closed then $E = \bar{E}$ and hence $y \in \bar{E} = E$.

Construction of the CANTOR set.

To construct the Cantor set, we proceed as follows:
Write $F_1 = [0, 1]$. From F_1, delete the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ which is an open middle third of F_1.
Write $F_2 = [0, 1] - \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$.
Now from F_2, delete the middle thirds of two pieces.

Write $F_3 = F_2 - \left\{\left(\frac{1}{9},\frac{2}{9}\right) \cup \left(\frac{7}{9},\frac{8}{9}\right)\right\} = \left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{3}{9}\right] \cup \left[\frac{6}{9},\frac{7}{9}\right] \cup \left[\frac{8}{9},1\right]$

If we continue this process of deleting the open middle third of intervals, we obtain a sequence of closed sets F_n such that $F_n \supseteq F_{n+1} \supseteq ....$

Now write $F = \bigcap_{n=1}^{\infty} F_n$. This F is called the Cantor set.

<u>Note</u>: (i) By above construction, since each F_n is a finite union of closed intervals, we have that each F_n is closed. So, $F = \bigcap_{n=1}^{\infty} F_n$ is closed. Hence Cantor's set is closed.

(ii) Since we are deleting the open middle third intervals from each F_n finally F contains the end points of the closed intervals of F_n for each n. The end points of the closed intervals in F_1 are 0, 1. The end points of the closed intervals in F_2 are 0, 1/3, 2/3 and 1. The end points of the closed interval in F_3 are 0, 1/9, 2/9, 6/9, 7/9, 8/9, 1. Therefore F contains 0, 1/3, 2/3, 1/9, 2/9, ...

Therefore, there are some numbers in F other than the end points.

(iii) The cardinal number 0f F is c, the cardinal number of the continuum.

(iv) We can define a bijection f: [0, l) → F. For this, let $x \in [0, 1)$.

Suppose $x = 0.b_1b_2...$ be its binary expansion. Now each bn is either 0 or 1. Write $t_n = 2b_n$ for each n, and write $f(x) = 0.t_1t_2...$ Now consider $f(x) = 0.\ t_1t_2...$ .as a number of ternary expansion.

Now $f(x) \in F$. Now it can be verified that f is one to one and onto.

(v) Let us consider the sum of lengths of the open intervals removed at every stage. First stage we removed the open interval (1/3, 2/3) and its length is 1/3. Second stage we removed (1/9, 2/9) and (7/9, 8/9). The sum of the length of these two intervals is 1/9 + 1/9 = 2/9 and so continuing this way we obtain a sequence of lengths 1/3, 2/9, 4/27, ... These numbers form a geometric progression with first term 1/3 and common ratio 2/3.

Therefore, the sum is $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$

<u>Definition</u>: Let X be a metric space and $A \subseteq X$. Then the closure of A (denoted by $\bar{A}$) is defined by $\bar{A} = A \cup D(A)$ where D(A) is the set of all limit points of A.

<u>Result:</u> A is closed if and only if $A = \bar{A}$

<u>Proof:</u> (i) Suppose A is closed

$\Rightarrow$ all the limit points of A are in A $\Rightarrow D(A) \subseteq A$

$\Rightarrow \bar{A} = A \cup D(A) \subseteq A \Rightarrow \bar{A} \subseteq A$. Hence $A = \bar{A}$

Converse: Suppose $A = \bar{A}$

$\Rightarrow A \cup D(A) \subseteq A \Rightarrow D(A) \subseteq A$
$\Rightarrow$ all the limit points of A are in A $\Rightarrow$ A is closed.

Result: $\bar{A}$ is a closed superset of A which is contained in any closed superset of A (equivalently, (i) $A \subseteq \bar{A}$ (ii) $\bar{A} = \bar{\bar{A}}$; and (iii) B is a closed set such that $A \subseteq B$ then $\bar{A} \subseteq B$ (iv) $\bar{A}$ equals to the intersection of all closed supersets of A.
Proof: (i) By the definition of $\bar{A}$, we have that $A \subseteq \bar{A}$.
(ii) To show that $\bar{A} = \bar{\bar{A}}$; Clearly $\bar{A} \subseteq \bar{\bar{A}}$. Let $x \in \bar{\bar{A}}$.
Then either $x \in \bar{A}$ or $x \in D(\bar{A})$ = the set of all limit points of $\bar{A}$.
If $x \in \bar{A}$, it is clear. Suppose $x \in D(\bar{A}) \Rightarrow$ x is a limit point of $\bar{A}$.
If $x \in A$ then clearly $x \in A \subseteq \bar{A}$.
Suppose $x \notin A$, Consider $S_r(x)$ and $r > 0$. Since it is a limit point of $\bar{A}$ there exists y $\in \bar{A} \cap S_r(x)$ such that $x \neq y$. $y \in S_r(x) \Rightarrow d(x, y) < r$.
Now $y \in \bar{A} = A \cup D(A)$. If $y \in A$ then $y \in A \cap S_r(x)$. If $y \notin A$ then $y \in D(A) \Rightarrow$ y is a limit point of A. Put $s = r - d(x, y)$. Then $s > 0$ and $\exists\, z \in A \cap S_s(y)$ and $z \neq y$.
Now $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + s = r$.
$\therefore$ either y or z is in A and also is in $S_r(x)$.
$\therefore$ x is a limit point of A which implies that $x \in D(A) \subseteq A \cup D(A) = \bar{A}$.
$\therefore \bar{\bar{A}} \subseteq \bar{A}$. . Hence $\bar{A} = \bar{\bar{A}}$.
(iii) Let B be a closed set such that $A \subseteq B$.
Now we wish to show that $\bar{A} \subseteq B$.
For this, let $x \in \bar{A} \Rightarrow x \in A$ or $x \in D(A)$ (since $A = A \cup D(A)$].
If $x \in A$ then $x \in B$ (since $A \subseteq B$).
If $x \in D(A)$, then since $D(A) \subseteq D(B)$ we have $x \in D(B)$
$\Rightarrow x \in B \cup D(B) = \bar{B} = B$ (since B is closed).
Hence $x \in \bar{A} \Rightarrow x \in B$. This shows that $\bar{A} \subseteq B$.
(iv) Let $\{B_i \,/\, i \in I\}$ is the collection of all closed supersets of A.
By (iii), $\bar{A} \subseteq B$ for all $i \in I \Rightarrow \bar{A} \subseteq \cap B_i$
Since $\bar{A}$ is closed and $A \subseteq \bar{A}$, we have that $\bar{A}$ belongs to the collection $\{B_i \,/\, i \in I\}$
$\Rightarrow \cap B_i \subseteq \bar{A}$.
Hence $\bar{A} = \cap B_i$.

Definition: Let X be a metric space, $A \subseteq X$. $x \in X$ is said to be a ***boundary point*** of A if each open sphere centred on the point x intersects both A and A′. The set of all boundary points of A is called the ***boundary*** of A.

Note: (i) The boundary of A equals
(ii) The boundary of A is a closed set. (iii) A is closed iff it contains its boundary.

Example: Consider R with usual metric. Write $x = 0 \in R$, $A = (0, l)$, $B = [0, l]$. x is a boundary point of both A and B. $x \notin A$ and $x \in B$. Therefore, a boundary point x of a set X need not be in the set X.

Result: Let $x \notin A$. Then x is a limit point of A iff x is a boundary point of A.
Proof: Suppose $x \notin A$ and x is a limit point of A. $\Rightarrow x \in D(A)$ and $x \in A'$
$\Rightarrow$ for every $r > 0$, the nbd $S_r(x)$ intersects both A and A'.
$\Rightarrow$ x is a boundary point of A.
Conversely, suppose x is a boundary point of A.
$\Rightarrow S_r(x)$ intersects A for every $r > 0$. Also given $x \notin A$. $\therefore$ x is a limit point of A.

CONVERGENCE, COMPLETENESS AND BAIRE'S THEOREM

Definition: Let (X, d) be a metric space and $\{x_n\}$ be a sequence of points in X. Then $\{x_n\}$ ***converges*** if $\exists$ a point $x \in X \ni$ for each $\varepsilon > 0$ $\exists$ a positive integer m $\ni d(x_n, x) < \varepsilon\ \forall\ n \geq m$. This fact is denoted by $x_n \to x$ or $\lim x_n = x$.
or equivalently, for each open sphere $S_\varepsilon(x)$ $\exists\ m \in \mathbb{Z}^+ \ni x_n \in S_\varepsilon(x)\ \forall\ n \geq m$.

Note: The following two conditions are equivalent: (i) $\{x_n\}$ converges to x in a metric space (X, d) and (ii) $\{d(x_n, x)\}$ converges to a real number 0.

Problem: Let X be a metric space. If $\{x_n\}$, $\{y_n\}$ are sequences in X $\ni x_n \to x$ and $y_n \to y$ then $d(x_n, y_n) \to d(x, y)$.
Solution: Let $\varepsilon > 0$. Since $x_n \to x$ $\exists\ k_1 \in \mathbb{Z}^+ \ni d(x_n, x) < \varepsilon/2\ \forall\ n \geq k_1$.
Since $y_n \to y$ $\exists\ k_2 \in \mathbb{Z}^+ \ni d(y_n, y) < \varepsilon/2\ \forall\ n \geq k_2$.
Now take $k = \max\{k_1, k_2\}$.
Then $d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$
$\Rightarrow d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y, y_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon\ \forall\ n \geq k$...(i)
Again $d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$
$\Rightarrow d(x, y) - d(x_n, y_n) \leq d(x, x_n) + d(y_n, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon\ \forall\ n \geq k$...(ii)
From (i) and (ii) $|d(x_n, y_n) - d(x, y)| < \varepsilon\ \forall\ n \geq k$. Hence $d(x_n, y_n) \to d(x, y)$.

Definition: A sequence $\{x_n\}$ of points in a metric space (X, d) is said to be a ***Cauchy sequence*** if for each $\varepsilon > 0$ $\exists\ k \in \mathbb{Z}^+ \ni d(x_n, x_m) < \varepsilon\ \forall\ n, m \geq k$.

Note: (i) Every convergent sequence is a Cauchy sequence (ii) Is the converse true? Justify your answer.

Proof: (i) Let the sequence $\{x_n\}$ converge to x.

Let $\varepsilon > 0$. Then corresponding to $\varepsilon/2 > 0 \; \exists \; k \in \mathbb{Z}^+ \ni \; d(x_n, x) < \varepsilon/2 \; \forall \; n \geq k$.

Take $n, m \geq k$. Now $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

(ii) The converse is not true.

Write $X = (0, 1]$. Consider the usual metric of real numbers on X. Then (X, d) is a metric space. Write $x_n = 1/n$ for each $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.

For, let $\varepsilon > 0$. Take $k \in \mathbb{Z}^+ \ni k > 1/\varepsilon$. Let $n \geq m \geq k$.

Then $|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \frac{1}{m} - \frac{1}{n} < \frac{1}{m} < \frac{1}{k} < \varepsilon$.

The sequence $\{x_n\} = \left\{\frac{1}{n}\right\} \to 0$ but $0 \notin X$.

Hence $\{x_n\}$ is not a convergent sequence in X.

Note: Let (X, d) be a metric space and $\{x_n\}$ be a sequence in $X \ni x_n \to x$ and $x_n \to x'$ in X. Then $x = x'$.

Definition: A metric space X is said to be complete if every Cauchy sequence in X is convergent.

Theorem: If a convergent sequence in a metric space has infinitely many distinct points then its limit is a limit point of the set of points of the sequence.

Proof: Let X be a metric space and $\{x_n\}$ be a convergent sequence in X. Suppose $x \in X \ni x_n \to x$. Write $A = \{x_n / n \geq 1\}$. Then A is an infinite set. If possible suppose x is ant a limit point of A.

Then $\exists \; r > 0 \ni S_r(x) \cap A \setminus \{x\} = \phi$.

$\Rightarrow S_r(x) \cap A = \phi$ or $S_r(x) \cap A = \{x\}$. $\Rightarrow S_r(x) \cap A \subseteq \{x\}$.

Since $r > 0$ and $x_n \to x$, $\exists \; k \in \mathbb{Z}^+ \ni d(x_n, x) < r \; \forall \; n \geq k$.

$\Rightarrow x_n \in S_r(x) \Rightarrow x_n \in S_r(x) \cap A \Rightarrow x_n \in S_r(x) \cap A \subseteq \{x\} \Rightarrow x_n = x \; \forall \; n \geq k$.

$\therefore A = \{x_1, x_2, ..., x_{k-1}, x\}$.

$\Rightarrow$ A is finite which is a contradiction. Hence X is a limit point of A.

Theorem: Let X be a complete metric space and Y be a subspace of X. Then Y is complete iff Y is closed.

Proof: Let Y be complete. Let $y \in X$ be a limit point of Y.

$\therefore$ For each $n \in \mathbb{N}$, $\exists \; y_n \in S_{\frac{1}{n}}(y) \cap Y \setminus \{y\}$.

Claim: $\{y_n\} \to y$. Let $\varepsilon > 0$. Take $k \in \mathbb{Z}^+ \ni k > 1/\varepsilon$. Let $n \geq k$.

Then $d(y_n, y) < 1/n < 1/k < \varepsilon$. $\therefore y_n \to y$. $\therefore \{y_n\}$ is a Cauchy sequence in Y. Since Y

is complete, $y_n \to y'$ for some y' in Y.
Since $y_n \to y$ and $y_n \to y'$, we have $y = y' \in Y$. Hence Y is closed.

Converse: Suppose Y is closed. Let $\{y_n\}$ be a Cauchy sequence in Y. Since Y $\subseteq$ X, $\{y_n\}$ is a Cauchy sequence in X.
Since X is complete, $\exists\, y \in X \ni y_n \to y$.
Case (i) If the sequence $\{y_n\}$ contains only a finite number of elements then y is a member of the sequence which repeats infinite number of times.
So, $y = y_m$ for some m and hence $y = y_m \in Y$.
Case (ii): Suppose $\{y_n\}$ contains infinite number of distinct elements.
Then y is a limit point of $\{y_n / n \geq 1\}$. Since $\{y_n / n \geq 1\} \subseteq Y$, y is a limit point of Y.
Since Y is closed, $y \in Y$. Hence $\{y_n\} \to y$ for some $y \in Y$.
Hence Y is complete.

Definition: A sequence $\{A_n\}$ of subsets of a metric space is called a decreasing sequence if $A_1 \supseteq A_2 \supseteq \ldots\ldots \supseteq A_n \supseteq A_{n+1} \supseteq \ldots$

Cantor Intersection Theorem: Let X be a complete metric space, and $\{F_n\}$ be a decreasing sequence of non – empty closed subsets of X such that $d(F_n) \to 0$. Then $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.
Proof:
Claim: F can not contain more than one element.
If possible suppose $x, y \in F \ni x \neq y$. $\therefore\ \varepsilon = d(x, y) > 0$.
Since $d(F_n) \to 0$, $\exists\, k \in \mathbb{Z}^+ \ni d(F_n) < \varepsilon\ \forall\, n \geq k$.
Since $x, y \in F \subseteq F_n$, $\varepsilon = d(x, y) < d(F_n) < \varepsilon$, a contradiction.
Hence F can not contain more than one element.
Claim: F contains at least one point.
Choose x_n in $F_n\ \forall\, n \geq 1$. Now we show that $\{x_n\}$ is a Cauchy sequence.
Let $\varepsilon > 0$. Since $d(F_n) \to 0$, $\exists\, k \in \mathbb{Z}^+ \ni d(F_n) < \varepsilon\ \forall\, n \geq k$. Now take $m, n \geq k$.
W.L.G. we may assume that $m \geq n$. Now $x_m \in F_m \subseteq F_k$; $x_n \in F_n \subseteq F_k$.
Hence $x_m, x_n \in F_k$ and so $d(x_m, x_n) \leq d(F_k) < \varepsilon$. $\therefore$ $\{x_n\}$ is a Cauchy sequence.
Since X is complete, $\exists\, x \in X \ni x_n \to x$.
Case (i): Suppose $\{x_n: n \geq 1\}$ contains only a finite number of elements. Then x repeats in the sequence on and after certain stage. Ie. $\exists\, k \in \mathbb{Z}^+ \ni x_n = x\ \forall\, n \geq k$.
Since $F_1 \supseteq F_2 \supseteq \ldots$, $x \in F_n\ \forall\, n$. $\therefore$ $x \in \bigcap_{n=1}^{\infty} F_n$.
Case (ii): Suppose $\{x_n: n \geq 1\}$ contains infinite number of elements. Then x is a limit point of the set $\{x_n / n \geq 1\}$. Clearly x is a limit point of $\{x_n / n \geq k\}\ \forall\, k$. $\Rightarrow$ x is a limit point of $\{x_n / n \geq k\} \subseteq F_k$. $\Rightarrow x \in F_k$ since each F_k is closed. This is true for all k. $\therefore$ $x \in \bigcap_{n=1}^{\infty} F_n$.

Definition: Let (X, d) be a metric space and $A \subseteq X$. A is said to be nowhere dense if Int $(\bar{A}) = \phi$.

Result: Let X be a metric space and $A \subseteq X$. Then the following are equivalent.

(i) A is a nowhere dense set.
(ii) A does not contain any non – empty open set.
(iii) Each non – empty open set has a non – empty open subset disjoint from $\bar{A}$.
(iv) Each non – empty open set has a non – empty open subset disjoint from A.
(v) Each non – empty open set contains a open sphere disjoint from A.

Proof: (i) $\Rightarrow$ (ii). Suppose A is nowhere dense. $\Rightarrow$ Int $(\bar{A}) = \phi$.
If A contains a non – empty open set G then $\phi \neq G \subseteq$ Int (A) $\subseteq$ Int $(\bar{A}) = \phi$, a contradiction.

(ii) $\Rightarrow$ (iii). Let G be a non – empty open subset.
By (ii) $G \nsubseteq \bar{A} \Rightarrow \exists\, x \in G \setminus \bar{A} \Rightarrow x \in G \cap (\bar{A})'$ Put $H = G \cap (\bar{A})'$
Since $\bar{A}$ is closed $(\bar{A})'$ is open and hence $H = G \cap (\bar{A})'$ is a open set and $x \in H$.
$\therefore$ G contains a non – empty open set such that $H \cap \bar{A} = \phi$.

(iii) $\Rightarrow$ (iv). Let G be a non – empty open set.
By (iii) $\exists$ a non – empty open subset H of G $\ni H \cap \bar{A} = \phi$.
Now $H \cap A \subseteq H \cap \bar{A} = \phi \Rightarrow H \cap A = \phi$.

(iv) $\Rightarrow$ (v). Let G be a non – empty open set.
By (iv) $\exists$ a non – empty open subset H of G with $H \cap A = \phi$.
Let $x \in H$. Since H is open $\exists\, r > 0$ such that $S_r(x) \subseteq H$.
Now $S_r(x) \cap A \subseteq H \cap A = \phi \Rightarrow S_r(x) \cap A = \phi$.
Hence G contains a non – empty open sphere $S_r(x)$ such that $S_r(x) \cap A = \phi$.

(v) $\Rightarrow$ (1): Suppose each non – empty open set contain a open sphere disjoint from A. If possible suppose Int $(\bar{A}) \neq \phi$. Write G = Int $(\bar{A})$.
By (v) $\exists\, x \in X$, $r > 0$ such that $S_r(x) \subseteq G$ and $S_r(x) \cap A = \phi$.
Now $S_r(x) \cap A = \phi \Rightarrow x \notin A$ and x is not a limit point of A. $\Rightarrow x \notin \bar{A}$.
On the other hand, $x \in S_r(x) \subseteq G \subseteq$ Int $(\bar{A}) \subseteq \bar{A}$ a contradiction.
Hence Int $(\bar{A}) = \phi$.

Problem: Show that a closed set A is nowhere dense iff its complement is everywhere dense.

Proof: Suppose A is closed and nowhere dense.
Since A is nowhere dense Int $(\bar{A}) = \phi$.

$\Rightarrow$ Int (A) = ϕ since A is closed.
Let U be any open set with U $\cap$ A′ = ϕ $\Rightarrow$ U $\subseteq$ A.
$\Rightarrow$ U $\subseteq$ Int (A) since U is open $\Rightarrow$ U = ϕ.
Ie. the only open set disjoint from A′ is ϕ. Hence $\bar{A}'$ = X.
Conversely suppose A′ is dense.
Int $(\bar{A})$ = Int (A) $\subseteq$ A $\Rightarrow$ (Int $(\bar{A})$) $\cap$ A′ = ϕ
$\Rightarrow$ Int $(\bar{A})$ = ϕ since the only open set disjoint from A′ is ϕ.
Hence A is nowhere dense.

Baire's Theorem: If $\{A_n\}$ is a sequence of nowhere dense sets in a complete metric space X, then there exists a point in X which is not in any of the A_n's.
Proof: Since X is a non – empty open set and A_1 is a nowhere dense set, $\exists$ an open sphere $S_r(x) \ni S_r(x) \cap A_1 = \phi$.
Let $0 < t_1 < 1$. Let $r_1 = \min\{r, t_1\}$.
Clearly $r_1 < 1$. Since $S_{r_1}(x) \subseteq S_r(x)$ we have $S_{r_1}(x) \cap A_1 = \phi$.
Put $G_1 = S_{r_1}(x)$. Define $F_1 = S_{r_1/2}[x]$. Clearly $d(F_1) < 1$.
Now F_1 is closed, G_1 is open and $F_1 \subseteq G_1$.
Also Int $(F_1) = S_{\frac{r_1}{2}}(x)$ is open and A_2 is nowhere dense, there exists an open sphere $G_2 \subseteq$ Int (F_1) and $G_2 \cap A_2 = \phi$.
Suppose $G_2 = S_{r_2}(x_1)$. Define $F_2 = S_{r_2/2}[x_1]$. Clearly $d(F_2) < 1/2$.
Now F_2 is closed, G_2 is open and $F_2 \subseteq G_2$.
Also, Int $(F_2) = S_{\frac{r_2}{2}}(x_1)$ is open and A_3 is nowhere dense, $\exists$ an open sphere $G_3 \subseteq$ Int (F_2) and $G_3 \cap A_3 = \phi$.
If we continue this process, we get $G_1 \supseteq F_1 \supseteq G_2 \supseteq F_2 \supseteq G_3 \supseteq \ldots \ni d(F_n) \to 0$, F_n is closed, G_n is open, $G_n \cap A_n = \phi$.
Since $d(F_n) \to 0$ and each F_n is closed, by Cantor's intersection theorem, we have that $\bigcap_{n=1}^{\infty} F_n \neq \phi$.
Let $a \in \bigcap_{n=1}^{\infty} F_n$. Since $a \in F_n$ for each n, $F_n \subseteq G_n$, and $G_n \cap A_n = \phi$, we have that $a \notin A_n$ for any n.
Hence $a \in X$ and $a \notin A_n$ for all n.

Theorem: If a complete metric space is the union of a sequence of its subsets then the closure of at least one set in the sequence must have non – empty interior.
Proof: Let X be a complete metric space and $X = \bigcup_{i=1}^{\infty} A_i$. If possible, suppose that Int $(\bar{A}_i) = \phi\ \forall$ i. Each A_i is a nowhere dense. So $\{A_n\}$ is a sequence of nowhere dense sets. By a Baire's theorem, $\exists\ a \in X \ni a \notin \bigcup_{i=1}^{\infty} A_i$, a contradiction to the fact that $X = \bigcup_{i=1}^{\infty} A_i$. Hence Int $(A_i) \neq \phi$ for some i.

Note: A subset A of a metric space X is said to be of first category if it can be represented as the union of sequence of nowhere dense sets. A is said to be second category if it is not first category. Every complete metric space is second category.

CONTINUOUS MAPPINGS

Definition: Let X and Y be metric spaces with metrics d_1 and d_2. Let f be a mapping of X into Y. F is said to be continuous at a point x_0 in X if either of the following two conditions is satisfied.

(i) for each $\varepsilon > 0$, $\exists\ \delta > 0 \ni d_1(x, x_0) < \delta \Rightarrow d_2(f(x), f(x_0)) < \varepsilon$.

(ii) for each open sphere $S_\varepsilon(f(x_0))$ centered on $f(x_0)$, $\exists$ an open sphere $S_\delta(x_0)$ centred on $x_0 \ni f(S_\delta(x_0)) \subseteq S_\varepsilon(f(x_0))$.

Theorem: Let X and Y be metric spaces and f is a mapping of X into Y.
Then f is continuous at x_0 if and only if $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$.

Proof: Suppose f is continuous at x_0. Let $\{x_n\}$ be a sequence in X $\ni x_n \to x_0$.
Let $S_\varepsilon(f(x_0)$ be an open sphere centred at $f(x_0)$.
Since f is continuous at x_0, $\exists$ an open sphere $S_\delta(x_0) \ni f(S_\delta(x_0)) \subseteq S_\varepsilon(f(x_0))$.
Since $x_n \to x_0$, $\exists\ k \in \mathbb{Z}^+ \ni x_n \in S_\delta(x_0)\ \forall\ n \geq k$.
Then $x_n \in S_\delta(x_0) \Rightarrow f(x_n) \in f(S_\delta(x_0)) \subseteq S_\varepsilon(f(x_0))\ \forall\ n \geq k$.
$\therefore\ f(x_n) \to f(x_0)$.

Converse: Suppose $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$.
If possible, suppose that f is not continuous at x_0.
Then $\exists\ \varepsilon > 0 \ni S_\varepsilon(f(x_0))$ does not contain $f(S_\delta(x_0))$ for any $\delta > 0$.
For $n \in \mathbb{N}$, $\frac{1}{n} > 0 \Rightarrow f\left(S_{\frac{1}{n}}(x_0)\right) \nsubseteq S_\varepsilon(f(x_0))$.
Take $x_n \in S_{\frac{1}{n}}(x_0) \ni f(x_n) \notin S_\varepsilon(f(x_0))$.
Now $\{x_n\}$ is a sequence of points from X and $x_n \to x_0$.
Since $x_n \to x_0$, we have $f(x_n) \to f(x_0)$ by hypothesis.
Since $\varepsilon > 0$, $\exists\ k \in \mathbb{Z}^+ \ni d_Y(f(x_n), f(x_0)) < \varepsilon\ \forall\ n \geq k$.
$\Rightarrow f(x_n) \in S_\varepsilon(f(x_0))\ \forall\ n \geq k$, a contradiction. Hence f is continuous.

Definition: Let X and Y be metric spaces. A mapping f: $X \to Y$ is said to be continuous if f is continuous at every point of X.

Theorem: Let X and Y be metric spaces and f a mapping of X into Y. Then f is continuous if and only if $x_n \to x \Rightarrow f(x_n) \to f(x)$.
Proof: f is continuous iff f is continuous at x $\forall$ x $\in$ X iff $x_n \to x \Rightarrow f(x_n) \to f(x)$ $\forall$ x $\in$ X. (by above theorem).

Theorem: Let X and Y be metric spaces and f is a mapping of X into Y. Then f is continuous iff $f^{-1}(G)$ is open in X whenever G is open in Y.
Proof: Suppose f is continuous. Let G be an open set in Y.
Let $p \in f^{-1}(G) \Rightarrow f(p) \in G$.
Since G is open $\exists$ $\varepsilon > 0$, $S_\varepsilon(f(p)) \subseteq G$.
Since f is continuous $\exists$ $\delta > 0$, $\ni f(S_\delta(p)) \subseteq G$
$\Rightarrow S_\delta(p) \subseteq f^{-1}(G)$.
$\Rightarrow$ p is an interior point of $f^{-1}(G)$.
$\therefore$ every point of $f^{-1}(G)$ is an interior point. Hence $f^{-1}(G)$ is open.
Converse: Suppose $f^{-1}(G)$ is open for all open sets G in Y. Let p $\in$ X.
Let $\varepsilon > 0$. Since $S_\varepsilon(f(p))$ is open in Y, $f^{-1}\big(S_\varepsilon f(p)\big)$ is open in X.
Since $p \in f^{-1}\big(S_\varepsilon f(p)\big)$ $\exists$ $\delta > 0 \ni S_\delta(p) \subseteq f^{-1}\big(S_\varepsilon f(p)\big) \Rightarrow f(S_\delta(p)) \subseteq S_\varepsilon f(p)$.
This shows that f is continuous at p. Since p is an arbitrary point in X, f is continuous on X.

Problem: Let X and Y be metric spaces and $\phi \neq A \subseteq X$. If f, g are continuous mappings from X to Y $\ni$ f(x) = g(x) $\forall$ x $\in$ A then f(y) = g(y) $\forall$ y $\in \bar{A}$.
Solution: Let y $\in \bar{A}$. If y $\in$ A then g(y) = f(y).
If y $\notin$ A, then since $A \neq X$, $y \in \bar{A} \Rightarrow$ y is a limit point of A.
Let $y_n \in A \cap S_{\frac{1}{n}}(y) \setminus \{y\}$. Consider $\{y_n\}$. Since for each n, $y_n \in S_{\frac{1}{n}}(y)$, $y_n \to y$.
Since f is continuous, $f(y_n) \to f(y)$. Since g is continuous, $g(y_n) \to g(y)$.
Hence $f(y) = \lim f(y_n) = \lim g(y_n) = g(y)$.

Definition: Let f be a mapping from metric space (X, d_1) to a metric space (Y, d_2). Then f is said to be uniformly continuous on X if given $\varepsilon > 0$, $\exists$ $\delta > 0 \ni$ $d_1(x, x') < \delta \Rightarrow d_2(f(x), f(x')) < \varepsilon$.

Theorem: Let X be a metric space, Y be a complete metric space and let A be a dense subspace of X. If f is uniformly continuous mapping of A into Y, then f can be extended uniquely to uniformly continuous mapping g: X $\to$ Y.

Proof: If A = X, then the conclusion is obvious. Assume that A $\neq$ X.
Then $\exists$ point in X which is not in A.

Define $g : X \to Y$ as follows. If $x \in A$ then $g(x) = f(x)$.
If $x \notin A$, then since $A \neq X$, $x \in \bar{A} \Rightarrow x$ is a limit point of some convergent sequence $\{x_n\}$ in $A \Rightarrow \{x_n\}$ is a Cauchy sequence in $X \Rightarrow \{f(x_n)\}$ is a Cauchy sequence in Y since f is uniformly continuous. Since Y is complete $\{f(x_n)]$ is convergent sequence in Y. Define $g(x) = \lim f(x_n)$.

Claim: g is well defined. Let $\{x_n\}$, $\{y_n\}$ be sequences in $A \ni x_n \to x$, $y_n \to x$. We know that $d_1(x_n, y_n) \to d_1(x, x) = 0$. Since f is uniformly continuous, $d_2(f(x_n), f(y_n)) \to 0$. Since $d_2(f(x_n), f(y_n)) \to d_2(\lim f(x_n), \lim f(y_n)) = 0$, we have $\lim f(x_n) = \lim f(y_n)$. Hence g is well defined.

Claim: g is uniformly continuous.
Let $\varepsilon > 0$. Since f is uniformly continuous on A, $\exists\ \delta > 0 \ni d_1(a, a') < \delta$
$\Rightarrow d_2(f(a), f(a')) < \varepsilon\ \forall\ a, a' \in A$... (i).
Let $x, x' \in X$ with $d_1(x, x') < \delta/3$.
We show that $d_2(g(x), g(x')) < 3\varepsilon$.
If $x, x' \in A$, then clearly $d_1(x, x') < \delta \Rightarrow d_2(f(x), f(x')) < \varepsilon \Rightarrow d_2(g(x), g(x')) < \varepsilon$.
Suppose $x, x' \notin A$. Then $x, x' \in \bar{A}$.
$\Rightarrow \exists$ sequences $\{x_n\}$, $\{x_n'\}$ in $A \ni x_n \to x$ and $x_n' \to x'$.
Since f is uniformly continuous on A, it follows that $g(x) = \lim f(x_n)$ and
$g(x') = \lim f(x_n')$.
Since $x_n \to x$, $x_n' \to x'$, $\exists\ k \ni \forall\ n \geq k$, $d_1(x_n, x) < \delta/3$ and $d_1(x_n', x') < \delta/3$.
Now $d_1(x_n, x_n') \leq d_1(x_n, x) + d_1(x, x') + d_1(x', x_n') < \delta/3 + \delta/3 + \delta/3 = \delta$.
$\Rightarrow d_2(f(x_n), f(x_n')) < \varepsilon$ by (i) $\forall\ n \geq k$.
Now $d_2(g(x), g(x')) \leq d_2(g(x), f(x_n)) + d_2(f(x_n), f(x_n')) + d_2(f(x_n'), g(x')) < 3\varepsilon$ for sufficiently large n. Hence $d_1(x, x') < \delta \Rightarrow d_2(g(x), g(x')) < 3\varepsilon$.
This is true for all $\varepsilon > 0$. Hence g is uniformly continuous.

Claim: g is unique.
Let g_1, g_2 be two extensions of f.
If possible, suppose $g_1 \neq g_2$.
Since $g_1(a) = f(a) = g_2(a)\ \forall\ a \in A$. $g_1(x) \neq g_2(x)$ for some $x \in X \backslash A$.
Since $x \in X = \bar{A}$, $\exists$ sequence $\{x_n\}$ in $A \ni x_n \to x$.
Let S_1 and S_2 be two disjoint spheres with the centers $g_1(x)$ and $g_2(x)$ respectively.
Since g_1 and g_2 are uniformly continuous, they are continuous.
$\therefore\ g_1^{-1}(S_1)$ and $g_2^{-1}(S_2)$ are open sets in $X \ni$ that $x \in g_1^{-1}(S_1) \cap g_2^{-1}(S_2)$.
$\therefore\ g_1^{-1}(S_1) \cap g_2^{-1}(S_2)$ is open and $x_n \to x\ \exists\ k \ni x_n \in g_1^{-1}(S_1) \cap g_2^{-1}(S_2)\ \forall n \geq k$.

Since $x_n \in A$, $g_1(x_n) = f(x_n) = g_2(x_n) \in S_1 \cap S_2$, a contradiction as $S_1 \cap S_2 = \phi$.
Hence $g_1 = g_2$.

Definition: Let (X, d_1), (Y, d_2) be two metric spaces and $f: X \to Y$ a bijection.
f is said to be an isometry if for any $x, x' \in X$, $d_2(f(x), f(x')) = d_1(x, x')$.

SPACES OF CONTINUOUS FUNCTIONS

A normed linear space X is a linear space in which there is defined a real number $\|x\|$ for every element x satisfying, (i) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$ (ii) $\|x + y\| \leq \|x\| + \|y\|$, (iii) $\|ax\| = |a|\|x\|$ scalar a and $x, y \in X$.

Definition: A Banach space is a normed linear space which is complete as a metric space.

Lemma: If f and g are continuous real functions defined on a metric space (X, d) then f + g and af are also continuous, where a is any real number.

Proof: Let $\varepsilon > 0$. Take $x_0 \in X$. Since $\varepsilon/2 > 0$ and f is continuous, $\exists\ \delta_1 > 0 \ni$
$x \in X$, $d(x, x_0) < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon/2$.
Since g is continuous $\exists\ \delta_2 > 0 \ni x \in X$, $d(x, x_0) < \delta_2 \Rightarrow |g(x) - g(x_0)| < \varepsilon/2$.
Take $\delta = \min\{\delta_1, \delta_2\}$. Now $d(x, x_0) < \delta \Rightarrow |(f + g)(x) - (f + g)(x_0)| \leq |f(x) - f(x_0)| + |g(x) - g(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\therefore$ f + g is continuous.

Let $\varepsilon > 0$. Corresponding to $\varepsilon' = \varepsilon / |a| > 0$, since f is continuous, $\exists\ \delta > 0 \ni$
$x \in X$, $d(x, x_0) < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon'$.

Now let $x \in X$, $d(x, x_0) < \delta$
Then $|af(x) - af(x_0)| = |a||f(x) - f(x_0)| < |a|\,\varepsilon' = \varepsilon$.
Hence af is continuous.

Note: Consider a non – empty set X. Write $L = \{f \,/\, f: X \to \mathbb{R}\}$.

Define $(f + g)(x) = f(x) + g(x)$ for $f, g \in L$ and for any a in $\mathbb{R}$, $f \in L$,
$(af)(x) = a\{f(x)\}$. With these operations L is a linear space over $\mathbb{R}$.
Let $B = \{f \,/\, f : X \to \mathbb{R}$, f is bounded$\}$. Then B is a linear subspace of L.
Write $C(X, \mathbb{R}) = \{f \,/\, f : X \to \mathbb{R}$, f is continuous and bounded$\}$ where (X, d) is a metric space. Clearly $C(X, \mathbb{R}) \subseteq B$.

Lemma: C(X, ℝ) is a closed subset of the metric space B.

Proof: Clearly C(X, ℝ) ⊆ B. Let f ∈ $\overline{C(X,\mathbb{R})}$.

Let ε > 0 and x_0 ∈ X. Let d be the metric on X.

Since ε/3 > 0 and f ∈ $\overline{C(X,\mathbb{R})}$ ∃ f_0 ∈ C(X, ℝ) ∋ $\|f - f_0\|$ < ε/3.

Now for any x ∈ X, $|f(x) - f_0(x)|$ ≤ sup {$|f(x) - f_0(x)|$ / x ∈ X} = $\|f - f_0\|$ < ε/3.

Since f_0 ∈ C(X, ℝ) it is continuous at x_0.

∴ ∃ δ > 0 ∋ d(x, x_0) < δ ⇒ $|f_0(x) - f_0(x_0)|$ < ε/3.

Now d(x, x_0) < δ ⇒ $|f(x) - f(x_0)| \le |f(x) - f_0(x)| + |f_0(x) - f_0(x_0)| + |f_0(x_0) - f(x_0)|$ < ε/3 + ε/3 +ε/3= ε. ∴ f is continuous at x_0.

Since x_0 is arbitrary we have that f ∈ C(X, ℝ). ∴ $\overline{C(X,\mathbb{R})}$ = C(X, ℝ).

Hence C(X, ℝ) is closed.

Theorem: The set C(X, ℝ) of all bounded and continuous real functions defined on a metric space X is a real Banach space with respct to pointwise addition and scalar multiplication, and the norm defined by $\|f\|$ = sup $|f(x)|$.

Proof: By a lemma f + g, af ∈ C(X, ℝ) for any f, g ∈ C(X, ℝ) and a ∈ℝ.

With respect to these operations C(X, ℝ) is a linear space.

Define $\|f\|$ = sup $|f(x)|$ for any f ∈ C(X, ℝ). This is a norm.

∴ C(X, ℝ) is a normed linear space.

If we define d(f, g) = $\|f - g\|$ then d is a metric on C(X, ℝ). With respect to this metric C(X, ℝ) is a closed subset of B. Since B is complete and C(X, ℝ) is closed subset of **B,** C(X, ℝ) is complete. Hence C(X, ℝ) is a Banach space.

Note: Let (X, d) be a metric space. Write C(X, ℂ)= {f: f : X→ ℂ, f is bounded and continuous}. Define $\|f\|$ = sup $|f(x)|$ for any f ∈ C(X, ℂ). Then C(X, ℂ) is a normed complex linear space.

Theorem: The set C(X, ℂ) of all bounded and continuous complex functions defined on a metric space X is a complex Banach space with respct to pointwise addition and scalar multiplication, and the norm defined by $\|f\|$ = sup $|f(x)|$.

EUCLIDEAN AND UNITARY SPACES.

Note: Let n be a fixed positive integer. Then $\mathbb{R}^n$ = {(x_1, x_2, ..., x_n)/ x_i ∈ ℝ, 1 ≤ i ≤ n}. Clearly $\mathbb{R}^n$ is a linear space over ℝ. For x = (x_1, x_2, ..., x_n) define Euclidean norm $\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$.

Lemma: (Cauchy Inequality) Let x = $(x_1, x_2, ..., x_n)$ and y = $(y_1, y_2, ..., y_n)$ be two n – tuples of real (or complex) numbers. Then $\sum_{i=1}^{n}|x_i y_i| \le \|x\|\|y\|$.

Ie $\sum_{i=1}^{n}|x_i y_i| \le (\sum_{i=1}^{n}|x_i|^2)^{\frac{1}{2}} (\sum_{i=1}^{n}|y_i|^2)^{\frac{1}{2}}$

Proof: Let a, b be any two non – negative real numbers.

Then $(a – b)^2 \ge 0 \Rightarrow a^2 + b^2 \ge 2ab$

$\Rightarrow (a + b)^2 \ge 4ab$

$\Rightarrow \frac{a+b}{2} \ge (ab)^{\frac{1}{2}}$...(i).

If x = 0 or y = 0 then $\sum_{i=1}^{n}|x_i y_i| = 0 = \|x\|\|y\|$.

Assume x ≠ 0 and y ≠ 0.

Take $a_i = \frac{|x_i|^2}{\|x\|^2}$ and $b_i = \frac{|y_i|^2}{\|y\|^2}$.

From (i) $\frac{|x_i||y_i|}{\|x\|\|y\|} \le \frac{\frac{|x_i|^2}{\|x\|^2}+\frac{|y_i|^2}{\|y\|^2}}{2}$ for $1 \le i \le n$.

Now summing $\sum_{i=1}^{n}\frac{|x_i||y_i|}{\|x\|\|y\|} \le \sum_{i=1}^{n}\frac{\frac{|x_i|^2}{\|x\|^2}+\frac{|y_i|^2}{\|y\|^2}}{2}$

$= \frac{\sum_{i=1}^{n}\frac{|x_i|^2}{\|x\|^2}+\sum_{i=1}^{n}\frac{|y_i|^2}{\|y\|^2}}{2} = \frac{\frac{\sum_{i=1}^{n}|x_i|^2}{\|x\|^2}+\frac{\sum_{i=1}^{n}|y_i|^2}{\|y\|^2}}{2} = \frac{\frac{\|x\|^2}{\|x\|^2}+\frac{\|y\|^2}{\|y\|^2}}{2} = \frac{1+1}{2} = 1.$

$\therefore \frac{\sum_{i=1}^{n}|x_i y_i|}{\|x\|\|y\|} \le 1 \Rightarrow \sum_{i=1}^{n}|x_i y_i| \le \|x\|\|y\|$

Lemma: Minkowski's inequality. Let x = $(x_1, x_2, ..., x_n)$ and y = $(y_1, y_2, ..., y_n)$ be two n – tuples of real (or complex) numbers. Then $\|x + y\| \le \|x\| + \|y\|$. Or in other words $(\sum_{i=1}^{n}|x_i + y_i|^2)^{1/2} \le (\sum_{i=1}^{n}|x_i|^2)^{1/2} + (\sum_{i=1}^{n}|y_i|^2)^{1/2}$.

Proof: If $\|x + y\| = 0$ then clearly $\|x + y\| \le \|x\| + \|y\|$.

Suppose $\|x + y\| \ne 0$.

Then $\|x + y\|^2 = \sum_{i=1}^{n}|x_i + y_i|^2$

$= \sum_{i=1}^{n}|x_i + y_i||x_i + y_i|$

$\le \sum_{i=1}^{n}|x_i + y_i|(|x_i| + |y_i|)$ since $|x_i + y_i| \le |x_i| + |y_i|$

$= \sum_{i=1}^{n}|x_i + y_i||x_i| + \sum_{i=1}^{n}|x_i + y_i||y_i|$

$\le \|x + y\|\|x\| + \|x + y\|\|y\|$ by Cauchy's inequality.

$= \|x + y\|(\|x\| + \|y\|)$.

ie. $\|x + y\|^2 \le \|x + y\|(\|x\| + \|y\|)$.

Hence $\|x + y\| \le \|x\| + \|y\|$ since $\|x + y\| \ne 0$.

Problem: Show that Int F = ϕ where F is the Cantor's set.

Fax: 08816-227318 off: 08816-224072, 224119,228342 Mobile: 9491334119
DANTULURI NARAYANA RAJU COLLEGE
(Autonomous)
BHIMAVARAM, W.G.DIST, ANDHRA PRADESH, INDIA, PIN- 534202.
(Accredited at 'B++' level by NAAC)
(Affiliated to Adikavi Nannaya University, Rajamahendravaram)

E – CONTENT

PAPER: M 104, TOPOLOGY

M. Sc. I YEAR, SEMESTER - I

UNIT – II: TOPOLOGICAL SPACES

PREPARED BY

K, C. TAMMI RAJU, M. Sc.

HEAD OF THE DEPARTMENT

DEPARTMENT OF MATHEMATICS,

PG COURSES

DNR COLLEGE (A),

BHIMAVARAM – 534202

<u>TOPOLOGICAL SPACES</u>.

<u>(104: TOPOLOGY, UNIT II)</u>

<u>Definition</u>: Let X be a non – empty set. A family τ of subsets of X is called a topology on X if it satisfies the following conditions:

(i) τ is closed under unions, and
(ii) τ is closed under finite intersections.

If τ is a topology on X, then (X, τ) is called a topological space. The members of τ are called open sets.

<u>Note</u>: Since the union of empty class of sets is empty, $\phi \in \tau$.
Since the intersection of empty class of sets is X, $X \in \tau$.
Hence in any topology τ on X, $\phi, X \in \tau$.

<u>Definition</u>: Let X be a non – empty set and τ be the family of all subsets of X . Then τ is a topology on X and it is called the discrete topology on X, and (X, τ) is called discrete topological space.

<u>Note</u>: in this case every subset of X is open.

<u>Definition</u>: Let X be a non – empty set and $\tau = \{\phi, X\}$. Then τ is a topology on X and it is called the indiscrete topology on X, and (X, τ) is called indiscrete topological space.

<u>Note</u>: in this case the only open sets are ϕ and X.

<u>Example:</u> Let $X = \{a, b, c\}$ where a, b, c are distinct and (i) $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then τ is a topology on X. (ii) $\tau = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. Then τ is a topology on X. (iii) $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Then τ is a topology on X. (iv) $\tau = \{\phi, \{a\}, X\}$. Then τ is a topology on X. (v) $\tau = \{\phi, \{a\}, \{b\}, X\}$. Then τ is not a topology on X.(vi) $\tau = \{\phi, \{a, c\}, \{b, c\}, \{a, b\}, X\}$. Then τ is not a topology on X. (vii) $\tau = \{\phi, \{a\}, \{a, c\}, \{a, b\}, X\}$. Then τ is a topology on X.

Example: Let (X, d) be any metric space. Let $\mathfrak{I}$ be the set of all open sets with respect to metric d. Then $\mathfrak{I}$ is a topology called usual topology on the metric space (X, d).

Definition: A metrizable space is a topological space X with the property that there exists at least one metric on the set X whose class of generated open sets is precisely the given topology.

Problem: Let X be a non – empty set and $\mathfrak{I}$ be the discrete topology on X. Show that (X, $\mathfrak{I}$) is a metrizable space.

Proof: Define d : X × X → ℝ by d(x, y) = 0 if x = y, and d(x, y) = 1 if x ≠ y.
Then (X, d) is a metric space. Here $S_{1/2}(x)$ is an open set and $S_{1/2}(x) = \{x\}$.
∴ {x} is open ∀ x ∈ X. For any subset A of X, since A = $\bigcup_{a\in A}\{a\}$, A is open in (X, d). Hence every subset of X is open in (X, d).
∴ The open sets in (X, $\mathfrak{I}$) and open sets in (X, d) are same.
Hence (X, $\mathfrak{I}$) is metrizable.

Problem: Let X be a non – empty set ∍ $|X| \geq 2$ and $\mathfrak{I}$ be indiscrete topology on X. Show that (X, $\mathfrak{I}$) is not metrizable space.

Proof: Given $\mathfrak{I} = \{\phi, X\}$. ∴ The only open sets in (X, $\mathfrak{I}$) are ϕ and X.
If possible, suppose (X, $\mathfrak{I}$) is metrizable.
⇒ ∃ a metric d on X ∍ the open sets in (X, d) are precisely the open sets in (X, $\mathfrak{I}$).
Since $|X| \geq 2$, ∃ a, b ∈ X ∍ a ≠ b. Take r = d(a, b) > 0.
Then $S_{r/2}(a)$ and $S_{r/2}(b)$ are disjoint non – empty open sets.
Now $\phi \neq S_{r/2}(a) \in \mathfrak{I} = \{\phi, X\} \Rightarrow S_{r/2}(a) = X$
⇒ b ∈ X = $S_{r/2}(a)$, a contradiction.
Hence (X, $\mathfrak{I}$) is not metrizable.

Theorem: Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be two topologies on a non – empty set X. Show that $\mathfrak{I}_1 \cap \mathfrak{I}_2$ is a topology on X.

Proof: Let $\{G_i\}_{i\in I}$ be an arbitrary collection of elements from $\mathfrak{I}_1 \cap \mathfrak{I}_2$.
Since $\mathfrak{I}_1 \cap \mathfrak{I}_2 \subseteq \mathfrak{I}_1$ and $\mathfrak{I}_1 \cap \mathfrak{I}_2 \subseteq \mathfrak{I}_2$, $\{G_i\}_{i\in I}$ is a collection of elements from $\mathfrak{I}_1$ as well as $\mathfrak{I}_2$. Since $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are topologies $\cup G_i \in \mathfrak{I}_1$ and $\cup G_i \in \mathfrak{I}_2$
$\Rightarrow \cup G_i \in \mathfrak{I}_1 \cap \mathfrak{I}_2$.
∴ $\mathfrak{I}_1 \cap \mathfrak{I}_2$ is closed under arbitrary unions.

Let G_i , $1 \le i \le n$ be a finite collection of elements from $\Im_1 \cap \Im_2$.
Since $\Im_1 \cap \Im_2 \subseteq \Im_1$ and $\Im_1 \cap \Im_2 \subseteq \Im_2$, G_i , $1 \le i \le n$ is a collection of elements from $\Im_1$ as well as $\Im_2$.
Since $\Im_1$ and $\Im_2$ are topologies $\bigcap_{i=1}^{n} G_i \in \Im_1$ and $\bigcap_{i=1}^{n} G_i \in \Im_2$
$\Rightarrow \bigcap_{i=1}^{n} G_i \in \Im_1 \cap \Im_2$.
$\therefore$ $\Im_1 \cap \Im_2$ is closed under finite intersections.
Hence $\Im_1 \cap \Im_2$ is a topology on X.

Definition: Let (X, $\Im$) be a topological space and Y be a non – empty subset of X. Let $\Im_Y = \{A / A = Y \cap G, G \in \Im\}$. Then (Y, $\Im_Y$) is a topological space and $\Im_Y$ is called the *relative topology* on Y, and (Y, $\Im_Y$) is called *subspace* of (X, $\Im$).

Example: Let X = {a, b, c} of distinct elements and $\Im = \{\phi, \{a\}, \{a, c\}, \{a, b\}, X\}$. Then $\Im$ is a topology on X. Let Y = {a, b}.
Then $\Im_Y = \{Y \cap G / G \in \Im\} = \{\phi, \{a\}, Y\}$ is a relative topology on Y. So (Y, $\Im_Y$) is

a *subspace* of (X, $\Im$).

Problem: Verify that a subspace (Y, $\Im_Y$) of topological space (X, $\Im$) is itself a topological space.

Solution: Let $\{H_\alpha : \alpha \in \Delta\}$ be a collection of elements from $\Im_Y$.
$\therefore$ for each α, $H_\alpha = Y \cap G_\alpha$ for some $G_\alpha \in \Im$.
Since $\cup G_\alpha \in \Im$, $\cup H_\alpha = \cup (Y \cap G_\alpha) = Y \cap (\cup G_\alpha) \in \Im_Y$.
Hence $\Im_Y$ is closed under arbitrary unions.

Let H_i, $1 \le i \le n$ be a finite collection of elements from $\Im_Y$.
Then $H_i = Y \cap G_i$ for some $G_i \in \Im$ for $1 \le i \le n$.
Since $\bigcap_{i=1}^{n} G_i \in \Im$, $\bigcap_{i=1}^{n} H_i = \bigcap_{i=1}^{n}(Y \cap G_i) = Y \cap \bigcap_{i=1}^{n} G_i \in \Im_Y$.
$\therefore$ $\Im_Y$ is closed under finite intersections also.
Hence $\Im_Y$ is itself a topology on Y.

Problem: Let X be an infinite set and $\Im$ consist of empty set together with all the subsets of X whose complements are finite. Show that (X, $\Im$) is a topological space. This topology is called the topology of finite complements.

Solution: Given $\Im = \{\phi\} \cup \{A \subseteq X \ni X \setminus A \text{ is finite}\}$.
(i) Let $\{G_\alpha\}$ be any class of sets from $\Im$.
If each G_α is empty, then $\cup G_\alpha$ is also empty and hence $\cup G_\alpha \in \Im$.

Now suppose $\exists\ \alpha_0 \ni G_{\alpha_0} \neq \phi$.
Then $X \setminus (\cup G_\alpha) \subseteq X \setminus G_{\alpha_0}$ since $G_{\alpha_0} \subseteq \cup\, G_\alpha$
$\therefore$ $X \setminus (\cup G_\alpha)$ is finite since $X \setminus G_{\alpha_0}$ is finite.
$\therefore$ $\cup G_\alpha \in \mathfrak{I}$. $\Rightarrow \mathfrak{I}$ is closed under arbitrary unions.
(ii) Let $G_i \in \mathfrak{I}$ for $1 \leq i \leq n$. Let $G = \bigcap_{i=1}^{n} G_i$.
If at least one $G_i = \phi$ then $G = \bigcap_{i=1}^{n} G_i = \phi \in \mathfrak{I}$.
Suppose $G_i \neq \phi\ \forall\ i \ni 1 \leq i \leq n$.
Since $\phi \neq G_i \in \mathfrak{I}$, $X \setminus G_i$ is finite $\forall\ i \ni 1 \leq i \leq n$.
Now $X \setminus G = X \setminus \bigcap_{i=1}^{n} G_i = \bigcup_{i=1}^{n} X \setminus G_i$ is finite since finite union of finite sets is finite.
$\Rightarrow G \in \mathfrak{I}$.
Hence $\mathfrak{I}$ is closed under finite intersections.
Hence $(X, \mathfrak{I})$ is a topological space.

Problem: Let X be an uncountable set and $\mathfrak{I}$ consist of empty set together with all the subsets of X whose complements are countable. Show that $(X, \mathfrak{I})$ is a topological space.

Solution: Let X be an uncountable set.
Given $\mathfrak{I} = \{\phi\} \cup \{A \subseteq X \ni X \setminus A \text{ is countable}\}$.
(i) Let $\{G_\alpha\}$ be any class of sets from $\mathfrak{I}$.
If each G_α is empty, then $\cup G_\alpha$ is also empty and hence $\cup G_\alpha \in \mathfrak{I}$.
Now suppose $\exists\ \alpha_0 \ni G_{\alpha_0} \neq \phi$.
Then $X \setminus (\cup G_\alpha) \subseteq X \setminus G_{\alpha_0}$ since $G_{\alpha_0} \subseteq \cup\, G_\alpha$
$\therefore$ $X \setminus (\cup G_\alpha)$ is countable since $X \setminus G_{\alpha_0}$ is countable.
$\therefore$ $\cup G_\alpha \in \mathfrak{I}$. $\Rightarrow \mathfrak{I}$ is closed under arbitrary unions.
(ii) Let $G_i \in \mathfrak{I}$ for $1 \leq i \leq n$. Let $G = \bigcap_{i=1}^{n} G_i$.
If at least one $G_i = \phi$ then $G = \bigcap_{i=1}^{n} G_i = \phi \in \mathfrak{I}$.
Suppose $G_i \neq \phi\ \forall\ i \ni 1 \leq i \leq n$.
Since $\phi \neq G_i \in \mathfrak{I}$, $X \setminus G_i$ is countable $\forall\ i \ni 1 \leq i \leq n$.
Now $X \setminus G = X \setminus \bigcap_{i=1}^{n} G_i = \bigcup_{i=1}^{n} X \setminus G_i$ is countable since finite union of countable sets is countable.
$\Rightarrow G \in \mathfrak{I}$.
Hence $\mathfrak{I}$ is closed under finite intersections.
Hence $(X, \mathfrak{I})$ is a topological space.

Definition: Let X and Y be topological spaces and f a mapping of X into Y.
f is called a ***continuous*** mapping if f^{-1} (G) is open in X whenever G is open in Y.
f is said to be an ***open* mapping** if f(G) is open in Y whenever G is open in X.
If f is continuous, then f(X) is called continuous image of X. If f is a bijection, continuous mapping and open mapping then f is called a ***homeomorphism***.
If f : X → Y is a homeoporphism then X and Y are said to be ***homeomorphic***.
In this Y is called a ***homeorphic image*** of X.

ELEMENTARY CONCEPTS

Definition: A *closed set* in a topological space is a set whose complement is open.

Theorem: Let (X, $\Im$) be a topological space. Then (i) any intersection of closed sets in X is closed and (ii) any finite union of closed sets in X is closed.

Proof: (i) Let $\{F_i\}$ be a class of closed sets in X $\Rightarrow F_i' \in \Im$ for all $i \in I$.
$\Rightarrow \bigcup_{i\in I} F_i' \in \Im$
$\Rightarrow (\bigcup_{i\in I} F_i')'$ is a closed set
$\Rightarrow [(\bigcap_{i\in I} F_i)']' = \bigcap_{i\in I} F_i$ is closed.
$\therefore$ any intersection of closed sets in X is closed
(ii) Let F_i, $1 \le i \le n$ be closed sets
$\Rightarrow F_i' \in \Im$ for $1 \le i \le n$.
$\Rightarrow \bigcap_{i=1}^{n} F_i' \in \Im$
$\Rightarrow (\bigcap_{i=1}^{n} F_i')'$ is a closed set.
$\Rightarrow [(\bigcup_{i=1}^{n} F_i)']' = \bigcup_{i=1}^{n} F_i$ is closed.
$\therefore$ any finite union of closed sets in X is closed.

Definition: Let (X, $\Im$) be a topological space and $A \subseteq X$.
The intersection of all closed super sets of A, is called the closure of A denoted by $\bar{A}$.

Note: A is closed iff $A = \bar{A}$.

Suppose A is closed. Clearly $A \subseteq \bar{A}$.
$\bar{A} \subseteq A$ since A is a closed superset of A and $\bar{A}$ is the intersection of all closed super sets of A.
Hence $A = \bar{A}$.
Conversely suppose $A = \bar{A}$.

$\bar{A}$ is closed since intersection of closed sets is closed and $\bar{A}$ is the intersection of all closed supersets of A.
$\therefore$ A is closed.

Definition: A subset A of X, where (X, $\Im$) is a topological space, is called ***dense (everywhere dense)*** if $\bar{A}$ = X.
A topological space X is said to be a ***separable*** space if it has a countable dense subset.

Theorem: Let X be a topological space. If A and B are arbitrary subsets of X, then the operation of forming closure has the following four properties.
(i) $\bar{\phi} = \phi$ (ii) A $\subseteq \bar{A}$. (iii) $\bar{\bar{A}} = \bar{A}$ and (iv) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
Proof: (i) Since X is open X′ = ϕ is closed so that $\bar{\phi} = \phi$.
(ii) Since $\bar{A}$ is the intersection of all closed supersets of A, A $\subseteq \bar{A}$.
(iii) Since $\bar{A}$ is closed $\bar{\bar{A}} = \bar{A}$.
(iv) A $\cup$ B $\subseteq \bar{A} \cup \bar{B}$ since A $\subseteq \bar{A}$ and B $\subseteq \bar{B}$.
Ie. $\bar{A} \cup \bar{B}$ is a closed superset of A $\cup$ B.
$\Rightarrow \overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$.
Again A $\subseteq$ A $\cup$ B $\subseteq \overline{A \cup B}$
ie. $\overline{A \cup B}$ is a closed super set of A
and since $\bar{A}$ is the intersections of all closed super sets of A, $\bar{A} \subseteq \overline{A \cup B}$.
Similarly $\bar{B} \subseteq \overline{A \cup B}$.
$\therefore \bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$
Hence $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Note: A $\subseteq$ B $\Rightarrow \bar{A} \subseteq \bar{B}$.
Proof: Since A $\subseteq$ B, A $\cup$ B = B
$\therefore \bar{B} = \overline{A \cup B} = \bar{A} \cup \bar{B} \Rightarrow \bar{A} \subseteq \bar{B}$.

Definition: A nbd of x $\in$ X, where (X, $\Im$) is a topological space, is G $\in \Im$ (an open set G) $\ni$ x $\in$ G. A class of nbds of a point x $\in$ X is called an ***open base*** for the point if for each nbd G of x $\exists$ a nbd H in this class $\ni$ H $\subseteq$ G.

Example:

Theorem: Let (X, $\Im$) be a topological space and A be an arbitrary subset of X. Then $\bar{A}$ = {x / each neighbourhood of x intersects A}

Proof: Let $x \in \bar{A}$.

If possible, suppose $x \notin \{x$ / each neighbourhood of x intersects $A\}$.

$\Rightarrow \exists$ a nbd G of $x \ni G \cap A = \phi$.

$\Rightarrow A \subseteq G'$.

$\Rightarrow \bar{A} \subseteq \overline{G'} = G'$ since G' is closed.

$\Rightarrow x \in G'$, a contradiction.

$\Rightarrow x \in \{x$ / each neighbourhood of x intersects $A\}$

Conversely suppose $x \in \{x$ / each neighbourhood of x intersects $A\}$.

If possible, suppose $x \notin \bar{A}$.

$\Rightarrow x \in (\bar{A})'$ and $(\bar{A})'$ is open.

$\Rightarrow (\bar{A})' \cap A \neq \phi$, a contradiction. Hence $x \in \bar{A}$.

Definition: Let X be a topological space and $A \subseteq X$. A point x in A is said to be an ***isolated point*** of A if $\exists$ nbd G of $x \ni (G \cap A) \setminus \{x\} = \phi$. A point $x \in X$ is said to be a ***limit point*** of A if $(G \cap A) \setminus \{x\} \neq \phi$ for every nbd G of x.

The ***derived set*** denoted by $D(A)$ is the set of all limit points of A.

Theorem: Let X be a topological space and $A \subseteq X$. Then (i) $\bar{A} = A \cup D(A)$ and (ii) A is closed iff $D(A) \subseteq A$.

Proof: Suppose $x \in A \cup D(A)$.

If possible suppose $x \notin \bar{A}$.

$\Rightarrow \exists$ nbd. G of $x \ni G \cap A = \phi$.

$\Rightarrow x \notin A$ and $(G \cap A)\setminus\{x\} = \phi$.

$\Rightarrow x \notin A$ and $x \notin D(A) \Rightarrow x \notin A \cup D(A)$ a contradiction. $\therefore x \in \bar{A}$.

Conversely suppose $x \in \bar{A}$.

If possible, suppose $x \notin A \cup D(A)$. $\Rightarrow x \notin A$ and $x \notin D(A)$.

$\Rightarrow x \notin A$ and x is not a limit point of A.

$\Rightarrow x \notin A$ and $\exists$ nbd. G of $x \ni (G \cap A)\setminus\{x\} = \phi$.

$\Rightarrow G \cap A = \phi \Rightarrow x \notin \bar{A}$, a contradiction.

$\therefore x \in A \cup D(A)$. Hence $\bar{A} = A \cup D(A)$.

(ii) A is closed iff $A = \bar{A}$ iff $A = A \cup D(A)$ iff $D(A) \subseteq A$.

Problem: Let f : X → Y be a mapping of one topological space into another. Show that (i) f is continuous, iff (ii) $f^{-1}(F)$ is closed in X whenever F is closed in Y, iff (iii) $f(\bar{A}) \subseteq \overline{f(A)}$ for every subset A of X.

Proof: (i) ⇒ (ii). Assume (i).
Let F be a closed set in Y
⇒ F′ is open
⇒ $f^{-1}(F') = [f^{-1}(F)]'$ is open in X, since f is continuous.
⇒ $f^{-1}(F)$ is closed.
(ii) ⇒ (i). Assume (ii).
Let G be open in Y.
⇒ G′ is closed
⇒ $f^{-1}(G') = [f^{-1}(G)]'$ is closed by (ii).
⇒ $f^{-1}(G)$ is open in X. Hence f is continuous.
(ii) ⇒ (iii) Assume (ii).
Let A ⊆ X. $\overline{f(A)}$ is closed in Y.
⇒ $f^{-1}\overline{f(A)}$ is closed in X.
Since $A \subseteq f^{-1}\overline{f(A)}$, $\bar{A} \subseteq f^{-1}\overline{f(A)}$
⇒ $f(\bar{A}) \subseteq \overline{f(A)}$
(iii) ⇒ (ii). Assume (iii).
Let F be a closed set in Y.
Write $A = f^{-1}(F)$ ⇒ f(A) = F
⇒ $\overline{f(A)} = \bar{F}$ = F (since F is closed) = f(A). By (iii), $f(\bar{A}) \subseteq \overline{f(A)}$ = f(A)
⇒ $\bar{A} \subseteq A$.
∴ $A = \bar{A}$. ⇒ $A = f^{-1}(F)$ is closed.

Theorem: Let X be a non – empty set and there be given a class of subsets of X which is closed under the formation of arbitrary intersections and finite unions. Then the class of all complements of these sets is a topology on X whose closed sets are precisely those initially given.

Proof: Suppose $\{F_i\}$ is the collection of given sets which is closed under arbitrary intersections and finite unions.
Write $\mathfrak{I} = \{F_i' \,/\, i \in \Delta\}$. Let $\{F_i'\}_{i\in I}$ where $I \subseteq \Delta$ be a collection of sets from $\mathfrak{I}$.
Now $\cup F_i' = (\cap F_i)'$ and since $\cap F_i \in \{F_i\}_{i\in\Delta}$, $(\cap F_i)' \in \mathfrak{I}$. ie. $\cup F_i' \in \mathfrak{I}$.
Hence $\mathfrak{I}$ is closed under arbitrary unions.
Let $F_1', \ldots, F_n' \in \mathfrak{I}$. ⇒ $F_1, \ldots, F_n \in \{F_i\}_{i\in\Delta}$.

Since the collection is closed under finite unions, $F_1 \cup \ldots \cup F_n$ is in this collection $\Rightarrow (F_1 \cup \ldots \cup F_n)' \in \mathfrak{I}. \Rightarrow F_1' \cap F_2' \cap \ldots \cap F_n' \in \mathfrak{I}.$

Hence $\mathfrak{I}$ is closed under finite intersections.

Hence $\mathfrak{I}$ is a topology.

Let F be a closed set in $(X, \mathfrak{I})$ iff F' is open iff $F' \in \mathfrak{I}$ iff $F = (F')' \in \{F_i\}_{i \in \Delta}$ iff F is in the collection.

Hence the closed sets in $(X, \mathfrak{I})$ are precisely the elements in the given collection.

<u>Theorem</u>: Let X be a non – empty set and there be given a closure operation which assigns to each subset A of X a subset $\bar{A}$ of X in such a manner that (i) $\bar{\phi} = \phi$ (ii) $A \subseteq \bar{A}$. (iii) $\bar{\bar{A}} = \bar{A}$ and (iv) $\overline{A \cup B} = \bar{A} \cup \bar{B}$. If a closed set A is defined to be one for which $A = \bar{A}$, then the class of all complements of such sets is a topology on X whose closure operation is precisely that initially given.

<u>Proof</u>: Write $\mathcal{G} = \{A : A \subseteq X \text{ and } A = \bar{A}\}$. It suffices if we prove that $\mathcal{G}$ is closed under arbitrary intersections and finite unions. Let $A_i \in \mathcal{G}$ for $1 \leq i \leq n$.

By (iii) $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2} = A_1 \cup A_2$. $\mathcal{G}$ is closed under unions when n = 2.

Let $\mathcal{G}$ be closed under unions when n = k – 1.

Assume $\overline{A_1 \cup A_2 \cup \ldots \cup A_{k-1}} = A_1 \cup A_2 \cup \ldots \cup A_{k-1}$

Now $\overline{A_1 \cup A_2 \cup \ldots \cup A_k} = \overline{A_1 \cup A_2 \cup \ldots \cup A_{k-1}} \cup \overline{A_k} = A_1 \cup A_2 \cup \ldots \cup A_{k-1} \cup A_k$

$\therefore$ By induction $\overline{A_1 \cup A_2 \cup \ldots \cup A_n} = A_1 \cup A_2 \cup \ldots \cup A_n$ $\forall$ integral values of n.

$\therefore$ $\mathcal{G}$ is closed under finite unions.

Now let $\{A_i\}_{i \in I}$ be a collection of elements from $\mathcal{G}$.

Then $A_i = \overline{A_\iota}$ for each $i \in I$.

Now $\bigcap_{i \in I} A_i \subseteq \overline{\bigcap_{\iota \in I} A_\iota}$ since by (ii) $A \subseteq \bar{A}$ for each subset A of X.

Again since $\bigcap_{i \in I} A_i \subseteq A_i$ for each i, $\overline{\bigcap_{\iota \in I} A_\iota} \subseteq \overline{A_\iota} = A_i$ for each i.

$\Rightarrow \overline{\bigcap_{\iota \in I} A_\iota} \subseteq \bigcap_{i \in I} A_i \;\therefore\; \overline{\bigcap_{\iota \in I} A_\iota} = \bigcap_{i \in I} A_i \;\therefore\; \bigcap_{i \in I} A_i \in \mathcal{G}$.

Hence $\mathcal{G}$ is closed under arbitrary intersections.

$\therefore \mathfrak{I} = \{ A' / A \in \mathcal{G}\}$ is a topology on X.

Now A is closed in X w. r. t. $\mathfrak{I}$ iff $A' \in \mathfrak{I}$ iff $A = (A')' \in \mathcal{G}$ iff $A = \bar{A}$ and A is closed in the given sense.

OPEN BASES AND OPEN SUBBASES

<u>Definition</u>: An open base for X where X is a topological space is a class β of open sets in X with the property that every open set in X is a union of sets from β.

Equivalently, if G is a non – empty open set and $x \in G$ then $\exists\, B \in \beta \ni x \in B \subseteq G$.

Example: Let (X, d) be a metric space and $\mathfrak{I}$ be the induced topology on X.
If β is the set of all open spheres in X, then β is an open base for (X, $\mathfrak{I}$).

Note: If β, β' are two collections of open sets of (X, $\mathfrak{I}$), β is open base and $\beta \subseteq \beta'$ then β' is also an open base for X.

Definition: A topological space (X, $\mathfrak{I}$) which has a countable open base is said to be a second countable space.

Note: Show that the two conditions are equivalent
(i) β is a class of open sets in X with the property that every open set in X is a union of sets from β and
(ii) β is a class of open sets in X $\ni$ G is a non – empty open set and $x \in G \Rightarrow \exists B \in \beta \ni x \in B \subseteq G$.

Solution: Claim: (i) $\Rightarrow$ (ii).
Assume (i). Let G be a non – empty set and $x \in G$.
Since G is open by (i) $\exists B_i \in \beta \ni G = \bigcup_{i \in I} B_i$.
$\therefore$ $x \in G = \bigcup_{i \in I} B_i \Rightarrow \exists B_i \in \beta$ for some i $\ni x \in B_i$ and hence $x \in B_i \subseteq G$, for $B_i \in \beta$.
Claim: (ii) $\Rightarrow$ (i). Let G be a open set in X.
If G is empty then G is a union of empty class of open sets from β.
Let G be non – empty and $x \in G$.
Then by (ii) $\exists B_x \in \beta \ni x \in B_x \subseteq G$. $\therefore \bigcup_{x \in G} B_x \subseteq G$.
Again $x \in G \Rightarrow x \in B_x \subseteq \bigcup_{x \in G} B_x$ $\therefore$ $G \subseteq \bigcup_{x \in G} B_x$.
Hence $G = \bigcup_{x \in G} B_x$.

LINDELOF'S Theorem: Let X be a second countable space. If a non – empty open set G in X is represented as the union of a class $\{G_i\}_{i \in I}$ of open sets, then G can be represented as a countable union of G_i's.
Proof: Since X is a second countable space, X has a countable open base, say, $\{B_n\}$. Given that G be a nonempty open set $\ni G = \bigcup_{i \in I} G_i$.
Let $x \in G$. Then $x \in G_{i(x)}$ for some $i(x) \in I$.
Since $G_{i(x)}$ is open and $x \in G_{i(x)}$ $\exists n(x) \ni x \in B_{n(x)} \subseteq G_{i(x)}$.
Since $\{B_n\}$ is a countable class and $\{B_{n(x)}\}_{x \in G}$ is a subclass of $\{B_n\}$ we have that $\{B_{n(x)}\}_{x \in G}$ is a countable class.
For every $x \in G$, corresponding to each $B_{n(x)}$ we have $G_{i(x)}$.
$\therefore$ $\{G_{i(x)}\}$ is also a countable class.

Now $\bigcup_{x\in G} G_{i(x)} \subseteq \bigcup_{i\in I} G_i = G$.
Let y $\in$ G $\Rightarrow$ y $\in G_{i(y)} \subseteq \bigcup_{x\in G} G_{i(x)}$.
Hence G = $\bigcup_{x\in G} G_{i(x)}$ and $\{G_{i(x)}\}_{x\in G}$ is a countable class.

Theorem: Let X be a second countable space. Then any open base for X has a countable subclass which is also an open base.
Proof: Given X is second countable.
Let $\{B_n\}$ be a countable open base for X.
Let $\{B_i\}$ be any open base for X. Since each B_n can be written as union of some B_i's (because B_n is open and $\{B_i\}$ is an open base), by Lindelof's theorem, for each non-empty B_n, $\exists$ a countable subclass $\{(B_i)_{n_k}\}$ of the class $\{B_i\}$ $\ni$ $B_n = \bigcup_k (B_i)_{n_k}$.
Now the class $\{(B_i)_{n_k} / n \geq 1, k \geq 1\}$ is a countable subclass of $\{B_i\}$.
We now show that the class B = $\{(B_i)_{n_k} / n\geq 1, k\geq 1\}$ is an open base for X.
Let G be any nonempty open set and x $\in$ G.
Since $\{B_n\}$ is an open base $\exists$ n $\ni$ x $\in B_n \subseteq$ G.
We know that x $\in B_n = \bigcup_k (B_i)_{n_k} \subseteq$ G and so B is an open base.
$\therefore$ B = $\{(B_i)_{n_k} / n\geq 1, k\geq 1\}$ is a countable subclass of $\{B_i\}$ which is also an open base for X.
Note: The axiom "topological space has a countable open base at each of its points" is called first axiom of countability. A topological space which satisfies this axiom is called a first countable space.

Theorem: Every second countable space is separable.
Proof: Let X be second countable space. Let $\{B_n\}$ be a countable open base for X.
Choose a point x_n from each non – empty set B_n.
Since $\{B_n\}$ is countable, A = $\{x_n / n \geq 1\}$ is countable.
Claim: $\bar{A}$ = X. Clearly $\bar{A} \subseteq$ X. Let x $\in$ X and G be a nbd of x.
Now $\exists$ a basic open set $B_n \ni x \in B_n \subseteq$ G.
If x = x_n then x $\in$ A $\subseteq \bar{A}$.
If x $\neq x_n$ for any n, then x, $x_n \in B_n \subseteq$ G and so $x_n \in G \cap A \setminus \{x\}$.
$\therefore$ for any nbd G of x, $G \cap A \setminus \{x\} \neq \phi$. $\Rightarrow$ x is a limit point of A and so x $\in \bar{A}$.

$\therefore X \subseteq \bar{A}$. Hence $\bar{A}$ = X.
Since, A is countable and $\bar{A}$ = X, X is separable.

Note: The converse of the above theorem need not be true.

For example, Consider $\mathbb{R}$ with topology $\mathfrak{I}$ of finite complements. Let F be a closed set in $(\mathbb{R}, \mathfrak{I})$. Then F' is open $\Rightarrow F' = \phi$, or $F' = G$ where G' is a finite set $\Rightarrow F = \mathbb{R}$ or F is a finite set. $\therefore$ Q is neither open nor closed. (Since Q is not a finite set and Q' is not a finite set Q is not open and not closed.) Since the only closed set containing Q is $\mathbb{R}$ we have $\bar{Q} = \mathbb{R}$. $\therefore$ $\mathbb{R}$ is a separable space.

Claim: $\mathbb{R}$ is not second countable. If possible, suppose $\mathbb{R}$ is second countable.
Then $\exists$ a countable open base $\{B_i\}_{i\in I}$.
Consider $A = \bigcup_{i=1}^{\infty} B_i{}'$.
Since each $B_i{}'$ is finite A is countable union of finite sets.
$\therefore$ A is countable. Since $\mathbb{R}$ is not countable $\mathbb{R} \not\subset A$.
$\therefore \exists\, y \in \mathbb{R} \setminus A$. Now write $G = \mathbb{R} \setminus \{y\}$.
Since $G' = \{y\}$ is finite, $G \in \mathfrak{I}$.
Let $z \in G$. Since $\{B_i\}$ is an open base, $\exists\, B_k \ni z \in B_k \subseteq G$ for some $k \in I$. $B_k \subseteq G$
$\Rightarrow B_k{}' \supseteq G' = \{y\}$
$\Rightarrow y \in B_k{}' \subseteq \bigcup_{i\in I} B_i{}'$
$\Rightarrow y \in A$, a contradiction to the selection of y.
Hence $\mathbb{R}$ is not second countable.

Theorem: Every separable metric space is second countable.

Proof: Let X be a separable metric space.
Let A be a countable dense subset of X.
Consider Q the set of rational numbers. We know that Q is countable.
Consider $\{S_r(a) \,/\, r \in Q\}$ for any $a \in A$.
Clearly this is a countable class of open spheres around $a \in A$.
Since A is countable $\mathcal{B} = \bigcup_{a\in A}\{S_r(a)/r \in Q\} = \{S_r(a) \,/\, a \in A, r \in Q\}$ is a countable union of countable class of sets.
Hence $\mathcal{B}$ is a countable class of sets.

Claim: $\mathcal{B}$ is an open base for X.
Let G be an open set and $x \in G$.
Since G is open $\exists$ a nbd $S_r(x)$ with some radius $r \ni x \in S_r(x) \subseteq G$.
Consider the open sphere $S_{r/3}(x)$.
Since A is dense, $\bar{A} = X$ and so every point of X is a limit point of A.
$\therefore$ x is a limit point of A and so $S_{r/3}(x) \cap A \neq \phi$.
Choose $r_1 \in Q \ni \frac{r}{3} < r_1 < \frac{2r}{3}$.

Now take a $\in Sr_{/3}(x) \cap$ A.
Then $S_{r_1}(a) \in \mathcal{B}$ and d(a, x) < r/3 < r_1.
$\Rightarrow$ x $\in S_{r_1}(a)$.

To show that $S_{r_1}(a) \subseteq S_r(x)$, take y $\in S_{r_1}(a)$.
Then d(x, y) $\leq$ d(x, a) + d(a, y) $\leq \frac{r}{3} + r_1 < \frac{r}{3} + \frac{2r}{3}$ = r.
$\Rightarrow$ y $\in S_r(x)$.
$\therefore S_{r_1}(a) \subseteq S_r(x) \subseteq$ G.
$\therefore$ x $\in S_{r_1}(a) \subseteq$ G.
$\therefore \mathcal{B}$ is an open base for X and $\mathcal{B}$ is countable.
Hence X is second countable.

Definition: Let X be a topological space. An ***open subbase*** is a class of open subsets of X whose finite intersections form an open base.
This open base is called the ***open base generated by the open subbase***.
The sets in an open subbase are called ***subbasic open sets***.

Note: Let (X, $\mathfrak{I}$) be a topological space and $\{B_i\}$ be an open subbase (say S = $\{B_i / i \in I\}$).
Then S* = $\{A_i / A_i = \bigcap_{k=1}^{n} B_{i_k}$ where n $\in \mathbb{N}$, $B_{i_k} \in$ S for $1 \leq k \leq n\}$ is the open base generated by S. Now $\mathfrak{I}$ = {G / G = $\cup A_i$ where $\{A_i\}$ is a collection of elements from S*}.

Example: Consider $\mathbb{R}$. Write S = $\{(a, \infty) / a \in \mathbb{R}\} \cup \{(-\infty, b) / b \in \mathbb{R}\}$.
Then S* = S $\cup\{\phi, \mathbb{R}\} \cup \{(a, b) / a, b \in \mathbb{R}\}$.
Now $\mathfrak{I}$ = {G / G = $\cup A_i$ where each A_i is from S*} = {G/ G is a union of open intervals of $\mathbb{R}$}. Clearly this $\mathfrak{I}$ is a topology on X induced by the usual metric on $\mathbb{R}$.
Hence S is an open subbase and S* is an open base generated by S.

Example:

Theorem: Let X be a non – empty set and let $\mathcal{C}$ be an arbitrary class of subsets of X. Then $\mathcal{C}$ can serve an open sub-base for a topology $\mathfrak{I}$ on X (in the sense that the class of all unions of finite intersections of sets in $\mathcal{C}$ forms a topology on X).

Proof: **Write $\mathcal{B}$ = the class of all finite intersections of sets of $\mathcal{C}$.**
Write $\mathfrak{I}$ = the class of all arbitrary unions of sets from $\mathcal{B}$.
If $\mathcal{C} = \phi$, then $\mathcal{B} = \{X\}$ and $\mathfrak{I} = \{\phi, X\}$.

In this case clearly $\mathfrak{I}$ is a topology on X.

Now assume that $\mathcal{C} \neq \phi$.

<u>Claim</u>: $\mathcal{B}$ is closed under finite intersections:

For this we prove $B_1, B_2, ..., B_k \in \mathcal{B} \Rightarrow B_1 \cap B_2 \cap ... \cap B_k \in \mathcal{B}$ for all integral values of k using induction.

Suppose for k = 2, B_1. $B_2 \in \mathcal{B}$.

Then $B_1 = P_1 \cap P_2 \cap ... \cap P_n$ and $B_2 = Q_1 \cap Q_2 \cap ... \cap Q_m$, where $P_i, Q_j \in \mathcal{C}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

Now $B_1 \cap B_2 = P_1 \cap P_2 \cap ... \cap P_n \cap Q_1 \cap Q_2 \cap ... \cap Q_m \in \mathcal{B}$.

Assume for k = n – 1ie. $B_1, B_2, ..., B_{n-1} \in \mathcal{B} \Rightarrow B_1 \cap B_2 \cap ... \cap B_{n-1} \in \mathcal{B}$.

Let $B_1, B_2, ..., B_n \in \mathcal{B}$

$\Rightarrow B_1 \cap B_2 \cap ... \cap B_n = (B_1 \cap B_2 \cap ... \cap B_{n-1}) \cap B_n \in \mathcal{B}$.

By induction $B_1, B_2, ..., B_k \in \mathcal{B} \Rightarrow B_1 \cap B_2 \cap ... \cap B_k \in \mathcal{B}$ for all integral values of k.

Hence $\mathcal{B}$ is closed under finite intersections.

Next we show that for any $x \in G \in \mathfrak{I}\ \exists\ B \in \mathcal{B} \ni x \in B \subseteq G$.

For this suppose $G \in \mathfrak{I}$.

Then by definition of $\mathfrak{I}$, $G = \bigcup_{i \in I} B_i$ where and $B_i \in \mathcal{B}$.

Now $x \in G \Rightarrow x \in B_i$ for some $i \in I$.

$\therefore \exists\ B_i \in \mathcal{B} \ni x \in B_i \subseteq G$.

To show that $\mathfrak{I}$ is closed under finite intersections.

Let $G_1, G_2, ..., G_n \in \mathfrak{I}$ and write $G^* = G_1 \cap G_2 \cap ... \cap G_n$.

Let $x \in G^*$. Then $x \in G_i$ for $1 \leq i \leq n$

$\Rightarrow \exists\ B_i \in \mathcal{B},\ 1 \leq i \leq n \ni x \in B_i \subseteq G_i$, for $1 \leq i \leq n$.

Write $B_x = B_1 \cap B_2 \cap ... \cap B_n \in \mathcal{B}$

$\therefore\ x \in G^* \Rightarrow \exists\ B_x \in \mathcal{B} \ni x \in B_x \subseteq G^*$. Hence $G^* = \bigcup_{x \in G^*} B_x \in \mathfrak{I}$.

$\therefore\ \mathfrak{I}$ is closed under finite intersections.

To show that $\mathfrak{I}$ is closed under arbitrary unions:

Let $\{G_i\}_{i \in I}$ be a collection of elements from $\mathfrak{I}$.

For each i, $G_i \in \mathfrak{I} \Rightarrow \exists \left\{B_{i_j}\right\} \ni G_i = \bigcup_j B_{i_j}$ and $\left\{B_{i_j}\right\} \in \mathcal{B}$.

Now $\bigcup_i G_i = \bigcup_i \bigcup_j B_{i_j} \in \mathfrak{I}$.

$\therefore\ \mathfrak{I}$ is a topology.

Already we have shown that for any $G \in \mathfrak{I}$, $x \in G\ \exists\ B \in \mathcal{B} \ni x \in B \subseteq G$.

$\therefore$ ***$\mathcal{B}$* is an open base for $\mathfrak{I}$.**

By construction of $\mathcal{B}$, **$\mathcal{C}$ is open sub-base.**

Definition: Let X be a non-empty set and $\mathcal{C}$ be any class of subsets of X. Write $\mathcal{B}$ = the class of all finite intersections of sets of $\mathcal{C}$. Write $\mathfrak{I}$ = the class of all arbitrary unions of sets from $\mathcal{B}$. Then $\mathfrak{I}$ is a topology on X called ***topology generated by the class $\mathcal{C}$***.

WEAK TOPOLOGIES

Definition: If $\mathfrak{I}_1$, $\mathfrak{I}_2$ are topologies on a set X such that $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$, then $\mathfrak{I}_1$ is said to be weaker than $\mathfrak{I}_2$

Note: Let X be any non-empty set. Then indiscrete topology is the weakest topology and discrete topology is the strongest topology on X.

Definition: A partially ordered set X is called a ***complete lattice*** if every non-empty subset of X has a greatest lower bound and least upper bound.

Theorem: Let X be a non-empty set. Then the family of all topologies on X is a complete lattice with respect to the relation "is weaker than". Furthermore, this lattice has a least member and a greatest member.

Proof: Let $\mathcal{G} = \{ \mathfrak{I} / \mathfrak{I}$ is a topology on X$\}$. Define a relation $\leq$ on $\mathcal{G}$ as $\mathfrak{I}_1 \leq \mathfrak{I}_2$ iff $\mathfrak{I}_1$ is weaker than $\mathfrak{I}_2$ ie. $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$. Then $(\mathcal{G}, \mathfrak{I})$ is a POset.

Claim: $(\mathcal{G}, \mathfrak{I})$ is a complete lattice. Let $\phi \neq \mathcal{G}_1 \subseteq \mathcal{G}$. Write $\mathfrak{I}_1 = \bigcap_{\mathfrak{I} \in \mathcal{G}_1} \mathfrak{I}$.

Then $\mathfrak{I}_1$ is a topology on X.

Since $\mathfrak{I}_1 \subseteq \mathfrak{I}\ \forall\ \mathfrak{I} \in \mathcal{G}_1$, $\mathfrak{I}_1 \leq \mathfrak{I}\ \forall\ \mathfrak{I} \in \mathcal{G}_1$.

$\therefore$ $\mathfrak{I}_1$ is a lower bound for $\mathcal{G}_1$.

Let $\mathfrak{I}^*$ be any lower bound of $\mathcal{G}_1$. Then $\mathfrak{I}^* \leq \mathfrak{I}\ \forall\ \mathfrak{I} \in \mathcal{G}_1$. $\Rightarrow \mathfrak{I}^* \subseteq \mathfrak{I}\ \forall\ \mathfrak{I} \in \mathcal{G}_1$.

$\Rightarrow \mathfrak{I}^* \subseteq \bigcap_{\mathfrak{I} \in \mathcal{G}_1} \mathfrak{I} = \mathfrak{I}_1 \Rightarrow \mathfrak{I}^* \leq \mathfrak{I}_1$

Hence $\mathfrak{I}_1$ is the glb of $\mathcal{G}_1$. Let $Y = \bigcup_{\mathfrak{I} \in \mathcal{G}_1} \mathfrak{I}$. Write $\mathcal{T} = \cap \{ \mathfrak{I} \in \mathcal{G} / Y \subseteq \mathfrak{I}\}$.

Since $\mathcal{T}$ is the intersection of a collection of topologies, $\mathcal{T}$ is a topology.

Since $\bigcup_{\mathfrak{I} \in \mathcal{G}_1} \mathfrak{I} \subseteq \mathcal{T}$, $\mathfrak{I} \subseteq \mathcal{T}\ \forall\ \mathfrak{I} \in \mathcal{G}_1 \Rightarrow \mathfrak{I} \leq \mathcal{T}\ \forall\ \mathfrak{I} \in \mathcal{G}_1$

$\Rightarrow$ $\mathcal{T}$ is an upper bound of $\mathcal{G}_1$.

Let $\mathcal{T}^*$ be any upper bound for $\mathcal{G}_1$. $\Rightarrow \mathfrak{I} \leq \mathcal{T}^*\ \forall\ \mathfrak{I} \in \mathcal{G}_1 \Rightarrow \mathfrak{I} \subseteq \mathcal{T}^*\ \forall\ \mathfrak{I} \in \mathcal{G}_1$

$\Rightarrow Y = \bigcup_{\mathfrak{I} \in \mathcal{G}_1} \mathfrak{I} \subseteq \mathcal{T}^*$.

$\Rightarrow \mathcal{T}^* \in \{ \mathfrak{I} \in \mathcal{G} / Y \subseteq \mathfrak{I}\}$ $\Rightarrow \mathcal{T} \subseteq \mathcal{T}^*$.

$\therefore$ $\mathcal{T} \leq \mathcal{T}^*$ for any upper bound $\mathcal{T}^*$ of $\mathcal{G}_1$.

Hence $\mathcal{T}$ is the least upper bound of $\mathcal{G}_1$.
Since every subset $\mathcal{G}_1$ of $\mathcal{G}$ has glb and lub, $\mathcal{G}$ is complete.

Note: Let X, Y be topological spaces. If $\mathfrak{I}$ is the discrete topology on X (ie. $\wp$(X)), then any mapping f: X $\to$ Y is continuous.

Definition: Let X be a non-empty set. Let $\{(X_i, \mathfrak{I}_i)_{i\in I}$ be a non-empty class of topological spaces. For each i $\in$ I, suppose f_i:X$\to$$X_i$ is a function. If $\wp$(X) is the topology on X then every f_i is continuous. Write $\mathfrak{I}^*$ = the intersection of all topologies on X which makes every $f_i : X \to X_i$ is continuous. This topology $\mathfrak{I}^*$ is called the weak topology generated by the f_i's.

THE FUNCTION ALGEBRAS **$\mathcal{C}$(X, $\mathbb{R}$), $\mathcal{C}$(X, $\mathfrak{C}$).**

Definition: An algebra is a linear space whose vectors can be multiplied in such a manner that (i) x(yz) = (xy)z; (ii) x(y + z) = xy + xz and (x + y)z = xz + yz and (iii) α(xy) = (αx)y = x(αy) for every scalar α.

If the scalars are real numbers then it is real algebra. If the scalar are complex numbers then the algebra is called complex algebra.

A commutative algebra is an algebra if xy = yx $\forall$ x, y.

An algebra with identity is an algebra satisfying the following property: $\exists$ a non-zero element denoted by 1 called the identity such that 1x = x = x1 for every x.

A subalgebra of an algebra is a linear subspace, which contains the product of each pair of its elements.

Lemma: If f and g are continuous real or complex functions defined on a topological space X, then f + g, af and fg are also continuous. Furthermore, if f and g are real, then $f \wedge g$ and $f \vee g$ are continuous.

Proof: (With suitable modifications in similar proof in metric spaces) we can prove that f + g, af are continuous.
Let $\varepsilon > 0$ and $x_0 \in X$. Take $\varepsilon_1 > 0 \ni \varepsilon_1\{|f(x_0)| + |g(x_0)|\} + \varepsilon_1^2 < \varepsilon$...(i).
Since f, g are continuous at x_0, corresponding to $\varepsilon_1 > 0$ $\exists$ nbds G_1 and $G_2 \ni x \in G_1$, $x \in G_2$, $|f(x) - f(x_0)| < \varepsilon_1$ and $|g(x) - g(x_0)| < \varepsilon_1$ respectively.
Then $G = G_1 \cap G_2$ is a nbd of x_0 and let $x \in G$.
Now $|(fg)(x) - (fg)(x_0)| = |f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)|$
$\leq |f(x)||g(x) - g(x_0)| +$

$|g(x_0)||f(x) - f(x_0)|$
$< \varepsilon_1 \, |f(x) - f(x_0)| + |f(x_0)| \, \varepsilon_1 + |g(x_0)| \, \varepsilon_1$
$< \varepsilon_1^2 + \varepsilon_1\{|f(x_0)| + |g(x_0)|\} < \varepsilon$ by (i).

$\therefore$ fg is continuous at x_0. Since this is true for any $x_0 \in X$, fg is continuous on X.
Put A = (a, ∞) and B = (- ∞, b).
Since f, g are continuous $f^{-1}(A)$, $g^{-1}(A)$, $f^{-1}(B)$, $g^{-1}(B)$ are open sets.

$$\text{Now } (f \vee g)^{-1}(A) = \{ x / (f \vee g)(x) \in A\} = \{x / \max [f(x), g(x)] > a\}$$
$$= \{x / f(x) > a\} \cup \{x / g(x) > a\}$$
$$= \{x / f(x) \in A\} \cup \{x / g(x) \in A\}$$
$$= f^{-1}(A) \cup g^{-1}(A) \text{ which is an open set.}$$

$$(f \vee g)^{-1}(B) = \{ x / (f \vee g)(x) \in B\} = \{x / \max [f(x), g(x)] < b\}$$
$$= \{x / f(x) < b\} \cap \{x / g(x) < b\}$$
$$= \{x / f(x) \in B\} \cap \{x / g(x) \in B\}$$
$$= f^{-1}(B) \cap g^{-1}(B) \text{ which is an open set.}$$

$\Rightarrow (f \vee g)^{-1}(A)$, $(f \vee g)^{-1}(B)$ are open sets on sub basic open sets A and B.

Hence $(f \vee g)$ is continuous. Similarly, we can show that $(f \wedge g)$ is continuous.

Lemma: Let X be a topological space, and $\{f_n\}$ be a sequence of real or complex functions defined on X which converges uniformly to a function f defined on X. If all the f_n's are continuous, then f is also continuous.

Theorem: Let $\boldsymbol{\mathcal{C}}(\mathbf{X}, \mathbb{R})$ be the set of all bounded continuous real functions defined on a topological space X. Then (i) $\boldsymbol{\mathcal{C}}(\mathbf{X}, \mathbb{R})$ is a real Banach space with respect to point wise addition and scalar multiplication and the norm defined by $\|f\| = \sup |f(x)|$; (ii) if multiplication is defined pointwise, then $\boldsymbol{\mathcal{C}}(\mathbf{X}, \mathbb{R})$ is a commutative real algebra with identity, in which $\|fg\| \leq \|f\|\|g\|$ and $\|1\| = 1$ and (iii) If $f \leq g$ is defined to mean that $f(x) \leq g(x)$ for all x, then $\boldsymbol{\mathcal{C}}(\mathbf{X}, \mathbb{R})$ is a lattice in which the greatest lower bound and least upper bound of a pair of functions f and g are given by $(f \wedge g)(x) = \min \{f(x), g(x)\}$ and $(f \vee g)(x) = \max \{f(x), g(x)\}$.

Proof: (i) follow the proof of "$\boldsymbol{\mathcal{C}}(\mathbf{X}, \mathbb{R})$ is a real Banach space" in metric spaces with relevant changes.
(ii) Let f, g, h $\in \boldsymbol{\mathcal{C}}(\mathbf{X}, \mathbb{R})$. Then for all $x \in X$, $f(gh)(x) = f(x)(gh)(x) = f(x)\{g(x)h(x)\} = \{f(x)g(x)\}h(x) = (fg)(x)h(x) = \{(fg)h\}(x)$

$\therefore$ f(gh) = (fg)h.
Similarly we can prove that f(g + h) = fg + fh; (f + g)h = fh + gh;
α(xy) = (αx)y = x(αy) for every scalar α and fg = gf.
$\therefore$ $\mathcal{C}$(X, ℝ) is a commutative algebra.
Define 1(x) = 1 ∀ x ∈ X.
Then for any f ∈ $\mathcal{C}$(X, ℝ) and x ∈ X, (f1)(x) = f(x)1(x) = f(x)1 = f(x) $\therefore$ f1 = f.
Similarly, 1f = f. $\therefore$ 1 is the identity element in $\mathcal{C}$(X, ℝ). Now $\|1\| = sup|1(x)|$ = sup |1| = 1.
Let f, g ∈ $\mathcal{C}$(X, ℝ). Then $\|fg\|$ = sup $|(fg)(x)|$ = sup $|f(x)||g(x)|$ ≤ sup $|f(x)|$ = sup $|g(x)|$ = $\|f\|\|g\|$

(iii) Define a relation ≤ on $\mathcal{C}$(X, ℝ) by f ≤ g iff f(x) ≤ g(x) ∀ x ∈ X.
Then clearly ≤ is a partial order on $\mathcal{C}$(X, ℝ).
By a lemma f ∨ g, f ∧ g are continuous ∍ $m_1 \le f(x) \le M_1$, $m_2 \le g(x) \le M_2$ ∀x∈X.
Take m = min $\{m_1, m_2\}$ and M = max $\{M_1, M_2\}$.
Then m ≤ (f ∧ g)(x) ≤ M and m ≤ (f ∨ g)(x) ≤ M ∀ x ∈ X.
$\therefore$ f ∧ g, f ∨ g are bounded continuous real valued on X.
⇒ f ∧ g, f ∨ g ∈ $\mathcal{C}$(X, ℝ).
Now it can be easily verified that f ∧ g = glb {f, g} and f ∨ g = lub {f, g}.
Hence $\mathcal{C}$(X, ℝ) is a lattice.

<u>Theorem</u>: Let $\boldsymbol{\mathcal{C}}$**(X, ℭ)** be the set of all bounded continuous real functions defined on a topological space X. Then (i) $\boldsymbol{\mathcal{C}}$**(X, ℭ)** is a complex Banach space with respect to pointwise addition and scalar multiplication and the norm defined by $\|f\|$ = sup $|f(x)|$; (ii) if multiplication is defined point wise, then $\boldsymbol{\mathcal{C}}$**(X, ℭ)** is a commutative complex algebra with identity, in which $\|fg\| \le \|f\|\|g\|$ and $\|I\|$ = 1 and
(iii) If $\bar{f}$ is defined by $\bar{f}(x) = \overline{f(x)}$ the complex conjugate of f(x), then f →$\bar{f}$ is a mapping of the algebra $\boldsymbol{\mathcal{C}}$**(X, ℭ)** into itself which has the following properties:
$\overline{f+g} = \bar{f} + \bar{g}$; $\overline{\alpha f} = \bar{\alpha}\bar{f}$; $\overline{fg} = \bar{f}\bar{g}$; $\bar{\bar{f}} = f$; $\|f\| = \|\bar{f}\|$.

<u>Proof</u>: (i), (ii) Similar proof as in the above theorem.
(iii) Let f ∈ $\boldsymbol{\mathcal{C}}$**(X, ℭ)** and define $\bar{f}$(x) = $\overline{f(x)}$ ∀ x ∈ X.
If f(x) = a + ib then $\overline{f(x)}$ = a – ib. $\therefore$ $\left|\overline{f(x)}\right| = \sqrt{a^2 + b^2} = \left|\bar{f}(x)\right|$.
$\therefore$ f is a bounded function from X to **ℭ.**
Clearly $\left|\bar{f}(x) - \bar{f}(x_0)\right| = \left|\overline{f(x)} - \overline{f(x_0)}\right| = \left|\overline{f(x) - f(f_0)}\right| = |f(x) - f(x_0|$.
Let ε> 0. Since f is continuous, ∃ a nbd G of x_0 ∍ x ∈ G. ⇒ $|f(x) - f(x_0)|$ < ε.

$\therefore \left|\bar{f}(x) - \bar{f}(x_0)\right| = |f(x) - f(x_0| < \varepsilon$. This is true $\forall$ x $\in$ G.
Hence $\bar{f}$ is continuous. $\therefore$ $\bar{f}$ is bounded and continuous. $\therefore$ $\bar{f} \in \mathcal{C}(X, \mathfrak{C})$.
So, f $\to \bar{f}$ is a mapping from $\mathcal{C}(X, \mathfrak{C})$ into itself.
$\overline{(f+g)}(x) = \overline{(f+g)(x)} = \overline{f(x)+g(x)} = \overline{f(x)} + \overline{g(x)} = \bar{f}(x) + \bar{g}(x)$
$= (\bar{f} + \bar{g})(x)$ $\forall$ x $\in$ X.
$\therefore$ $\overline{f+g} = \bar{f} + \bar{g}$. Similarly $\overline{\alpha f} = \bar{\alpha}\bar{f}$; $\overline{fg} = \bar{f}\bar{g}$ and $\bar{\bar{f}} = f$.
Now $\|\bar{f}\| = \sup \left|\bar{f}(x)\right| = \sup \left|\overline{f(x)}\right| = \sup |f(x)| = \|f\|$.

Fax: 08816-227318 off: 08816-224072, 224119,228342 Mobile: 9491334119

DANTULURI NARAYANA RAJU COLLEGE

(Autonomous)

BHIMAVARAM, W.G.DIST, ANDHRA PRADESH, INDIA, PIN- 534202.

(Accredited at 'B++' level by NAAC)

(Affiliated to Adikavi Nannaya University, Rajamahendravaram)

E – CONTENT

PAPER: M 104, TOPOLOGY

M. Sc. I YEAR, SEMESTER - I

UNIT – III: COMPACTNESS

PREPARED BY

K, C. TAMMI RAJU, M. Sc.

HEAD OF THE DEPARTMENT

DEPARTMENT OF MATHEMATICS,

PG COURSES

DNR COLLEGE (A),

BHIMAVARAM – 534202

COMPACTNESS
104: TOPOLOGY; UNIT III

COMPACT SPACES

Definitions: (i) Let X be a topological space. A class $\{Gi\}$ of open subsets of X is said to be an open cover if each point in X belongs to at least one G_i. i.e. $X = \bigcup_i G_i$.

(ii) A subset of an open cover which is itself an open cover is called a subcover.

(iii) A compact space is a topological space in which every open cover has a finite subcover.

(iv) Let (Y, $\mathfrak{I}_Y$) is a subspace of (X, $\mathfrak{I}_X$). Y is said to be compact subspace of the topological space X, if Y is compact in its own rights.

Theorem: Any closed subspace of a compact space is compact.

Proof: Let X be a compact space and Y be a closed subspace of X.
Let $\{G_i\}_{i\in I}$ be an open cover of Y.
Then for each i, $\exists$ an open subset H_i of X $\ni G_i = H_i \cap Y$.
Now $Y \subseteq \bigcup_i G_i = \bigcup_{i\in I}(H_i \cap Y) \subseteq (\bigcup_i H_i) \cap Y$
So $X = Y \cup Y' \subseteq (\bigcup_i H_i) \cup Y'$.
$\therefore Y'$ together with H_i, $i \in I$ forms an open cover for X since Y' is open.
Since X is compact, $\exists$ a finite subcover $H_{i_1}, H_{i_2}, \dots, H_{i_n}, Y'$ of X such that
$X = H_{i_1} \cup H_{i_2} \cup \dots \cup H_{i_n} \cup Y'$.
Now $Y = Y \cap X = Y \cap (H_{i_1} \cup H_{i_2} \cup \dots \cup H_{i_n} \cup Y')$
$= (Y \cap H_{i_1}) \cup (Y \cap H_{i_2}) \cup \dots \cup (Y \cap H_{i_n}) \cup (Y \cap Y')$
$= G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_n} \cup \phi = G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_n}$.
$\therefore G_{i_1}, G_{i_2}, \dots, G_{i_n}$ forms a finite subcover to Y. Hence Y is compact.

Theorem: Any continuous image of a compact space is compact.
Proof: Let $f : X \to Y$ be a continuous mapping of a compact metric space X into a topological space Y.
Let $\{G_i\}_{i\in I}$ be an open cover of f(X). Ie. $f(X) \subseteq \bigcup_{i\in I} G_i$ ...(i)
Since f is continuous, $f^{-1}(G_i)$ is open in X for all $i \in I$.

From (i), X $\subseteq f^{-1}\{f(X)\} \subseteq f^{-1}(\cup_{i\epsilon I} G_i) = \cup_{i\epsilon I} f^{-1}(G_i)$.
$\therefore$ $\{f^{-1}(G_i)\}_{i\in I}$ is an open cover for X. Since X is compact, this open cover has a finite subcover.
Ie. $\exists\ f^{-1}(G_{i_1}), f^{-1}(G_{i_2}), \ldots, f^{-1}(G_{i_n}) \ni X \subseteq f^{-1}(G_{i_1}) \cup f^{-1}(G_{i_2}) \cup \ldots \cup f^{-1}(G_{i_n})$
$\Rightarrow$ f(X) $\subseteq G_{i_1} \cup G_{i_2} \cup \ \ldots \cup\ G_{i_n}$. $\therefore$ the open cover $\{G_i\}_{i\in I}$ of f(X) has a finite subcover $G_{i_1}, G_{i_2}, \ldots, G_{i_n}$. Hence f(X) is compact.

Definition: A class $\{A_i\}_{i\in I}$ of sets X is said to have the finite intersection property if every finite subclass $\{A_{i_1}, A_{i_2}, \ldots, A_{i_n}\}$ has a non – empty intersection.
Ie. $A_{i_1} \cap A_{i_2} \cap\ \ldots\ \cap A_{i_n} \neq \phi$.

Theorem: A topological space is compact iff every class of closed sets with empty intersection has a finite subclass with empty intersection.
Proof: Let X be compact. Let $\{F_i\}_{i\in I}$ be a class of closed sets such that $\cap_{i\in I} F_i = \phi$.
For each i $\in$ I, since F_i is closed, F_i' is open. $\therefore$ X = $\phi' = \{\cap_{i\in I} F_i\}' = \cup F_i'$, i $\in$ I.
Clearly $\{F_i'\}_{i\in I}$ is an open cover for X. Since X is compact, this open cover has a finite subcover. $\therefore \exists\ F_{i_1}', F_{i_2}', \ldots, F_{i_n}' \ni X = F_{i_1}' \cup F_{i_2}' \cup \ldots \cup F_{i_n}'$
$\Rightarrow \phi = X' = \left(F_{i_1}' \cup F_{i_2}' \cup \ldots \cup F_{i_n}'\right)' = F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_n}$
Hence $\exists$ a finite subclass $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$ of the class $\{F_i\}_{i\in I}$.
Conversely suppose that every class of closed sets with empty intersection has a finite subclass with empty intersection. Let $\{G_i\}_{i\in I}$ be an open cover for X. ie. X = $\cup G_i$ $\Rightarrow \phi = X' = (\cup G_i)' = \cap G_i'$. Since $\{G_i'\}_{i\in I}$ is a collection of closed sets whose intersection is empty, by assumption, $\exists$ a finite subclass $G_{i_1}', G_{i_2}', \ldots, G_{i_n}' \ni G_{i_1}' \cap G_{i_2}' \cap \ldots \cap F_{i_n}' = \phi$
$\Rightarrow X = \phi' = \left(G_{i_1}' \cap G_{i_2}' \cap \ldots \cap G_{i_n}'\right)' = G_{i_1} \cup G_{i_2} \cup \ldots \cup G_{i_n}$ $\therefore$ The cover $\{G_i\}_{i\in I}$ of X has a finite subcover. Hence X is compact.

Theorem: A topological space is compact if and only if every class of closed sets with finite intersection property has nonempty intersection.
Proof: Let X be compact. Let $\{F_i\}_{i\in I}$ be a class of closed sets with finite intersection property. In contrary suppose that $\cap_{i\in I} F_i = \phi$. By above theorem, $\exists$ a finite subclass $F_{i_1}, F_{i_2}, \ldots, F_{i_n} \ni F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_n} = \phi$, a contradiction to the finite intersection property. Hence $\cap_{i\in I} F_i \neq \phi$.
Conversely suppose that every class of closed sets with finite intersection property has nonempty intersection. If possible suppose that X is not compact. Then $\exists$ an open cover $\{G_i\}_{i\in I}$ which has no finite subcover. This means for any subcover

$G_{i_1}, G_{i_2}, ..., G_{i_n} \ni$ X $\supsetneq G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_n}$. $\Rightarrow \phi = X' \neq (G_{i_1} \cup G_{i_2} \cup \ldots \cup G_{i_n})' = G_{i_1}' \cap G_{i_2}' \cap ... \cap G_{i_n}'$. Now $\{G_i'\}_{i\in I}$ is a class of closed sets with finite intersection property. $\Rightarrow \cap G_i' \neq \phi$, $_{i\in I}$

$\Rightarrow (\cap G_i')' \neq \phi' = X \Rightarrow X \neq \cup G_i$ a contradiction, since $\{G_i\}_{i\in I}$ is an open cover of X. Hence X is compact.

Definition: Let X be a topological space. (i) an open cover of X whose sets are in some given open base is called a basic open cover. (ii) an open cover of X whose sets are in some given open subbase , is called a subbasic open cover.

Theorem: A topological space is compact if every basic open cover has a finite subcover.

Proof: Suppose every basic open cover has a finite subcover. Now to show X is compact, take an open cover $\{G_i\}_{i\in I}$ to X. Let $\{B_i\}_{j\in J}$ be an open base. By the definition of open base $G_i = \cup B_{j_k}$. Fix $k \in I$ and consider G_k. Since $\{B_j\}_{j\in J}$ is an open base $G_k = \cup B_j$, $j \in j_k$ for some subclass $\{B_j\}_{j\in jk}$. Now $X = \bigcup_{k\in I} G_k = \bigcup_{k\in I} \bigcup_{j\in j_k} B_j$. Now those B_j's form a basic open cover for X. By the hypothesis, this basic open cover has a finite subcover. $\therefore \exists$ $k_1, k_2, ..., k_n \in I$ and $j_1 \in j_{k_1}$, $J_2 \in j_{k_2}$,.., $j_n \in j_{k_n}$ such that $X = B_{j_1} \cup ... \cup B_{j_n}$ ... (i).

By the selection of j_k's $G_{k_1} = \bigcup_{j\in j_{k_1}} B_j \supseteq B_{j_1}$ (since $j_1 \in j_{k_1}$), $G_{k_2} = \bigcup_{j\in j_{k_2}} B_j \supseteq B_{j_2}$ since $j_2 \in j_{k_2}$,..., $G_{k_n} = \bigcup_{j\in j_{k_n}} B_j \supseteq B_{j_n}$ (since $j_n \in j_{k_2}$).

By (i) X = $B_{j_1} \cup ... \cup B_{j_n} \subseteq G_{k_1} \cup ... \cup G_{k_n}$ which is a finite subcover of $\{G_i\}_{i\in I}$.

Theorem: A topological space is compact if every subbasic open cover has a finite subcover or equivalently if every class of subbasic closed sets with finite intersection property has non – empty intersection.

Proof: Proof is out of the scope of this book.

Heine – Borel theorem: Every closed and bounded subspace of the real line is compact. (M. Imp).

Proof: First we prove that any closed interval [a, b] of the real line is compact.

Consider A = { [a, d) / a < d < b} $\cup$ {(c , b] / a < c < b}

We show that B = {(c, d) / a ≤ c < d ≤ b}} forms an open base for [a, b].

Let G be an open set in [a, b], and $x \in G$. Since G is open $\exists$ r > 0 $\ni$ $S_r(x) \subseteq G$.

$\Rightarrow (x - r, x + r) \subseteq G$. Now $(x - r, x + r) \subseteq G \subseteq [a, b] \Rightarrow a \le x - r < x + r \le b$. now if we write $c = x - r$, $d = x + r$ then $(c, d) \in B$ and $x \in (c, d) \subseteq G$. Hence B is an open base for [a, b].

If we take [a, d) and (c, b] then $[a, d) \cap (c, b] = \phi$ or (c, d). Therefore, every basic open set in B can be written as intersection of finite sets in A. Hence A is an open subbase for [a, b].

Consider $F = \{Y' / Y \in A\} = \{[a, b] \setminus [a, d) / a < d < b\} \cup \{[a, b] \setminus (c, b] / a < c < b\} = \{[d, b] / a < d < b\} \cup \{[a, c] / a < c < b\}$. Since A is an open subbase, we have that F is a closed subbase. These closed subbasic sets are of the form [a, c] or [d, b].

Consider $G = \{[a, c_i]\}_{i \in I} \cup \{[d_j, b]\}_{j \in J}$ be a collection of subbasic closed sets with finite intersection property. To prove [a, b] is compact it is enough to prove that the intersection of the collection G is non - empty.

If G contains only the sets of the form $[a, c_i]$ then their intersection contain a. If G contains only the sets of the form $[d_j, b]$ then their intersection contain b. Now we assume that G contains both the forms.

Write $d = \sup \{d_j / [d_j, b] \in G\}$. Since $d \geq d_j$, we have $d \in [d_j, b]$ for all $j \in J$. Now we wish to show that $d \in [a, c_i]$ for all $[a, c_i] \in G$.
In a contrary way suppose $d \notin [a, c_{i_o}]$ for some $[a, c_{i_o}] \in G$. Then $d > c_{i_o}$. Since d is the supremum, and $c_{i_o} < d$, we have that there exists d_{j_o} such that $c_{i_o} < d_{j_o} < d$ and $[d_{j_o}, b] \in G$. Now $[a, c_{i_o}] \cap [d_{j_o}, b] = \phi$, a contradiction to finite intersection property.
Hence $d \in [a, c_i]$ for all i. Therefore, the intersection of sets in G is non – empty.
$\therefore$ [a, b] is compact.
Let E be a bounded and closed subset of $\mathbb{R}$. Since E is bounded, $\exists$ an upper bound b and a lower bound a for E. This implies that $E \subseteq [a, b]$. Since [a, b] is compact and E is a closed subset of [a, b], we have that E is compact.

PRODUCT SPACES

Definition: Let (X_1, T_1) and (X_2, T_2) be topological spaces and for the product $X = X_1 \times X_2$, consider the class S of all subsets of X of the form $G_1 \times X_2$ and $X_1 \times G_2$ where G_1 and G_2 are open subsets of X_1 and X_2 respectively. The class T of all unions of finite intersections of sets in S is a topology on S called product topology on X. Her S is open subbase of T.

Definition: Let $(X_i, \mathfrak{I}_i)i \in I$ be a collection of topological spaces then P_iX_i is the Cartesian product of X_i. Here p_i: $P_iX_i \to X_i$ is defined by $p_i(\{x_j\}, j \in j) = x_i$. Here S = $\{Pi - 1\ (G_i)/\ G_i \in \mathfrak{I}_i\}$. S* = $\{P_iG_i$/where $G_i \in \mathfrak{I}_i$ and $G_i = X_i$ for all but a finite number of i∈I}. Here P_iG_i = {{x_i} / where $x_i \in G_i$ for some finite number of i's and there is no restriction on the other coordinates xi} now S* is the open base for (X, $\mathfrak{I}$). S* is the open base generated by the open subbase S.

Definition: The class defined above is called the defining open subbase for the product topology. F = { F / F′∈ S} = the class of all products of the form P_iF_i where F_i is a closed subset of X_i which equals to X_i for all i's but one, is called the defining closed subbase.

Definition: S* defined above is called the defining open base for the product topology. Ie. the defining open base is a tipical one of its sets consists of all points x = {x_i} in the product such that i^{th} coordinate x_i is required to lie in an open subset of G_i of X_i for the finite number of i's and all other coordinates being unrestricted.

Definition: The product of the non-empty class of topological spaces equipped with the product topology is called a product space.

Tychonoff's theorem: The product of any non – empty class of compact spaces is compact. (M. Imp).
Proof: Let {X_i} be a nonempty class compact spaces.
Let X = P_iX_i, i ∈I. Let {F_j}, j ∈J, be a nonempty subclass of the defining closed subbase with finite intersection property for the product topology on X.
This means that each F_j is a product of the form $F_j = P_iF_{ij}$, i ∈ I where F_{ij} is a closed subset of X_i which equals X_i for all i's but one.
For a fixed i, $\{F_{ij}\}_{j\epsilon J}$ is a class of closed subsets of X_i.
We now show that this class $\{F_{ij}\}_{j\epsilon J}$ has finite intersection property.
Let $F_{ij_1}, F_{ij_2}, \dots, F_{ij_n}$ be a finite number of sets in the class $\{F_{ij}\}_{j\epsilon J}$.
Since the class $\{F_j\}_{j\epsilon J}$ has the finite intersection property, $F_{j_1} \cap F_{j_2} \cap \dots \cap F_{j_n} \neq \phi$.
Let x $\in F_{j_1} \cap F_{j_2} \cap \dots \cap F_{j_n}$.
Then x $\in F_{j_k}$ for k = 1, 2, …, n.
∴ x(i) ∈ $\bigcap F_{ij_k}$ for k = 1, 2, …, n.
⇒ x(i) $\in F_{ij_1} \cap F_{ij_2} \cap \dots \cap F_{ij_n}$
⇒ $F_{ij_1} \cap F_{ij_2} \cap \dots \cap F_{ij_n} \neq \phi$.
∴ the class $\{F_{ij}\}_{j\epsilon J}$ has finite intersection property.
Since X_i is compact, $\bigcap_{j\in J} F_{ij} \neq \phi$.
Let $y_i \epsilon \bigcap_{j\in J} F_{ij}$, j ∈ J then $y_i \epsilon F_{ij}$ ∀ j ∈ J.

Define y by y(i) = y_i.
Then y(i) ϵF_{ij} ∀ j ∈ J.
Now y = {y(i)} ∈$P_i F_{ij} = F_j$ ∀ j ∈ J.
⇒ y ∈$\bigcap_{j\in J} F_j$. ∴$\bigcap_{j\in J} F_j \neq \phi$.
Hence X is compact.

Generalised Heine Borel theorem: Every closed and bounded subspace of $\mathbb{R}^n$ is compact. (Imp).

Proof: For 1 ≤ i ≤ n, consider $X_i = [a_i, b_i]$, the closed interval with endpoints a_i, and b_i with $a_i < b_i$. Now $X_i = \prod_{i=1}^{n} X_i = \prod_{i=1}^{n}[a_i, b_i] = \{(x_1, x_2, ..., x_n) / a_i < x_i < b_i$ for $1 \leq i \leq n\}$ is a closed rectangle in $\mathbb{R}^n$. First, we show that this closed rectangle X is compact. Since each $[a_i, b_i]$ is a closed and bounded interval of ℝ, by Heine – Borel theorem, we have $X_i = [a_i, b_i]$ is compact for 1 ≤ i ≤ n.
∴ By Tychonoff's theorem, X = $\prod_{i=1}^{n} X_i$ is compact.

Let E be a closed and bounded subspace of $\mathbb{R}^n$. Since E is bounded, ∃ a_i, b_i ∈ ℝ for 1 ≤ i ≤ n, such that $E \subseteq \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n / a_i < x_i < b_i$ for $1 \leq i \leq n\}$ = X say. Now E is a closed subset of X. By above part X is compact. Since E is a closed subset of the compact space X, by theorem we have that E is compact. ∴ Every closed and bounded subspace of $\mathbb{R}^n$ is compact.

Definition: A topological space is said to be locally compact if each of its points has a neighbourhood with compact closure (compact closure means for any x ∈ X, there exists a nbd G_x such that x ∈ G_x, $\overline{G_x}$ is a compact set.

COMPACTNESS FOR METRIC SPACES

Definition: A metric space is said to have the Bolzano – Weierstrass property if every infinite subset has a limit point. (ii) A metric space is said to be sequentially compact if every sequence in it has a convergent subsequence.

Theorem: A metric space is sequentially compact if and only if it has the Bolzano Weierstrass property. (M. Imp)

Proof: Let X be a metric space. Assume that X is sequentially compact.
Let A be an infinite subset of X. Let a1 be any point of A.
Having chosen $a_1, a_2, a_3, ... a_{n-1}$, consider the set $A - \{a_1, a_2, a_3, ... a_{n-1}\}$.
Since A is infinite and so choose an element $a_n \in A - \{a_1, a_2, a_3, ... a_{n-1}\}$.
By induction we get a sequence $\{a_n\}$ of distinct points from A.
Since X is sequentially compact, the sequence $\{a_n\}$ has a convergent subsequence

$\{ a_{n_k}\}$ *of distinct points* converging to a (say).
By a theorem, a is a limit point of the set $\{ a_{n_k} : k \geq 1\}$.
Since the set $\{ a_{n_k} : k \geq 1\} \subseteq A$, a is a limit point of A.
Hence X has Bolzano – Weierstrass property.
Conversely suppose that X has the Bolzano – Weierstrass property.
Let $\{a_n\}$ be a sequence in X.
Let A be the set of points of the sequence $\{a_n\}$.ie. $A = \{a_n / n \geq 1\}$.
Case (i): Suppose A is finite. Then ∃ a in A which repeats infinite times.
So ∃ a subsequence $\{ a_{n_k} \}$ of $\{a_n\}$such that $a_{n_1} = a_{n_2} = \ldots = a$.
Then clearly the sequence $\{ a_{n_k} \}$ converges to a.
Case (ii): Assume that A is infinite. By hypothesis, A has a limit point say a.
Take $r_1 = 1$. Now the open sphere $S_{r_1}(a)$ contains a point of A.
∴ ∃ a positive integer $n_1 \ni a_{n_1} \in S_{r_1}(a)$. Ie. $d(a, a_{n_1}) < r_1 = 1$.
Take $r_2 = \min \{ d(a, a_{n_1}), 1/2\}$.
Since $S_{r_2}(a) \cap A \neq \emptyset$, ∃ $n_2 > n_1 \ni a_{n_2} \in S_{r_2}(a)$. Ie. $d(a, a_{n_2}) < \frac{1}{2}$.
Having chosen $n_1, n_2, \ldots, n_{k-1}$, choose $n_k \ni n_k > n_{k-1}$ and $d(a, a_{n_k}) < \frac{1}{k}$.
By induction, we get a subsequence $\{ a_{n_k} \} \ni d(a, a_{n_k}) < \frac{1}{k}\ \forall$ k.
Clearly the subsequence $\{ a_{n_k} \}$ converges to a.

Theorem: Every compact metric space has the Bolzano Weierstrass property (less imp).
Proof: Let X be a compact metric space. let A be an infinite subset of X. In a contrary way, suppose A has no limit point. If a is a point of X then a in not a limit point of A and hence there is an open sphere $S_{r_a}(a)$ such that $S_{r_a}(a) \cap A - \{a\} = \phi$. i.e. $S_{r_a}(a) \cap A \subseteq \{a\}$. i.e $S_{r_a}(a) \cap A = \{a\}$ or $S_{r_a}(a) \cap A = \phi$.

Consider the class $\left\{ S_{r_a}(a) / a \in X \right\}$ of all these open spheres. Clearly this is an open cover for X. Ie. $X = \bigcup_{a \in X} S_{r_a}(a)$. Since X is compact, this open cover has a finite subcover, say, $S_{r_{a_1}}(a_1),\ S_{r_{a_2}}(a_2),\ \ldots,\ S_{r_{a_m}}(a_m)$ where $a_1, a_2, \ldots, a_m \in X$. ∴ $A = A \cap X = A \cap \left\{ S_{r_{a_1}}(a_1) \cup S_{r_{a_2}}(a_2) \cup \ldots \cup S_{r_{a_m}}(a_m) \right\}$.
$= \left\{ A \cap S_{r_{a_1}}(a_1) \right\} \cup \left\{ A \cap S_{r_{a_2}}(a_2) \right\} \cup \ldots \cup \left\{ A \cap S_{r_{a_m}}(a_m) \right\}$
$\subseteq \{a_1\} \cup \{a_2\} \cup \ldots \cup \{a_m\} = \{a_1, a_2, \ldots, a_m\}$. ⇒ A is finite which is a contradiction to the fact that A is infinite.
∴ A has a limit point. Hence X has the Bolzano -Weierstrass property.

Definition: Let $\{G_i\}$ be an open cover of a metric space X. A real number $a > 0$ is called a Lebesgue number for the given open cover $\{G_i\}$, if each subset A of X with $d(A) < a$ is contained in at least one G_i. Ie. a is the Lebesgue number, if $A \subseteq X$, $d(A) < a \Rightarrow A \subseteq G_i$ for some i.

Definition: Suppose X is a metric space and $\{G_i\}_{i \in I}$ be an open cover. A subset A of X is said to be big if $A \nsubseteq G_i$ for any $i \in I$.

Note: (i) Singleton subsets are not big sets. (ii) If A is a big set then A contains at least two points.
Sol: (i) Let $x \in X$. Write $A = \{x\}$. Now $x \in X \subseteq \cup G_i \Rightarrow x \in G_i$ for some $i \Rightarrow A \subseteq G_i$. So A is not big.

Example: Let $X = \{a, b, c\}$. Define $d : X \times X \to \mathbb{R}$ by $d(x, y) = 0$ if $x = y$ and 1 if $x \neq y$. Then d is a metric on X. Every subset of X is open in X. Write $B = \{\{a, b\}, \{b, c\}\}$. Then B is an open cover for X. If $A = \{a, c\}$ then A is a big set. Also $\{a\}, \{b\}, \{c\}$ are not big sets. Let $0 < s \leq 1$. We show that s is a Lebesgue number for B. Let G be any subset of X such that $d(G) < s$. Then $d(G) < 1$. $\Rightarrow$ G is a singleton set. If $G = \{a\}$ or $\{b\}$ then $G \subseteq \{a, b\}$. If $G = \{c\}$ then $G \subseteq \{b, c\}$. This shows that s is a Lebesgue number for B. Let $s > 1$. Then $d(X) = 1 < s$. But $X \nsubseteq \{a, b\}$ and $X \nsubseteq \{b, c\}$. $\therefore$ any real number $s > 1$ is not a Lebesgue number.

Theorem: Lebesgue's covering lemma: In a sequentially compact metric space every open cover has a Lebesgue number. (M. Imp)
Proof: Let X be sequentially compact metric space and $\{G_i\}_{i \in I}$ be an open cover of X. Case (i) Suppose X contains no big sets. In this case, we will show that every positive real number is Lebesgue number for the open cover $\{G_i\}_{i \in I}$. Let $a > 0$ be a real number. Let A be a subset of X such that $d(A) < a$. Since X contains no big sets, A is not a big set. $\therefore \exists\, i \in I$ such that $A \subseteq G_i$. Hence a is a Lebesgue number for $\{G_i\}$.
Case (ii): Step (i): Suppose X contains big sets. Let $a' = \text{glb}\,\{d(A) / A \text{ is a big set}\}$. Clearly $0 \leq a' < \infty$. Now we show that $a' > 0$. If possible, suppose $a' = 0$. Now we construct an infinite sequence $\{x_n\}$ of distinct points. For this consider 1. Since $1 > 0$, and $a' = 0 = \text{glb}\,\{d(A) / A \text{ is a big set}\}$, there is a big set B_1 such that $0 < d(B_1) < 1$. Let $x_1 \in B_1$. Since $½ > 0$ $\exists$ a big set B_2 such that $0 < d(B_2) < ½$. Since B_2 is a big set containing atleast two points, take $x_2 \in B_2 \backslash \{x_1\}$. Clearly $x_1 \neq x_2$. Write $r_3 = \min\,\{1/3, d(\{x_1, x_2\})\}$. Since $x_1 \neq x_2$, we have $d(x_1, x_2) \neq 0$. $\therefore$ $r_3 > 0$.now $\exists$ a big set B_3

such that $0 < d(B_3) < r_3$. Now if $x_1 \in B_3$ then $x_2 \notin B_3$ (if both $x_1, x_2 \in B_3$ then $d(B_3) \geq d(x_1, x_2) \geq r_3 > d(B_3)$, a contradiction). Now x_1, x_2 and x_3 are distinct points.
After constructing $\{x_1, x_2, ..., x_n\}$, write $r_{n+1} = \min\{1/n, d(\{x_1, x_2, ..., x_n\})\}$. Since $r_{n+1} > 0$ $\exists$ a big set B_{n+1} such that $0 < d(B_{n+1}) < r_{n+1}$. Let $x_{n+1} \in B_{n+1} \setminus \{x_1, x_2, ..., x_n\}$.
In this way we construct a sequence $\{x_n\}$ of distinct points. Note that for each n, we have $d(B_n) < 1/n$.
Step (ii): Since X is sequentially compact, $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, which converges to a point $x \in X$. So, $x \in G_i$ for some $i \in I$. Since G_i is open, $\exists$ $r > 0$ such that $S_r(x) \subseteq G_i$. Consider the open sphere $S_{r/2}(x)$. Since $x_{n_k} \to x$, $\exists$ m such that $x_{n_m} \in S_{r/2}(x)$ for some $m \geq k$. Let $m \geq k$ and $0 < \frac{1}{n_m} < \frac{r}{2}$.
Take $y \in B_{n_m}$. Now $y, x_{n_m} \in B_{n_m} \Rightarrow d(y, x_{n_m}) < \frac{1}{n_m} < \frac{r}{2}$.
$\therefore$ $d(x, y) \leq d(x, x_{n_m}) + d(x_{n_m}, y) < \frac{r}{2} + \frac{r}{2} = r \Rightarrow y \in S_r(x) \subseteq G_i$. Hence $y \in B_{n_m}$ $\Rightarrow y \in G_i$. $\therefore$ $B_{n_m} \subseteq G_i$, a contradiction to the fact that B_{n_m} is a big set.
So $a' \neq 0$ Hence $a' > 0$.
Now we show that a' is a big number. Let Y be any subset of X with $d(Y) < a'$. Then Y is not a big set(If Y is a big set, then $a' \leq d(Y)$ (by construction of a') and so $a' \leq d(Y) < a'$, a contradiction]. This means that $Y \subseteq G$, for some $i \in I$. $\therefore$ a' is a Lebesgue number.

Definition: (i) Let X be a metric space and $\varepsilon > 0$. A subset A of X is called an ε – net if A is finite and $X = \bigcup_{a \in A} S_\varepsilon(a)$.
(ii) X is said to be totally bounded if it has an ε - net for each $\varepsilon > 0$.

Theorem: Every sequentially compact metric space is totally bounded. (M. Imp)
Proof: Let X be a sequentially compact metric space. If possible suppose X is not totally bounded. Ie. X has no ε - net for some $\varepsilon > 0$. Take this ε. Let $a_1 \in X$. Since $\{a_1\}$ is not an ε - net for X, $X \not\subseteq S_\varepsilon(a_1)$. Let $a_2 \in X \setminus S_\varepsilon(a_1)$. Clearly $d(a_1, a_2) \geq \varepsilon$.
Consider $\{a_1, a_2\}$. Since this is not an ε - net, $\exists$ $a_3 \in X \setminus \{S_\varepsilon(a_1) \cup S_\varepsilon(a_2)\}$. Clearly $d(a_1, a_3) \geq \varepsilon$, $d(a_3, a_2) \geq \varepsilon$.
Having chosen $a_1, a_2, ..., a_n$ select $a_{n+1} \in X \setminus \{S_\varepsilon(a_1) \cup S_\varepsilon(a_2) \cup ... \cup S_\varepsilon(a_n)\}$.
Continuing this process $\{a_n\}$ is a sequence of distinct points $\ni$ $d(a_i, a_j) \geq \varepsilon$ for $i \neq j$.
Since X is sequentially compact $\exists$ a convergent subsequence $\{a_{n_k}\}$ of $\{a_n\}$.
Since it is convergent it is also Cauchy sequence.
Since $\varepsilon > 0$, $\exists$ a positive integer k $\ni$ $d(a_{n_i}, a_{n_j}) < \varepsilon$ for all $n_i, n_j \geq k$, a contradiction to the fact that $d\left(a_{n_i}, a_{n_j}\right) \geq \varepsilon$. Hence X is totally bounded.

Theorem: Every sequentially compact metric space is compact. (very imp).
Proof: Let X be a sequentially compact metric space.
Let $\{G_i\}_{i\in I}$ be an open cover of X.
Since X is sequentially compact, by Lebesgue covering lemma, the open cover has a Lebesgue number a say.
Take $\varepsilon = a / 3 > 0$.
Since X is sequentially compact, X is totally bounded, and hence X has an ε - net, say $A = \{x_1, x_2, x_3, \ldots, x_n\}$.
$\Rightarrow X = \cup_{k=1}^{n} S_\varepsilon(x_k)$. We know that $d(S_\varepsilon(x_k)) \leq 2\varepsilon = 2a/3 < a$ for each $1 \leq k \leq n$. Since a is a Lebesgue number for the open cover $\{G_i\}$ and $d(S_\varepsilon(x_k)) < a$, we have that $S_\varepsilon(x_k) \subseteq G_{i_k}$ for some $i_k \in I$.
$\therefore X = \cup_{k=1}^{n} S_\varepsilon(x_k) \subseteq \cup_{k=1}^{n} G_{i_k} \subseteq X$.
$\Rightarrow X = \cup_{k=1}^{n} G_{i_k}$
Thus the open cover $\{G_i\}$ has finite subcover $\{G_{i_k}\}$, k = 1, 2, ... , n.
Hence X is compact.

Theorem: Any continuous mapping of a compact metric space into a metric space is uniformly continuous. (Imp).
Proof: Let f: X → Y be a continuous mapping of a compact metric space X into a metric space Y. Let d_1 and d_2 be the metrics on X and Y respectively. We prove that f is uniformly continuous. Let $\varepsilon > 0$. For any $x \in X$, consider the open sphere $S_{\frac{\varepsilon}{2}}\{f(x)\}$ with center f(x) and radius $\varepsilon/2$ in Y. Since f is continuous, we have that $f^{-1}\left[S_{\frac{\varepsilon}{2}}\{f(x)\}\right]$ is open in X. This is true for any $x \in X$.
Consider the family $\mathfrak{A} = \left\{f^{-1}\left[S_{\frac{\varepsilon}{2}}\{f(x)\}\right] / x \in X\right\}$.
It is clear that $\mathfrak{A}$ is a family of open sets in X which forms an open cover for X.
Since X is compact, it is sequentially compact.
So by Lebesgue covering lemma, the open cover $\mathfrak{A}$ has a Lebesgue number, say δ.
Suppose $x, x' \in X$ such that $d_1(x, x') < \delta \Rightarrow d(\{x, x'\}) < \delta$.
Since δ is a Lebesgue number, $\{x, x'\} \subseteq \left\{f^{-1}\left[S_{\frac{\varepsilon}{2}}\{f(y)\}\right]\right\}$ for some $y \in X$
$\Rightarrow f(x), f(x') \in S_{\frac{\varepsilon}{2}}\{f(y)\}$. $\Rightarrow d_2(f(x), f(y)) < \varepsilon / 2$ and $d_2(f(x'), f(y)) < \varepsilon / 2$.
Consider $d_2(f(x), f(x'))$.
Now $d_2(f(x), f(x')) \leq d_2(f(x), f(y)) + d_2(f(y), f(x')) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
So $\exists\, \delta > 0$ for any $x, x' \in X$ such that $d_1(x, x') < \delta \Rightarrow d_2(f(x), f(x')) < \varepsilon$.
Hence f is uniformly continuous.

Theorem: A metric space is compact if and only if it is complete and totally bounded. (very imp)

Proof: Suppose (X, d) be a metric space. Suppose X is compact.
Then X is sequentially compact. $\Rightarrow$ X is totally bounded.
Claim: X is complete. Let $\{x_n\}$ be a Cauchy sequence in X. Since X is sequentially compact $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. By a problem $\{x_n\}$ is convergent. Hence X is complete.
Conversely suppose that X is complete and totally bounded.
Claim: X is sequentially compact.
Claim: Every sequence has a Cauchy subsequence.
Let $S_1 = \{x_{11}, x_{12}, x_{13}, ...\}$ be an arbitrary sequence in X. If the set of points S_1 is finite, then there exists an element which repeats infinite number of times.
$\therefore$ S_1 has a constant subsequence which is convergent. Suppose the set of points of S_1 is infinite. Since X is totally bounded X has an ½ -net say $\{y_1, y_2, ..., y_n\}$
$\Rightarrow$ $X = \cup_{i=1}^{n} S_{\frac{1}{2}}(y_i)$

Then $S_1 = S_1 \cap X = S_1 \cap \left\{\cup_{i=1}^{n} S_{\frac{1}{2}}(y_i)\right\} = \cup_{i=1}^{n} \left\{S_1 \cap S_{\frac{1}{2}}(y_i)\right\}$

Since S_1 is infinite, $S_1 \cap S_{\frac{1}{2}}(y_i)$ is infinite for at least one i.

$\therefore$ S_1 has a subsequence, $S_2 = \{x_{21}, x_{22}, x_{23}, ...\}$ and all of the points of S_2 lie in the same open sphere of radius ½. We continue like this we have $S_1, S_2, ..., S_n, ...$ such that S_n is a subsequence of S_{n-1} and all of the points of S_n lie in some open sphere of radius 1/n.
Then $S = \{x_{11}, x_{22}, x_{33}, ...\}$ is a diagonal subsequence of S_i, i = 1, 2, ...
Claim: S is a Cauchy subsequence.
Let $\varepsilon > 0$. We can choose an integer $M > 0$ such that $2/M < \varepsilon$. Since S_i is a subsequence of S_{i-1}, for all $n, m \geq M$, $x_{nn}, x_{mm} \in S_M$. $\Rightarrow$ $x_{nn}, x_{mm} \in S_{1/M}(y)$ for some $y \in X$.
$\Rightarrow d(x_{nn}, y) < 1/M$, $d(x_{mm}, y) < 1/M \Rightarrow d(x_{nn}, x_{mm}) < 2/M < \varepsilon$.
$\therefore$ S is a Cauchy subsequence of S_1. Since X is complete, S is convergent sequence.
$\therefore$ S is a convergent subsequence of S_1. $\therefore$ X is sequentially compact.
Hence X is compact.
Theorem: A closed subspace of a complete metric space is compact iff it is totally bounded.

<u>Definition:</u> Let X be a compact metric space with metric d and let A be a nonempty set of continuous real or complex valued functions defined on X. A is said to be **equicontinuous** if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x, x' \in X$, $d(x, x') < \delta \Rightarrow |f(x) - f(x')| < \varepsilon$ for all $f \in A$.

<u>ASCOLI'S THEOREM.</u>

<u>Theorem</u>: If X is a compact metric space, then a closed subspace F of $C(X, \mathbb{R})$ or $C(X, \mathbb{C})$ is compact iff it is bounded and equicontinuous.

<u>Proof</u>: Let X be a compact metric space and F be a closed subspace of $C(X, \mathbb{R})$ or $C(X, \mathbb{C})$.

Suppose F is compact.

Since F is compact subspace of $C(X, \mathbb{R})$ or $C(X, \mathbb{C})$,] F is bounded.

Since X is compact every $f \in C(X, \mathbb{R})$ or $C(X, \mathbb{C})$ is uniformely continuous.

<u>Claim</u>: F is equicontinuous.

Let $\varepsilon > 0$.

Since F is compact, F is sequentially compact and hence F is totally bounded.

$\therefore$ F has an $\varepsilon/3$ – net, say $\{f_1, f_2, f_3, ..., f_n\}$. $\Rightarrow F = \bigcup_i^n S_{\varepsilon/3}(f_i)$.

Let $f \in F$.

$\Rightarrow f \in S_{\varepsilon/3}(f_k)$ for some k.

$\Rightarrow \|f - f_k\| < \varepsilon/3$.

$\Rightarrow |f(x) - f_k(x)| < \varepsilon/3$ for all $x \in X$...(i)

Since each $f_k \in F$ is uniformly continuous, for each k = 1, 2, 3, ..., n, $\exists\, \delta_k > 0$ such that $d(x, x') < \delta_k \Rightarrow |f_k(x) - f_k(x')| < \varepsilon/3$

Let $\delta = \min\{\delta_1, \delta_2, ..., \delta_n\}$.

Suppose $d(x, x') < \delta$.

$\Rightarrow d(x, x') < \delta_k$ for all k = 1, 2, ..., n.

$\Rightarrow |f_k(x) - f_k(x')| < \varepsilon/3$ for k = 1, 2, ..., n ... (ii)

Now $|f(x) - f(x')| \le |f(x) - f_k(x)| + |f_k(x) - f_k(x')| + |f(x') - f_k(x')|$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Hence F is equicontinuous.

<u>Converse</u>: Suppose F is bounded and equicontinuous.

Claim: F is sequentially compact.

Part (i): $C(X, \mathbb{R})$ is complete and F is closed in $C(X, \mathbb{R})$.

$\therefore$ F is complete.

Since X is compact it is separable.

$\therefore$ X has a countable dense subset say $A = \{x_2, x_3, ..., x_n, ...\}$, say.

Part II: Let $S_1 = \{f_{11}, f_{12}, f_{13}, ...\}$ be an arbitrary sequence.

Since F is bounded $\exists$ a real number k > 0 such that $\|f\| \leq$ k for all f $\in$ F.
$\Rightarrow |f(x)| < k$ for all f $\in$ F and x $\in$ X.
Then $\{f_{1j}(x_2)\}$ is a bounded sequence of real numbers.
$\therefore$ This sequence has a convergent subsequence.
Let $S_2 = \{f_{21}, f_{22}, f_{23}, \ldots\}$ be a subsequence of S_1 such that $\{ f_{2j}(x_2) \}$ is convergent.
[Then $\{f_{2j}(x_3)\}$is a bounded sequence of real numbers.
As above this sequence has a convergent subsequence $S_3 = \{f_{31}, f_{32}, f_{33}, \ldots\}$ of S_2 such that $\{f_{3j}(x_3)\}$ is convergent.]
Continuing in this way we have $S_1 = \{f_{11}, f_{12}, f_{13}, \ldots\}$, $S_2 = \{f_{21}, f_{22}, f_{23}, \ldots\}$, ..., $S_i = \{f_{i1}, f_{i2}, f_{i3}, \ldots\}$... such that S_i is a subsequence of S_{i-1}, and $\{ f_{ij}(x_i) \}$ is convergent.
Part III: Then $S = \{f_{11}, f_{22}, f_{33}, \ldots\}$ is a diagonal sequence of S_i, i = 1, 2, ... and subsequence of S_1.
Write $f_n = f_{nn}$. $\therefore$ $\{ f_n(x_i) \}$ is a convergent subsequence for each $x_i \in$ A.
Claim: S is a Cauchy sequence.
Let $\varepsilon > 0$.
Since F is equicontinuous, $\exists\ \delta > 0 \ni d(x, x') < \delta \Rightarrow |f_n(x) - f_n(x')| < {}^{\varepsilon}/_{3}$...(i)
Since A is dense in X, $\mathcal{B} = \{ S_\delta(x_i) : x_i \in A \}$ is an open cover for X.
[For x $\in$ X = $\bar{A} \Rightarrow S_\delta(x) \cap A \neq \phi \Rightarrow x_i \in S_\delta(x)$ for some $x_i \in A \Rightarrow d(x, x_i) < \delta$
$\Rightarrow$ x $\in S_\delta(x_i) \Rightarrow$ x $\in \cup S_\delta(x_i)$.]
Since X is compact X $\subseteq \cup_{i=2}^{t} S_\delta(x_i)$ for some positive integer t.
Since $\{f_n(x_i)\}$ is a convergent subsequence for each $x_i \in$ A,
$\{f_n(x_i)\}$ is a Cauchy sequence for each $x_i \in$ A.
$\therefore \{f_n(x_i)\}$ is a Cauchy sequence for each $x_2, x_3, \ldots, x_t$.
$\Rightarrow$ For each i = 2, 3, ..., t $\exists$ integer $M_i \ni |f_n(x_i) - f_m(x_i)| < {}^{\varepsilon}/_{3}$ $\forall$ n, m $\geq M_i$.
Write M = max $\{M_i$, i = 2, 3, ..., t$\}$. Then $|f_n(x_i) - f_m(x_i)| < {}^{\varepsilon}/_{3}$ $\forall$ n, m $\geq$ M.
Let x $\in$ X $\subseteq \cup_{i=2}^{t} S_\delta(x_i)$
$\Rightarrow$ x $\in S_\delta(x_i)$ for some i, 2 $\leq$ i $\leq$ t
$\Rightarrow$ d(x, x') < δ for n, m $\geq$ M.
$\therefore |f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(x)|$
$< {}^{\varepsilon}/_{3} + {}^{\varepsilon}/_{3} + {}^{\varepsilon}/_{3} = \varepsilon$ for all n, m $\geq$ M.
$\Rightarrow$ sup $\{|f_n(x) - f_m(x)|\} < \varepsilon \Rightarrow \|f_n - f_m\| < \varepsilon$ for all n, m $\geq$ M.
$\therefore$ S is a Cauchy sequence.
$\therefore$ S is convergent subsequence of S_1.
$\therefore$ F is sequentially compact and hence F is compact.

Fax: 08816-227318 off: 08816-224072, 224119,228342 Mobile: 9491334119

DANTULURI NARAYANA RAJU COLLEGE

(Autonomous)

BHIMAVARAM, W.G.DIST, ANDHRA PRADESH, INDIA, PIN- 534202.

(Accredited at 'B++' level by NAAC)

(Affiliated to Adikavi Nannaya University, Rajamahendravaram)

E – CONTENT

PAPER: M 104, TOPOLOGY

M. Sc. I YEAR, SEMESTER - I

UNIT – IV: SEPARATION

PREPARED BY

K, C. TAMMI RAJU, M. Sc.

HEAD OF THE DEPARTMENT

DEPARTMENT OF MATHEMATICS,

PG COURSES

DNR COLLEGE (A),

BHIMAVARAM – 534202

M 104: TOPOLOGY

UNIT IV – SEPARATION

T_1 - SPACES AND HAUSDORFF SPACES

<u>Definition</u>: A T_1 - space is a topological space in which given any pair of distinct elements, each has a neighbourhood which does not contain the other.
(equivalently, if x and y are elements such that $x \neq y$ then there exists neighbourhoods G and H of x and y respectively such that $y \notin G$ and $x \notin H$.

<u>Example</u> (i): Suppose $X = \{a, b, c\}$, $\Im = \{\phi, \{a\}, \{a, b\}, X\}$. Then X is not a T_1 – space.
(ii) Let X be an infinite set. Write $\Im = \{A \subseteq X: A' \text{ is finite}\} \cup \{\phi\}$.
Then X is a T_1 - space. Let $x, y \in X$ such that $x \neq y$.
Then $\{x\}'$ and $\{y\}'$ are open sets in X; and $x \in \{y\}'$ and $y \in \{x\}'$ but $x \notin \{x\}'$, $y \notin \{y\}'$. Hence X is a T_1 Space.

Note: Every discrete topological space is a T_1 - Space.
Remark: Every subspace of a T_1-Space is also a T_1-Space.
<u>Proof</u>: Let X be a T_1-space and Y be any subspace of X.
Let y_1, y_2 where $y_1 \neq y_2$ be any two - points in Y.
$\because Y \subseteq X$, X is a T_1-space, $\exists$ an open sets G and H in X $\ni y_1 \in G$, $y_2 \notin G$, $y_2 \in H$, $y_1 \notin H$. Put $A = G \cap Y$ and $B = H \cap Y$.
Then A and B are open sets in Y $\ni y_1 \in A$, $y_2 \notin A$, $y_2 \in B$ and $y_1 \notin B$.
$\therefore$ Y is a T_1-Space.
Thus every subspace of a T_1-Space is also a T_1-Space.

<u>Theorem:</u> A topological space is a T_1-space if and only if each point is a closed set.
<u>Proof</u>: Let X be a topological space. Assume that X is a T_1 space.
Let $x \in X$. Now we show that $\{x\}$ is a closed set.
To prove this, it is enough to prove $\{x\}'$ is open.
Let $y \in \{x\}'$. Then $y \neq x$. Since X is a T_1-space and $x, y \in X$ such that $x \neq y$, there exists neighbourhood H of y such that H does not contain x.
Now $y \in H \subseteq \{x\}'$. This shows that y is an interior point of $\{x\}'$.
Hence $\{x\}'$ is open.
Converse: Suppose that each point is a closed set.

Let x, y be any two points of X such that $x \neq y$.
Put $G = \{y\}'$ and $H = \{x\}'$. By hypothesis, G and H are open sets such that $x \in G$, $y \notin G$ and $y \in H$, $x \notin H$. Therefore, X is a T_1-space.

Definition: A Hausdorff space is a topological space in which each pair of distinct points can be separated by open sets (equivalently, if $x \neq y$ are distinct points, then there exists open sets G and H such that $x \in G$, $y \in H$ and $G \cap H = \phi$).

Result: (i) Every discrete topological space is a Hausdorff space.
Proof: Let $(X, \mathfrak{I})$ be a discrete topological space.
Let $x, y \in X$ and $x \neq y$.
Then $\{x\}, \{y\}$ are open such that $x \in \{x\}$, $y \in \{y\}$ and $\{x\} \cap \{y\} = \phi$.
$\therefore (X, \mathfrak{I})$ is Hausdorff space.

Result (ii) Every metric space is a Hausdorff space.
Proof: Let (X, d) be a metric space. Let $x, y \in X$ and $x \neq y$.
Then $d(x, y) > 0$. Let $r = d(x, y)$.
Then $S_{\frac{r}{2}}(x), S_{\frac{r}{2}}(y)$ are open sets, $x \in S_{\frac{r}{2}}(x)$, $y \in S_{\frac{r}{2}}(y)$ and $S_{\frac{r}{2}}(x) \cap S_{\frac{r}{2}}(y) = \phi$
$\therefore$ (X, d) is Hausdorff space.

Result (iii): Every Hausdorff space is a T_1-space. But the converse need not be true.
Proof: Let $(X, \mathfrak{I})$ be a Hausdorff space and $x, y \in X \ni x \neq y$.
Then $\exists$ open sets G and H $\ni x \in G$, $y \in H$ and $G \cap H = \phi$. Clearly $y \notin G$ and $x \notin H$. $\therefore$ Every Hausdorff space is a T_1 – space.
Converse need not be true. For this consider the following example.
Let X be an infinite set. Write $\mathfrak{I} = \{A \subseteq X: A' \text{ is finite}\} \cup \{\phi\}$.
Then $(X, \mathfrak{I})$ is a T_1 - Space (see example).
Now we will show that X is not a Hausdorff space.
In a contrary way, suppose that X is a Hausdorff space.
Take $x, y \in X$ such that $x \neq y$. Since X is Hausdorff there exists neighbourhoods G and H of x and y respectively such that $G \cap H = \phi$ (by def.).
Since G and H are non-empty open sets, we have G' and H' are finite.
Now $G \cap H = \phi \Rightarrow (G \cap H)' = \phi' \Rightarrow G' \cup H' = X$.
This shows that X is finite, a contradiction. Hence X is not Hausdorff.

Result (iv): Every subspace of a Hausdorff space is a Hausdorff space.

Proof: Let X be a Hausdorff space and Y be any subspace of X.
Let $y_1 \neq y_2$ be two - points in Y.
$\because$ X is Hausdorff, $\exists$ open sets G and H in X $\ni$ $y_1 \in$ G, $y_2 \in$ H, and G $\cap$ H = ϕ.
Put A = G $\cap$ Y and B = H $\cap$ Y.
Then A and B are open sets in Y. Clearly $y_1 \in$ A, $y_2 \in$ B and A $\cap$ B $\subseteq$ G $\cap$ H = ϕ.
$\therefore$ Y is a Hausdorff space.
Hence every subspace of a Hausdorff space is a Hausdorff space.

Theorem: The product of any non-empty class of Hausdorff spaces is Hausdorff.
Proof: Let $\{X_i\}$ be a non-empty class of Hausdorff spaces.
Let X = P_iX_i be the Product of X_i's.
Let x = $\{x_i\}$ and y = $\{y_i\}$ be any two distinct points in X.
Then $x_{i_0} \neq y_{i_0}$ for at least one index i_0.
Since X_{i_0} is a Hausdorff space and $x_{i_0} \neq y_{i_0}$ are distinct points in X_{i_0} there exists open sets G_{i_0} and H_{i_0}, in X_{i_0} such that $x_{i_0} \in G_{i_0}$, $y_{i_0} \in H_{i_0}$ and $G_{i_0} \cap H_{i_0} = \phi$.
Define A = P_iA_i where $A_i = X_i$ for $i \neq i_0$ and $A_{i_0} = G_{i_0}$ and B = P_iB_i where $B_i = X_i$ for $i \neq i_0$ and $B_{i_0} = H_{i_0}$.
Now A and B are open sets in X such that A $\cap$ B = ϕ, x $\in$ A and y $\in$ B.
Hence X is Hausdorff.

Theorem: In a Hausdorff space, any point and a disjoint compact subspace can be separated by open sets. In the sense that they have disjoint neighbourhoods (that is, if x is any point and if C is a compact subspace such that x $\notin$ C then there exists disjoint open sets G and H such that x $\in$ G and C $\subseteq$ H).

Proof: Let X be a Hausdorff space. Let x be any point in X, and let C be any disjoint compact subspace. Now if y $\in$ C, then x $\neq$ y (since x $\notin$ C). Since X is a Hausdorff space, there exists open sets G_y and H_y such that x $\in G_y$, y $\in H_y$, and $G_y \cap H_y = \phi$.
Now $\{H_y\}_y$ is a class of open sets such that C $\subseteq \bigcup_{y \epsilon C} H_y$. Since C is compact, there exists a finite subclass of $\{H_y\}$, which we denote by $\{H_{y_1}, H_{y_2}, \dots, H_{y_n}\}$ such that C $\subseteq H_{y_1} \cup H_{y_2} \cup \dots \cup H_{y_n}$. Let $G_{y_1}, G_{y_2}, \dots, G_{y_n}$ be open sets which corresponds to the sets $H_{y_1}, H_{y_2}, \dots, H_{y_n}$. Put G = $\bigcap_{i=1}^{n} G_{y_i}$, and H = $\bigcup_{i=1}^{n} H_{y_i}$.
Now for $1 \leq i \leq n$, consider G $\cap H_{y_i} \subseteq G_{y_i} \cap H_{y_i} = \phi$. (since $G_y \cap H_y = \phi$)
$\Rightarrow$ G $\cap H_{y_i} = \phi$. Therefore G $\cap$ H = G $\cap \left[\bigcup_{i=1}^{n} H_{y_i}\right] = \bigcup_{i=1}^{n}\left[G \cap H_{y_i}\right] = \phi$.
Hence G and H are disjoint open sets such that x $\in$ G and C $\subseteq$ H.

Theorem: Every compact subspace of a Hausdorff space is closed.
Proof: Let C be a compact subspace of a Hausdorff space X. To prove C is closed, it is enough to prove that C′ is open.
If C′ is empty then clearly it is open. We assume that C' is non-empty. Let $x \in C'$. Then $x \notin C$. By above theorem, there exists disjoint open sets G and H such that $x \in G$ and $C \subseteq H$. Since $G \cap H = \phi$, we have $G \subseteq H'$ and $H' \subseteq C'$ (since $C \subseteq H$). Therefore $G \subseteq C'$ and $x \in G \subseteq C'$. Therefore C′ is open which implies that C is closed.

Theorem 8*: A one – to – one continuous mapping of a compact space onto a Hausdorff space is a homeomorphism.
Proof: Let $f: X \to Y$ be a one - to - one continuous mapping of a compact metric space X onto a Hausdorff space Y. We must show that f(G) is open in Y whenever G is open in X. To prove this, we first show that f(F) is closed in Y whenever F is closed in X.
If F is empty, then $f(F) = \phi$ and hence it is closed. Assume that F is non-empty. Since X is compact, we have F is compact. Since f is continuous, f(F) is compact. Therefore, by a theorem, f(F) is closed. Thus, we proved that f(F) is closed in Y whenever F is closed in X.
If G is open in X, then G′ is closed in X. Now f(G′) is closed in Y. But $f(G') = (f(G))'$. Therefore $(f(G))'$ is closed in Y
$\Rightarrow f(G) = [\{f(G)\}']'$ is open in Y. Thus, f is a homeomorphism.

COMPLETELY REGULAR SPACES AND NORMAL SPACES

Definition: A normal space is a T_1-space in which each pair of disjoint closed sets can be separted by open sets. In the sense that they have disjoint neighbourhoods.
Remark: Every normal space is Hausdorff.
Proof: Let X be a normal space. Let x and y be distinct points in X.
Now $\{x\}$ and $\{y\}$ are disjoint closed sets. Since X is normal, there exists disjoint open sets G and H such that $\{x\} \subseteq G$ and $\{y\} \subseteq H$.
Now G and H are disjoint neighbourhoods of x and y respectively.
Therefore, X is Hausdorff. Hence every normal space is Hausdorff.

Theorem: (11*) Every compact Hausdorff space is normal.
Proof: Let X be a compact Hausdorff space. Since X is Hausdorff, it is a T_1-space. Let A and B be a pair of disjoint closed sets. If either of the closed sets is empty, we can take the empty set as a neighbourhood of it, and the full space as the neighbourhood of the other. So, we may assume that both A and B are non-empty

sets. Since X is compact, we have that A and B are compact sets. Let $x \in A$. Now $x \in X$ and B is a compact subspace such that $x \notin B$. Since X is Hausdorff, we have that x and B have disjoint neighbourhoods, say G_x and H_x respectively.
Therefore $\{G_x\}_{x\in A}$, is a class of open sets such that $A \subseteq \cup G_x$, $x \in A$. Since A is compact, there exists a finite subclass of the class of $\{G_x\}_{x\in A}$, which we denote by $\{G_{x_1}, G_{x_2}, \ldots, G_{x_n}\}$ such that $A \subseteq G_{x_1} \cup G_{x_2} \cup \ldots \cup G_{x_n}$. Let H_{x_1}, $H_{x_2}, \ldots, H_{x_n}$ be the neighbourhoods of B which corresponds to $G_{x_1}, G_{x_2}, \ldots, G_{x_n}$
Put $G = \bigcup_{i=1}^{n} G_{x_i}$ and $H = \bigcap_{i=1}^{n} H_{x_i}$.
Now G and H are neighbourhoods of A and B respectively, such that
$G \cap H = \left(\bigcup_{i=1}^{n} G_{x_i}\right) \cap H = \bigcup_{i=1}^{n}\left(G_{x_i} \cap H\right) \subseteq \bigcup_{i=1}^{n}\left(G_{x_i} \cap H_{x_i}\right) = \phi$.
Therefore $G \cap H = \phi$. Hence X is normal.

Problem: (1*): Let X be a T_1 - space. Show that X is normal if and only if each neighbourhood of a closed set F contains the closure of some neighbourhood of F (that is, if O is a neighbourhood of F then there exists neighbourhood G of F such that $F \subseteq G \subseteq \bar{G} \subseteq O$.
Solution: Assume that X is normal. Let O be a neighbourhood of F.
Then $F \cap O' = \phi$ (since $F \subseteq O$). Now F and O' are disjoint closed sets.
Since X is normal, $\exists$ disjoint open sets G and H $\ni F \subseteq G$ and $O' \subseteq H$.
Since $G \cap H = \phi$, we have $G \subseteq H'$. Now $O' \subseteq H \Rightarrow H' \subseteq (O')' = O$.
Since H' is closed, we have that $\bar{G} \subseteq H'$. $\therefore$, $F \subseteq G \subseteq \bar{G} \subseteq H' \subseteq O$.
Hence $F \subseteq G \subseteq \bar{G} \subseteq O$, and G is open.
Conversely, suppose that X has the stated property. Let A and B be disjoint closed sets. Since $A \cap B = \phi$, we have $A \subseteq B'$. ie B' is a neighbourhood of A.
Now by converse hypothesis, there exists an open set G such that $A \subseteq G \subseteq \bar{G} \subseteq B'$.
Since $\bar{G} \subseteq B'$, we have $(B')' \subseteq \bar{G}'$. $\Rightarrow B \subseteq \bar{G}'$.
Since $\bar{G}'$ is a neighbourhood of B, again by converse hypothesis, there exists an open set H such that $B \subseteq H \subseteq \bar{H} \subseteq \bar{G}'$.
Now consider $G \cap H \subseteq \bar{G} \cap \bar{H} = \phi$. (since $\bar{H} \subseteq \bar{G}'$) $\Rightarrow$ G and H are disjoint.
Thus, G and H are disjoint neighbourhoods of A and B. Hence X is normal.

URYSHONS LEMMA AND TETZE EXTENSION THEOREM

URYSOHN'S LEMMA: Let X be a normal space and let A and B be disjoint closed subspaces of X. Then there exists continuous real valued function f on X, all of whose values lie in the closed unit interval [0, 1] such that $f(A) = 0$ and $f(B) = 1$.
Proof: For each pair of rational numbers r, s we define an open set G_r such that $r < s \Rightarrow \bar{G}_r \subseteq G_s$, if $r < 0$, define $G_r = \phi$; if $r > 1$, define $G_r = X$.

Let $\{r_1, r_2, \ldots, r_n, \ldots\}$ be a listing of rational numbers in [0, 1] with $r_1 = 0$ and $r_2 = 1$.
Define $G_{r_2} = B'$.
Then G_{r_2} is a neighbourhood of A (since $A \cap B = \phi$, we have $A \subseteq B'$).
By hypothesis ($\because$ X is normal), there exists an open set $G_{r_1} \ni A \subseteq G_{r_1} \subseteq \overline{G_{r_1}} \subseteq G_{r_2}$.
Suppose we have defined $G_{r_1}, G_{r_2}, \ldots, G_{r_{n-1}}$
We now define G_{r_n} as follows: Choose largest r_i and smallest r_j such that $i, j < n$ and $r_i < r_n < r_j$. Now $r_i < r_j \Rightarrow \bar{G}_{r_i} \subseteq G_{r_j}$.

$A \subseteq G_{r_1} \subseteq \overline{G_{r_1}} \subseteq G_{r_2}$

$\overline{G_{r_1}} \subseteq G_{r_3} \subseteq \overline{G_{r_3}} \subseteq G_{r_2}$;

$\overline{G_{r_1}} \subseteq G_{r_4} \subseteq \overline{G_4} \subseteq G_{r_3} \subseteq \overline{G_{r_3}} \subseteq G_{r_5} \subseteq \overline{G_5} \subseteq G_{r_2}$

$\overline{G_{r_1}} \subseteq G_{r_6} \subseteq \overline{G_{r_6}} \subseteq G_{r_4} \subseteq \overline{G_4} \subseteq G_{r_7} \subseteq \overline{G_{r_7}} \subseteq G_{r_3} \subseteq \overline{G_{r_3}} \subseteq G_{r_8} \subseteq \overline{G_{r_8}} \subseteq G_{r_5} \subseteq \overline{G_5} \subseteq G_{r_9} \subseteq \overline{G_{r_9}} \subseteq G_{r_2}$

$\overline{G_0} \subseteq G_{\frac{1}{2}} \subseteq \overline{G_{\frac{1}{2}}} \subseteq G_1$;

$\overline{G_0} \subseteq G_{\frac{1}{4}} \subseteq \overline{G_{\frac{1}{4}}} \subseteq G_{\frac{1}{2}} \subseteq \overline{G_{\frac{1}{2}}} \subseteq G_{\frac{3}{4}} \subseteq \overline{G_{\frac{3}{4}}} \subseteq G_1$;

$\overline{G_0} \subseteq G_{\frac{1}{8}} \subseteq \overline{G_{\frac{1}{8}}} \subseteq G_{\frac{1}{4}} \subseteq \overline{G_{\frac{1}{4}}} \subseteq G_{\frac{3}{8}} \subseteq \overline{G_{\frac{3}{8}}} \subseteq G_{\frac{1}{2}} \subseteq \overline{G_{\frac{1}{2}}} \subseteq G_{\frac{5}{8}} \subseteq \overline{G_{\frac{5}{8}}} \subseteq G_{\frac{3}{4}} \subseteq \overline{G_{\frac{3}{4}}} \subseteq G_{\frac{7}{8}} \subseteq \overline{G_{\frac{7}{8}}} \subseteq G_1$;

$\overline{G_{r_1}} \subseteq G_{r_4} \subseteq \overline{G_4} \subseteq G_{r_3} \subseteq \overline{G_{r_3}} \subseteq G_{r_5} \subseteq \overline{G_5} \subseteq G_{r_2}$

$\overline{G_{r_1}} \subseteq G_{r_6} \subseteq \overline{G_{r_6}} \subseteq G_{r_4} \subseteq \overline{G_4} \subseteq G_{r_7} \subseteq \overline{G_{r_7}} \subseteq G_{r_3} \subseteq \overline{G_{r_3}} \subseteq G_{r_8} \subseteq \overline{G_{r_8}} \subseteq G_{r_5} \subseteq \overline{G_5} \subseteq G_{r_9} \subseteq \overline{G_{r_9}} \subseteq G_{r_2}$

Again, by hypothesis, $\exists$ an open set $G_{r_n} \ni$ $\bar{G}_{r_i} \subseteq G_{r_n} \subseteq \bar{G}_{r_n} \subseteq G_{r_j}$.
By induction for each rational number r_n $\exists$ an open set $G_{r_n} \ni r_n < r_m \Rightarrow \bar{G}_{r_n} \subseteq G_{r_m}$
Define $f: X \to R$ by $f(x) = \text{Inf}\{r: x \in G_r\}$.
We now show that $f(x) \in [0, 1]$ for all $x \in X$.
Let x be any arbitrary point in X.
By the definition of G_r's, $x \in G_r \Rightarrow r \geq 0$. Therefore $f(x) \geq 0$.
If $f(x) > 1$, then choose a rational number 'r' such that $f(x) > r > 1$.
Now $r > 1 \Rightarrow G_r = X$. Let $x \in X \Rightarrow x \in G_r \Rightarrow f(x) \leq r$, a contradiction to $f(x) > r$.
Thus, for $x \in X$, $0 \leq f(x) \leq 1$. Therefore $f(x) \in [0, 1]$.
If $a \in A$, then $a \in G_{r_1} \Rightarrow f(a) \leq r_1 \Rightarrow f(a) \leq 0 = r_1 \Rightarrow f(a) = 0$ (since $f(a) \geq 0$).
Therefore $f(A) = 0$. Suppose $b \in B$. Then $b \in G_r \Rightarrow r \geq 1$, for if $r < 1 = r_2$ then $\bar{G}_r \subseteq G_{r_2}$ which $\Rightarrow b \in G_{r_2} = B'$, a contradiction.
$\therefore$, $f(b) \geq 1$. But $f(b) \leq 1$ (since $f(x) \leq 1$ for all x). Hence $f(b) = 1$.

Since b $\in$ B is arbitrary, we have that f(B) = 1.
We show that f is continuous: All the intervals of the form (a, b) where a and b are real, form an open base for the real number system R.
$\therefore$, to show f is continuous, it suffices to show f^{-1}(a, b) is open, for any reals a, b.
For this, first we show that f(x) < b $\Leftrightarrow$ x $\in$ G_r for some r < b. Suppose f(x) < b.
By def. of f(x) there exists a rational number r such that x $\in$ G_r, and r < b.
Conversely suppose that x $\in$ G_r for some r < b. Then f(x) $\leq$ r and r < b $\Rightarrow$ f(x) < b.
Consider $f^{-1}[(-\infty, b)]$ = {x $\in$ X: f(x) < b} = $\cup_{r<b} G_r \Rightarrow f^{-1}[(-\infty, b)]$ is open.
Similarly, we can prove that $f^{-1}[(a, \infty)] = \cup_{r>a} (\bar{G}_r)'$.
Therefore $f^{-1}[(a, \infty)]$ is open. Now $f^{-1}[(a, b)] = f^{-1}[(-\infty, b)] \cap f^{-1}[(a, \infty)]$.
Hence $f^{-1}[(a, b)]$ is open. Thus, f is continuous.

Definition: A completely regular space is a T_1-space X with the property that if x is any point and 'F' is any closed subspace which does not contain x, then there exists a real continuous function f on X, all of whose values lie in [0, 1] such that f(x) = 0 and f(F) = 1.

Theorem (1*): Every normal space is completely regular.
Proof: Let X be a normal space. Then X is T_1-space. Let x $\in$ X and F be any closed subspace of X which does not contain x. Put A = {x}. Now A and F are disjoint closed subspaces. By Uryshon's lemma, there exists a continuous real function f, all of whose values lie in the closed interval [0, 1] such that f(A) = 0, f(F) = 1. Therefore f(x) = 0 & f(F) = 1. Hence X is completely regular.

Theorem: Every completely regular space is Hausdroff.
Proof: Let X be a completely regular space.
Let x and y be any two distinct elements in X.
Put F = {y}. Now x $\in$ X and F is a closed subspace, which does not contain x.
Since X is completely regular, there exists a continuous function f: X$\to$ R such that f(x) = 0 and f(F) = 1. Let r be any real number such that 0 < r < 1.
Now {z $\in$ X: f{z) > r}, and {z $\in$ X: f(z) < r} are disjoint neighbourhoods of 'y' and 'x' respectively. Therefore, X is Hausdorff.

Theorem: Every subspace of a completely regular space is completely regular.
Proof: Let X be a completely regular space and let Y be a subspace of X.
Let x $\in$ Y, and F be a closed subspace of Y, which does not contain x.
Then F = Y $\cap$ H, where H is a closed subspace of X. Also, x $\notin$ H. Since X is completely regular, there exists a continuous function f: X $\to$ R, all of whose values lie in [0, 1], such that f(x) = 0 and f{H) = 1. Define 'g' to be the restriction

of f to Y. Then g: Y → R is continuous and g(y) = f(y) ∈ [0, 1] for all y ∈ Y.
Since x ∈ Y, we have 0 = f(x) = g(x) ⇒ g(x) = 0. Now y ∈ F = Y ∩ H
⇒ y ∈ Y and y ∈ H ⇒ g(y) = f(y) and f(y) = 1 ⇒ g(y) = 1. Therefore g(F) = 1.
Hence Y is completely regular.

<u>Theorem</u>: (9*) (TIETZE EXTENSION THEOREM)
Let X be a normal space 'F' a closed subspace of X, and f a continuous real function defined on F whose values lie in the closed interval [a, b]. Then f has a continuous extension f^1 defined on all of X whose values also lie in [a, b].
<u>Proof</u>: Step (i): If a = b then the function f^1 defined by $f^1(x)$ = a for all x ∈ X is a continuous function of X into [a, b] such that $f^1(x)$ = f(x) for all x ∈ F.
Step (ii): Suppose a < b. Assume that [a, b] is the smallest closed interval containing the range of f and without loss of generality a = – 1 and b = 1.
Write f_0 = f. Then the domain of f_0 is F.
Now we define two subsets A_0 and B_0 of F as $A_0 = \left\{x\epsilon F: f_0(x) \le -\frac{1}{3}\right\}$ and
$B_0 = \left\{x\epsilon F: f_0(x) \ge \frac{1}{3}\right\}$. Since [– 1, 1] is the smallest closed interval containing the range of f_0, we have A_0 and B_0 are non-empty. Clearly A_0 and B_0 are disjoint.
Since f_0 is continuous, we have that $A_0 = {f_0}^{-1}\left[-1, \frac{1}{3}\right]$ and $B_0 = {f_0}^{-1}\left[\frac{1}{3}, 1\right]$ are closed in F. Since F is a closed subspace of X, we have A_0 and B_0 are closed in X.
[Now X is a normal space, A_0, B_0 are disjoint closed subspaces of X, and $\left[-\frac{1}{3}, \frac{1}{3}\right]$ is a closed interval.] Then by the Uryshon's lemma, there exists continuous function g_0: X → $\left[-\frac{1}{3}, \frac{1}{3}\right]$ such that $g_0(A_0) = -\frac{1}{3}$ and $g_0(B_0) = \frac{1}{3}$.
Write $f_1 = f_0 - g_0$. Then f_1 is a continuous function of F and $|f_1(x)| < \frac{2}{3}$ ∀ x ∈ F.
Next, we define two subsets A_1 and B_1 of F as $A_1 = \left\{x\epsilon F: f_1(x) \le \left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)\right\}$} and
$B_1 = \left\{x\epsilon F: f_1(x) \ge \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\right\}$}. Then A_1 and B_1 are non-empty disjoint closed subsets of F and hence A_1 and B_1 are disjoint closed subspaces of X.
Since X is normal by Urysohn's lemma, there exists a continuous function
g_1: X → $\left[\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right), \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\right]$ ∋ $g_1(A_1) = \left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)$, and $g_1(B_1) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$.
Write $f_2 = f_1 - g_1 = f_0 - (g_0 + g_1)$. Then f_2 is a continuous function on F, and
$|f_2(x)| \le \left(\frac{2}{3}\right)^2$ for all x ∈ F.
If we continue this process, we get a sequence $\{f_n\}$ of continuous functions defined on F and $\{g_n\}$ of continuous functions defined on X with the property that:
$f_n = f_0 - (g_0 + g_1 + ... + g_{n-1})$ and $|f_n(x)| \le \left(\frac{2}{3}\right)^n$ ∀ x ∈ F and $|g_n(x)| \le \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n$.

Step (iii). Write $s_n = g_0 + g_1 + ... + g_{n-1}$. Then $\{s_n\}$ is a sequence of partial sums of an infinite series of functions of C(X, R).

C(X, R) is complete and $|g_n(x)| \leq \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n$.

Now $\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n = 1$. By Cauchy's criterion for uniform convergence, $\sum g_n(x)$ converges uniformly to a bounded continuous real function f^1 defined on X such that $|f^1(x)| \leq 1$. That is, $\{s_n\}$ converges uniformly to f^1 on X.

ie,, $\lim s_n = f^1$ on X … (i).

Since the sequence $\left\{\left(\frac{2}{3}\right)^n\right\}$ converges to 0, for $\varepsilon > 0$ there exists a positive integer N such that $\left(\frac{2}{3}\right)^n < \varepsilon$ for all $n \geq N$. $\Rightarrow |f_n(x)| < \varepsilon$ for all $n \geq N$ and for all $x \in F$

$\Rightarrow f_n \to 0$ uniformly on F $\Rightarrow \lim s_n = f_0$ on F … {ii)

From (i) and (ii) $f_0 = f^1$ on F.

That is, $f^1/F = f_0$, that is, $f^1/F = f$.

This shows that f^1 is a continuous extension of f on X.

Note: If X is a normal space which contains only a finite number of points, then the topology on X is the discrete topology.

<u>Problem</u>: Deduce the Urysohn's lemma from Tietze extension theorem.

<u>Proof</u>: Let A, B be two disjoint closed subsets of a normal space X. Since A, B are closed, we have that $F = A \cup B$ is also a closed subset of X.

Define f: $F \to [0, 1]$ by $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Since $A \cap B = \phi$, we have that f is well defined. Clearly f is a constant function on A and also on B. So, f is continuous on both A and B and hence f is continuous on $F = A \cup B$ (since $A \cap B = \phi$).

Now by Tietze extension theorem, there exists a continuous function $f':X \to [0, 1]$ such that f' is an extension of f. Now $f'(A) = f(A) = 0$ and $f'(B) = f(B) = 1$.

This completes the proof.

THE URYSOHN'S IMBEDDING THEOREM

<u>Definition</u>: A topological space X is said to be metrizable if and only if there exists a metric 'd' for X which induces the same topology as the topology of X.

<u>Note</u>: If X is a metric space with finite number of points then the topology on X induced by the given metric is the discrete topology on X.

Verification: Let (X, d) be a metric space with finite number of points.

So, take $X = \{x_1, x_2, \ldots, x_n\}$. Write $r = \min\{d(x_i, x_j): i \neq j, 1 \leq i \leq n; 1 \leq j \leq n\}$.

Then for any $x_i \in X$, $S_r(x_i) = \{x_i\}$ which is an open set.
This shows that singleton sets are open in X.
Hence the topology on X is the discrete topology on X.

URYSOHN'S IMBEDDING THEOREM: (3*)
If X is a second countable normal space then there exists a homeomorphism f of X onto a subspace of R^∞, and therefore X is metrizable.
Proof: we may assume that X has infinitely many points, for otherwise it would be finite and discrete, and clearly homeomorphic to any subspace of R^∞ with the same number of points.
Since X is second countable, X has a countable infinite open base $B = \{G_1, G_2, ...\}$ whose members are different from ϕ and X. Let $G_j \in B$ and $x \in G_j$. Then $\{x\}$ is a closed set. Since X is normal, there exists $G_i \in B$ such that $x \in G_i \subseteq \bar{G_i} \subseteq G_j$.
So, for a given G_j and $x \in G_j$, we have a pair (G_i, G_j) of open sets in B such that $\bar{G_i} \subseteq G_j$. The set of all ordered pairs (G_i, G_j) is countably infinite.
So, we can arrange them as a sequence $P_1, P_2, ...$, for any arbitrary n, $P_n = (G_i, G_j)$.
By Urysohn's lemma there exist continuous functions $f_n: X \to [0, 1]$ such that $f_n(\bar{G_i}) = 0$ and $f_n(G_j') = 1$.
Now define $f:X \to R^\infty$ by setting $f(x) = \{f_1(x), \frac{f_2(x)}{2}, \frac{f_3(x)}{3}, \dots \}$ for all $x \in X$.
For any integer $n \geq 1$, $f_n(x) \in [0, 1] \Rightarrow 0 \leq f_n(x) \leq 1 \Rightarrow \frac{f_n(x)}{n} \leq \frac{1}{n}$
$\Rightarrow \sum_{n=1}^{\infty} \left|\frac{f_n(x)}{n}\right|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$
Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we have $\sum_{n=1}^{\infty} \left|\frac{f_n(x)}{n}\right|^2$ is also convergent.
So, $f(x) = \{f_1(x), \frac{f_2(x)}{2}, \frac{f_3(x)}{3}, \dots\} \in R^\infty$. It is clear that $f: X \to R^\infty$ is a function.
Next we will show that f is one-one: Let $x, y \in X$ such that $x \neq y$.
Since X is a T_1 - space, $\exists\, G_j \in B \ni x \in G_j$ and $y \notin G_j$. That is, $x \in G_j$ and $y \in G_j'$.
By the above fact we have an ordered pair $P_n = (G_i, G_j)$ such that $x \in G_i \subseteq \bar{G_i} \subseteq G_j$.
$\Rightarrow f_n(\bar{G_i}) = 0$ and $f_n(G_j') = 1$. So, $f_n(x) = 0$ and $f_n(y) = 1$.
$\Rightarrow f_n(x) \neq f_n(y) \Rightarrow f(x) \neq f(y)$. Therefore, f is $1 - 1$.
Now we show that f is continuous: Let $x \in X$, and $\varepsilon > 0$.
Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, $\exists$ a positive integer $N \ni \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \frac{\varepsilon^2}{4}$ …(i).
For $n = 1, 2, ..., N$, f_n is continuous $\Rightarrow \exists$ an open set H_n containing $x \ni y \in H_n$
$\Rightarrow |f_n(x) - f_n(y)| < \frac{n\varepsilon}{\sqrt{2N}}$ for = 1, 2, …, N
$\Rightarrow \left|\frac{f_n(x) - f_n(y)}{n}\right|^2 < \frac{\varepsilon^2}{2N}$…(ii) for n = 1, 2, ..., N.
Write $G = \bigcap_{n=1}^{N} H_n$. Then G is an open set containing x.

Let y ∈ G. Consider $\|f(x) - f(y)\|^2 = \sum_{n=1}^{\infty} \left|\frac{f_n(x)-f_n(y)}{n}\right|^2$
$\leq \sum_{n=1}^{N} \left|\frac{f_n(x)-f_n(y)}{n}\right|^2 + \sum_{n=N+1}^{\infty} \left|\frac{f_n(x)}{n}\right|^2 + \sum_{n=N+1}^{\infty} \left|\frac{f_n(y)}{n}\right|^2$
$< \sum_{n=1}^{N} \frac{\varepsilon^2}{2N} + \sum_{n=N+1}^{\infty} \frac{1}{n^2} + \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4} = \varepsilon^2$.
$\Rightarrow \|f(x) - f(y)\|^2 < \varepsilon^2 \Rightarrow \|f(x) - f(y)\| < \varepsilon$. So, y ∈ G ⇒ $\|f(x) - f(y)\| < \varepsilon$.
This shows that f is continuous at x.
Since x is arbitrary, we have that f is continuous on X.
Now we show that f is an open mapping: Let G_j be any basic open set.
Now we claim that $f(G_j)$ is open in f(X).
Let z ∈ $f(G_j)$ ⇒ z = f(x) for some x ∈ G_j.
x ∈ G_j ⇒ There exists G_i ∈ B such that x ∈ $G_i \subseteq \bar{G_i} \subseteq G_j$.
Write $P_{n_0} = (G_i, G_j) \Rightarrow f_{n_0}(\bar{G_i}) = 0$, $f_{n_0}(G_j{}') = 1$. Choose ε such that $0 < \varepsilon < \frac{1}{2n_0}$.
Consider $S_\varepsilon(z)$, the open set in R^∞. Then $S_\varepsilon(z) \cap f(X)$ is open in f(X).
Let f(y) ∈ $S_\varepsilon(z) \cap f(X) \Rightarrow \|f(x) - f(y)\| < \varepsilon \Rightarrow \left[\sum_{n=1}^{\infty} \left|\frac{f_n(x)-f_n(y)}{n}\right|^2\right]^{1/2} < \varepsilon$.
$\Rightarrow \sum_{n=1}^{\infty} \left|\frac{f_n(x)-f_n(y)}{n}\right|^2 < \varepsilon^2 < \left(\frac{1}{2n_0}\right)^2$
$\Rightarrow \left|\frac{f_{n_0}(x)-f_{n_0}(y)}{n_0}\right|^2 < \left(\frac{1}{2n_0}\right)^2 \Rightarrow \left|\frac{f_{n_0}(x)-f_{n_0}(y)}{n_0}\right| < \frac{1}{2n_0} \Rightarrow |f_{n_0}(x) - f_{n_0}(y)| < \frac{1}{2}$
$\Rightarrow |f_{n_0}(y)| < \frac{1}{2}$ (since x ∈ G_i and $f_{n_0}(G_i) = 0$). ⇒ y ∈ G_j. (If y ∉ G_j, then y ∈ $G_j{}'$
$\Rightarrow f_{n_0}(y) = 1 \Rightarrow 1 < ½$ a contradiction). So, y ∈ G_j ⇒ f(y) ∈ $f(G_j)$.
This shows that $S_\varepsilon(z) \cap f(X) \subseteq f(G_j)$: Thus, we have an open set $G = S_\varepsilon(z) \cap f(X)$ in f(X) such that z ∈ G ⊆ $f(G_j)$ ⇒ $f(G_j)$ is open in f(X).
Consequently f: X → f(X) is open mapping.
Hence X is homeomorphic to a subspace f(X) of R^∞. Thus, X is metrizable.

STONE-CECH COMPACTIFICATION.

<u>Theorem</u>: Let X be an arbitrary completely regular space. Then there exists a compact Hausdorff space β(X) with the following properties: (i) X is dense subspace of β(X); (ii) every bounded continuous real function defined on X has a unique extension to a bounded continuous real function defined on β(X).