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LEBESGUE THEORY

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UNIT - I

e-content

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OUTER MEASURE

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Definition: Let I be an interval in $\mathbb{R}$. The length l(I) of I is defined by

$$l(I) = \begin{cases} +\infty \; if \; I \; is \; unbounded \\ b - a \; if \; I = (a,\, b) or \; (a,\, b],\; or \; [a, b) or \; [a, b] \end{cases}$$

Definition: Let A be any subset of real numbers.

Let $\{I_n\}$ be a countable collection of open intervals that cover A.

Let $l(I_n)$ be the length of the interval I_n.

Outer measure of A is defined as

$$m^*(A) = \inf_{A \subseteq \cup I_n} \{\textstyle\sum_{n=1}^{\infty} l(I_n)\}.$$

ie. $m^*(A) = \inf\{\sum_{n=1}^{\infty} l(I_n) \,/ A \subseteq \cup_{n=1}^{\infty} I_n \; where \; each \; I_n is \; an \; open \; interval\}$.

Observations:

(i) If $A \subseteq I$, where I is an open interval, then $m^*(A) \leq l(I)$.

(ii) For $\varepsilon > 0$ $\exists$ a sequence of open intervals $\{I_n\}$ $\ni$ $A \subseteq \cup_n I_n$ and

$\sum_n l(I_n) < m*(A) + \varepsilon$

Properties:

(i) $m^*(A) \geq 0 \; \forall \; A \subseteq \mathbb{R}$.

(ii) $m^*(\phi) = 0$

(iii) If $A \subseteq B$ then $m^*(A) \leq m^*(B)$

(iv) $m^*(\{a\}) = 0 \; \forall \; a \in \mathbb{R}$.

Proof: (i) Let $A \subseteq \mathbb{R}$. $Let \; A \subseteq \cup_{n=1}^{\infty} I_n \; where \; each \; I_n is \; an \; open \; interval\}$

Then $\sum_{n=1}^{\infty} l(I_n) \geq 0$

$m^*(A) = \inf_{A \subseteq \cup I_n} \{\sum_{n=1}^{\infty} l(I_n)\} \geq 0.$

(ii) By (i) $m^*(\phi) \geq 0 \ldots(1)$

Clearly $\phi \subseteq \left(-\frac{1}{n}, \frac{1}{n}\right) \forall$ n $\in \mathbb{N}$

$\therefore m^*(\phi) \leq l\left(-\frac{1}{n}, \frac{1}{n}\right) \forall$ n $\in \mathbb{N}$

Ie. $m^*(\phi) \leq \frac{2}{n} \forall$ n $\in \mathbb{N}$

$\therefore m*(\phi) \leq 0$ $as \frac{2}{n} \to 0$ as n $\to \infty \ldots(2)$

From (1) and (2)

From (1) and (2) $m^*(\phi) = 0$.

(iii) $Let \{I_n\}$ *be a seq of open interval intervals* $\ni B \subseteq \bigcup_{n=1}^{\infty} I_n$

$$Then\ A \subseteq \bigcup_{n=1}^{\infty} I_n$$

$m^*(A) \leq \sum_{n=1}^{\infty} l(I_n)$ $\forall\{(I_n)\}$ with $B \subseteq \cup I_n$

$m^*(A) \leq \inf_{B \subseteq \cup I_n} \{\sum_{n=1}^{\infty} l(I_n)\} = m^*(B).$

(iv) By (i) $m^*(\{a\}) \geq 0 \ldots(1)$

Clearly $\{a\} \subseteq \left(a - \frac{1}{n}, a + \frac{1}{n}\right) \forall$ n $\in \mathbb{N}$

$m*(\{a\}) \leq l\left(a - \frac{1}{n}, a + \frac{1}{n}\right) \forall$ n $\in \mathbb{N}$

Ie. $m*(\{a\}) \leq \frac{2}{n} \forall$ n $\in \mathbb{N}$

$\therefore m*(\{a\}) \leq 0$ $as \frac{2}{n} \to 0$ as n $\to \infty \ldots(2)$

From (1) and (2) $m^*(\{a\}) = 0$

Proposition: m* is countably sub additive. Ie. If $\{A_n\}$ is a countable collection of subsets of real numbers, then $m^*(\cup A_n) \leq \sum m^*(A_n)$.

Proof: Let $\{A_n\}$ is a countable collection of subsets of real numbers.

If $m^*(A_n) = \infty$ for some n, then the inequality is trivial.

Suppose $m^*(A_n) < \infty$ $\forall$ n. Let $\varepsilon > 0$. Fix n. Then $\exists$ a countable collection of open intervals, $\{I_{n,i}\}$ such that $A_n \subseteq \bigcup_i I_{n,i}$ and $\sum_i l(I_{n,i}) <$ m*(A_n) + $\frac{\varepsilon}{2^n}$.

Now $\bigcup_{n,i} I_{n,i} = \bigcup_n \left(\bigcup_i I_{n,i}\right)$ is the union of a countable collection of open intervals.

$\therefore \bigcup_n A_n \subseteq \bigcup_n \left(\bigcup_i I_{n,i}\right)$

$\therefore$ By definition, m*($\bigcup_n A_n$) $\leq \sum_{n,i} l(I_{n,i}) = \sum_n \left[\sum_i l(I_{n,i})\right] \leq \sum_n \left\{m^*(A_n) + \frac{\varepsilon}{2^n}.\right\}$

$= \sum_n \{m^*(A_n)\} + \sum_n \frac{\varepsilon}{2^n} = \sum_n \{m^*(A_n)\} + \varepsilon \sum_n \frac{1}{2^n} = \sum_n \{m^*(A_n)\} + \varepsilon. \therefore$

$\Rightarrow$ m*($\bigcup_n A_n$) $\leq \sum_n \{m^*(A_n)\} + \varepsilon$ $\forall$ $\varepsilon > 0$.

Hence $m^*(\bigcup A_n) \leq \sum m^*(A_n)$.

Corollary: If A is countable then m*(A) = 0.

Proof: Let A = $\{a_1, a_2, \ldots \}$ be a countable set in $\mathbb{R}$.

Then A = $\bigcup_{n=1}^{\infty} \{a_n\}$.

$\therefore$ $0 \leq$ m*(A) $\leq \sum_{n=1}^{\infty} m^*(\{a_n\}) = 0$ since m*($\{a_n\}$) = 0.

Corollary: The set [a, b] is uncountable for all a, b $\in \mathbb{R}$ with a < b.

Proof: If possible suppose [a, b] is countable for a, b $\in \mathbb{R}$ with a < b.

Then by above corollary m*[a, b] = 0.

But m*[a, b] = l([a, b]) = b – a $\neq$ 0 which is a contradiction.

This is due to our assumption [a, b] is countable.

$\therefore$ our assumption is wrong.

Hence the set [a, b] is uncountable for all a, b $\in \mathbb{R}$ with a < b.

Proposition: Given any set A and any $\varepsilon > 0$, (i) there is an open set G such that A $\subseteq$ G and $m^*(G) < m^*(A) + \varepsilon$. (ii) There is a G $\in G_\delta$ such that A $\subseteq$ G and $m^*(A) = m^*(G)$.

Proof: Let A be any set and $\varepsilon > 0$.

Case (i): Suppose m*(A) = ∞. Take G = $\mathbb{R}$. Then m*(G) = m*($\mathbb{R}$) = ∞ = m*(A).

Case (ii): Assume m*(A) < ∞.

For ε > 0 ∃ a countable collection of open intervals $\{I_n\}$ ∋ A ⊆ $\bigcup_n I_n$ and

$\sum_n l(I_n) < m^*(A) + \varepsilon$. Take G = $\bigcup_n I_n$. Then G is open and A ⊆ G.

Also m*(G) = m*($\bigcup_n I_n$) ≤ $\sum_n m^*(I_n)$ = $\sum_n l(I_n)$ < m*(A) + ε.

(ii) If m*(A) = ∞ then it is true as in (i).

Suppose m*(A) < ∞.

Take ε = $\frac{1}{n}$. Then for each positive integer n, ∃ an open set G_n such that A ⊆ G_n

and $m^*(G_n) < m^*(A) + \frac{1}{n}$.

Put G = ∩ G_n. Then G is a G_δ set and A ⊆ G.

So m*(A) ≤ m*(G) … (1)

Also m*(G) ≤ m*(G_n) (since G ⊆ G_n) < $m^*(A) + \frac{1}{n}$ ∀ n ∈ ℕ.

⇒ m*(G) ≤ m*(A) … (2).

From (1) and (2) m*(A) = m*(G).

Note: (i) Let A be a set of all rational numbers between 0 and 1. Let $\{I_i\}$, 1 ≤ i ≤ n be a finite collection of open intervals that covers A. Then $\sum l(I_n)$ ≥ 1

(ii) m* is translation invariant.

(iii) If m*(A) = 0 then m*(A∪ B) = m*(B).

Proof: (i) Given A = {r ∈ Q / r ∈ (0, 1)} and A ⊆ $\bigcup_{i=1}^n I_i$

⇒ $\bar{A} \subseteq \overline{\bigcup_{i=1}^n I_\iota} = \bigcup_{i=1}^n \overline{I_\iota}$

∴ 1 = ℓ [0, 1] = m*[0,1] = m*($\bar{A}$) ≤ m*($\bigcup_{i=1}^n \overline{I_\iota}$) ≤ $\sum_{i=1}^n m^*(\overline{I_\iota}) = \sum_{i=1}^n l(\overline{I_\iota})$ = $\sum_{i=1}^n l(I_i)$

(ii) Let ε > 0. Then ∃ a countable collection of open intervals $\{I_n\}$ ∋ A ⊆ $\bigcup_n I_n$ and

$\sum_n l(I_n) < m*(A) + \varepsilon$.

Then A + x ⊆ ($\bigcup_n I_n$) + x = $\bigcup_n (I_n + x)$

∴ m*(A + x) ≤ $\sum l(I_n + x)$ = $\sum l(I_n)$ < m*(A) + ε ∀ ε > 0

∴ m*(A + x) ≤ m*(A)…(1)

Put B = A + x. Then by above argument m*(B + - x) ≤ m*(B)

Now m*(A) = m*(A + x + -x) = m*(B + -x) ≤ m*(B) = m*(A + x)

Ie. $m^*(A) \le m^*(A + x)$… (2)

From (1) and (2) $m^*(A + x) = m^*(A)$

(iii) Suppose $m^*(A) = 0$.

$m^*(B) \le m^*(A \cup B) \le m^*(A) + m^*(B) = 0 + m^*(B) = m^*(B)$

$\Rightarrow m^*(A \cup B) = m^*(B)$ if $m^*(A) = 0$.

Proposition: The outer measure of an interval is its length.

Proof: Let I be an interval.

Case (i) : Let I be a closed and finite interval say $[a, b]$ where $a, b \in \mathbb{R}$.

Then $I \subseteq (a - \varepsilon, b + \varepsilon)\ \forall\ \varepsilon > 0$.

$\therefore m^*(I) \le \ell(a - \varepsilon, b + \varepsilon) = b - a + 2\varepsilon\ \forall\ \varepsilon > 0$.

$\Rightarrow m^*(I) \le b - a$ … (1)

Let $\{I_i\}$ be a countable collection of open intervals $\ni I \subseteq \cup I_i$.

Then I is compact.

$\therefore \exists$ a finite subcover $\{I_1, I_2, \dots, I_n\}$ of $\{I_i\} \ni I \subseteq I_1 \cup I_2 \cup \dots \cup I_n$.

If I_k is infinite interval for some k, then $\sum_{j=1}^{n} I_j = \infty \ge b - a$.

So assume that I_i is finite interval say (a_i, b_i) for $i = 1, 2, \dots, n$.

Then $a \in I \subseteq \bigcup_{i=1}^{n} I_i$

$\Rightarrow a \in I_j$ for some j.

W.l.g. assume that $a \in I_1 = (a_1, b_1)$. Ie. $a_1 < a < b_1$.

Then either $b < b_1$ or $b_1 \le b$.

Assume $b < b_1$. Then $b - a \le b_1 - a_1 = \ell(I_1) \le \sum_{j=1}^{n} I_j \le \sum_{j=1}^{\infty} I_j$.

Suppose $b_1 \le b$. Then $b_1 \in [a, b] = I \subseteq \bigcup_{i=1}^{n} I_i$

W. l. g. assume that $b_1 \in I_2 = (a_2, b_2)$. Ie. $a_2 < b_1 < b_2$.

Again either $b < b_2$ or $b_2 \le b$.

If $b < b_2$ then $\ell(I_1) + \ell(I_2) = b_1 - a_1 + b_2 - a_2 = b_2 - (a_2 - b_1) - a_1 \ge b_2 - a_1 \ge b - a$.

$\Rightarrow b - a \le \ell(I_1) + \ell(I_2) \le \sum_{j=1}^{n} I_j \le \sum_{j=1}^{\infty} I_j$.

Suppose $b_2 \le b$. Then $b_2 \in [a, b] = I \subseteq \bigcup_{i=1}^{n} I_i$.

W. l. g. assume that $b_2 \in I_3 = (a_3, b_3)$. Ie $a_3 < b_2 < b_3$.

Continue this process. It will terminate in a finite number of steps. Ie. $\exists$ k $\leq$ n $\ni$ b < b_k.

Then we have $b - a \leq \ell(I_1) + \ell(I_2) + \ldots + \ell(I_k) \leq \sum_{j=1}^{n} l(I_j) \leq \sum_{j=1}^{\infty} l(I_j)$.

$\Rightarrow$ b – a is a lower bound of $\{\sum_{n=1}^{\infty} l(I_n) : I \subseteq \cup I_n\}$

$\Rightarrow b - a \leq \inf_{I \subseteq \cup I_n} \{\sum_{n=1}^{\infty} l(I_n)\} = m^*(I)\ldots$ (2).

From (i) and (ii) $m^*(I) = b - a$.

Case (ii): Let I be any finite interval.

Let $\varepsilon > 0$.

Then $\exists$ a closed interval $J \subseteq I \ni \ell(J) \geq \ell(I) - \varepsilon$.

$\therefore \ell(I) - \varepsilon \leq \ell(J) = m^*(J) \leq m^*(I) \leq m^*(\bar{I}) = \ell(\bar{I}) = \ell(I) \ \forall \ \varepsilon > 0$.

Ie. $\ell(I) \leq m^*(I) \leq \ell(I) \Rightarrow m^*(I) = \ell(I)$.

Case (iii): Let I be an infinite interval.

Let $\varepsilon > 0$.

Then $\exists$ a closed interval $J \subseteq I \ni \ell(J) = \varepsilon$.

$\therefore m^*(I) \geq m^*(J) = \ell(J) = \varepsilon \ \forall \ \varepsilon > 0$.

$\Rightarrow m^*(I) = \infty = \ell(I)$.

Hence the proposition.

MEASURABLE SETS AND LEBESGUE MEASURE.

Definition: A set E is said to be measurable if for each set A we have

$m^*(A) = m^*(A \cap E) + m^*(A \cap \bar{E})$.

Remark: Let $E \subseteq \mathbb{R}$. (i) A set E is measurable if for each set $A \subseteq \mathbb{R}$ we have

$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \bar{E})$.

(ii) If E is measurable then $\bar{E}$ is measurable.

(iii) ϕ and $\mathbb{R}$ are measurable.

Proof: (i) For each set $A \subseteq \mathbb{R}$, let $m^*(A) \geq m^*(A \cap E) + m^*(A \cap \bar{E})$...(1)

Clearly A = A $\cap$ $\mathbb{R}$ = A $\cap$ (E $\cup$ $\bar{E}$) = (A $\cap$ E) $\cup$ (A $\cap$ $\bar{E}$).

$\therefore$ m*(A) $\leq$ m*(A $\cap$ E) + m*(A $\cap$ $\bar{E}$)...(2)

From (1) and (2) m*(A) = m*(A $\cap$ E) + m*(A $\cap$ $\bar{E}$)

Hence E is measurable.

(ii) Let E be a measurable set.

Let A be any set of real numbers.

Then m*(A) = m*(A $\cap$ E) + m*(A $\cap$ $\bar{E}$).

$\quad$ = m*(A $\cap$ $\bar{\bar{E}}$) + m*(A $\cap$ $\bar{E}$).

$\Rightarrow$ m*(A) = m*(A $\cap$ $\bar{E}$) + m*(A $\cap$ $\bar{\bar{E}}$) $\forall$ A ($\subseteq$ $\mathbb{R}$)

$\Rightarrow$ $\bar{E}$ is measurable

Hence $\bar{E}$ is measurable whenever E is measurable.

(iii) For any set A, we have

m*(A $\cap$ ϕ) + m*(A $\cap$ $\bar{\phi}$) = m*(ϕ) + m*(A $\cap$ $\mathbb{R}$) = 0 + m*(A) = m*(A).

Ie. m*(A) = m*(A $\cap$ ϕ) + m*(A $\cap$ $\bar{\phi}$) $\forall$ A $\subseteq$ $\mathbb{R}$.

Hence ϕ is measurable.

Since ϕ is measurable, by (ii) $\bar{\phi}$ is measurable. Ie. $\mathbb{R}$ is measurable.

Lemma: If, for a set E, m*(E) = 0 then E is measurable. ie. a set of measure zero is measurable.

Proof: Let m*(E) = 0 for a set E and A be any set of real numbers.

Since A $\cap$ E $\subseteq$ E, m*(A $\cap$ E) $\leq$ m*(E) = 0.

$\Rightarrow$ m*(A $\cap$ E) = 0.

Again m*(A $\cap$ $\bar{E}$) $\leq$ m*(A) since A $\cap$ $\bar{E}$ $\subseteq$ A.

$\therefore$ m*(A $\cap$ E) + m*(A $\cap$ $\bar{E}$) = 0 + m*(A) = m*(A).

Ie. m*(A) = m*(A $\cap$ E) + m*(A $\cap$ $\bar{E}$) $\forall$ A ($\subseteq$ $\mathbb{R}$).

$\Rightarrow$ E is measurable.

Hence E is measurable if m*(E) = 0.

Lemma: If E_1 and E_2 are measurable then so is $E_1 \cup E_2$.

Proof: Let E_1 and E_2 be any two measurable subsets of real numbers.

Let A be any set of real numbers.

Then clearly $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap \overline{E_1})$.

$\therefore\ m^*[A \cap (E_1 \cup E_2)] \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap \overline{E_1})$

So $m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap \overline{E_1 \cup E_2}]$

$\leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap \overline{E_1}) + m^*[A \cap \overline{E_1} \cap \overline{E_2}]$

$= m^*(A \cap E_1) + m^*(A \cap \overline{E_1}) = m^*(A)$.

$\therefore\ m^*(A) \geq m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap \overline{E_1 \cup E_2}]\ \ \forall A\ (\subseteq \mathbb{R})$.

$\Rightarrow E_1 \cup E_2$ is measurable.

Hence $E_1 \cup E_2$ is measurable if E_1 and E_2 are measurable.

Note: If $E_1, E_2, \ldots, E_n$ are measurable then $\bigcup_{i=1}^{n} E_i$ is measurable.

Corollary: The family $\mathfrak{M}$ of measurable sets is an algebra of sets.

Proof: $\phi \in \mathfrak{M}$.

$\therefore\ \mathfrak{M} \neq \phi$.

$E_1 \cup E_2 \in \mathfrak{M}$ whenever $E_1, E_2 \in \mathfrak{M}$.

$\bar{E} \in \mathfrak{M}$ whenever $E \in \mathfrak{M}$.

Hence $\mathfrak{M}$ is an algebra of sets.

Lemma: Let A be any set, and $E_1, E_2, \ldots, E_n$ be a finite sequence of pair wise disjoint measurable sets. Then $m^*(A \cap \bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*(A \cap E_i)$...(I)

Proof: If n = 1, then the statement (I) is clearly true.

Let n > 1 and assume that (I) is true for n – 1.

viz. we assume that

$$m^*\left(A \cap \bigcup_{i=1}^{n-1} E_i\right) = \sum_{i=1}^{n-1} m^*(A \cap E_i)$$

whenever $\{E_1, E_2, \ldots E_{n-1}\}$is a finite sequence of pair wise disjoint measurable sets.

Let $\{E_1, E_2, \ldots E_n\}$ be a finite sequence of pair wise disjoint measurable sets.

Observe that $E_i \cap \overline{E_n} = E_i$ for $1 \leq i \leq n - 1$, and ϕ for $i = n$.

*Since $E_i \cap E_n = \phi$ for $1 \le i \le n-1$, $E_i \subseteq \overline{E_n}$ so that $E_i \cap \overline{E_n} = E_i$ for $1 \le i \le n-1$, and ϕ for $i = n$.

$\therefore$ $(\bigcup_{i=1}^{n} E_i) \cap \overline{E_n} = \bigcup_{i=1}^{n}(E_i \cap \overline{E_n}) = \bigcup_{i=1}^{n-1} E_i$ and $(\bigcup_{i=1}^{n} E_i) \cap E_n = E_n$.

Since E_n is measurable we have

$$m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = m^*\left\{\left(A \cap \bigcup_{i=1}^{n} E_i\right) \cap E_n\right\} + m^*\left\{\left(A \cap \bigcup_{i=1}^{n} E_i\right) \cap \overline{E_n}\right\}$$

$$= m^*(A \cap E_n) + m^*\left(A \cap \bigcup_{i=1}^{n-1} E_i\right) == m^*(A \cap E_n) + \sum_{i=1}^{n-1} m^*(A \cap E_i)$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i)$$

Ie. (I) is true for n.

Hence by induction $m^*(A \cap \bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*(A \cap E_i)$ for all integral values of n whenever $\{E_1, E_2, \dots E_n\}$is a finite sequence of pair wise disjoint measurable sets.

Theorem: The collection $\mathfrak{M}$ of all measurable sets is a σ - algebra of sets.

Proof: $\mathfrak{M} \neq \phi$ since $\phi \in \mathfrak{M}$.

Let $E \in \mathfrak{M}$ Then $\bar{E}$ is measurable so that $\bar{E} \in \mathfrak{M}$.

Let $\{E_i\}$ be a countable collection of measurable sets and $E = \bigcup_{i=1}^{\infty} E_i$.

Then $\exists$ a disjoint sequence $\{F_i\}$ of measurable sets such that $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$

Put $G_n = \bigcup_{i=1}^{n} F_i$ for n = 1, 2, ...

Then $G_n \in \mathfrak{M}$ for all n.

Also $G_n = \bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i \subseteq \bigcup_{i=1}^{\infty} E_i = E$

$\Rightarrow \overline{G_n} \supseteq \bar{E} \Rightarrow A \cap \overline{G_n} \supseteq A \cap \bar{E}$ ...(1).

And since $A \cap E = A \cap F = A \cap \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty}(A \cap F_i)$,

$m^*(A \cap E) \le \sum_{i=1}^{\infty} m^*(A \cap F_i)$ ... (2).

$\therefore$ For any set A, $m^*(A) = m^*(A \cap G_n) + m^*(A \cap \overline{G_n})$

$\ge m^*(A \cap G_n) + m^*(A \cap \bar{E})$ by (1)

$= m^*(A \cap \bigcup_{i=1}^{n} F_i) + m^*(A \cap \bar{E}) = \sum_{i=1}^{n} m^*(A \cap F_i) + m^*(A \cap \bar{E})$.

Viz. m*(A) ≥ $\sum_{i=1}^{n} m^*(A \cap F_i)$ + m* (A ∩ $\bar{E}$) ∀ n.

⇒ m*(A) ≥ $\sum_{i=1}^{\infty} m^*(A \cap F_i)$ + m* (A ∩ $\bar{E}$)

≥ m*(A ∩ E) + m*(A ∩ $\bar{E}$) by (2).

Ie. m*(A) ≥ m*(A ∩ E) + m*(A ∩ $\bar{E}$) ∀ A ⊆ ℝ.

∴ $\bigcup_{i=1}^{\infty} E_i$ is measurable if {E_i} is a countable collection of measurable sets.

Hence 𝔐 is a σ - algebra of sets.

Lemma: The interval (a, ∞) is measurable.

Proof: Let A be any set. Let A_1 = A ∩(a, ∞), A_2 = A ∩ $\overline{(a,\infty)}$ = A ∩ (– ∞, a].

Claim: m*(A_1) + m*(A_2) ≤ m*(A).

Let m*(A) < ∞ and ε > 0. Then ∃ a countable collection { I_n } of open intervals such that A ⊆ $\bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} l(I_n)$ ≤ m*(A) + ε

Let I_n' = I_n ∩ (a, ∞) and I_n'' = I_n ∩ (– ∞, a]

Then I_n' and I_n'' are either intervals or empty.

Also I_n = I_n' ∪ I_n'' and I_n' ∩ I_n'' = ∞.

∴ l(I_n) = l(I_n') + l(I_n'') = m*(I_n') + m*(I_n'') ...(1)

But A_1 = A ∩ (a, ∞) ⊆ $\bigcup_{n=1}^{\infty} I_n$ ∩ (a, ∞) = $\bigcup_{n=1}^{\infty}\{I_n \cap (a,\infty)\}$ = $\bigcup_{n=1}^{\infty} I_n{}'$

∴ m*(A_1) ≤ $\sum_{i=1}^{\infty} m^*(I_n')$....(2).

Similarly A_2 ⊆$\bigcup_{n=1}^{\infty} I_n{}''$ and m*(A_2) ≤ $\sum_{i=1}^{\infty} m^*(I_n'')$....(3).

From (2) and (3), m*(A_1) + m*(A_2) ≤ $\sum_{i=1}^{\infty} m^*(I_n')$ + $\sum_{i=1}^{\infty} m^*(I_n'')$

= $\sum_{i=1}^{\infty}\{m^*(I_n') + m^*(I_n'')\}$ = $\sum_{n=1}^{\infty} l(I_n)$ ≤ m*(A) + ε.

Ie m*(A_1) + m*(A_2) ≤ m*(A) + ε ∀ ε > 0

∴ The interval (a, ∞) is measurable.

Theorem: Every Borel set is measurable. In particular each open set and closed set is measurable.

Proof: For each real a, we have proved (a, ∞) is measurable

⇒ $\overline{(a,\infty)}$ = (– ∞, a] is measurable.

Now for any real b, $(-\infty, b) = \bigcup_{n=1}^{\infty} \left]-\infty, b-\frac{1}{n}\right]$ is a countable union of measurable sets.

$\Rightarrow (-\infty, b)$ is measurable.

Now for any real a and b such that $a < b$ we have $(a, b) = (-\infty, b) \cap (a, \infty)$ is measurable.

Since any open set is a countable union of open intervals, that every open set is also measurable.

Since Borel field is the smallest σ - algebra generated by the set of all open sets, each Borel set is measurable.

F is closed $\Rightarrow \bar{F}$ is open .

Since every open set is measurable, $\bar{F}$ is measurable.

$\Rightarrow \bar{\bar{F}}$ = F is measurable. $\Rightarrow$ Every closed set is measurable.

<u>Definition</u>: If E is a measurable set, then define **Lebesgue measure m(E)** to be the outer measure of E.

<u>Proposition</u>: Let $\{E_i\}$ be a sequence of measurable sets. Then $m(\cup E_i) \le \sum m(E_i)$

If the sets E_i are pairwise disjoint, then $m(\cup E_i) = \sum m(E_i)$

<u>Proof</u>: $m(\cup E_i) = m^*(\cup E_i) \le \Sigma m^*(E_i) = \Sigma\, m(E_i)$

Thus $m(\cup E_i) \le \Sigma\, m(E_i)$.

Suppose $\{E_i\}$ are pairwise disjoint.

Then $m(\cup E_i) \ge$

$m(\bigcup_{i=1}^{n} E_i) = m^*(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*(E_i) = \sum_{i=1}^{n} m(E_i)\ \forall$ n.

$\Rightarrow m(\bigcup_{i=1}^{\infty} E_i) \ge \sum_{i=1}^{\infty} m(E_i)$.

Hence $m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$.

Proposition 15: Let $\{E_i\}$ be an infinite decreasing sequence of measurable sets and let $m(E_1)$ be finite. Then $m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n)$

Proof: Put $E = \bigcap_{i=1}^{\infty} E_i$ and $F_i = E_i - E_{i+1}$.

Since, each E_i is measurable, each F_i is also measurable.

Claim: $E_1 \setminus E = \cup F_i$

Let $x \in E_1 \setminus E$. $\Rightarrow x \in E_1$, $x \notin E = \cap E_i$

$\Rightarrow x \notin E_i$ for some i. Assume that i is the least number such that $x \notin E_i$.

Then $x \in E_{i-1} \Rightarrow x \in E_{i-1} \setminus E_i = F_{i-1} \subseteq \cup F_i$.

$\therefore E_1 \setminus E \subseteq \cup F_i$.

Now let $x \in \cup F_i$

$\Rightarrow x \in F_i = E_i \setminus E_{i+1}$ for some i.

$\Rightarrow x \notin E_{i+1}$ and $x \in E_i \subseteq E_1$.

$\Rightarrow x \notin \cap E_i = E$ and $x \in E_1 \Rightarrow x \in E_1 \setminus E$.

$\therefore \cup F_i \subseteq E_1 \setminus E$

Hence $E_1 \setminus E = \cup F_i$

Claim: $F_i \cap F_j = \phi$ if $i \neq j$.

W.L.G assume that $i < j$.

Then $F_i \cap F_j = (E_i \setminus E_{i+1}) \cap (E_j \setminus E_{j+1}) = E_i \cap \tilde{E}_{i+1} \cap E_j \cap \tilde{E}_{j+1} \subseteq E_j \cap \tilde{E}_{i+1}$

$\subseteq E_{i+1} \cap \tilde{E}_{j+1} = \phi$.

$\therefore F_i \cap F_j = \phi$ if $i \neq j$.

Suppose $B \subseteq A$. Then $A = B \cup (A \setminus B) \Rightarrow m^*(A) = m^*(B) + m^*(A \setminus B)$

$\Rightarrow m^*(A \setminus B) = m^*(A) - m^*(B)$.

$\therefore m(E_1 \setminus E) = m(E_1) - m(E)$ and $m(E_i \setminus E_{i+1}) = m(E_i) - m(E_{i+1})$

Consider $m(E_1) - m(E) = m(E_1 \setminus E) = m(\cup F_i) = \sum_{i=1}^{\infty} m(F_i)$

$= \sum_{i=1}^{\infty}\{m(E_i) - m(E_{i+1})\} = \lim_{n\to\infty} [\sum_{i=1}^{n}\{m(E_i) - m(E_{i+1})\}]$

$= m(E_1) - \lim_{n\to\infty} m(E_n)$

$\therefore m(E_1) - m(E) = m(E_1 \setminus E) = m(E_1) - \lim_{n\to\infty} m(E_n)$

$$\Rightarrow m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n)$$

<u>Proposition</u>: Let E be a given set. The following five statements are equivalent:

(i) E is measurable.

(ii) Given ε > 0, there is an open set $O \supseteq E$ with $m^*(O \sim E) < \varepsilon$

(iii) Given ε > 0, there is a closed set $F \subseteq E$ with $m^*(E \sim F) < \varepsilon$

(iv) There is a G in G_δ with $E \subseteq G$, $m^*(G \sim E) = 0$.

(v) There is a F in F_σ with $F \subseteq E$, $m^*(E \sim F) = 0$.

If m*(E) is finite, the above statements are equivalent to:

(vi) Given ε > 0, there is a finite union U of open intervals such that $m^*(U \Delta E) < \varepsilon$.

<u>Proof</u>: Claim: (i) ⇒ (ii). Assume (i).

Case (i) Suppose $m^*(E) < \infty$ and $\varepsilon > 0$.

Then ∃ an open set $O \supseteq E \ni m^*(O) \leq m^*(E) + \varepsilon/2 < m^*(E) + \varepsilon$.

$\therefore$ $m^*(O \setminus E) = m^*(O) - m^*(E) < \varepsilon$...(I)

Case (ii): Suppose m*(E) is infinite.

Clearly $\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$, where $I_n = (-n, n)$.

Put $E_n = E \cap I_n$.

Then $E = E \cap \mathbb{R} = E \cap \bigcup_{n=1}^{\infty} I_n = \bigcup_{n=1}^{\infty} (E \cap I_n) = \bigcup_{n=1}^{\infty} E_n$

Also $m^*(E_n) \leq m^*(I_n) = 2n < \infty$ for all n and E_n is measurable for all n.

By case (i) ∃ open set $O_n \supseteq E_n \ni m^*(O_n \setminus E_n) < \varepsilon / 2^{n+1}$.

Put $O = \bigcup_{n=1}^{\infty} O_n$.

Then O is an open set such that $O \supseteq E$.

Now $O \setminus E \subseteq \bigcup_{n=1}^{\infty} (O_n - E_n)$.

$\therefore$ $m^*(O \setminus E) \leq \sum_{n=1}^{\infty} m^*(O_n \setminus E_n) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \varepsilon / 2 < \varepsilon$.

Thus we have proved that (i) ⇒ (ii).

Claim: (ii) $\Rightarrow$ (iv)

Let us assume (ii).

Ie. for each positive integer n $\exists$ an open set $O_n \supseteq E \ni m^*(O_n \setminus E) < \frac{1}{n}$.

Then $G = \bigcap_{n=1}^{\infty} O_n$ is an G_δ set and $G \supseteq E$.

So $0 \leq m^*(G \setminus E) \leq m^*(O_n \setminus E) < \frac{1}{n} \; \forall$ n.

Hence $m^*(G \setminus E) = 0$

Thus we have proved that (ii) $\Rightarrow$ (iv).

Claim: (iv) $\Rightarrow$ (i).

Ie. $\exists$ a set $G = \cap G_i$ where each G_i is open (ie. G in ζ_δ) $\ni E \subseteq G$ and $m^*(G \setminus E) = 0$

$\Rightarrow$ $G \setminus E$ is measurable.

Since each G_i is open G_i is measurable and hence G is measurable.

$\Rightarrow E = G \setminus (G \setminus E)$ is measurable.

Thus we have proved that (iv) $\Rightarrow$ (i).

Claim: (i) $\Rightarrow$ (iii)

Assume (i). Ie. Let E be measurable.

$\Rightarrow \bar{E}$ is measurable.

Let $\varepsilon > 0$. Since (i) $\Rightarrow$ (ii), $\exists$ an open set $O \supseteq \bar{E} \ni \; m^*(O \setminus \bar{E}) < \in$.

Put $F = \bar{O}$

Then F is closed $F \subseteq E$.

$\therefore \; m^*(E \setminus F) = m^*(O \setminus \bar{E}) < \in$.

Thus we have proved that (i) $\Rightarrow$ (iii).

Claim: (iii) $\Rightarrow$ (v)

Assume (iii). Let $n \in \mathbb{N}$. Then by (iii) $\exists$ a closed set $F_n \subseteq E \ni m^*(E \setminus F_n) < \frac{1}{n}$

Put F = ∪ F_n. Then F ∈ G_σ, F ⊆ E and m*(E \ F) ≤ m*(E \ F_n) < $\frac{1}{n}$ ∀ n > 0.

∴ m*(E \ F) = 0.

Now m*(E) = m*[(E \ F) ∪ F] = m*(E \ F) + m*(F) = 0 + m*(F) = m*(F)

Thus we have proved that (iii) ⇒ (v).

Claim: (v) ⇒ (i).

Assume (v).

Ie. ∃ a set F = ∪ F_i, F_i closed, (ie. F ∈ G_σ) ∋ F ⊆ E and m*(E \ F) = 0.

⇒ E \ F is measurable.

Since F is union of closed sets, F is measurable.

Hence E = (E \ F) ∪ F is measurable.

Thus we have proved that (v) ⇒ (i).

Claim: (i) ⇒ (vi).

Assume (i). Suppose E is measurable. Let ε > 0. Since m*(E) < ∞, ∃ a sequence {I_n} of open intervals ∋ E ⊆ $\bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} l(I_n) < m^*(E) + \frac{\varepsilon}{2}$

∴ $\sum_{n=1}^{\infty} l(I_n)$ is a convergent series and hence ∃ m∈ℕ∋ $\sum_{n=m+1}^{\infty} l(I_n) < \frac{\varepsilon}{2}$

Put U = $\bigcup_{n=1}^{m} I_n$. Then m*(E \ U) ≤ $m^*(\bigcup_{n=m+1}^{\infty} I_n)$ ≤ $\sum_{n=m+1}^{\infty} l(I_n) < \frac{\varepsilon}{2}$.

Also U \ E = $\bigcup_{n=1}^{m} I_n$ \ E ⊆ $\bigcup_{n=1}^{\infty} I_n$\ E

∴m*(U \ E) ≤ m*($\bigcup_{n=1}^{\infty} I_n$) – m*(E) ≤ $\sum_{n=1}^{\infty} l(I_n)$ – m*(E) < $\frac{\varepsilon}{2}$.

∴ U is a finite union of open intervals and

m*(U Δ E) ≤ m*(U \ E) + m*(E \ U) < $\frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ = ε.

Claim: (vi) ⇒ (i).

Assume (vi). It is enough to show that (ii) holds.

Let ε > 0. Then ∃ a finite union U of open intervals ∋ m*(U Δ E) < $\frac{\varepsilon}{2}$

Since m*(E) < ∞, ∃ a sequence {I_n} of open intervals ∋ E ⊆ $\bigcup_{n=1}^{\infty} I_n$ and

$\sum_{n=1}^{\infty} l(I_n) < m^*(E) + \frac{\varepsilon}{2}$

Put G = U $\cup$ $(\cup_{n=1}^{\infty} I_n)$

Then G is open, E $\subseteq$ G and m*(G \ E) $\le$ m*(U \ E) + m*$[(\cup_{n=1}^{\infty} I_n) \setminus E]$

$\le$ m*(U \ E) + $\sum_{n=1}^{\infty} l(I_n)$ $-$m*(E)

$\le \frac{\varepsilon}{2}$ + m*(E) + $\frac{\varepsilon}{2}$ – m*(E) = ε

$\Rightarrow$ (ii) holds

A NON MEASURABLE SET.

Definition: If x and y are any real numbers in [0, 1), we define the sum modulo 1 of x and y denoted by $x \overset{\circ}{+} y$ to be x + y, if x + y < 1 and x + y – 1 if x + y $\ge$ 1. Note: $\overset{\circ}{+}$ is commutative and associative operation taking pairs of numbers in [0, 1).

If E is a subset of [0, 1) then define the translate modulo 1 of E to be the set

$$E \overset{\circ}{+} y = \left\{ x \overset{\circ}{+} y \text{ for some } x \in E \right\}$$

Lemma: Let E $\subseteq$ [0, 1] be a measurable set. Then for each y$\in$ [0, 1) the set $E \overset{\circ}{+} y$ is measurable.

Definition: Define x ~ y if x – y is a rational number for x , y $\in$[0, 1). This is an equivalence relation and hence partitions [0, 1) into equivalence classes. By the axiom of choice there is a set P which contains exactly one element from each equivalence class.

Theorem: There exists a non-measurable set.

Proof: Let x, y $\in$ [0, 1).

Define x ~ y iff x – y is a rational number.

Then ~ is an equivalence relation.

Observe that any two elements in the same class differ by a rational number.

[x] = {y $\in$ [0, 1) / x ~ y}

$= \{y \in [0, 1) / x - y$ or $y - x$ is a positive rational number$\}$.

$= \{y \in [0, 1) / y = x + r$ or $y = x - r$ for some rational r in $[0, 1)\}$

Thus the class containing [0] is the set of all rationals in [0, 1). By axiom of choice we can choose one element from each equivalence class. Let P be the set which contains exactly one element from each equivalence class.

Let $\{r_i\}_{i=0}^{\infty}$ be the enumeration of all rationals in [0, 1) with $r_0 = 0$.

Put $P_i = P \dotplus r_i$.

Then $P_0 = P$ (Since $P_0 = P + r_0 = P + 0 = P$)

Claim: $\cup P_i = [0, 1)$ and P_i's are disjoint.

Let $x \in [0, 1)$. Then $x \sim x_k$ for some $x_k \in P$.

$\Rightarrow x - x_k$ or $x_k - x$ is a rational in $[0, 1)$

$\Rightarrow x = x_k + r_i$ or $x = x_k - r_i$ for some i.

Suppose $x = x_k + r_i$.

Now $x = x_k + r_i = x_k \dotplus r_i$ (since $x < 1$) $\in P \dotplus r_i = P_i \subseteq \cup P_i$.

If $x = x_k - r_i$, then put $r_j = 1 - r_i$.

Now $x_k + r_j = x_k + 1 - r_i = x + 1 \geq 1$

$\Rightarrow x_k \dotplus r_j = x_k + r_j - 1 = x_k + 1 - r_i - 1 = x_k - r_i = x$.

$\therefore x = x_k \dotplus r_j \in P \dotplus r_j = P_j \subseteq \cup P_i$.

So $\cup P_i = [0, 1)$.

Claim: P_i's are pair wise disjoint.

Let $n \neq m$.

Suppose $z \in P_n \cap P_m$.

Then $z = x_\alpha + r_n$, $z = x_\beta + r_m$ where $x_\alpha, x_\beta \in P$. $\Rightarrow x_\alpha - x_\beta$ is a rational $\Rightarrow x_\alpha \sim x_\beta$

$\Rightarrow \alpha = \beta$ since P contains exactly one element from each equivalence class.

$x_\alpha + r_n = x_\beta + r_m \Rightarrow r_n = r_m \Rightarrow n = m$, a contradiction. This shows that P_i's are disjoint.

Claim: P is a non - measurable set.

$m^*(P_n) = m^*(P + r_n) = m^*(P)$.

Suppose P is a measurable set.

Then each P_i is measurable.

Now 1 = m*[0, 1) = $m^*(\bigcup_{i=1}^{\infty} P_i)$ = $\sum_{i=1}^{\infty} m^*(P_i)$ = $\sum_{i=1}^{\infty} m^*(P)$ = 0 if m*(P) = 0 or ∞ if m*(P) is positive which is a contradiction.

Thus P is a non - measurable set.

Theorem: If m is a countably additive, translation invariant measure defined on a σ – algebra containing the set P, then m[0, 1) is either zero or infinite.

MEASURABLE FUNCTIONS:

Proposition 18: Let f be an extended real valued function whose domain E is measurable. Then the following statements are equivalent.

(i) For each real number α the set { x : f(x) > α } is measurable.

(ii) For each real number α the set { x : f(x) ≥ α } is measurable.

(iii) For each real number α the set { x : f(x) < α } is measurable.

(iv) For each real number α the set { x : f(x) ≤ α } is measurable.

These statements imply

(v) For each real number α the set { x : f(x) = α } is measurable.

Proof:

Claim: (i) ⇔ (iv)

Assume (i). For each α, { x / f(x) ≤ α} = E \ {x / f(x) > α}.

Since E is measurable by (i), {x / f(x) > α} is measurable. Since the difference of two measurable sets is also measurable we get { x / f(x) ≤ α} is measurable.

Thus we have proved that (i) ⇒ (iv).

Assume (iv). For each α, { x / f(x) > α} = E \ {x / f(x) ≤ α}.

Since E is measurable, by (iv), {x / f(x) ≤ α} is measurable.

Since the difference of two measurable sets is also measurable we get {x / f(x) > α} is measurable.

Thus we have proved that (iv) ⇒ (i).

Claim: (ii) $\Leftrightarrow$ (iii)

Assume (i). For each α, $\{x / f(x) < \alpha\} = E \setminus \{x / f(x) \geq \alpha\}$.

Since E is measurable by (ii), $\{x / f(x) \geq \alpha\}$ is measurable. Since the difference of two measurable sets is also measurable we get $\{ x / f(x) < \alpha\}$ is measurable.

Thus we have proved that (ii) $\Rightarrow$ (iii).

Assume (iii). For each α, $\{ x / f(x) \geq \alpha\} = E \setminus \{x / f(x) < \alpha\}$.

Since E is measurable, by (iii), $\{x / f(x) < \alpha\}$ is measurable.

Since the difference of two measurable sets is also measurable we get

$\{ x / f(x) \geq \alpha\}$ is measurable.

Thus we have proved that (iii) $\Rightarrow$ (ii).

Claim: (i) $\Leftrightarrow$ (ii)

Assume (i). For each α, $\{ x / f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x : f(x) > \alpha - \frac{1}{n}\right\}$.

Since E is measurable by (i), $\left\{x : f(x) > \alpha - \frac{1}{n}\right\}$ is measurable for all n > 0.

$\therefore \left\{x : f(x) > \alpha - \frac{1}{n}\right\}$ is measurable.

we get $\{ x / f(x) \geq \alpha\}$ is measurable for all α.

Thus we have proved that (i) $\Rightarrow$ (ii).

Now assume (ii). For each α, $\{ x / f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \left\{x : f(x) \geq \alpha + \frac{1}{n}\right\}$..

Since E is measurable, by (ii), $\left\{x : f(x) \geq \alpha + \frac{1}{n}\right\}$ is measurable for all n > 0.

$\therefore \bigcup_{n=1}^{\infty} \left\{x : f(x) \geq \alpha + \frac{1}{n}\right\}. We$ get $\{x / f(x) > \alpha\}$ is measurable.

Thus we have proved that (ii) $\Rightarrow$ (i).

$\therefore$ (i) $\Leftrightarrow$ (ii).

Claim: $\{x / f(x) = \alpha\}$ is measurable assuming any one of the conditions (i) to (iv) is true.

Let α be any extended real number and assuming any one of the conditions (i) to (iv) is true.

Case (i) α is real.

We have $\{x / f(x) = \alpha\} = \{x / f(x) \le \alpha\} \cap \{x / f(x) \ge \alpha\}$.

By our assumption $\{x / f(x) \le \alpha\}$, $\{x / f(x) \ge \alpha\}$ are measurable and hence their intersection.

$\therefore \{x / f(x) = \alpha\}$ is measurable.

Case (ii). Let $\alpha = \infty$.

Clearly $\{x / f(x) = \infty\} = \bigcap_{n=1}^{\infty}\{x : f(x) > n\}$.

But by (i) $\{x / f(x) > n\}$ is measurable for all n.

$\therefore \bigcap_{n=1}^{\infty}\{x : f(x) > n\}$ is measurable.

Hence $\{x / f(x) = \infty\}$ is measurable.

Similarly we can prove it in the case $\alpha = -\infty$.

Definition: An extended real – valued function f is said to be Lebesgue measurable if its domain is measurable and if it satisfies one of the first four statements of the proposition.

Proposition: Let c be a constant and f and g be two measurable real valued functions defined on the same domain. Then the functions (i) $f + c$, (ii) cf, (iii) $f + g$, (iv) $g - f$, and (v) fg are also measurable.

Proof: Let D be the domain of f and g and c be any constant.

(i) For any real α, $\{x \in D : (f + c)(x) > \alpha\} = \{x \in D : f(x) + c > \alpha\}$

$= \{x \in D : f(x) > \alpha - c\}$ is measurable since f is measurable

$\therefore$ the function $f + c$ is measurable.

(ii) Claim: cf is measurable.

Let $c > 0$.

Now $\{x \in D : (cf)(x) > \alpha\} = \{x \in D : c\, f(x) > \alpha\} = \{x \in D : f(x) > \alpha / c\}$ is measurable since f is measurable and α/c is real.

$\therefore$ when $c > 0$, cf is measurable.

Let $c < 0$.

Now $\{x \in D : (cf)(x) > \alpha\} = \{x \in D : c\, f(x) > \alpha\} = \{x \in D : f(x) < \alpha/c\}$ is measurable since f is measurable and α/c is real.

$\therefore$ when $c < 0$, cf is measurable.

Let $c = 0$. Then $cf = 0$ is constant function.

$\therefore$ cf is measurable for any c.

(iii) for any α, $\{x \in D : (f + g)(x) < \alpha\} = \{x \in D : f(x) + g(x) < \alpha\}$

If $f(x) + g(x) < \alpha$, then $f(x) < \alpha - g(x)$

$\therefore \exists$ a rational number r such that $f(x) < r < \alpha - g(x)$

Ie. $\exists$ a rational number r such that $f(x) < r$ and $g(x) < \alpha - r$

$\therefore$ $\{x \in D : f(x) + g(x) < \alpha\} = \cup\, [\{x / f(x) < r\} \cap \{x / g(x) < \alpha - r\}]$

But $\{x / f(x) < r\}$ and $\{x / g(x) < \alpha - r\}$ are measurable.

$\therefore$ $\{x / f(x) < r\} \cap \{x / g(x) < \alpha - r\}$ is measurable.

Since the rationals are countable, $\cup\, [\{x / f(x) < r\} \cap \{x / g(x) < \alpha - r\}]$ is countable.

Hence $f + g$ is measurable.

(iv) Since g is measurable by (ii) $-g$ is measurable. Now by (iii) $f + (-g)$ is measurable. Ie. $f - g$ is measurable.

(v). Let α be a real and $\alpha \geq 0$.

Then $\{x / f^2(x) > \alpha\} = \{x / f(x) > \sqrt{\alpha}\} \cup \{x / f(x) < -\sqrt{\alpha}\}$ which is the union of measurable sets and so measurable. Hence f^2 is measurable.

Let α be a real and $\alpha < 0$.

Since $f^2(x) \geq 0$ for all $x \in D$, $\{x / f^2(x) > \alpha\} = D$ which is measurable.

Hence f^2 is measurable.

Thus applying above results $(f + g)^2$, $-f^2$, $-g^2$ are measurable.

$\therefore$ $fg = \frac{1}{2}\{(f+g)^2 - f^2 - g^2\}$ is measurable.

Theorem: Let $\{f_n\}$ be a sequence of measurable functions (with the same domain of definition). Then the functions (i) $\sup\{ f_1, f_2, ..., f_n\}$, (ii) $\inf\{ f_1, f_2, ..., f_n\}$, (iii) $\sup_n f_n$ (iv) $\inf_n f_n$ (v) $\overline{lim} f_n$ and (vi) $\underline{lim} f_n$ are measurable.

Proof: Let D be the domain of the sequence of functions $\{f_n\}$.

(i) Define $g(x) = \sup\{ f_1, f_2, ..., f_n\}(x)$

$= \sup\{ f_1(x), f_2(x), ..., f_n(x)\}$.

Now for each real α, $\{x / g(x) > \alpha\} = \bigcup_{i=1}^{n}\{x/f_i(x) > \alpha\}$.

Since each f_i is measurable, $\{x / f_i(x) > \alpha\}$ is measurable $\forall\ \alpha \in \mathbb{R}$ and for each $i = 1,2, ..., n$.

$\Rightarrow \bigcup_{i=1}^{n}\{x/f_i(x) > \alpha\}$ is measurable.

$\Rightarrow \{x / g(x) > \alpha\}$ is measurable $\forall\ \alpha \in \mathbb{R}$.

$\Rightarrow \sup\{ f_1, f_2, ..., f_n\}$ is measurable.

(ii) Define $h(x) = \inf\{ f_1, f_2, ..., f_n\}(x)$

$= \inf \{ f_1(x), f_2(x), ..., f_n(x)\}$.

Now for each real α, $\{x / h(x) > \alpha\} = \bigcap_{i=1}^{n}\{x : f_i(x) > \alpha\}$.

Since each f_i is measurable, $\{x / f_i(x) > \alpha\}$ is measurable $\forall\ \alpha \in \mathbb{R}$ and for each $i = 1,2, ..., n$.

$\Rightarrow \bigcap_{i=1}^{n}\{x : f_i(x) > \alpha\}$ is measurable.

$\Rightarrow \{x / h(x) > \alpha\}$ is measurable $\forall\ \alpha \in \mathbb{R}$.

$\Rightarrow \inf\{ f_1, f_2, ..., f_n\}$ is measurable.

(iii) Define $G(x) = \{\sup_n f_n\}(x) = \sup_n f_n\,(x)$

Now for each real α, $\{x / G(x) > \alpha\} = \bigcup_{i=1}^{\infty}\{x/f_i(x) > \alpha\}$.

Since each f_i is measurable, $\{x / f_i(x) > \alpha\}$ is measurable $\forall\ \alpha \in \mathbb{R}$ and for each $i = 1,2, ...$

$\Rightarrow \bigcup_{i=1}^{\infty}\{x/f_i(x) > \alpha\}$ is measurable.

$\Rightarrow \{x / G(x) > \alpha\}$ is measurable $\forall\ \alpha \in \mathbb{R}$.

$\Rightarrow \sup\limits_{n} f_n$ is measurable.

(iv) Define $H(x) = \{\inf\limits_{n} f_n\}(x) = \inf\limits_{n} f_n(x)$

Now for each real α, $\{x / H(x) > \alpha\} = \bigcap_{i=1}^{\infty}\{x : f_i(x) > \alpha\}$.

Since each f_i is measurable, $\{x / f_i(x) > \alpha\}$ is measurable $\forall\ \alpha \in \mathbb{R}$ and for each $i = 1,2, \ldots$

$\Rightarrow \bigcap_{i=1}^{\infty}\{x : f_i(x) > \alpha\}$ is measurable.

$\Rightarrow \{x / H(x) > \alpha\}$ is measurable $\forall\ \alpha \in \mathbb{R}$.

$\Rightarrow \inf\limits_{n} f_n$ is measurable.

(v) $\overline{lim} f_n = \inf\limits_{n}\{\sup\limits_{k \geq n} f_k\}$.

Write $g_n = \sup\limits_{k \geq n} f_k$

Since each f_k is measurable for all $k \geq n$, by (iii) g_n is measurable for all n.

By (iv) $\inf\limits_{n} g_n$ is measurable. Ie. $\overline{lim} f_n$ is measurable.

(vi) $\underline{lim} f_n = \sup\limits_{n}\{\inf\limits_{k \geq n} f_k\}$.

Write $h_n = \inf\limits_{k \geq n} f_k$

Since each f_k is measurable for all $k \geq n$, by (iv) h_n is measurable for all n.

By (iii) $\sup\limits_{n} h_n$ is measurable. Ie. $\underline{lim} f_n$ is measurable.

Definition: A property is said to hold almost everywhere (abbreviated a.e.) if the set of points where it fails to hold is a set of measure zero.

Proposition: If f is a measurable function and f = g a.e., then g is measurable.

Proof: Let E be a measurable set and f, g be defined on E.

Let $\alpha \in \mathbb{R}$. Write $E_1 = \{ x \in E / f(x) = g(x)\}$, $E_2 = \{ x \in E / f(x) \neq g(x)\}$.

Also $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \phi$ and $E \setminus E_2 = E_1$ and so E_1 is measurable.

Since f = g a.e. $m(E_2) = 0$.

$\Rightarrow E_2$ is measurable.

Write $A = \{x \in E / g(x) > \alpha\}$.

Then $A \cap E_1 = \{x / g(x) > \alpha\} \cap E_1 = \{x / f(x) > \alpha\} \cap E_1$

Since f and E_1 are measurable, $A \cap E_1$ is measurable.

Now $m(A \cap E_2) \leq m(E_2) = 0$.

$\Rightarrow m(A \cap E_2) = 0 \Rightarrow A \cap E_2$ is measurable.

$\therefore A = A \cap E = A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2)$.

Since $A \cap E_1$ and $A \cap E_2$ are measurable, A is measurable $\forall\ \alpha$.

Hence g is measurable.

Proposition 22: Let f be a measurable function defined on an interval [a, b], and assume that f takes the values $\pm\infty$ only on a set of measure zero. Then given $\varepsilon > 0$, we can find a step function g and a continuous function h such that $|f - g| < \epsilon$ and $|f - h| < \epsilon$ .except on a set of measure less than ε; ie. $m\{x : |f(x) - g(x)| \geq \varepsilon\} < \varepsilon$ and $m\{x : |f(x) - h(x)| \geq \varepsilon\} < \varepsilon$

If in addition $m \leq f \leq M$, then we may choose the functions g and h so that $m \leq g \leq M$ and $m \leq h \leq M$.

Definition: If A is any set, we define the characteristic function χ_A of the set A to be the function given by $\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$.

Result: The function χ_A is measurable iff A is measurable.

Proof: Suppose χ_A is measurable. Then clearly $A = \{x / \chi_A(x) > ½\}$

Since χ_A is measurable, $\{x / \chi_A(x) > ½\}$ is measurable.

Conversely suppose that A is measurable.

If $\alpha < 0$, then $\{x / \chi_A(x) > \alpha\} = \mathbb{R}$. is measurable.

If $0 \leq \alpha < 1$, then $\{x / \chi_A(x) > \alpha\} = A$ is measurable.

If $\alpha \geq 1$, then $\{x / \chi_A(x) > \alpha\} = \phi$ is measurable.

Thus for any α, $\{x / \chi_A(x) > \alpha\}$ is a measurable set.

Hence χ_A is measurable.

Note: Existence of a non measurable set implies the existence of a non-measurable function.

Definition: A real valued function ϕ is called simple if it is measurable and assumes only a finite number of values. If ϕ is simple and has the values $\alpha_1, \alpha_2, ..., \alpha_n$ then

$\phi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ where $A_i = \{ x : \phi(x) = \alpha_i\}$.

Note: The sum, product, and difference of two simple functions are simple.

LITTLEWOOD'S THREE PRINCIPLES.

There are three principles, roughly expressible in the following terms:

Every (measurable) set is nearly a finite union of intervals,

Every (measurable) function is nearly continuous;

Every convergent sequence of (measurable) functions is nearly uniformly convergent.

Various forms of the first principle are given by Proposition 15, One version of second principle is given by Proposition 22.

The following proposition gives one version of the third principle.

Proposition 23: Let E be a measurable set of finite measure, and $\{f_n\}$ a sequence of measurable functions defined on E. Let f be a real valued function such that for each x in E we have $f_n(x) \to f(x)$. Then given $\varepsilon > 0$ and $\delta > 0$, there is a measurable set $A \subseteq E$ with $m(A) < \delta$ and an integer N such that $\forall\, x \notin A$ and $\forall\, n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

Proof: Let $\varepsilon > 0$ and $\delta > 0$.

For each positive integer write, $G_n = \{x \in E / |f_n(x) - f(x)| \geq \varepsilon\}$.

Since each f_n is measurable, we have that $f = \lim f_n$ is measurable.

So $(f_n - f)$ is measurable. So each G_n is measurable.

Put $E_n = \bigcup_{m=n}^{\infty} G_m = \{x \in E : |f_m(x) - f(x)| \geq \varepsilon \text{ for some } m \geq n\}$.

Now $E_{n+1} \subseteq E_n \ \forall\ n$ and so $\{E_n\}$ is a decreasing sequence of measurable sets.

Claim: $\cap E_n = \phi$.

Suppose $x \in \cap E_n$.

$\Rightarrow x \in E_n \ \forall\ n$. Since each $E_n \subseteq E$, $x \in E$.

By hypothesis, $f_n(x) \to f(x)$.

$\therefore$ $\exists$ an integer N such that $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

So $x \notin E_N$, which is a contradiction. Thus $\cap E_n = \phi$.

Now $E_1 \subseteq E \Rightarrow m(E_1) \leq m(E) < \infty$.

$\therefore \lim\limits_{n\to\infty} m(E_n) = m(\bigcap_{n=1}^{\infty} E_n) = m(\phi) = 0$.

$\Rightarrow$ $\exists$ an integer N such that $m(E_n) < \delta \ \forall\ n \geq N$.

Write $A = E_N$. Then A is a measurable subset of E and $m(A) < \delta$.

Now $\tilde{A} = \widetilde{E_N} = \{x \in E : |f_m(x) - f(x)| < \varepsilon \ \forall\ m \geq n\}$.

$\therefore x \in \tilde{A} \Rightarrow |f_m(x) - f(x)| < \varepsilon \ \forall\ m \geq n$.

Hence the Theorem.

Proposition 24: Let E be a measurable set of finite measure, and $\{f_n\}$ a sequence of measurable functions that converge to a real valued function f a.e. on E. Then given $\varepsilon > 0$ and $\delta > 0$, there is a measurable set $A \subseteq E$ with $m(A) < \delta$ and an integer N such that for all $x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

Proof: since $f_n \to f$ a.e. on E, $\exists$ a measurable sudset $B \subseteq E$ with $m(B) = 0$ and

$\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in E \setminus B$.

Since each fn is measurable, f is measurable function.

$\therefore$ $\exists$ a measurable set $A \subseteq E \setminus B$ with $m(A) < \delta$ and an integer N such that

$|f_n(x) - f(x)| < \varepsilon$ for all $x \in (E \setminus B) \setminus A = E \setminus (A \cup B)$ and for all $n \geq N$.

Now $A \cup B$ is measurable and $A \cup B \subseteq E$.

$m(A \cup B) \leq m(A) + m(B) = m(A) < \delta$, since $m(B) = 0$.

Now for all x ∈ E \ (A ∪ B) and for all n ≥ N, $|f_n(x) - f(x)| < \varepsilon$.

Egoroff's Theorem: If $\{f_n\}$ a sequence of measurable functions that converge to a real valued function f a.e. on a measurable set E of finite measure,. Then given η > 0 there is a measurable set A ⊆ E with m(A) < η and $f_n \to f$ uniformly on E \ A.

Proof: By the above proposition, for each positive integer n, there exists a measurable subset $A_n \subseteq E$ with $m(A_n) \leq \frac{\eta}{2^n}$ and an integer k_n such that for all $x \notin A_n$,

$|f_m(x) - f(x)| < \frac{1}{n}$ for all $m \geq k_n$.

Write $A = \cup A_n$.

Clearly A is measurable subset of E and $m(A) = m(\cup A_n)$

$\leq \sum_{n=1}^{\infty} m(A_n) < \sum_{n=1}^{\infty} \frac{\eta}{2^n} = \eta \sum_{n=1}^{\infty} \frac{1}{2^n} = \eta$.

∴ m(A) < η.

Let ε > 0. Choose n ∍ $\frac{1}{n} < \varepsilon$.

Then for x ∈ E \ A and $m \geq k_n$, $|f_m(x) - f(x)| < \frac{1}{n} < \varepsilon$.

∴ $f_n \to f$ uniformly on E \ A.

Lusin's Theorem: Let f be a measurable real – valued function on an interval [a, b]. Then given δ > 0, there is a continuous function ϕ on [a, b] such that m{x: f(x) ≠ ϕ(x)} < δ.

Proof: Take δ > 0, and a measurable function f on [a, b].

∴ to each n ∈ℕ, ∃ a continuous function h_n and a measurable set $A_n \subseteq [a, b]$ ∍

$|h_n(x) - f(x)| < \frac{\delta}{2^{n+1}}$ ∀ $x \notin A_n$ and $m(A_n) < \frac{\delta}{2^{n+1}}$....(i)

Write $E = \bigcap_{n=1}^{\infty} \overline{A_n}$.

Clearly E is a measurable set.

Also $E \subseteq \overline{A_n} \subseteq [a, b] \Rightarrow m(E) < \infty$ and for all x ∈ E, $|h_n(x) - f(x)| < \frac{\delta}{2^{n+1}}$ ∀ n

$\therefore \lim_n h_n = f(x)\ \forall\ x \in E.$

Observe that each h_n is a measurable function.

So by egoroff's theorem $\exists$ a measurable set $A \subseteq E$ with $m(A) < \delta/4$...(ii)

and $h_n \to f$ uniformly on $E \setminus A$.

Since E is measurable and A is measurable, $E \setminus A$ is measurable.

$\therefore \exists$ a closed set $F \subseteq E \setminus A \ni m((E \setminus A) \setminus F) < \delta / 4$ ... (iii).

Since f is the uniform limit of a sequence of continuous functions on the set E\A, we have that f is continuous on $E \setminus A$.

Thus f is continuous on a closed set F.

Since $F \subseteq [a, b]$ and f is continuous on F, we have f has unique continuous extension g on [a, b].

Now $\{x/ f(x) \neq g(x)\} \subseteq \bar{F}$ and so $m\{x / f(x) \neq g(x)\} \leq m(\bar{F})$.

Since $F \subseteq E \setminus A$, $\bar{F} \subseteq (\bar{E} \cup A) \cup (E \setminus A) \setminus F)$.

$\Rightarrow m(\bar{F}) \leq m(\bar{E} \cup A) + m((E \setminus A) \setminus F) \leq m(\bar{E}) + m(A) + m((E \setminus A) \setminus F)$

$= m(\bar{E}) + \delta / 2$ from (ii) and (iii).

Since $E = \bigcap_{n=1}^{\infty} \overline{A_n}$, we have $\bar{E} = \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow m(\bar{E}) \leq \sum_{n=1}^{\infty} m(A_n) < \sum_{n=1}^{\infty} \frac{\delta}{2^{n+1}} = \delta / 2.$

By substituting $m(\bar{F}) < \frac{\delta}{2} + \frac{\delta}{2} + \frac{\delta}{4} = \delta.$

$\therefore m\{x/ f(x) \neq g(x)\} < \delta.$

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E – CONTENT

PAPER: M 302,

LEBESGUE THEORY

M. Sc. II YEAR, SEMESTER - III

UNIT – II: THE LEBESGUE INTEGRAL

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302, LEBESGUE THEORY
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UNIT II

Riemann Integral

Let f be a bounded real – valued function defined on the interval [a, b] and let a = $\xi_0 < \xi_1 < \ldots < \xi_n = b$ be a subdivision of [a, b].

Define U(P, f) = S = $\sum_{i=1}^{n} M_i(\xi_i - \xi_{i-1})$ and L(P, f) = s = $\sum_{i=1}^{n} m_i(\xi_i - \xi_{i-1})$ where $M_i = \sup\limits_{\xi_{i-1}<x<\xi_i} f(x)$ and $m_i = \inf\limits_{\xi_{i-1}<x<\xi_i} f(x)$. Then we define the upper integral of f by $R\int_a^{\bar{b}} f(x)dx = \inf S$ with the infimum taken over all possible subdivisions of [a, b]

the lower integral of f by $R\int_{\bar{a}}^{b} f(x)dx = \sup s$ with the supreum taken over all possible subdivisions of [a, b]. If upper and lower integrals are equal then we say that f is Riemann Integrable and call the common value the Riemann integral of f and is denoted by $R\int_a^b f(x)dx$

Definition: By a step function we mean a function ψ which has the form

Problem: 3*: Define f(x) = 0 if x is irrational and 1 if x is rational. Then prove that f is not R-integrable but Lebesgue integrable.

Solution: Let a = $x_0 < x_1 < \ldots < x_n = b$ be a subdivision of [a, b].

$M_i = \sup\limits_{x_{i-1}<x<x_i} f(x)$ = Sup {0, 1} = 1,

and $m_i = \inf\limits_{x_{i-1}<x<x_i} f(x)$ = Inf {0, 1} = 0.

U(P, f) = $\sum_{i=1}^{n} M_i(x_i - x_{i-1})$ = 1 $\sum_{i=1}^{n}(x_i - x_{i-1})$ = b – a.

L(P, f) = $\sum_{i=1}^{n} m_i(x_i - x_{i-1})$ = 0 $\sum_{i=1}^{n}(x_i - x_{i-1})$ = 0.

$R\int_a^{\bar{b}} f(x)dx = \inf\{U(P,f): P \in \wp[a,b]\}$ = inf {b – a} = b – a.

$R\int_{\bar{a}}^{b} f(x)dx = \text{Sup}\ \{L(P,f): P \in \wp[a,b]\}$ = sup { 0 } = 0.

$\therefore R\int_a^{\bar{b}} f(x)dx \neq R\int_{\bar{a}}^{b} f(x)dx$ so that f is not Riemann integrable.

Let A be the set of all rationals in [0, 1]. Then f = $1\cdot\chi_A + 0\cdot\chi_{\bar{A}}$ ∴ f = χ_A

And m(A) = 0. $\int f dx = 1 \times m(A) + 0 \times m(\bar{A})$ = 0.

Definition: A function defined on R is called a ***simple*** if it is measurable and assumes only finite number of values.

Definition: Canonical representation: If φ is simple taking nonzero values a_1, a_2, …, a_n, then $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$ where $A_i = \{x: \varphi(x) = a_i\}$, a_i's are distinct and A_i's are disjoint.

Definition: Lebesgue Integral of a simple function: Let φ be a simple function and vanishes outside a set of finite measure and $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$ is canonical representation. Then Lebesgue Integral of φ is defined as $\int \varphi(x)dx = \sum_{i=1}^{n} a_i m(A_i)$

Lemma: If $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ is a simple function where E_i's are pairwise disjoint, then $\int \varphi(x)dx = \sum_{i=1}^{n} a_i m(E_i)$.

Proposition: 1*: Let φ and ψ be simple functions which vanish outside a set of finite measure. Then (i) $\int(a\varphi + b\psi) = a\int\varphi + b\int\psi$ (ii) $\varphi \geq \psi \Rightarrow \int\varphi \geq \int\psi$.

Proof: Let $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$ and $\psi = \sum_{j=1}^{m} b_j \chi_{B_j}$ be canonical representations.

Then $E = \cup_{i=0}^{n} A_i = \cup_{j=0}^{m} B_j$.

Let $A_0 = \{x: \varphi(x) = 0\}$ and $B_0 = \{x: \psi(x) = 0\}$.

Then $\varphi = \sum_{i=0}^{n} a_i \chi_{A_i}$ and $\psi = \sum_{j=0}^{m} b_j \chi_{B_j}$ where $a_0 = 0$, $b_0 = 0$.

Let $C_{ij} = A_i \cap B_j$. Then $C_{ij} \cap C_{kl} = \phi$ if $i \neq k$ or $j \neq l$.

Also, $A_i = A_i \cap E = A_i \cap \cup_{j=0}^{m} B_j = \cup_{j=0}^{m}(A_i \cap B_j) = \cup_{j=0}^{m} C_{ij}$

Similarly, $B_j = \cup_{i=0}^{n} C_{ij}$

$\varphi = \sum_{i=0}^{n} a_i \chi_{A_i} = \sum_{i=0}^{n} a_i \chi_{\cup_{j=0}^{m} C_{ij}} = \sum_{i=0}^{n} a_i \sum_{j=0}^{m} \chi_{C_{ij}} = \sum_{i=0}^{n} \sum_{j=0}^{m} a_i \chi_{C_{ij}}$

Similarly, $\psi = \sum_{i=0}^{n} \sum_{j=0}^{m} b_j \chi_{C_{ij}}$

$\therefore a\varphi + b\psi = \sum_{i=0}^{n} \sum_{j=0}^{m} (aa_i + bb_j) \chi_{C_{ij}}$

Then $\int(a\varphi + b\psi) = \sum_{i=0}^{n} \sum_{j=0}^{m} (aa_i + bb_j) m(C_{ij}) = \sum_{i=0}^{n} \sum_{j=0}^{m} (aa_i) m(C_{ij}) + \sum_{i=0}^{n} \sum_{j=0}^{m} (bb_j) m(C_{ij}) = a \sum_{i=0}^{n} a_i \sum_{j=0}^{m} m(C_{ij}) + b \sum_{j=0}^{m} b_j \sum_{i=0}^{n} m(C_{ij})$

$= a \sum_{i=0}^{n} a_i m(A_i) + b \sum_{j=0}^{m} b_j m(B_j) = a \sum_{i=1}^{n} a_i m(A_i) + b \sum_{j=1}^{m} b_j m(B_j)$

$= a\int\varphi + b\int\psi$

(ii) Let $\varphi \geq \psi$. Then $\varphi - \psi \geq 0 \Rightarrow \int(\varphi - \psi) \geq 0 \Rightarrow \int\varphi - \int\psi \geq 0 \Rightarrow \int\varphi \geq \int\psi$.

Proposition: 10*: Let f be defined and bounded on a measurable set E with finite measure. In order that $\inf_{f \leq \psi} \int_E \psi(x)dx = \sup_{\varphi \leq f} \int_E \varphi(x)dx$ for all simple functions ψ and φ it is necessary and sufficient that f be measurable.

Proof: Suppose f is measurable.

Since f is bounded $\exists\, M > 0 \ni |f(x)| \leq M\ \forall\, x \in E$. ie. $f(x) \in [-M, M]$.

For each k define $E_k = \left\{x: \frac{(k-1)M}{n} < f(x) \leq \frac{kM}{n}\right\}$ for any $n \geq 1$.

Then clearly $[-M, M] \subseteq \cup_{k=-n}^{n} \left]\frac{(k-1)M}{n}, \frac{kM}{n}\right]$.

If $x \in E$, then $f(x) \in [-M, M] \subseteq \cup_{k=-n}^{n} \left]\frac{(k-1)M}{n}, \frac{kM}{n}\right] \Rightarrow \exists$ unique integer $k \ni$

$\frac{(k-1)M}{n} < f(x) \leq \frac{kM}{n}$ $\Rightarrow$ x $\in$ E_k for some k.

Thus, E = $\bigcup_{k=-n}^{n} E_k$ so that m(E) = $\sum_{k=-n}^{n} m(E_k)$

Now for x $\in f^{-1}\left]\frac{(k-1)M}{n}, \frac{kM}{n}\right]$ define $\varphi_n(x) = \frac{(k-1)M}{n}$ and $\psi_n(x) = \frac{kM}{n}$.

φ_n, ψ_n are simple since each of them takes only finitely many values and

Definition: Let f be a bounded measurable function on a set of finite measure E.

$$\int_E f(x)dx = \inf_{\substack{\psi \geq f \\ \psi\ simple}} \int \psi(x)dx = \sup_{\substack{\varphi \leq f \\ \varphi\ simple}} \int \varphi(x)dx$$

Proposition: 9*: Let f be a bounded function defined on [a, b]. If f is R-integrable on [a, b], then it is measurable and R-$\int_a^b f(x)dx = \int_a^b f(x)dx$.

Proof: Let f be R-integrable on [a, b]. Let ψ' be a step function such that $\psi' \geq f$.

Then $\inf_{\substack{\psi \geq f \\ \psi\ simple}} \int \psi(x)dx \leq \int_{\substack{\psi' \geq f \\ \psi'\ step\ fun}} \psi'(x)dx$

$\Rightarrow \inf_{\substack{\psi \geq f \\ \psi\ simple}} \int \psi(x)dx \leq \inf_{\substack{\psi' \geq f \\ \psi' stepfun}} \int \psi'(x)dx$

Similarly, if φ' is a step function such that $\varphi' \leq f$.

then $\sup_{\substack{\varphi \leq f \\ \varphi\ simple}} \int \varphi(x)dx \geq \sup_{\substack{\varphi' \leq f \\ \varphi' stepfun}} \int \varphi'(x)dx$

Now by def of Riemann integral, $R \underline{\int_a^b} f(x)dx = \sup_{\substack{\varphi' \leq f \\ \varphi' stepfun}} \int \varphi'(x)dx$

$$\leq \sup_{\substack{\varphi \leq f \\ \varphi\ simple}} \int \varphi(x)dx \leq \inf_{\substack{\psi \geq f \\ \psi\ simple}} \int \psi(x)dx \leq \inf_{\substack{\psi' \geq f \\ \psi' stepfun}} \int \psi'(x)dx = R \int_a^{\bar{b}} f(x)dx$$

Since f is Riemann integrable $R \int_{\underline{a}}^b f(x)dx = R \int_a^{\bar{b}} f(x)dx$

$$\therefore R \int_a^b f(x)dx = \sup_{\substack{\varphi \leq f \\ \varphi\ simple}} \int \varphi(x)dx = \inf_{\substack{\psi \geq f \\ \psi\ simple}} \int \psi(x)dx$$

$\Rightarrow$ f is Riemann integrable and $\int_a^b f(x)dx = R \int_a^b f(x)dx$

Proposition: Let f and g be bounded measurable functions on a set of finite measure E and a, b $\in$R. Then (i) $\int_E (af + bg) = a\int_E f + b\int_E g$

(ii) If f = g a. e. then $\int_E f = \int_E g$

(iii) If f ≤ g a. e. then $\int_E f \le \int_E g$. In particular $\left|\int_E f\right| \le \int_E |f|$

(iv) a and b are constants such that a ≤ f(x) ≤ b ⇒ a m(E) ≤ $\int_E f(x)dx$ ≤ b m(E)

(v) If A and B are disjoint measurable sets of finite measure then $\int_{A\cup B} f = \int_A f + \int_B f$

<u>Proof</u>: (i) Claim: $\int_E (af) = a\int_E f$

If a = 0 then it is trivial. Let a > 0. Let ψ be a simple function ∍ ψ ≥ af.

$\therefore \int_E af = \inf\limits_{\substack{\psi \ge af \\ \psi\ simple}} \int \psi = \inf\limits_{\substack{\frac{\psi}{a} \ge f \\ \psi\ simple}} \int a\frac{\psi}{a} = \inf\limits_{\substack{\psi' \ge f \\ \psi'\ simple}} \int a\psi' = \inf\limits_{\substack{\psi' \ge f \\ \psi'\ simple}} a\int \psi' =$

$a \inf\limits_{\substack{\psi' \ge f \\ \psi'\ simple}} \int \psi' = a\int_E f$

Let a < 0.

Then $\int_E af = \inf\limits_{\substack{\psi \ge af \\ \psi\ simple}} \int \psi = \inf\limits_{\substack{\frac{\psi}{a} \le f \\ \psi\ simple}} \int a\frac{\psi}{a} = \inf\limits_{\substack{\psi' \le f \\ \psi'\ simple}} \int a\psi' = \inf\limits_{\substack{\psi' \le f \\ \psi'\ simple}} a\int \psi' =$

$a \sup\limits_{\substack{\psi' \le f \\ \psi'\ simple}} \int \psi' = a\int_E f$

Claim: $\int_E (f+g) = \int_E f + \int_E g$

Let ψ, ψ′ be simple functions ∍ ψ ≥ f and ψ′ ≥ g. Then $\psi + \psi'$ is a simple function ∍ $\psi + \psi'$≥ f + g

$\int_E (f+g) \le \int_E (\psi + \psi') = \int_E \psi + \int_E \psi' \le \inf\limits_{\substack{\psi \ge f, \psi' \ge g \\ \psi + \psi'\ simple}} (\int_E \psi + \int_E \psi')$

$= \inf\limits_{\substack{\psi \ge f \\ \psi\ simple}} (\int \psi) + \inf\limits_{\substack{\psi' \ge g \\ \psi'\ simple}} \int_E \psi' = \int_E f + \int_E g$ Thus, $\int_E (f+g) \le \int_E f + \int_E g$

Similarly using definition $\int_E f(x)dx = \sup\limits_{\substack{\varphi \le f \\ \varphi\ simple}} \int \varphi(x)dx$ we get

$\int_E (f+g) \ge \int_E f + \int_E g$. Hence $\int_E (f+g) = \int_E f + \int_E g$

(ii) Let f = g a.e. So, f – g = 0 a.e.

Let ψ be a simple function ∍ ψ ≥ f – g ⇒ ψ ≥ 0 a.e. $\int_E \psi$ ≥ 0 ∀ ψ ≥ f – g

In particular $\inf\limits_{\substack{\psi \ge f-g \\ \psi\ simple}} (\int \psi) \ge 0 \Rightarrow \int_E (f+g) \ge 0 \Rightarrow \int_E f - \int_E g \ge 0 \Rightarrow \int_E f \ge \int_E g$

By interchanging f and g we get $\int_E f \le \int_E g$. Hence $\int_E f = \int_E g$

(iii) Let f ≤ g a.e. Then g – f ≥ 0 a.e.
Let ψ be a simple function ∋ ψ ≥ g – f ⇒ ψ ≥ 0 a.e.
By (ii) $\int_E (g-f) \geq 0 \Rightarrow \int_E g - \int_E f \geq 0 \Rightarrow \int_E g \geq \int_E f$
Since $-|f| \leq f \leq |f|$, $-\int_E |f| \leq \int_E f \leq \int_E |f| \Rightarrow \left|\int_E f\right| \leq \int_E |f|$
(iv) Let a and b are constants such that a ≤ f(x) ≤ b ⇒ $\int_E a \leq \int_E f \leq \int_E b$
$\Rightarrow a\int_E 1 \leq \int_E f \leq b\int_E 1$
$\Rightarrow a\, m(E) \leq \int_E f(x)dx \leq b\, m(E)$
(v) Let A and B are disjoint measurable sets of finite measure.
Then $\int_{A\cup B} f = \int_E f\chi_{A\cup B} = \int_E f(\chi_A + \chi_B) = \int_E f(\chi_A + f\chi_B) = \int_E f\chi_A + \int_E f\chi_B) =$
$\int_A f + \int_B f$

Bounded Convergence Theorem: Let $\{f_n\}$ be a sequence of measurable functions defined on a set E of finite measure and suppose that there is a real number M such that $|f_n(x)| \leq M$ for all x and for all n. If f(x) = $\lim_n f_n(x)$ for each x in E, then $\int_E f = \lim_n \int_E f_n$.

Proof: Let ε > 0. By Littlewood's third principle, corresponding to $\varepsilon_1 = \frac{\varepsilon}{2m(E)}$ and $\delta = \frac{\varepsilon}{4M}$ ∃ N ∈ ℕ and a measurable set A ⊆ E with $m(A) < \frac{\varepsilon}{4M}$…(i) ∋
$|f_n(x) - f(x)| < \frac{\varepsilon}{2m(E)}$ … (ii) ∀ n ≥ N and ∀ x ∈ E \ A.
Now $\left|\int_E f_n - \int_E f\right| = \left|\int_E (f_n - f)\right| \leq \int_E |f_n - f| = \int_A |f_n - f| + \int_{E\setminus A} |f_n - f|$
$\leq \int_A (|f_n| + |f|) + \int_{E\setminus A} |f_n - f| < \int_A (M+M) + \int_{E\setminus A} \frac{\varepsilon}{2m(E)}$
$= 2M\, m(A) + \frac{\varepsilon}{2m(E)} m(E \setminus A) < 2M \frac{\varepsilon}{4M} + \frac{\varepsilon}{2m(E)} m(E\setminus A) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$
i.e. given ε > 0, ∃ N ∈ ℕ ∋ $\left|\int_E f_n - \int_E f\right| < \varepsilon$ ∀ n ≥ N.
$\therefore \lim_n \int_E f_n = \int_E f.$

INTEGRAL OF NON – NEGATIVE FUNCTION.
***Definition**: If f is a non – negative measurable function defined on a measurable set E, f vanishes outside a set of finite measure if there exists measurable set $E_0 \subseteq E$ with $m(E_0) < \infty$ and f = 0 on $E - E_0$.

Definition: If f is a non – negative measurable function defined on a measurable set E, then Lebesgue integral of f over E is defined as

$$\int_E f(x)dx = \sup_{h\le f}\{\int_E h : h \text{ is a bounded measurable function} \ni m\{x: h(x) \neq 0\} < \infty\}.$$

Proposition: Let f and g be non-negative measurable functions on a set of finite measure E and a, b $\in$R. Then (i) $\int_E (cf) = c\int_E f$ if c > 0

(ii) $\int_E (f+g) = \int_E f + \int_E g$

(iii) If f $\le$ g a.e. then $\int_E f \le \int_E g$.

Proof: (i) Let c > 0. Let h be any bounded measurable function such that m{x: h(x) $\neq$ 0} < ∞ and h $\le$ cf.

$$\therefore \int_E cf = \sup_{h\le cf}\int h = \sup_{\frac{h}{c}\le f}\int c\frac{h}{c} = \sup_{h'\le f}\int ch' = \sup_{h'\le f} c\int h' = c\sup_{h'\le f}\int h' = c\int_E f$$

(ii) $\int_E (f+g) = \int_E f + \int_E g$

Let h, h′ be bounded measurable functions such that m{x: h(x) $\neq$ 0} < ∞ and h $\le$ f and k $\le$ g. Then $h+k$ is a bounded measurable function $\ni$ $h+k \le$ f + g and m{x: (h + k)(x) $\neq$ 0} < ∞.

$\int_E (h+k) \le \int_E (f+g) \Rightarrow \int_E h + \int_E k \le \int_E (f+g)$ $\forall$ h $\le$ f, k $\le$ g

$\Rightarrow \sup_{h\le f, k\le g}(\int_E h + \int_E k) \le \int_E (f+g). \Rightarrow \int_E f + \int_E g \le \int_E (f+g)$ … (1)

Let l be a bounded measurable function such that l $\le$ f + g and m (E_0) < ∞ where E_0 ={x: l(x) $\neq$ 0} and l(x) = 0 on E – E_0.

Define h(x) = min {f(x), l(x) and k(x) = l(x) – h(x).

Clearly h(x) $\le$ f(x) and h(x) = 0 on E – E_0. Also, k(x) = 0 on E – E_0.

Now if h(x) = f(x), then k(x) = l(x) – f(x) $\le$ g(x) and

if h(x) = l(x), then k(x) = 0 $\le$ g(x) ie k(x) $\le$ g(x)

$\therefore$ h and k are bounded measurable functions such that h(x) $\le$ f(x) and k(x) $\le$ g(x) and vanish outside E_0.

$\Rightarrow \int_E h \le \int_E f$ and $\int_E k \le \int_E g$

$\Rightarrow \int_E h + \int_E k \le \int_E f + \int_E g$

$\Rightarrow \int_E (h+k) \le \int_E f + \int_E g$

Taking sup over all such l we get $\int_E (f+g) \le \int_E f + \int_E g$ … (2)

From (1) and (2) $\int_E (f+g) = \int_E f + \int_E g$

(iii) Let f $\le$ g a.e. Then $\exists$ measurable set E_0 with m(E_0) = 0 and f $\le$ g on E – E_0.

Then $\int_E (g-f) = \int_{E-E_0} (g-f) = \int_{E_0} (g-f) + \int_{E-E_0} (g-f) = \int_{E-E_0} (g-f) \ge 0$
since g – f ≥ 0 on E – E_0.
$\therefore\ 0 \le \int_E (g-f) = \int_{E-E_0} (g-f) = \int_{E-E_0} g - \int_{E-E_0} f = \int_E g - \int_E f$
$\Rightarrow \int_E f \le \int_E g$

<u>Fatou's Lemma:</u> If $\{f_n\}$ is a sequence of non-negative measurable functions defined on a measurable set E and if $f_n(x) \to$ f(x) a. e. on E, then $\int_E f \le \underline{\lim}_n \int_E f_n$.

<u>Proof</u>: Since integrals over sets of measure zero are equal to zero, we may assume without loss of generality that the convergence is everywhere on E.
ie. $f_n(x) \to$ f on E.
Let h be a bounded measurable function ∋ E_0 = {x: h(x) ≠ 0}is a set of finite measure, and h(x) ≤ f(x).
Since h is bounded, ∃ M $\in \mathbb{R}^+$ ∋ h(x) ≤ M for all x.
Define $h_n(x)$ = min {h(x), $f_n(x)$} for n = 1, 2, 3, …
Also h_n is bounded by M and $h_n(x) = 0\ \ \forall x \in E_0'$.
Also, $h_n(x) \to$ h(x) for each x ∈ E_0.
Now by Bounded convergence theorem, we have $\int_{E_0} h = \lim_n \int_{E_0} h_n$
$\Rightarrow \int_E h = \lim_n \int_E h_n$ since h = 0, h_n = 0 on E – E_0
Since $0 \le h_n(x) \le f_n(x)$ ∀ n, ∀ x, $\underline{\lim}_n \int_E h_n \le \underline{\lim}_n \int_E f_n$,
$\Rightarrow \lim_n \int_E h_n \le \underline{\lim}_n \int_E f_n$
$\Rightarrow \int_E h \le \underline{\lim}_n \int_E f_n$ ∀ h ≤ f.
This being true for every h(x) ≤ f(x), taking supremum over all such h,
$\sup_{h \le f} \int_E h \le \underline{\lim}_n \int_E f_n$
$\Rightarrow \int_E f \le \underline{\lim}_n \int_E f_n$

<u>Proof</u>: Since $f_n(x) \to$ f a. e. on E, ∃ a measurable set A ⊆ E ∋ m(A) = 0 and $f_n(x) \to$ f ∀ x ∈ E \ A.
Then $\int_A f$ = 0, $\int_A f_n$ = 0 ∀ n since m(A) = 0.
$\therefore \int_E f = \int_A f + \int_{E \setminus A} f$.

Similarly, $\int_E f_n = \int_{E\setminus A} f_n$

So, it is enough to show that $\int_{E\setminus A} f \leq \underline{lim}_{n} \int_{E\setminus A} f_n$.

Let h be a bounded measurable function defined on E \ A ∋ h ≤ f and h vanishes outside a set of finite measure.

Put $\Delta_h = \{x \in E \,/\, h(x) \neq 0\}$. Then $m(\Delta_h) < \infty$.

Now we show that $\int_{E\setminus A} h \leq \lim_{n} \int_{E\setminus A} f_n$.

Define $h_n(x) = \min\{h(x), f_n(x)\}$.

Now $x \notin \Delta_h \Rightarrow h(x) = 0. \Rightarrow h_n(x) = 0 \Rightarrow x \notin \Delta_{h_n}$ where $\Delta_{h_n} = \{x \in E: h_n(x) \neq 0\}$

$\therefore\ \Delta_{h_n} \subseteq \Delta_h\ \forall\ n. \Rightarrow m(\Delta_{h_n}) \leq m(\Delta_h) < \infty\ \forall\ n.$

Now $h_n \leq h$ and h is bounded $\Rightarrow h_n$ is bounded. Also, $h_n \leq f_n\ \forall$ n.

Since $\lim_{n} f_n(x) = f(x)\ \forall\ x \in E \setminus A$, we have $\lim_{n} h_n(x) = \lim_{n} \min\{h(x), f_n(x)\}$

$= \min\{h(x), \lim_{n} f_n(x)\} = \min\{h(x), f(x)\} = h(x)\ \forall\ x \in E \setminus A.$

Given $\varepsilon > 0$, $\exists$ N ∋ $|h_n(x) - h(x)| < \varepsilon\ \forall\ n \geq N$.

Since $\Delta_h \subseteq E \setminus A$ we have that $\lim_{n} h_n(x) = h(x)\ \forall\ x \in \Delta_h$.

$\therefore$ $\{h_n\}$ is a sequence of bounded measurable functions ∋ $\lim_{n} h_n(x) = h(x)\ \forall\ x \in \Delta_h$ and $m(\Delta_h) < \infty$.

Now by Bounded convergence theorem, we have $\int_{\Delta_h} h = \lim_{n} \int_{\Delta_h} h_n$.

$\therefore\ \int_{E\setminus A} h = \lim_{n} \int_{E\setminus A} h_n \ldots (1)$ since $h(x) = 0$ and $h_n(x) = 0\ \forall\ x \notin \Delta_h$.

Now $h_n \leq f_n$ on E \ A and hence on Δ_h.

$\Rightarrow \int_{\Delta_h} h_n \leq \int_{\Delta_h} f_n \ldots (2)$

Consider $\int_{E\setminus A} h = \lim_{n} \int_{E\setminus A} h_n \leq \underline{lim}_{n} \int_{E\setminus A} f_n$ by (2)

$\Rightarrow \operatorname{Sup}_{h\leq f} \int_{E\setminus A} h \leq \underline{lim}_{n} \int_{E\setminus A} f_n$

$\Rightarrow \int_E f \leq \underline{lim}_{n} \int_E f_n$

<u>Monotone Convergence Theorem:</u> Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions defined on a measurable set E and let $f = \lim_{n} f_n$.

Then $\int_E f = \lim_{n} \int_E f_n$.

<u>Proof</u>: Since $\{f_n\}$ be an increasing sequence $f = \lim_{n} f_n$, we have $f_n \leq f\ \forall$ n.

Since each f_n is measurable, f is also measurable. $\therefore\ \int_E f_n \leq \int_E f$

$\Rightarrow \overline{\lim_n} \int_E f_n \leq \int_E f \ldots$ (i)

By Fatou's Lemma, $\int_E f \leq \underline{\lim_n} \int_E f_n \ldots$ (ii).

From (i) and (ii), $\int_E f \leq \underline{\lim_n} \int_E f_n \leq \overline{\lim_n} \int_E f_n \leq \int_E f$.

$\Rightarrow \underline{\lim_n} \int_E f_n = \overline{\lim_n} \int_E f_n = \int_E f$.

$\therefore \lim_n \int_E f_n$ exists and $\int_E f = \lim_n \int_E f_n$.

<u>Proposition</u>: 2*: Let f be a non-negative function which is measurable over a measurable set E. Then given ε > 0, there is a δ > 0 such that $\int_A f$ < ε for every set A ⊆ E with m(A) < δ.

Proof: Define $f_n(x) = \begin{cases} f(x), if\ f(x) \leq n \\ n, otherwise \end{cases}$

Then $|f_n(x)| \leq n\ \forall\ n$

$\therefore f_n$ is bounded for all n.

Since f is measurable, f_n is measurable and $f_n \leq f_{n+1}$ ∀ n.

$Also\ \lim_n f_n = f$ (since f(x) ≤ ∞ ⇒ $f_n(x)$ = f(x).)

By Monotone convergence theorem, $\int_E f = \lim_n \int_E f_n$

Since $f_n(x) \leq f(x)\ \forall\ n$ and f is integrable over E, $f_n$is integrable over E.

Now $\int_E (f - f_n) = \int_E f - \int_E f_n \ldots$ (i)

$\therefore$Given ε > 0, ∃ N ∈ ℕ ∋ $\left|\int_E f - \int_E f_n\right| < \frac{\varepsilon}{2}$ ∀ n ≥ N.

But $\int_E f = \lim_n \int_E f_n \geq \int_E f_n$ since f_n are increasing.

$\therefore \int_E f - \int_E f_n < \frac{\varepsilon}{2}$. Ie. $\int_E (f - f_n) < \frac{\varepsilon}{2}$ by (i).

Choose δ ∋ δ < $\frac{\varepsilon}{2N}$.

Let A ⊆ E ∋ m(A) < δ. Then $\int_A f = \int_A (f - f_N + f_N)$

$= \int_A (f - f_N) + \int_A f_N \leq \int_E (f - f_N) + \int_A f_N < \frac{\varepsilon}{2} + N\, m(A) < \frac{\varepsilon}{2} + N\delta < \varepsilon$.

Hence the result.

4. THE GENRAL LEBESGUE INTEGRAL

By a positive part f^+ of a function f we mean $f^+(x) = \begin{cases} f(x) & if\ f(x) \geq 0 \\ 0 & if\ f(x) < 0 \end{cases}$
= max {f(x), 0}.

Similarly, by a negative part f^- of a function f we mean

$f^-(x) = \begin{cases} -f(x) & if\ f(x) \leq 0 \\ 0 & if\ f(x) > 0 \end{cases}$ = max {-f(x), 0} = - min {f(x), 0}.

Note: (i) Both $f^+$and f^- are non – negative functions.

(ii) $f^+ - f^- = f$ and $f^+ + f^- = |f|$

Definition: A measurable function f is said to be integrable on E, if $f^+$and f^- are integrable and integral of f over E is defined as $\int_E f = \int_E f^+ - \int_E f^-$

Proposition: Let f and g be integrable functions on a set of finite measure E. Then

(i) $\int_E (cf) = c\int_E f$.

(ii) $\int_E (f+g) = \int_E f + \int_E g$

(iii) If f ≤ g a.e. then $\int_E f \leq \int_E g$.

(iv) If A and B are disjoint measurable sets of finite measure then

$\int_{A\cup B} f = \int_A f + \int_B f$

Proof: (i) Claim: $\int_E (cf) = c\int_E f$ if c > 0

Let c ∈ R. If c = 0 then it is trivial.

Let c > 0. Then $(cf)^+(x) = \begin{cases} cf(x) & if\ cf(x) \geq 0 \\ 0 & if\ cf(x) < 0 \end{cases}$ = c$\begin{cases} f(x) & if\ f(x) \geq 0 \\ 0 & if\ f(x) < 0 \end{cases}$ = c$(f)^+$

Similarly $(cf)^-(x) = cf^-(x)$.

$\int_E (cf) = \int_E (cf)^+ - \int_E (cf)^- = \int_E c(f^+) - \int_E c(f^-)$ = c{$\int_E f^+ - \int_E f^-$} = c$\int_E f$

Let c < 0. Then $(cf)^+(x) = \begin{cases} cf(x) & if\ cf(x) \geq 0 \\ 0 & if\ cf(x) < 0 \end{cases}$ = -c$\begin{cases} -f(x) & if\ f(x) \leq 0 \\ 0 & if\ f(x) > 0 \end{cases}$

= -c$(f)^-$

Similarly $(cf)^-(x) = \begin{cases} -cf(x) & if\ cf(x) \leq 0 \\ 0 & if\ cf(x) > 0 \end{cases}$ = -c$\begin{cases} f(x) & if\ f(x) \geq 0 \\ 0 & if\ f(x) < 0 \end{cases}$ $= -cf^+(x)$.

$\int_E (cf) = \int_E (cf)^+ - \int_E (cf)^- = \int_E -c(f^-) - \int_E -c(f^+)$ = c{$\int_E f^+ - \int_E f^-$}

= c$\int_E f$

(ii) Claim: If f_1 and f_2 are non-negative integrable functions such that f = $f_1 - f_2$, then $\int_E f = \int_E f_1 - \int_E f_2$.

Let $f = f_1 - f_2$ where f_1, f_2 are non-negative integrable functions.

$f^+ - f^- = f = f_1 - f_2 \Rightarrow f^+ + f_2 = f_2 + f^- \Rightarrow \int_E (f^+ + f_2) = \int_E (f_2 + f^-)$

$\Rightarrow \int_E f^+ + \int_E f_2 = \int_E f_2 + \int_E f^- \Rightarrow \int_E f = \int_E f_1 - \int_E f_2$.

Now f + g = $f^+ - f^- + g^+ - g^- = f^+ + g^+ - (f^- + g^-)$

$\Rightarrow \int_E (f+g) = \int_E (f^+ + g^+) - \int_E (f^- + g^-)$ by claim

$= \int_E f^+ + \int_E g^+ - \int_E f^- - \int_E g^- = \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^-$

$= \int_E f + \int_E g$

(iii) Let f ≤ g a.e. Then g – f ≥ 0 a.e.

By (ii) $\int_E (g - f) \geq 0 \Rightarrow \int_E g - \int_E f \geq 0 \Rightarrow \int_E g \geq \int_E f$

(iv) Let A and B are disjoint measurable sets.

Then $\int_{A\cup B} f = \int_E f\chi_{A\cup B} = \int_E f(\chi_A + \chi_B) = \int_E f(\chi_A + f\chi_B) = \int_E f\chi_A + \int_E f\chi_B) = \int_A f + \int_B f$

<u>**Lebesgue Convergence Theorem**</u>: Let g be an integrable function over E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq$ g on E and f(x) = $\lim_n f_n(x)$ a. e. on E. Then $\int_E f = \lim_n \int_E f_n$.

<u>**Proof**</u>: Given that 0 ≤ $|f_n| \leq$ g on E.

So, g is non-negative, – g ≤ f_n ≤ g for all n.

Since g is integrable, f_n is integrable for all n.

Now $f_n \leq$ g ⇒ $\lim_n f_n \leq$ g ⇒ f = $\lim_n f_n \leq$ g a. e. ⇒ f ≤ g a. e.

Now g is integrable ⇒ f is integrable. Since g ≥ f_n, g – f_n ≥ 0 on E.

∴ {g – f_n} is a sequence of non-negative measurable functions ∍ {g – f_n} converges to g – f a. e. on E.

∴By Fatou's lemma, $\int_E g - \int_E f = \int_E (g - f) \leq \underline{\lim}_n \int_E (g - f_n) = \int_E g - \overline{\lim}_n \int_E f_n. \Rightarrow \overline{\lim}_n \int_E f_n \leq \int_E f$. … (i)

{g + f_n} is a sequence of non-negative measurable functions ∍ {g + f_n} converges to g + f a. e. on E.

∴By Fatou's lemma, $\int_E g + \int_E f = \int_E (g + f) \leq \underline{\lim}_n \int_E (g + f_n) = \int_E g + \underline{\lim}_n \int_E f_n$. $\int_E f \leq \underline{\lim}_n \int_E f_n$ … (ii).

From (i) and (ii), $\int_E f \leq \underline{\lim}_n \int_E f_n \leq \overline{\lim}_n \int_E f_n \leq \int_E f$.

$\Rightarrow \underline{\lim}_n \int_E f_n = \overline{\lim}_n \int_E f_n = \int_E f$.

$\therefore \lim_n \int_E f_n$ exists and $\int_E f = \lim_n \int_E f_n$.

<u>Proposition:</u> 6*: Let $\{f_n\}$ be a sequence of measurable functions that convergences in measure to f. Then, there is a subsequence of $\{f_n\}$ which converges to f almost everywhere.

Theorem: Generalized Lebesgue Convergence theorem

Let $\{g_n\}$ be a sequence of integrable functions converge to an integral function g a. e. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \le g_n$ $\forall$ n and $f_n(x) \to f(x)$ a. e. If $\int_E g = \lim_n \int_E g_n$. Then $\int_E f = \lim_n \int_E f_n$.

<u>Proof:</u> Given that $0 \le |f_n| \le g_n$ on E.

So, g_n is non-negative, $-g_n \le f_n \le g_n$ for all n.

Since g_n is integrable, f_n is integrable for all n.

Since $f_n \to f$ and $g_n \to g$ and $f_n \le g_n$; $f \le g$ a. e.

Now g is integrable $\Rightarrow$ f is integrable. Since $g_n \ge f_n$, $g_n - f_n \ge 0$ on E.

$\therefore$ $\{g_n - f_n\}$ is a sequence of non-negative measurable functions $\ni$ $\{g_n - f_n\}$ converges to g – f a. e. on E.

$\therefore$By Fatou's lemma, $\int_E g - \int_E f = \int_E (g - f) \le \underline{\lim}_n \int_E (g_n - f_n)$

$= \underline{\lim}_n \int_E g_n - \overline{\lim}_n \int_E f_n = \lim_n \int_E g_n - \overline{\lim}_n \int_E f_n = \int_E g - \overline{\lim}_n \int_E f_n$

ie. $\int_E g - \int_E f \le \int_E g - \overline{\lim}_n \int_E f_n \Rightarrow \overline{\lim}_n \int_E f_n \le \int_E f$. … (i)

$\{g_n + f_n\}$ is a sequence of non-negative measurable functions $\ni$ $\{g_n + f_n\}$ converges to g + f a. e. on E.

$\therefore$By Fatou's lemma, $\int_E g + \int_E f = \int_E (g + f) \le \underline{\lim}_n \int_E (g_n + f_n)$

$= \underline{\lim}_n \int_E g_n + \underline{\lim}_n \int_E f_n = \lim_n \int_E g_n + \underline{\lim}_n \int_E f_n = \int_E g + \underline{\lim}_n \int_E f_n$

$\Rightarrow \int_E f \le \underline{\lim}_n \int_E f_n$ … (ii).

From (i) and (ii), $\int_E f \le \underline{\lim}_n \int_E f_n \le \overline{\lim}_n \int_E f_n \le \int_E f$.

$\Rightarrow \underline{\lim}_n \int_E f_n = \overline{\lim}_n \int_E f_n = \int_E f$.

$\therefore \lim_n \int_E f_n$ exists and $\int_E f = \lim_n \int_E f_n$.

CONVERGENCE IN MEASURE:

Definition: A sequence $\{f_n\}$ of measurable functions is said to converge to f in measure if $\forall\ \varepsilon > 0$, $m\{x : |f_n(x) - f(x)| \geq \varepsilon\} \to 0$
Ie. $\forall\ \varepsilon > 0, \forall\ \eta > 0\ \exists\ k \in \mathbb{N} \ni m\{x : |f_n(x) - f(x)| \geq \varepsilon\} < \eta\ \forall\ n \geq k$.

Result: If $\{f_n\}$ is a sequence of measurable functions on E with finite measure and $f_n \to f$ a. e, then $f_n \to f$ in measure.
Proof: Suppose $f_n \to f$ a.e. Let $\varepsilon > 0$, and $\eta > 0$. By Little Woods third principle $\exists$ a measurable set $A \subseteq E$ with $m(A) < \eta$ and $k \in \mathbb{N} \ni |f_n(x) - f(x)| < \varepsilon\ \forall\ n \geq k$ and $\forall\ x \in A^c$.
$\Rightarrow |f_n(x) - f(x)| \geq \varepsilon$ then $x \in A$.
$\Rightarrow \{x : |f_n(x) - f(x)| \geq \varepsilon\ \forall\ n \geq k\} \subseteq A$.
$\Rightarrow m\{x : |f_n(x) - f(x)| \geq \varepsilon\} \leq m(A)\ \forall\ n \geq k$
$\Rightarrow m\{x : |f_n(x) - f(x)| \geq \varepsilon\} < \eta\ \forall\ n \geq k$.
$\Rightarrow f_n \to f$ in measure.

Note: Converse is not true.

Proposition: Let $\{f_n\}$ be a sequence of measurable functions that converges to f in measure. Then there is a subsequence $\{f_{n_k}\}$ that converges to f a. e.
Proof: Given that the sequence $\{f_n\}$ converges to f in measure.
By definition, corresponding to $\frac{1}{2}$, $\exists\ n_1 \in \mathbb{N} \ni m\left\{x : |f_n(x) - f(x)| \geq \frac{1}{2}\right\} < \frac{1}{2}$ $\forall\ n \geq n_1$. And so on, having chosen n_{k-1}, choose n_k as follows.
Corresponding to $\frac{1}{2^k}$, $\exists\ n_k \in \mathbb{N} \ni m\left\{x : |f_n(x) - f(x)| \geq \frac{1}{2^k}\right\} < \frac{1}{2^k}\ \forall\ n \geq n_k$.
Then $\{f_{n_k}\}$ is a subsequence of $\{f_n\}$.
Write $E_k = \left\{x : \left|f_{n_k}(x) - f(x)\right| \geq \frac{1}{2^k}\right\}$. Then $m(E_k) < \frac{1}{2^k}\ \forall$ k;
If $x \notin \bigcup_{k=j}^{\infty} E_k$, then $x \notin E_k\ \forall\ k \geq j$.
$\Rightarrow \left|f_{n_k}(x) - f(x)\right| < \frac{1}{2^k}\ \forall\ k \geq j$.
$\Rightarrow f_{n_k}(x) \to f(x)$,
$\Rightarrow f_{n_k}(x) \to f(x)\ \forall\ x \notin \bigcup_{k=j}^{\infty} E_k$ … (i)
Put $A = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k$.

If x $\notin$ A, then x $\notin \bigcup_{k=j}^{\infty} E_k$ for some j.

$\therefore$ by (i) $f_{n_k}(x) \to f(x)$

$\Rightarrow f_{n_k}(x) \to f(x) \ \forall \ x \ \notin A$.

Then $m\left(\bigcup_{k=j}^{\infty} E_k\right) \leq \sum_{k=j}^{\infty} m(E_k) \leq \sum_{k=j}^{\infty} \frac{1}{2^k} = \frac{1}{2^{j-1}}$ $\forall$ j.

But A $\subseteq \bigcup_{k=j}^{\infty} E_k$

$\Rightarrow$ m(A) $\leq$ m($\bigcup_{k=j}^{\infty} E_k$) < $\frac{1}{2^{j-1}}$ $\forall$ j.

$\Rightarrow$ m(A) = 0

$\therefore$ the subsequence $\{f_{n_k}\}$ of the sequence $\{f_n\}$ converges to f a. e.

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E – CONTENT

PAPER: M 302:

LEBESGUE THEORY

M. Sc. II YEAR, SEMESTER - III

UNIT – III: DIFFERENTIATION AND INTEGRATION

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302: LEBESGUE THEORY
UNIT III
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DIFFERENTIATION AND INTEGRATION

SECTION – 1: DIFFERENTIATION OF MONOTONE FUNCTIONS

Definition: Let $\mathcal{I}$ be a collection of intervals in $\mathbb{R}$. $\mathcal{I}$ covers a set E in the sense of Vitali if for each $\varepsilon > 0$ and x in E, there exists an interval I in $\mathcal{I}$ with $l(I) < \varepsilon$ and $x \in I$.

Vitali Covering Lemma: Let E be a set of finite measure and $\mathcal{I}$, a collection of intervals which cover E in the sense of Vitali. Then given $\varepsilon > 0$, there is a finite disjoint collection $\{I_1, I_2, \ldots, I_n\}$ of intervals in $\mathcal{I}$ such that $m^*(E \sim \bigcup_{i=1}^{n} I_i) < \varepsilon$.

Proof: We may assume that each interval in $\mathcal{I}$ is closed. Since $m^*(E) < \infty$ there exists an open set O containing E with $m^*(O) < \infty$. Write $\mathcal{I}_0 = \{ I \in \mathcal{I} / I \subseteq O\}$. Then $\mathcal{I}_0$ covers E in the sense of Vitali. So we can assume that each interval in $\mathcal{I}$ is contained in the open set O.

Let I_1 be any interval from $\mathcal{I}$. Suppose $I_1, I_2, \ldots, I_n$ were chosen. Now we can choose I_{n+1} as follows: Put $k_n = \sup \{l(I): I \in \mathcal{I}, I \cap \bigcup_{i=1}^{n} I_i = \phi\}$....(i).
$I \subseteq O$ for all $I \in \mathcal{I} \Rightarrow m^*(I) \leq m^*(O) < \infty. \Rightarrow \ell(I) \leq m^*(O)$. This is true for all I belonging to the set given in (i).
$\therefore$ $m^*(O)$ is an upper bound for that set. So $k_n \leq m^*(O) < \infty$.
If $E \subseteq \bigcup_{i=1}^{n} I_i$, then the lemma is trivial.
Otherwise there exists an element $x \in E \sim \bigcup_{i=1}^{n} I_i$.
Since $x \notin \bigcup_{i=1}^{n} I_i$ $\exists$ a $\delta > 0 \ni N_\delta(x) \cap \bigcup_{i=1}^{n} I_i = \phi$.
Now corresponding to for $x \in E$ and $\delta > 0$, $\exists$ $I \in \mathcal{I} \ni x \in I$ and $\ell(I) < \delta$ since $\mathcal{I}$ covers E in the sense of Vitali
Clearly $I \subseteq N_\delta(x)$.
Hence, $I \cap (\bigcup_{i=1}^{n} I_i) \subseteq N_\delta(x) \cap (\bigcup_{i=1}^{n} I_i) = \phi \Rightarrow I \cap (\bigcup_{i=1}^{n} I_i) = \phi \Rightarrow \ell(I) \leq k_n$.
Also, $I \neq \phi$ since $x \in I$.
$\therefore$ $\ell(I) \neq 0$. Now $0 < k_n$.
So $\frac{k_n}{2} < k_n$. $\Rightarrow$ $\frac{k_n}{2}$ is not an upper bound of the set given in (i).
$\Rightarrow \exists$ an interval $I_{n+1} \in \mathcal{I}$ $\ni$ $\frac{k_n}{2} < \ell(I_{n+1})$ and $I_{n+1} \cap (\bigcup_{i=1}^{n} I_i) = \phi$. Thus by induction we get a sequence of disjoint intervals from I $\ni$ $k_n < 2\ell(I_{n+1})$. ... (ii).

Now $\bigcup_{i=1}^{n} I_i \subseteq$ O $\Rightarrow$ m*$(\bigcup_{i=1}^{n} I_i) \leq$ m*(O) < ∞

$\Rightarrow \sum_{i=1}^{\infty} l(I_n) < \infty$.

$\Rightarrow \sum_{i=1}^{\infty} l(I_n)$ converges.

$\therefore$ corresponding to ε/5 > 0, ∃ an integer N ∍ $\sum_{i=N+1}^{\infty} l(I_n) < \frac{\varepsilon}{5}$.

Put R = E ~ $\left(\bigcup_{i=1}^{N} I_i\right)$. Let x ∈ R.

Then x ∉ $\bigcup_{i=1}^{N} I_i$ and $\bigcup_{i=1}^{N} I_i$ is closed.

$\therefore$ ∃ δ > 0 ∍ $N_\delta(x) \cap \left(\bigcup_{i=1}^{N} I_i\right) = \phi$.

Also, x ∈ E and so ∃ I ∈ $\mathcal{J}$ ∍ x ∈ I and I $\cap \left(\bigcup_{i=1}^{N} I_i\right) = \phi$.

Let n be any positive integer such that I ∩ $I_i = \phi$ for all i ≤ n.

ie. I $\cap (\bigcup_{i=1}^{n} I_i) = \phi$.

Then ℓ(I) ≤ k_n by (i).

< 2 $\ell(I_{n+1})$ by (ii).

But $\ell(I_n) \to 0$ as n → ∞.

$\Rightarrow \ell$(I) = 0 which is a contradiction.

$\therefore$ ∃ a positive integer m ∍ I ∩ $I_m \neq \phi$.

Let n be the least positive integer such that I ∩ $I_n \neq \phi$.

Then I ∩ $I_i = \phi$ for all i ≤ n – 1.

$\Rightarrow$ I $\cap \left(\bigcup_{i=1}^{n-1} I_i\right) = \phi$.

$\Rightarrow \ell$(I) ≤ k_{n-1} by (i)

< 2$\ell(I_n)$ by (ii).

Let a_n be the midpoint of I_n.

Let y ∈ I ∩ I_n.

Now $|x - a_n| \leq |x - y| + |y - a_n| \leq l(I) + \frac{l(I_n)}{2} < 2l(I_n) + \frac{l(I_n)}{2} = \frac{5l(I_n)}{2}$.

Let J_n be the closed interval having a_n as its midpoint such that $l(\overline{J_n}) = 5l(I_n)$.

Then x ∈ J_n. Also I ∩ $I_n \neq \phi$ since n > N.

$\therefore$ given x ∈ R, ∃ n > N ∍ x ∈ J_n.

So, R $\subseteq \bigcup_{n=N+1}^{\infty} J_n$.

$\therefore$ m*(R) $\leq \sum_{n=N+1}^{\infty} m^*(J_n) = \sum_{n=N+1}^{\infty} l(J_n) = \sum_{n=N+1}^{\infty} 5l(I_n) =$
$5\sum_{n=N+1}^{\infty} l(I_n) < 5\frac{\varepsilon}{5} = \varepsilon$.

$\Rightarrow$ m*(R) < ε.

Definition: Let f be an extended real valued function defined for all x in an interval containing the point y. We define

$$\overline{\lim_{x\to y}} f(x) = \inf_{\delta>0}\left\{\sup_{0<|x-y|<\delta} f(x)\right\}$$

$$\overline{\lim_{x\to y+}} f(x) = \inf_{\delta>0}\left\{\sup_{0<x-y<\delta} f(x)\right\}$$

$$\overline{\lim_{x\to y-}} f(x) = \inf_{\delta>0}\left\{\sup_{0<y-x<\delta} f(x)\right\}$$

$$\underline{lim}_{x\to y} f(x) = \sup_{\delta>0}\left\{\inf_{0<|x-y|<\delta} f(x)\right\}$$

$$\underline{lim}_{x\to y+} f(x) = \sup_{\delta>0}\left\{\inf_{0<x-y<\delta} f(x)\right\}$$

$$\underline{lim}_{x\to y-} f(x) = \sup_{\delta>0}\left\{\inf_{0<y-x<\delta} f(x)\right\}$$

Definition: Let f be an extended real valued function defined on an interval containing a point x.

$D^+f(x) = \overline{\lim_{h\to 0+}} \frac{f(x+h)-f(x)}{h}$ is called the upper right derivative of f at x.

$D^-f(x) = \overline{\lim_{h\to 0+}} \frac{f(x)-f(x-h)}{h}$ is called the upper left derivative of f at x.

$D_+f(x) = \underline{lim}_{h\to 0+} \frac{f(x+h)-f(x)}{h}$ is called the lower right derivative of f at x.

$D_-f(x) = \underline{lim}_{h\to 0+} \frac{f(x)-f(x-h)}{h}$ is called the lower left derivative of f at x.

Note: $D^+f(x) \leq D_+f(x)$; $D^-f(x) \leq D_-f(x)$ for any function f.

Definition: If $D^+f(x) = D_+f(x) = D^-f(x) = D_-f(x) \neq \pm\infty$, then we say that f is differentiable at x

Exercise: If f is continuous on [a, b] and D^+ exists, every where and non-negative on [a, b], then f(b) ≥ f(a).

Solution: In contrary suppose f(b) < f(a). $\Rightarrow \frac{f(b)-f(a)}{b-a} < 0 \Rightarrow -\left[\frac{f(b)-f(a)}{b-a}\right] > 0$.

Choose ε > 0 such that $-\left[\frac{f(b)-f(a)}{b-a}\right] > \varepsilon > 0$.

$\therefore$ f(b) – f(a) + ε(b – a) < 0.

Define a function ϕ on [a, b] by ϕ(x) = f(x) – f(a) + ε (x –a).

Then ϕ(a) = 0. Let ξ be the largest value in (a, b] such that ϕ(ξ) = 0.

Then for all x ∈ (ξ, b), ϕ(x) < 0.

$D^+\phi(\xi) = \overline{\lim_{h\to 0+}} \frac{\phi(\xi+h)}{h} \leq 0$.

But $D^+\phi(\xi) = \overline{\lim}_{h\to 0+} \frac{\phi(\xi+h)}{h} = \overline{\lim}_{h\to 0+} \frac{f(\xi+h)-f(a)+\varepsilon(\xi+h-a)}{h} = \overline{\lim}_{h\to 0+} \frac{f(\xi+h)-f(\xi)+\varepsilon h}{h}$
since $\phi(\xi) = 0$.

$= \overline{\lim}_{h\to 0+} \frac{f(\xi+h)-f(\xi)}{h} + \varepsilon = D^+f(\xi) + \varepsilon \geq \varepsilon > 0$. Ie. $D^+\phi(\xi) > 0$, a contradiction.
Hence $f(b) \geq f(a)$.

Lebesgue Theorem: Let f be an increasing real valued function on [a, b]. Then f is differentiable almost everywhere. The derivative f ′ is measurable and
$\int_a^b f'(x)dx \leq f(b) - f(a)$.

Proof: We prove $D^+f(x) = D_+f(x) = D^-f(x) = D_-f(x) = f'(x)$ exists ever where.
Ie. to prove the sets where any two of the derivatives mentioned here are unequal, have measure zero.
Let $E = \{ x / D^+f(x) > D_-f(x) \}$. Now we prove that m(E) = 0.
The sets arising from other derivatives can be handled similarly.
For each pair of rationals u and v with u > v, write
$E_{u,v} = \{x / D^+f(x) > u > v > D_-f(x) \}$.
Then $E = \bigcup_{u>v} E_{u,v}$ is a countable union.
Clearly $m^*(E) \leq \sum_{u>v} m^*(E_{u,v})$
So it is enough if we prove that $m^*(E_{u,v}) = 0$. Put $s = m^*(E_{u,v})$.
Let $\varepsilon > 0$. Now $\exists$ an open set $O \supseteq E_{u,v}$ such that $m^*(O) < m^*(E_{u,v}) + \varepsilon = s + \varepsilon$.
Now for each $x \in E_{u,v}$ $\exists$ an arbitrarily small interval
$[x - h, x] \subseteq O$ such that $f(x) - f(x - h) \leq vh$ and $\{[x - h, x] : x \in E_{u,v}\}$ covers $E_{u,v}$ in the sense of Vitali.
$\therefore$ By Vitali covering lemma, $\exists$ a finite disjoint collection of intervals
$\{I_i = [x_i - h_i, x_i] \subseteq O\}$, i = 1, 2, ..., N $\ni m^*\left(E_{u,v} \setminus \bigcup_{i=1}^N I_i\right) < \varepsilon$.
Put $A = E_{u,v} \cap \bigcup_{i=1}^N (x_i - h_i, x_i)$. Then $A \subseteq E_{u,v}$.
Summing over these N intervals
$\sum_{n=1}^N [f(x_n) - f(x_n - h_n)] < v\sum_{n=1}^N h_n < v\, m(O) < v(s + \varepsilon)$...(ix).
Here $h_1, ..., h_N$ are so small that $h_1 + ... + h_N < m(O)$
Now for each $y \in A$ $\exists$ an arbitrarily small interval of the form $[y, y + k] \subseteq I_i$ for some $i \in \{1, 2, ... N\}$ such that $f(y + k) - f(y) > uk$.
The collection of intervals $\{(y, y + k) / y \in A\}$ forms a Vitali covering for the

set A. $\therefore$ By Vitali covering lemma, $\exists$ a finite disjoint collection of intervals
$\{J_i = (y_i, y_i + k_i) \subseteq O\}$, i = 1, 2, ...,M $\ni m^*\left(A \setminus \bigcup_{i=1}^M J_i\right) < \varepsilon$ ... (v).

Put A[1] = $A \cap \bigcup_{j=1}^{M}(y_j, y_j + k_j)$. Now s – ε $< m^*(A)$ (proved)

= $m^*(A') + m^*\left(A \setminus \left(\bigcup_{i=1}^{M} J_i\right)\right)$ since O_2 is measurable.

= m*(A[1]) + ε by (vii)

⇒ m*(A[1]) > s – 2ε... (viii)

Summing over these M intervals

$\sum_{j=1}^{M}[f(y_j + k_j) - f(y_j)]$ > u$\sum_{j=1}^{n} k_j$ > u(s - 2ε).

By (iv) we have that each interval J_i is contained in some I_n and if we sum over those i, for which $J_i \subseteq I_n$ we have

$\sum_{J_i \subseteq I_n}[f(y_i + k_i) - f(y_i)] \leq f(x_n) - f(x_n - h_n)$ (since f is an increasing function). Thus $\sum_{n=1}^{N}[f(x_n) - f(x_n - h_n)] \geq \sum_{j=1}^{M}[f(y_j + k_j) - f(y_j)$

and so v(s + ε) > u(s - 2ε) for all ε > 0.

∴ vs > us, a contradiction since u > v.

∴ s = 0.

Write $(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

The above means that f is differentiable a. e. whenever g is finite.

Define $g_n(x) = n\left\{f\left(x + \frac{1}{n}\right) - f(x)\right\}$ and set f(x) = f(b) for all x ≥ b.

Then $\{g_n\}$ is a sequence of non – negative functions since f is increasing.

Since f is measurable, each g_n is measurable and so $\lim_{n \to \infty} g_n(x)$ is measurable.

Now $\lim_{n \to \infty} g_n(x)$ = $\lim_{n \to \infty} n\left\{f\left(x + \frac{1}{n}\right) - f(x)\right\}$. =

$\lim_{\frac{1}{n} \to 0} \frac{f\left(x+\frac{1}{n}\right) - f(x)}{\frac{1}{n}}$ = $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ = g(x) a. e.

Since each g_n is measurable, lim g_n = g = f ′ is measurable.

∴ By Fatou's Lemma, $\int_a^b g \leq \underline{lim}_n \int_a^b g_n$ = $\underline{lim}_n\, n \int_a^b \left\{f\left(x + \frac{1}{n}\right) - f(x)\right\}$

= $\underline{lim}_n \left(n \int_b^{b+\frac{1}{n}} f - n \int_a^{a+\frac{1}{n}} f\right)$ = $\underline{lim}_n \left(f(b) - n \int_a^{a+\frac{1}{n}} f\right)$ ≤ f(b) – f(a)

since – f(x) ≤ – f(a)

ie. $\int_a^b f'(x)dx$ ≤ f(b) – f(a). This completes the proof.

SECTION – 2:

FUNCTIONS OF BOUNDED VARIATION

Definition: (i) If r is a real number, then we define $r^+ = \max\{r, 0\}$ and
$r^- = \max\{-r, 0\}$.

Clearly $r = r^+ - r^-$, $|r| = r^+ + r^-$

(ii) Let f be a real valued function defined on [a, b]. Let $a = x_0 < x_1 < ... < x_k = b$ be a subdivision of [a, b].

Define $p = \sum_{i=1}^{k}[f(x_i) - f(x_{i-1})]^+$, $n = \sum_{i=1}^{k}[f(x_i) - f(x_{i-1})]^-$

(iii) $t = n + p = \sum_{i=1}^{k}|f(x_i) - f(x_{i-1})|$

(iv) $p - n = \sum_{i=1}^{k}[f(x_i) - f(x_{i-1})]^+ - \sum_{i=1}^{k}[f(x_i) - f(x_{i-1})]^-$

$= \sum_{i=1}^{k}[\{f(x_i) - f(x_{i-1})\}^+ - \{f(x_i) - f(x_{i-1})\}^-] = \sum_{i=1}^{k}[f(x_i) - f(x_{i-1})]$

$= f(b) - f(a)$.

(v) Define P = sup p, N = sup n and T = sup t where supremums are taken over all possible subdivisions of [a, b].

(vi) Clearly $p < t$.

$\therefore$ sup p $\leq$ sup t $\Rightarrow$ $P \leq T$. Similarly $N \leq T$ and $T \leq P + N$. We call P, N and T the positive, negative and total variations of f over [a, b].

We denote them by P_a^b(f), N_a^b(f) and T_a^b(f) respectively.

Definition: A function f on an interval [a, b] is said to be a function of bounded variation over the interval [a, b] if $T_a^b(f) < \infty$.

Note: any monotonic function is of bounded variation.

Lemma: If f is a function of bounded variation on [a, b], then $T_a^b = P_a^b + N_a^b$ and $f(b) - f(a) = P_a^b - N_a^b$

Proof: Let f be a function of bounded variation on the interval [a, b].

By definition $T_a^b < \infty$.

For any partition of [a, b], we have

$p - n = \sum_{i=1}^{k}[\{f(x_i) - f(x_{i-1})\}^+ - \{f(x_i) - f(x_{i-1})\}^-]$

$\quad = \sum_{i=1}^{k}[f(x_i) - f(x_{i-1})] = f(b) - f(a)$ ...(i)

$\Rightarrow p = f(b) - f(a) + n \leq f(b) - f(a) +$
N.

$\Rightarrow P \leq f(b) - f(a) + N$.

$\Rightarrow P - N \leq f(b) - f(a)$.

Again $n = p + f(a) - f(b) \leq P + f(a) - f(b) \Rightarrow N \leq P + f(a) - f(b)$.

$\Rightarrow$ P – N ≥ f(b) – f(a)

$\therefore$ P – N = f(b) – f(a). ie. $P_a^b - N_a^b = f(b) - f(a)$

t = p + n = p + [p – {f(b) – f(a)}] = 2p – {f(b) – f(a)}

$\Rightarrow$ T ≥ t = 2p – {f(b) – f(a)} for all p.

$\Rightarrow$ T ≥ 2P – {f(b) – f(a)} = 2P – (P – N) = P + N. But we have T ≤ P + N

Hence T = P + N.

Theorem: A function f is of bounded variation on [a, b] if and only if f is the difference of two monotone real valued functions on the interval [a, b].

Proof: Suppose f is a function of bounded variation on the interval [a, b].

Define g and h as g(x) = P_a^x, h(x) = N_a^x.

Now $x_1 \le x_2 \Rightarrow P_a^{x_1} \le P_a^{x_2} \Rightarrow g(x_1) \le g(x_2) \Rightarrow$ g is increasing.

Similarly, h is increasing.

Also, by definition, $0 \le P_a^x \le T_a^x \le T_a^b < \infty$ and $0 \le N_a^x \le T_a^x \le T_a^b < \infty$.

Thus, both g and h are real valued functions.

Now g(x) – h(x) = $P_a^x - N_a^x$ = f(x) – f(a) by lemma

$\Rightarrow$ f(x) = g(x) – {h(x) – f(a)} $\forall$ x $\in$ [a, b] where g and h are increasing real valued functions.

Conversely suppose f = g – h where g and h are monotonic.

Let a = $x_0 < x_1 < \dots < x_n$ = b be a partition of [a, b].

Now t = $\sum_{i=1}^{n}|f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n}|\{g(x_i) - h(x_i)\} - \{g(x_{i-1}) - h(x_{i-1})\}| \le \sum_{i=1}^{n}|g(x_i) - g(x_{i-1})| + \sum_{i=1}^{n}|h(x_i) - h(x_{i-1})|$

= $|g(b) - g(a)| + |h(b) - h(a)|$ since g and h are monotonic.

< ∞.

$\Rightarrow T_a^b < |g(b) - g(a)| + |h(b) - h(a)| < \infty$ since each function is bounded.

Hence f is a function of bounded variation on [a, b].

Corollary: If f is a function of bounded variation on [a, b] then f is differentiable a.e.

Proof: Since f is a function of bounded variation on [a, b], f can be written as a difference of two monotone functions. Suppose f = g – h where g and h are monotonic functions. W. L. G. We may assume that f = g ± h where g and h are increasing functions. Then, we have that g and h are differentiable almost everywhere and hence f = g = h is also differentiable a. e.

Example: If a ≤ c ≤ b then (i) $T_a^b = T_a^c + T_c^b$; $T_a^c \le T_a^b$

(ii) $T_a^b(f+g) \le T_a^b(f) + T_a^b(g)$ (iii) $T_a^b(cf) = |c| T_a^b(f)$.

Proof: (i) Let f be a function defined on [a, b], $c \in [a, b]$, $a = x_0 < x_1 < ... < x_n = b$ be a partition of [a, b] and $\exists$ i ∍ $x_i \le c \le x_{i+1.}$

Now $a = x_0 < x_1 < ... < x_i \le c$ is a partition of [a, c] and $c \le x_{i+1} < x_{i+2} < ... < x_n = b$ is a partition of [c, b].

Write $t_1 = \sum_{j=1}^{i}\left|f\left(x_j\right) - f\left(x_{j-1}\right)\right| + |f(c) - f(x_i)|$

and $t_2 = \sum_{j=i+2}^{n}\left|f\left(x_j\right) - f\left(x_{j-1}\right)\right| + |f(x_{i+1}) - f(c)|$

$\therefore$ $t_1 \le T_a^c$ and $t_2 \le T_c^b$

Now $t = \sum_{i=1}^{n}|f(x_i) - f(x_{i-1})| = \sum_{j=1}^{i}\left|f\left(x_j\right) - f\left(x_{j-1}\right)\right| + |f(x_{i+1}) - f(x_i)|$ $+ \sum_{j=i+2}^{n}\left|f\left(x_j\right) - f\left(x_{j-1}\right)\right|$

$\le \sum_{j=1}^{i}\left|f\left(x_j\right) - f\left(x_{j-1}\right)\right| + |f(x_{i+1}) - f(c)| + |f(c) - f(x_i)|$

$+\sum_{j=i+2}^{n}\left|f\left(x_j\right) - f\left(x_{j-1}\right)\right| = t_1 + t_2 \le T_a^c + T_c^b$

$\therefore$ $t \le T_a^c + T_c^b$ Hence $T_a^b \le T_a^c + T_c^b$

Let $a = x_0 < x_1 < ... < x_m = c$ is a partition of [a, c] and $c = y_0 < y_1 < ... < y_k = b$ is a partition of [c, b].

Put $t' = \sum_{j=1}^{m}\left|f\left(x_j\right) - f\left(x_{j-1}\right)\right|$ and $t'' = \sum_{j=1}^{k}\left|f\left(y_j\right) - f\left(y_{j-1}\right)\right|$

Now $a = x_0 < x_1 < ... < x_m = c = y_0 < y_1 < ... < y_k = b$ is a partition of [a, b].

So $t = \sum_{j=1}^{m}\left|f\left(x_j\right) - f\left(x_{j-1}\right)\right| + \sum_{j=1}^{k}\left|f\left(y_j\right) - f\left(y_{j-1}\right)\right| = t' + t''$

But $T_a^b \ge t = t' + t''$.

$\Rightarrow T_a^b \ge \sup\{t' + t''\}$ $\Rightarrow$ $T_a^b \ge \sup\{t') + \sup\{t'')$

$\Rightarrow$ $T_a^c + T_c^b$ $\therefore T_a^b = T_a^c + T_c^b$

(ii) consider a partition $a = x_0 < x_1 < ... < x_n = b$ of [a, b].

Now $T_a^b(f + g) = \sum_{i=1}^{n}|(f + g)(x_i) - (f + g)(x_{i-1})|$

$= \sum_{i=1}^{n}|f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})|$

$\le \sum_{i=1}^{n}|f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n}|g(x_i) - g(x_{i-1})| = T_a^b(f) + T_a^b(g)$.

(iii) $T_a^b(cf) = \sup\{\sum_{i=1}^{n}|cf(x_i) - cf(x_{i-1})|\} = \sup\{\sum_{i=1}^{n}|c||f(x_i) - f(x_{i-1})|\} = |c| \sup\{\sum_{i=1}^{n}|f(x_i) - f(x_{i-1})|\} = |c|T_a^b(f)$.

Result: Show that if f' exists and is bounded on [a, b] then f is of bounded variation on [a, b].

Proof: Let $a = x_0 < x_1 < ... < x_n = b$ be a partition of [a, b]. Since f' is bounded there exists M such that $|f'(x)| \le M$ $\forall$ $x \in [a, b]$.

Now $t_a^b(f) = \sum_{i=1}^{n}|f(x_i) - f(x_{i-1})| \le \sum_{i=1}^{k}|f'(t_i)|(x_i - x_{i-1})$

$\leq M\sum_{i=1}^{k}(x_i - x_{i-1})$ = M(b – a)

$\therefore\ T_a^b(f)$ = sup $t_a^b(f) \leq$ M(b – a) < ∞.

SECTION – 3. DIFFERENTIATION OF AN INTEGRAL

Lemma: If f is integrable on [a, b], then the function F defined by
F(x) = $\int_a^x f(t)dt$ is a continuous function of bounded variation.

Proof: Since f is integrable, we have that $|f|$ is a non negative integrable function on [a, b].
Let ε > 0.
By a proposition ∃ δ > 0 ∍ $\int_A |f|$ < ε ∀ A ⊆ [a, b] with m(A) < δ.
Let x, y ∈ [a, b] ∍ $|x - y| < \delta$.
Without loss of generality, we assume that x < y.
Now $|F(y) - F(x)| = \left|\int_a^y f(t)dt - \int_a^x f(t)dt\right| = \left|\int_x^y f(t)dt\right|$
$\leq \int_x^y |f(t)|dt$ < ε.
Thus, F is uniformly continuous and hence F is continuous.
Let a = $x_0 < x_1 < ... < x_n$ = b be a partition of [a, b].
Now $t_a^b(F) = \sum_{i=1}^{n}|F(x_i) - F(x_{i-1})|$
$= \sum_{i=1}^{n}\left|\int_{x_{i-1}}^{x_i} f(t)dt\right| \leq \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}|f|dt = \int_a^b |f|dt$ [since $\int_E f = \sum_{i=1}^{n}\int_{E_i} f$,
where E = $\cup E_i$ and the union is disjoint.]
< ∞ since $|f|$ is integrable.
∴ $T_a^b(F)$ < ∞ and so F is of bounded variation on [a, b].

Lemma: If f is integrable on [a, b] and $\int_a^x f(t)dt$ = 0 ∀ x ∈ [a, b], then f(t) = 0 a. e. on [a, b].

Proof: Write E = {t ∈ [a, b] / f(t) > 0}.
Claim: m(E) = 0.
If possible, suppose m(E) > 0.
[Since E is measurable, ∃ F ∈$\mathcal{F}_\sigma$ ∍ F ⊆ E and m(E\F) = 0.
Now m(E) = m(F) + m(E\F) = m(F) + 0 = m(F).
Also, F ∈ $\mathcal{F}_\sigma$ ⇒ F = ∪ F_i, F_i is closed.
Now 0 < m(E) = m(F) = m(∪F_i) ≤ $\sum_{i=1}^{\infty} m(F_i)$
ie. $\sum_{i=1}^{\infty} m(F_i)$ > 0

$\Rightarrow$ m(F_i) > 0 for some i.]
So, $\exists$ a closed set $F_i \subseteq E \ni m(F_i) > 0$ since E is measurable.
Write O = (a, b) ~ F_i.
Since F_i is closed, O is open.
Suppose O = $\bigcup_n (a_n, b_n)$ where $\{(a_n, b_n)\}_n$ is a countable disjoint family of open intervals.
Clearly (a, b) = O $\cup$ F_i.
Now 0 = $\int_a^b f$ since $\int_a^x f = 0 \ \forall \ x \in [a, b]$.

$= \int_O f + \int_{F_i} f$

$\Rightarrow \int_O f = -\int_{F_i} f \neq 0$ since f > 0 on F_i, and m(F_i) > 0.

$\Rightarrow 0 \neq \int_O f = \sum_n \int_{a_n}^{b_n} f$ since O = $\bigcup_n (a_n, b_n)$ and the union is disjoint.

$\Rightarrow \int_{a_n}^{b_n} f \neq 0$ for some n.

$\Rightarrow 0 \neq \int_{a_n}^{b_n} f = \int_a^{b_n} f - \int_a^{a_n} f$

$\Rightarrow 0 \neq \int_a^{b_n} f$, or $\int_a^{a_n} f \neq 0$ a contradiction to the hypothesis.

$\therefore$ m(E) = 0.
Similarly we can show that m{t $\in$ [a, b] / f(t) < 0} = 0.
Hence f = 0 a. e. on [a, b].

Lemma: If f is bounded and measurable on [a, b] and F(x) = $\int_a^x f(t)dt$ + F(a) then F′(x) = f(x) for almost all x in [a, b].

Proof: By a lemma F is a function of bounded variation and continuous.
By a theorem F′ (x) exists a. e. Since f is bounded $\exists$ a real k such that $|f| \leq k$.
Write $f_n(x) = \frac{F(x+h)-F(x)}{h}$ with $h = \frac{1}{n}$.

Now $|f_n(x)| = \left|\frac{F(x+h)-F(x)}{h}\right| = \left|\frac{1}{h}\int_x^{x+h} f(t)dt\right| \leq \frac{1}{h}\int_x^{x+h}|f(t)|dt \leq \frac{1}{h}hk = k$.

So each f_n is bounded.

$\lim_{n\to\infty} f_n(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$ = F′(x) a. e.

By the bounded convergence theorem, for all x $\in$ [a, b],

$\int_a^x F'(t)dt = \lim_{n\to\infty} \int_a^x f_n(t)dt = \lim_n \left[\int_a^x \left[n\left\{F\left(t+\frac{1}{n}\right) - F(t)\right\}\right] dt\right]$

$= \lim_n n\left[\int_a^x F\left(t+\frac{1}{n}\right)dt - \int_a^x F(t)dt\right] = \lim_n \left[n\int_{a+\frac{1}{n}}^{x+\frac{1}{n}} F(t)dt - n\int_a^x F(t)dt\right] =$

$\lim_n \left[n\int_x^{x+\frac{1}{n}} F(t)dt - n\int_a^{a+\frac{1}{n}} F(t)dt\right]$ = F(x) – F(a) by mean value theorem. =

$\int_a^x f(t)dt$.

Thus, we have proved that $\int_a^x F'(t)dt = \int_a^x f(t)dt$.

$\Rightarrow \int_a^x (F'(t) - f(t))dt = 0 \ \forall \ x \in [a, b]$.

$\Rightarrow$ F′(t) – f(t) = 0 a. e.

$\Rightarrow$ F′(t) = f(t) a. e.

Theorem: Let f be an integrable function on [a, b] and F(x) = $\int_a^x f(t)dt$ + F(a). Then F′(x) = f(x) for almost all x in [a, b].

Proof: Assume that the theorem is true for all non – negative integrable functions. Let f be an integrable function. Then f^+ and f^- are non-negative integrable functions.

Define f_1 and f_2 by $f_1(x) = \int_a^x f^+(t)dt + f_1(a)$ where $f_1(a)$ = F(a).

$$f_2(x) = \int_a^x f^-(t)dt + f_2(a) \text{ where } f_2(a) = 0.$$

Then by the assumption, $f_1'(x) = f^+(x)$ a. e. and $f_2'(x) = f^-(x)$ a. e.

$\therefore \ f_1'(x) - f_2'(x) = f^+(x) - f^-(x) = f(x)$ a. e.

$\Rightarrow \left(\int_a^x f^+(t)dt + f_1(a)\right)' - \left(\int_a^x f^-(t)dt + f_2(a)\right)'$ = f(x) a. e.

$\Rightarrow \left(\int_a^x f^+(t)dt + F(a)\right)' - \left(\int_a^x f^-(t)dt\right)'$ = f(x) a. e.

$\Rightarrow \left(\int_a^x f(t)dt + F(a)\right)'$ = f(x) a. e.

$\Rightarrow$ F′(x) = f(x) a. e.

Assume that f is non – negative. i.e. f ≥ 0.

Define $f_n(x)$ = f(x) if f(x) ≤ n and n if f(x) > n.

Let f(x) = 10.9; Then $f_1(x) = 1$, $f_2(x) = 2$, …, $f_{10}(x) = 10$, $f_{11}(x) = f_{12}(x) = \ldots =$ 10.9

Then $|f_n(x)| \le n \ \forall$ n; also $|f_n(x)| \le |f(x)| \ \forall$ x and each f_n is measurable since f is measurable.

Since $f_n(x) = f(x) \ \forall \ f(x) \le n$, f_n is a sequence of bounded measurable functions such that $\lim_n f_n(x)$ = f(x).

Also, $f - f_n \ge 0$, since $f_n \le f \ \forall$ n.

Put $G_n(x) = \int_a^x (f - f_n)(t)dt$.

Then G_n is an increasing function of x since $f - f_n \ge 0$.

$\therefore$ G_n is differentiable a. e. and G_n is increasing.

$\Rightarrow G_n'(x)$ is positive.

Now F(x) = $\int_a^x f(t)dt$ + F(a) = $\int_a^x (f - f_n)(t)dt + \int_a^x f_n(t)dt + F(a)$

$= G_n(x) + \int_a^x f_n(t)dt + F(a)$

$\Rightarrow F'(x) = G_n'(x) + \left(\int_a^x f_n(t)dt + F(a)\right)' = G_n'(x) + f_n(x)$ a. e. by a lemma

$\geq f_n(x)$

ie. $F'(x) \geq f_n(x)$ a. e. $\forall$ n.

$\Rightarrow F'(x) \geq \lim_n f_n(x)$ a. e. = f(x) a. e.

i.e. $F'(x) \geq f(x)$ a. e.

$\Rightarrow \int_a^b F'(x)dx \geq \int_a^b f(x)dx$ = F(b) – F(a).

Since F is increasing by a theorem, $\int_a^b F'(x)dx \leq$ F(b) – F(a)...(ii)

From (i) and (ii), $\int_a^b F'(x)dx$ = F(b) – F(a) = $\int_a^b f(x)dx$

ie. $\int_a^b (F' - f)(x)dx = 0$. $\Rightarrow F'(x) - f(x) = 0$ a. e.

Hence $F'(x) = f(x)$ a. e.

SECTION 4: ABSOLUTE CONTINUITY.

Definition: A real valued function f defined on [a, b] is said to be absolutely continuous on [a, b] if given ε > 0, there is a δ > 0 э $\sum_{i=1}^n |f(x_i') - f(x_i)| < \varepsilon$ for every finite collection $\{(x_i, x_i') / 1 \leq i \leq n\}$ of non-overlapping intervals with $\sum_{i=1}^n |x_i' - x_i| < \delta$.

Note: Every absolutely continuous function is continuous.

Lemma: Every absolutely continuous function is of bounded variation.

Proof: let f be an absolutely continuous function.

Take ε = 1. Then ∃ δ > 0 э $\sum_{i=1}^n |f(x_i') - f(x_i)| < 1$ for every finite collection $\{(x_i, x_i') / 1 \leq i \leq n\}$ of non-overlapping intervals with $\sum_{i=1}^n |x_i' - x_i| < \delta$.

Choose an integer n э n > (b – a) / δ.

Let $a = x_0 < x_1 < x_2 < ... < x_n = b$ be a partition of [a, b] where $x_i - x_{i-1} = (b - a) / n$ for all i = 1, 2, ..., n.

Let $x_i = y_0 < y_1 < ... < y_k = x_{i+1}$ be a partition of $[x_i, x_{i+1}]$

Now $\{(y_0, y_1), (y_1, y_2), ..., (y_{k-1}, y_k)\}$ is a finite collection of non-overlapping intervals of $[x_i, x_{i+1}]$ such that $\sum_{i=1}^k (y_i - y_{i-1}) = x_{i+1} - x_i = \frac{b-a}{n} < \delta$.

Since f is absolutely continuous, we have $\sum_{i=1}^n |f(y_i) - f(y_{i-1})| < 1$.

Ie. $T_{x_i}^{x_{i+1}} \leq 1$.

$\therefore T_a^b = \sum_{i=0}^{n-1} T_{x_i}^{x_{i+1}} \leq n < \infty$. Hence f is a function of bounded variation.

Corollary: If f is absolutely continuous then f has a derivative a. e.

Proof: Let f be absolutely continuous.
⇒ f is a function of bounded variation.
⇒ f is differentiable a. e.
⇒ f′(x) exists a. e.

Lemma: If f is absolutely continuous on [a, b] and f ′(x) = 0 a. e. then f is constant.
Proof: let f be absolutely continuous on [a, b]and f ′ (x) = 0 a. e.
Claim: f(c) = f(a) ∀ c ∈ [a, b].
Let c ∈ [a, b]. Write E = (a, c) ∩ {x / f ′ (x) = 0}.
[Then E ⊆ [a, c]. Also [a, c] = E ∪ {[a, c] \ E}.
Let x ∈ [a, c] \ E. So, x ∈ [a, c] and x ∉ E. ⇒ f′(x) ≠ 0.
Ie. [a, c] \ E ⊆ {x : f′(x) ≠ 0}.
∴ m{[a, c] \ E} = 0. m(E) = m{[a, c]} + m{[a, c] \ E} = c – a + 0]
Then m(E) = c – a.
Let ε > 0 and η > 0.
Since f is absolutely continuous, corresponding to ε > 0, ∃ δ > 0 ∍
$\sum_{i=1}^{n}|f(x_i{}') - f(x_i)|$ < ε for every finite collection { $(x_i, x_i{}')$ / 1 ≤ i ≤ n } of non-overlapping intervals with $\sum_{i=1}^{n}|x_i{}' - x_i|$ < δ.
Now f ′(x) = 0 for every x ∈ E ⇒ $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ = 0.
⇒ ∃ arbitrarily small interval [x, x + h] ⊆ [a, c] ∍ $|f(x+h) - f(x)|$ < ηh ... (i).
Now take h < δ. So the collection of all such intervals [x, x + h] form a Vitali covering for the set E. By Vitali covering lemma, ∃ a finite disjoint collection of intervals, say $\{[x_1, x_1 + h_1], \ldots, [x_n, x_n + h_n]\}$ such that
$m(E\sim \bigcup_{i=1}^{n}[x_i, x_i + h_i]) < \delta$.
⇒ $m([a, c]\setminus \bigcup_{i=1}^{n}[x_i, x_i + h_i]) < \delta$.... (ii)
Now we label x_k such that $x_k \le x_{k+1}$.
So, we have $x_0 = a \le x_1 < x_1 + h_1 \le x_2 < x_2 + h_2 \le x_3 < x_3 + h_3 \le \ldots \le x_n < x_n + h_n \le c = x_{n+1}$.
Write $h_0 = 0$.
Now $\bigcup_{i=0}^{n}(x_i + h_i, x_{i+1}) = [a, c] \setminus \bigcup_{i=1}^{n}[x_i, x_i + h_i]$
⇒ $m[\bigcup_{i=0}^{n}(x_i + h_i, x_{i+1})] = m([a, c] \setminus \bigcup_{i=1}^{n}[x_i, x_i + h_i])$
⇒ $\sum_{i=0}^{n}|x_{i+1} - (x_i + h_i)| = m([a, c]\setminus \bigcup_{i=1}^{n}[x_i, x_i + h_i]) < \delta$
⇒ $\sum_{i=0}^{n}|f(x_{i+1}) - f(x_i + h_i)|$ < ε ... (iii) since f is absolutely continuous.
But $\bigcup_{i=0}^{n}[x_i, x_i + h_i] \subseteq [a, c]$
⇒ $\sum_{i=0}^{n} h_i \le c - a$....(iv).
Now $|f(c) - f(a)| = |\sum_{i=0}^{n} f(x_{i+1}) - f(x_i + h_i) + \sum_{i=1}^{n} f(x_i + h_i) - f(x_i)|$
$\le \sum_{i=1}^{n}|f(x_i + h_i) - f(x_i)| + \sum_{i=0}^{n}|f(x_{i+1}) - f(x_i + h_i)| < \sum_{i=1}^{n}\eta h_i + \varepsilon$

= $\eta\sum_{i=1}^{n} h_i$ + ε ≤ η(c – a) + ε.
Since εand η are arbitrarily small numbers, $|f(c) - f(a)| = 0$
∴ f(c) = f(a)
Hence f is constant.

Definition: If f is an integrable function on [a, b], then we define its indefinite integral to be the function F on [a, b] by F(x) = $\int_a^x f(t)dt$ + F(a).

Theorem: Any real valued function F defined on [a, b] is absolutely continuous if and only if it is an indefinite integral.

Proof: Assume that F is an indefinite integral. Then there is an integrable function f on [a, b] such that F(x) = = $\int_a^x f(t)dt$ + F(a).
Let ε > 0. Since f is integrable, $|f|$ is integrable as $|f|$ = $f^+ + f^-$.
∴ ∃ δ > 0 ∋ $\int_A |f(t)dt| < \varepsilon$ ... (i) ∀ measurable sets A ⊆ [a, b] with m(A) < δ.
Let {(a_1, b_1), (a_2, b_2), ... , (a_n, b_n)} be a finite collection of non-overlapping intervals of [a, b] such that $\sum_{i=1}^{n}|b_i - a_i|$ < δ.
Write A = $\bigcup_{i=1}^{n}(a_i, b_i)$ ⊆ [a, b].
Then m(A) = $m\{\bigcup_{i=1}^{n}(a_i, b_i)\}$ ≤ $\sum_{i=1}^{n}|b_i - a_i|$ < δ.
Hence $\int_A |f(t)dt| < \varepsilon$ by (i).
Now $\sum_{i=1}^{n}|F(b_i) - F(a_i)|$ = $\sum_{i=1}^{n}\left|\int_a^{b_i} f(t)dt + \mathrm{F(a)} - \int_a^{a_i} f(t)dt - \mathrm{F(a)}\right|$
= $\sum_{i=1}^{n}\left|\int_{a_i}^{b_i} f(t)dt\right| \le \sum_{i=1}^{n}\int_{a_i}^{b_i}|f(t)|dt = \int_A |f(t)dt| < \varepsilon.$
∴ F is absolutely continuous.

Converse: Suppose F is absolutely continuous.
⇒ F is a function of bounded variation. ⇒ F = $F_1 - F_2$, where F_1 and F_2 are two increasing real valued functions.
Since f is a function of bounded variation, F′(x) exists a. e. by a cor. and
$|F'(x)| \le |F_1'(x)| + |F_2'(x)| = F_1'(x) + F_2'(x)$
⇒ $\int |F'(x)|dx \le \int_a^b F_1'(x)dx + \int_a^b F_2'(x)dx$ ≤ $F_1(b) - F_1(a) + F_2(b) - F_2(a) < \infty$.
∴ F′ is an integrable function.
Let G(x) = $\int_a^x F'(t)dt$. Then G is absolutely continuous [by first part of this proof.]. Note that G′(x) = F′(x). Write f = F – G.
Since F and G are absolutely continuous, f is absolutely continuous.
Also f ′(x) = F′(x) - G′(x) = F′(x) - F′(x) = 0. a. e. by lemma.
∴ f is a constant function. So ∃ c ∋ f(x) = c ∀ x ∈ [a, b].

Now f(x) = F(x) – G(x).
$\Rightarrow$ F(x) = G(x) + f(x) = $\int_a^x F'(t)dt$ + c.
Now F(a) = $\int_a^a F'(t)dt$ + c = 0 + c = c.
$\therefore$ F(x) = $\int_a^x F'(t)dt$ + F(a).
$\therefore$ F is an integrable function.

<u>Corollary</u>: Every absolutely continuous function is the indefinite integral of its derivative.

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E – CONTENT

PAPER: M 302:

LEBESGUE THEORY

M. Sc. II YEAR, SEMESTER - III

UNIT – IV: L^p SPACES

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LEBESGUE THEORY UNIT IV

The Classical Banach Spaces

Section I: The L^p Spaces.

Definition: Let p be a positive real number. We define

$L^p = L^p[0,1] = \left\{ f \,/\, f : [0,1] \to \mathbb{R} \text{ is measurable and } \int_0^1 |f|^p < \infty \right\}$.

$L^1 = L^1[0,1] = \left\{ f \,/\, f : [0,1] \to \mathbb{R} \text{ is measurable and } \int_0^1 |f| < \infty \right\}$ = the set of all Legesgue integrable functions on [0, 1].

Lemma: L^p space is a linear space.

Proof: Let f, g $\in L^p[0,1]$.

Then $|f+g|^p \leq \max\ \{|f+f|^p, |g+g|^p\}$.

$= 2^p\ \max\{|f|^p,\ |g|^p\}$

$\leq 2^p(|f|^p + |g|^p) < \infty$.

Let $\alpha \in \mathbb{R}$, f $\in L^p[0,1]$.

$\therefore$ f is measurable and $\int_0^1 |f|^p < \infty$.

Then αf is measurable and $\int_0^1 |\alpha f|^p = |\alpha| \int_0^1 |f|^p < \infty$.

$\therefore$ αf $\in L^p[0,1]$.

Hence $L^p[0,1]$ is a linear space.

Definition: For a function f $\in L^p[0,1]$, *define* $\|f\|_p = \left(\int_0^1 |f|^p\right)^{\frac{1}{p}}$.

Definition: Two measurable functions f, g are said to be equivalent if there are equal almost everywhere. Ie. f ~ g iff f = g a. e.

Note: (i) $\|f\|_p \geq 0$.

Proof: Let f $\in L^p[0,1]$ and p be a positive real number.

Then $|f| \geq 0$.

$\Rightarrow |f|^p \geq 0$

$\Rightarrow \int_0^1 |f|^p \geq 0$

$\Rightarrow \left(\int_0^1 |f|^p\right)^{\frac{1}{p}} \geq 0$

$\Rightarrow \|f\|_p \geq 0$.

(ii) $\|f\|_p = 0$ iff f = 0.

Proof: Let f = 0.

Then $|f| = 0$.

$\Rightarrow |f|^p = 0$

$\Rightarrow \int_0^1 |f|^p = 0$

$\Rightarrow \left(\int_0^1 |f|^p\right)^{\frac{1}{p}} = 0$

$\Rightarrow \|f\|_p = 0$.

Conversely suppose $\|f\|_p = 0$

$\Rightarrow \left(\int_0^1 |f|^p\right)^{\frac{1}{p}} = 0$

$\Rightarrow \int_0^1 |f|^p = 0$

$\Rightarrow |f|^p = 0$ a. e.

$\Rightarrow |f| = 0\ a.e.$

$\Rightarrow$ f = 0 a. e.

(iii) Let $\alpha \in \mathbb{R}$, f $\in L^p[0,1]$. Then $\|\alpha f\|_p = |\alpha| \|f\|_p$.

Proof: $\|\alpha f\|_p = \left(\int_0^1 |\alpha f|^p\right)^{\frac{1}{p}}$

$$= \left(\int_0^1 |\alpha|^p |f|^p\right)^{\frac{1}{p}}$$

$$= |\alpha| \left(\int_0^1 |f|^p\right)^{\frac{1}{p}}$$

$= |\alpha| \|f\|_p$.

Definition: A real number M is said to be an *essential bound* for the function f if $|f(x)| \leq M\ a.e.$ on [0, 1]. A function f defined on [0, 1] is *essentially bounded* if it is bounded except possibly on a set of measure zero.

The *essential supremum of f on [0, 1]* is defined by inf {M : $|f(x)| \leq M\ a.e.$ on [0, 1] and denoted by 'ess sup $|f|'$.

Equivalently *ess sup* $|f|$ = inf {M: m({x∈E: $|f(x)| > M$}) = 0}.

If f does not have any essential bound then its essential suprimum is defined to be +∞.

We denote the class of all measurable functions defined on [0, 1] which are essentially bounded on [0, 1] by $L^\infty[0,1]$.

For f ∈ we define $\|f\|_\infty$= ess sup $|f|$.

Problem: If f ∈ L^1 and g ∈ L^∞, then $\int|fg| \leq \|f\|_1 \|g\|_\infty$

Solution: Put $\|g\|_\infty$= M′.

Then m({t : $|g(t)| > M'$}) = 0

⇒ $|g(t)| \leq M'$ a. e.

⇒ $|f(t)||g(t)| \leq |f(t)|M'$ a. e.

⇒$|f(t)g(t)| \leq |f(t)||M'|$ a. e.

⇒ $\int|fg| \leq M' \int|f|$

∴ $\int|fg| \leq \|f\|_1 \|g\|_\infty$.

Problem: Let f be a bounded measurable function on [0, 1]. Then

$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$

Proof: Put M = $\|f\|_\infty$. Then m({t : $|f(t)| > M$}) = 0

$\Rightarrow |f(t)| \leq M$ a. e.

$\Rightarrow |f(t)|^p \leq M^p$ a. e.

$\Rightarrow \int |f|^p \leq M^p$.

$\therefore \|f\|_p \leq M$ for all p.

$\overline{\lim}_{p\to\infty} \|f\|_p \leq M$...(i)

Suppose a < M. Then m({t : $|f(t)| > a\}) \neq 0$.

(if m({t : $|f(t)| > a$}) = 0 then by the definition of the norm in L∞, M ≤ a, a contradiction.)

Put A = {t ∈ [0, 1] / $|f(t)| > a$}.

Then m(A) ≠ 0. $\int_0^1 |f|^p \geq \int_A |f|^p \geq \int_A a^p = a^p\, m(A)$.

$\Rightarrow \left(\int_0^1 |f|^p\right)^{\frac{1}{p}} \geq a.\{m(A)\}^{\frac{1}{p}}$ for all p ≥ 1.

$\therefore \underline{\lim}_{p\to\infty} \|f\|_p \geq \underline{\lim}_{p\to\infty}\, a.\{m(A)\}^{\frac{1}{p}}$

$= a\ \underline{\lim}_{p\to\infty} \{m(A)\}^{\frac{1}{p}} = a$

$\therefore \underline{\lim}_{p\to\infty} \|f\|_p \geq a$ for all a such that a < M...(ii)

From (i) and (ii) $M \leq \underline{\lim}_{p\to\infty} \|f\|_p \leq \overline{\lim}_{p\to\infty} \|f\|_p \leq M$

$\therefore \lim_{p\to\infty} \|f\|_p = M = \|f\|_\infty$.

SECTION II : THE MINKOWSKI AND HOLDER INEQUALITIES

Lemma: Let α, β be non – negative real numbers and 0 < λ < 1. Then $\alpha^\lambda \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta$ with equality if α = β.

Proof: Define ϕ as ϕ(t) = (1 – λ) + λt – t^λ for all real numbers t.

Then ϕ(1) = 1 – λ + λ – **1 = 0.**

Also $\phi'(t) = \lambda - \lambda t^{\lambda - 1} = \lambda(1 - t^{\lambda - 1})$

$\phi''(t) = -\lambda(\lambda - 1)t^{\lambda - 2}$.

And $\phi'(t) = 0$ iff $t = 1$ and $\phi''(1) = -\lambda(\lambda - 1) > 0$.

$\therefore$ ϕ has local minimum at t = 1.

$\therefore$ $t < 1 \Rightarrow \phi$ is decreasing. Ie. $\phi(t) > \phi(1)$ ant $t > 1 \Rightarrow \phi$ is increasing

ie. $\phi(t) > \phi(1)$.

Thus $t \neq 1 \Rightarrow \phi(t) > \phi(1) \Rightarrow (1 - \lambda) + \lambda t - t^{\lambda} > 0 \Rightarrow t^{\lambda} < (1 - \lambda) + \lambda t$

$\therefore$ we may say that $t^{\lambda} \leq (1 - \lambda) + \lambda t$ for all t and with equality if t = 1...(i)

If $\beta \neq 0$ put $t = \alpha / \beta$ in (i).

Then $\left(\frac{\alpha}{\beta}\right)^{\lambda} \leq 1 - \lambda + \lambda\left(\frac{\alpha}{\beta}\right)$

$\Rightarrow \frac{\alpha^{\lambda}}{\beta^{\lambda}} \leq (1 - \lambda) + \frac{\lambda\alpha}{\beta}$

$\Rightarrow \alpha^{\lambda}\beta^{1-\lambda} \leq \lambda\alpha + (1 - \lambda)\beta$ with equality if $\alpha = \beta$.

HOLDER'S INEQUALITY:

If p and q are non – negative extended real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and if $f \in L^p$, $g \in L^q$ $, then\ fg \ \in L^1$and $\int|fg| \ \leq \|f\|_p \, \|g\|_q$ equality holds iff for some non – zero constants α and β, we have $\alpha|f|^p = \beta|g|^q \ a.e.$

Proof: If p 1, q = ∞, then the in equality holds. So assume that $1 < p < \infty$. First assume that $\|f\|_p$= 1 = $\|g\|_q$

Take $\alpha = \ |f(t)|^p, \beta = \ |g(t)|^q$and $\lambda = \frac{1}{p}, 1 - \lambda = \frac{1}{q}$.

Then by Lemma we get $|f(t)||g(t)| \ \leq \frac{1}{p}|f(t)|^p + \frac{1}{q}|g(t)|^q$ and equality holds if $\alpha = \beta$ ie. $|f(t)|^p = \ |g(t)|^q$...(i)

$\Rightarrow \int|fg| \ \leq \frac{1}{p}\int|f(t)|^p + \frac{1}{q}\int|g(t)|^q = \frac{1}{p}\|f\|^p + \frac{1}{q}\|g\|^q = \frac{1}{p} + \frac{1}{q} = 1$

Ie. $\int|fg| \ \leq 1$ = $\|f\|_p\|g\|_q$

Let f $\in L^p$, g $\in L^q$. Now if $\|f\| = 0$ or $\|g\| = 0$ then the inequality is obvious.

Assume that $\|f\| \neq 0$ and $\|g\| \neq 0$.

Then $\frac{f}{\|f\|} \in L^p, \frac{g}{\|g\|} \in L^q$.

Also $\left\|\frac{f}{\|f\|}\right\|$ = 1 and $\left\|\frac{g}{\|g\|}\right\|$ = 1.

So by the above case $\int \left|\frac{f}{\|f\|}\frac{g}{\|g\|}\right| \leq 1$ and equality holds iff $\left|\frac{f}{\|f\|}\right|^p = \left|\frac{g}{\|g\|}\right|^q$ iff

$\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q}$ iff $\|g\|_q^q|f|^p = \|f\|_p^p|g|^q$ a. e. ... (ii).

Now $\int \left|\frac{f}{\|f\|}\frac{g}{\|g\|}\right| \leq 1 \Rightarrow \frac{1}{\|f\|\|g\|}\int|fg| \leq 1 \Rightarrow \int|fg| \leq \|f\|_p\|g\|_q$

Also equality holds iff $\|g\|_q^q|f|^p = \|f\|_p^p|g|^q$ a. e. ie equality holds iff for some non – zero constants α = $\|g\|_q^q$ and β = $\|f\|_p^p$, we have $\alpha|f|^p = \beta|g|^q \ a.e.$

<u>MINKOWSKI'S INEQUALITY:</u>

If f, g ∈ $L^p$with 1 ≤ p ≤ ∞, then f + g ∈ L^p and $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

Proof: Let f, g ∈ $L^p$with 1 ≤ p ≤ ∞. Then f + g ∈ L^p since Lp is linear.

$$
\begin{aligned}
\text{Now } \|f+g\|_1 &= \int|f+g|dx \\
&\leq \int(|f|+|g|)dx \\
&= \int|f|\,dx + \int|g|dx \\
&= \|f\|_1 + \|g\|_1
\end{aligned}
$$

$$
\begin{aligned}
\text{Also } \|f+g\|_\infty &= \text{ess sup } |(f+g)(t)| \\
&\leq \text{ess sup } |f(t)| + \text{ ess sup } |g(t)| \\
&= \|f\|_\infty + \|g\|_\infty
\end{aligned}
$$

So, assume that 1 < p < ∞. Let q be the real number such that $\frac{1}{p} + \frac{1}{q} = 1$.

$$
\begin{aligned}
\text{Now } |f+g|^p &= |f+g|^{p-1} \cdot |f+g| \\
&\leq |f+g|^{p-1} \cdot (|f|+|g|) \\
&= |f+g|^{p-1} \cdot |f| + |f+g|^{p-1} \cdot |g| \text{ ... (i)}
\end{aligned}
$$

Claim: $|f+g|^{p-1} \in L^q$

Now $(|f+g|^{p-1})^q = |f+g|^{(p-1)q} = |f+g|^p$

Since f + g $\in$ L^p, we have $\int|f+g|^p < \infty$.

Now $\int(|f+g|^{p-1})^q = \int|f+g|^p < \infty$.

So we have $|f+g|^{p-1} \in L^q$. Since f, g $\in L^p$ and $|f+g|^{p-1} \in L^q$, we have by Holder's inequality, $\int|f||f+g|^{p-1} \leq \|f\|_p \|(f+g)^{p-1}\|_q$.

$$\int|g||f+g|^{p-1} \leq \|g\|_p \|(f+g)^{p-1}\|_q$$

But $\|(f+g)^{p-1}\|_q = \{\int(|f+g|^{p-1})^q\}^{\frac{1}{q}} = \left(\int|f+g|^{(p-1)q}\right)^{\frac{1}{q}} = (\int|f+g|^p)^{\frac{1}{q}} =$

$\left\{(\int|f+g|^p)^{\frac{1}{p}}\right\}^{\frac{p}{q}} = \|f+g\|_p^{\frac{p}{q}}$

$\therefore \int|f||f+g|^{p-1} \leq \|f\|_p \|f+g\|_p^{\frac{p}{q}}$ and

$\int|g||f+g|^{p-1} \leq \|g\|_p \|f+g\|_p^{\frac{p}{q}}$...(ii).

From (i) and (ii), $\int|f+g|^p \leq \|f\|_p \|f+g\|_p^{\frac{p}{q}} + \|g\|_p \|f+g\|_p^{\frac{p}{q}}$

$\Rightarrow \int|f+g|^p \leq (\|f\|_p + \|g\|_p) \cdot \|f+g\|_p^{\frac{p}{q}}$

$\Rightarrow \|f+g\|_p^p \leq (\|f\|_p + \|g\|_p) \cdot \|f+g\|_p^{\frac{p}{q}}$

$\Rightarrow \|f+g\|_p^{p-\frac{p}{q}} \leq \|f\|_p + \|g\|_p$

$\Rightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p$ since $p - \frac{p}{q} = p\left(1-\frac{1}{q}\right) = p\left(\frac{1}{p}\right) = 1$.

<u>Note</u>: $L^p[0,1]$ is a linear space. For a function f $\in L^p[0,1], define\ \|f\|_p =$

$\left(\int_0^1|f|^p\right)^{\frac{1}{p}}$. Then it satisfies

(i) $\|f\|_p \geq 0$. and $\|f\|_p = 0$ iff f = 0.

(ii) Let $\alpha \in \mathbb{R}$, f $\in L^p[0,1]$. Then $\|\alpha f\|_p = |\alpha|\|f\|_p$.

(iii) $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

$\therefore$ This is a norm on $L^p[0,1]$.

Hence $L^p[0,1]$ is a normed linear space with this norm

SECTION III: CONVERGENCE AND COMPLETENESS.

Definition: A series $\sum_{n=1}^{\infty} f_n$ in a normed linear space is said to be summable to a sum s if the partial sum sequence of the series converges to s.

Definition: A series is said to be absolutely summable if $\sum_{n=1}^{\infty}\|f_n\| < \infty$.

Theorem: A normed linear space X is complete iff every absolutely summable series is summable.

Proof: Assume that X is complete.

Let $\sum_{n=1}^{\infty} f_n$ be an absolutely summable series.

Then $\sum_{n=1}^{\infty}\|f_n\| < \infty$ by definition n.

So given ε > 0, ∃ a positive integer N such that $\sum_{n=N}^{\infty}\|f_n\| < \varepsilon$.

Let $\{g_n\}$ be the sequence of partial sum of the series of $\sum_{n=1}^{\infty} f_n$.

For $n > m \geq N$, $\|g_n - g_m\| = \|\sum_{k=m+1}^{n} f_k\|$

$$\leq \sum_{k=m+1}^{n} \|f_k\|$$

$$\leq \sum_{k=N}^{n} \|f_k\|$$

$$< \sum_{k=N}^{\infty}\|f_k\| < \varepsilon$$

$\therefore$ $\{g_n\}$ is a Cauchy sequence in X.

Since X is complete $\{g_n\}$ converges in X.

Ie. $\exists$ g $\in$ X $\ni \lim_{n} g_n = g$.

$\Rightarrow \sum_{n=1}^{\infty} f_n$ = g

$\Rightarrow$ the series $\sum_{n=1}^{\infty} f_n$ is summable.

Conversely suppose that every absolutely summable series is summable.

Let $\{f_n\}$ be a Cauchy sequence in X.

For each positive integer k $\exists$ an integer $n_k \ni \|f_n - f_m\| < \frac{1}{2^k}$ $\forall$ n, m $\geq n_k$.

Without loss of generality, assume that $n_k < n_{k+1}$ for all k.

Consider $\{f_{n_k}\}$, a subsequence of the sequence $\{f_n\}$.

Put $g_1 = f_{n_1}$, $g_2 = f_{n_2} - f_{n_1}$, ..., $g_k = f_{n_k} - f_{n_{k-1}}$, ... for k > 2.

Then $g_1 + g_2 + ... + g_k = f_{n_k}$.

$\therefore$ $\{f_{n_k}\}$ is a sequence of partial sums of the series $\sum_{k=1}^{\infty} g_k$,

and $\|g_k\| = \|f_{n_k} - f_{n_{k-1}}\| \leq \frac{1}{2^{k-1}}$ $for\ k > 1$ ... (i).

$\therefore \sum_{k=1}^{\infty} \|g_k\| = \|g_1\| + \sum_{k=2}^{\infty} \|g_k\|$

$\leq \|g_1\| + \sum_{k=2}^{\infty} \frac{1}{2^{k-1}}$ by (i)

$= \|g_1\| + 1 < \infty$

$\therefore$ $\sum_{k=1}^{\infty} g_k$ is absolutely summable.

$\therefore$ By assumption $\sum_{k=1}^{\infty} g_k$ is summable.

Suppose $\sum_{k=1}^{\infty} g_k$ = s

$\Rightarrow \lim_{k} f_{n_k}$= s since g_k is the partial sum of $\{f_{n_k}\}$.

Let ε > 0. Now $\{f_n\}$ is a Cauchy sequence, and there exists a subsequence $\{f_{n_k}\}$ which converges to s.

$\therefore$ $f_n \to s$.

This shows that X is complete.

Exercise: (i) Let $\{f_n\}$ be a sequence of functions in L^∞. Prove that $\{f_n\}$ converges to f in L^∞ iff there is a set E of measure zero such that f_n converges to f uniformely on E. (ii) Prove that L^∞ is complete.

RIESZ FISHER THEOREM

The L^p spaces are complete.

Proof: Let $1 \leq p < \infty$.

By a theorem it suffices if we prove that every absolutely summable series in L^p is summable.

Let $\sum_{n=1}^{\infty} f_n$ be an absolutely summable series in L^p.

Put $\sum_{n=1}^{\infty} \|f_n\| = M < \infty$ since $\sum_{n=1}^{\infty} f_n$ is absolutely summable.

We have to show that $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely.

Define $g_n(x) = \sum_{k=1}^{n} |f_k(x)|$.

Then $|g_n| = g_n$.

By Minkowski inequality, $\|g_n\| \leq \sum_{n=1}^{\infty} \|f_n\| \leq M$

$\Rightarrow (\int |g_n|^p)^{\frac{1}{p}} \leq M \Rightarrow \int |g_n|^p \leq M^p \Rightarrow \int {g_n}^p \leq M^p < \infty$.

$\therefore$ for each x, $\{g_n(x)\}$ is an increasing sequence of extended real numbers and so must converge to an extended real number say g(x).

Ie. $g(x) = \lim_n g_n(x)$ for all n.

Since each g_n is measurable, g is measurable.

Also $g^p(x) = \lim_n {g_n}^p(x)$ for all x.

Since $g_n(x) \geq 0$ for all n, by Fatou's lemma,

$$\int g^p \leq \underline{\lim}_n \int {g_n}^p \leq M^p < \infty.$$

$\Rightarrow \int g^p$ is finite.

$\Rightarrow g^p$ is an integrable function and g^p is finite almost everywhere.

$\therefore$ g is finite almost everywhere.

So for each x for which g(x) is finite $\lim_n g_n(x) < \infty$.

$\Rightarrow \sum_{k=1}^{\infty}|f_k(x)| < \infty$.

$\Rightarrow$ the series $\sum_{k=1}^{\infty}|f_k(x)|$ converges absolutely.

Thus for each x, the series $\sum_{k=1}^{\infty}|f_k(x)|$ is an absolutely summable series of real numbers and hence summable to real number.

Define s(x) = $\begin{cases} \sum_{k=1}^{\infty} f_k(x) & if\ g(x) is\ finite \\ 0 & if\ g(x) = \infty \end{cases}$

Claim: $\sum_{n=1}^{\infty} f_n$ = s.

Write $s_n = \sum_{k=1}^{n} f_k$.

Now $\{s_n\}$ is the sequence of partial sums of the series $\sum_{n=1}^{\infty} f_n$.

Now we have to show that $\lim_n s_n = s\ in\ L^p$.

If g(x) is finite, then s(x) = $\sum_{k=1}^{\infty} f_k(x)$ = $\lim_n \sum_{k=1}^{n} f_k$ = $\lim_n s_n$.

Since g is finite a.e., s = $\lim_n s_n$ a. e.

Since each s_n is measurable s is measurable.

Now $|s_n| = |\sum_{k=1}^{n} f_k| \leq \sum_{k=1}^{n}|f_k| = |g_n| \leq |g|$ = g since $g = \lim_n g_n$.

Ie. $|s_n| \leq$ g for all n.

$\Rightarrow |s| \ \leq g$

$\Rightarrow |s|^p \ \leq g^p$

$\Rightarrow \int |s|^p \ \leq \int g^p < \infty$.

$\therefore$ s $\in L^p$.

Now we show that $|s_n - s|^p \leq 2^p g^p$.

So $|s_n - s| \ \leq |s_n| + |s| < g + g = 2g$.

So $|s_n - s|^p \leq 2^p g^p$.

Ie. $\{|s_n - s|^p\}$ is a sequence of non – negative measurable functions such that $|s_n - s|^p \leq 2^p g^p$ where $2^p g^p$ is integrable and $\lim_n |s_n - s|^p = 0\ a.e.$ since s = $\lim_n s_n$ a. e.

$\therefore$ By Lebesgue convergence theorem, $\lim_n \int |s_n - s|^p = 0$

$\Rightarrow \lim_{n} \|s_n - s\|^p = 0.$

$\Rightarrow \lim_{n} \|s_n - s\| = 0.$

$\Rightarrow \lim_{n} s_n = s \text{ } in \text{ } L^p.$

$\therefore$ the given series is summable to the sum s.

Hence L^p is complete.