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E – CONTENT

PAPER: M 401,

MEASURE THEORY

M. Sc. II YEAR, SEMESTER - IV

UNIT – I

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401, MEASURE THEORY
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UNIT I: MEASURE SPACE

Definition: A collection $\mathcal{C}$ of subsets of an arbitrary space X is called an **algebra** of sets if (i) A ∪ B is in $\mathcal{C}$ whenever A and B are and (ii) A′ is in $\mathcal{C}$ whenever A is.

Definition: A class of subsets of an arbitrary space X is said to be a **σ - algebra**, if X and ϕ belong to the class and class is closed under the formation of countable unions and of complements.

Example: The class of Lebesgue measurable sets is a σ - algebra of subsets of ℝ.

Definition: A class of sets, R, is called a **ring** if whenever E ∈ R, F ∈ R then E ∪ F and E – F ∈ R.

Example: The class of finite unions of intervals of the form [a, b) forms a ring.

Definition: A ring is called a **σ - ring** if it is closed under the formation of countable unions.

Result: Every algebra is ring and every σ - algebra is a σ - ring but not conversely.

Definition: A pair (X, 𝔅) where 𝔅 is a σ - algebra of subsets of X, is called a measurable space. The sets of 𝔅 are called measurable sets.

Definition: A measure μ on a measurable space (X, 𝔅) is a non – negative set function defined for all sets of 𝔅 satisfying μ(ϕ) = 0 and $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$
for any sequence $\{E_i\}$ of disjoint measurable sets. i.e. μ is countably additive.

Definition: A measurable space (X, 𝔅) together with a measure μ defined as above on 𝔅 is called a measure space and it is denoted by a triple (X, 𝔅, μ).

Observation: If (X, 𝔅, μ) is a measure space then it is finitely additive i.e. $E_1, ..., E_n$ are sets in 𝔅 such that $E_i \cap E_j = \phi$ for i ≠ j, then $\mu(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mu(E_i)$.

Hint: Set $E_{n+1} = E_{n+2} = ... = \phi$. Then the family $\{E_i\}$ is a pair wise disjoint family of subsets from 𝔅 and $\mu(E_i) = 0$ for i ≥ n + 1 and the result follows from definition.

Example 1: $(\mathbb{R}, \mathfrak{M}, m)$ is a measure space where ℝ is the set of real numbers 𝔐 is Lebesgue measurable sets of real numbers and m is Lebesgue measure.

Example 2: $(\mathbb{R}, \mathfrak{B}, m)$ is a measure space where ℝ is the set of real numbers 𝔅 is the class σ - algebra of Borel subsets of ℝ and m is Lebesgue measure.

Example 3: $([0, 1], \mathfrak{M}, m)$ is a measure space where 𝔐 is measurable subsets of [0, 1] and m is Lebesgue measure.

Example 4: Let X be an uncountable set. 𝔅 = {A ⊆ X: A is countable or A′ is countable}. Define μ on 𝔅 by μ(A) = 0 if A ∈ 𝔅 is countable, μ(A) = 1 if A ∈ 𝔅 and A′ is countable. Show that $(X, \mathfrak{B}, \mu)$ is a measure space.

Solution: Given X is uncountable. ∴ ϕ = X′ is countable so that μ(X) = 1, μ(ϕ) = 0.
Claim: A, B ∈ 𝔅 such that A′, B′ are countable ⇒ A ∩ B ≠ ϕ
Let A, B ∈ 𝔅 and A′, B′ be countable.
A ∪ B can be expressed as a disjoint union of 3 sets as
A ∪ B = (A ∩ B′) ∪ (A ∩ B) ∪ (A′ ∩ B)...(i)
Observe that A ∪ B is uncountable, A ∩ B′ and A′ ∩ B are countable.
If A ∩ B = ϕ then RHS of (i) is countable and LHS is uncountable which is a contradiction.
∴ A ∩ B ≠ ϕ.
Hence it is enough if we verify countable additive property of μ in the following two cases. Let $\{A_i, i \in \mathbb{Z}^+\}$ consist of pair wise disjoint sets of 𝔅.
Case (i): A_i are all countable:
Then $\bigcup_{i=1}^{\infty} A_i$ is countable. Thus $\mu(A_i) = 0$ for each i and $\mu(\bigcup_{i=1}^{\infty} A_i) = 0$
This proves $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.
Case (ii): Only one of $A_i{}'$ say $A_1{}'$ is countable and the remaining A_i are countable.
Then $\bigcup_{i=1}^{\infty} A_i$ is uncountable. Thus $\mu(A_1) = 1$, $\mu(A_i) = 0$ for each i = 2, 3, 4, ..., and $\mu(\bigcup_{i=1}^{\infty} A_i) = 1$. This proves $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$
∴ μ is a measure on (X, 𝔅) and hence $(X, \mathfrak{B}, \mu)$ is a measure space.

Proposition: Let $(X, \mathfrak{B}, \mu)$ be measure space. If A, B ∈ 𝔅 and A ⊆ B then μ(A) ≤ μ(B). i.e μ is monotone.
Proof: Let A, B ∈ 𝔅 and A ⊆ B. Clearly B = A ∪ (B ~ A) is a disjoint union.
∴ μ (B) = μ (A) + μ(B ~ A) by finite additivity.
Hence μ(A) ≤ μ(B) since μ(B ~ A) ≥ 0. ∴ μ is monotone.

Note: If A ⊆ B then μ (B) = μ (A) + μ(B ~ A).

Theorem: Let $(X, \mathfrak{B}, \mu)$ be measure space. If A, B ∈ 𝔅 then μ(A ∪ B) ≤ μ (A) + μ(B).

Proof: Let A, B $\in \mathfrak{B}$.

A$\cup$ B can be expressed as a disjoint union of 3 sets as

A $\cup$ B = (A ~ B) $\cup$ (A $\cap$ B) $\cup$ (B ~ A).

By finite additivity of μ,

μ(A $\cup$ B) = μ(A ~ B) + μ(A $\cap$ B) + μ(B ~ A).

$\leq$ μ(A ~ B) + μ(A $\cap$ B) + μ(B ~ A) + μ(B $\cap$ A) since μ(B $\cap$ A) $\geq$ 0.

= μ(A) + μ(B) since A $\cap$ B $\subseteq$ A, A $\cap$ B $\subseteq$ B.

Proposition: Let (X, $\mathfrak{B}$, μ) be a measure space. If $E_i \in \mathfrak{B}$, $\mu(E_i) < \infty$ and $E_i \supseteq E_{i+1}$ then $\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} \mu(E_i)$.

Proof: Let $E_i \in \mathfrak{B}$, $\mu(E_i) < \infty$ and $E_i \supseteq E_{i+1}$.

Then clearly $E_i = E_{i+1} \cup (E_i \sim E_{i+1})$ is a disjoint union.

$\therefore \mu(E_i) = \mu(E_{i+1}) + \mu(E_i \sim E_{i+1})$ by finite additivity.

$\therefore \mu(E_i \sim E_{i+1}) = \mu(E_i) - \mu(E_{i+1})$...(i).

Set $E = \bigcap_{i=1}^{\infty} E_i$

Then clearly $E_1 = E \cup (E_1 \sim E_2) \cup (E_2 \sim E_3) \cup (E_3 \sim E_4) \cup \ldots$

$= E \cup \bigcup_{i=1}^{\infty}(E_i \sim E_{i+1})$ is a countable union of disjoint measurable sets.

$\therefore \mu(E_1) = \mu(E) + \sum_{i=1}^{\infty} \mu(E_i \sim E_{i+1}) = \mu(E) + \lim_{n\to\infty} \sum_{i=1}^{n-1}\{\mu(E_i) - \mu(E_{i+1})\}$

$= \mu(E) + \lim_{n\to\infty}\{\mu(E_1) - \mu(E_2) + \mu(E_2) - \mu(E_3) + \mu(E_3) - \mu(E_4) + \cdots + \mu(E_{n-1}) - \mu(E_n)\}$

$= \mu(E) + \lim_{n\to\infty}\{\mu(E_1) - \mu(E_n)\} = \mu(E) + \mu(E_1) - \lim_{n\to\infty} \mu(E_n)$

Hence $\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} \mu(E_i)$.

Proposition: Let (X, $\mathfrak{B}, \mu$) be measure space. If $E_i \in \mathfrak{B}$, then $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

Proof: Let $E_i \in \mathfrak{B}$.

Put $G_1 = E_1$, $G_2 = E_2 \sim E_1$, $G_3 = E_3 \sim (E_1 \cup E_2)$, ..., $G_n = E_n \sim \bigcup_{i=1}^{n-1} E_i$, ...

Then $\{G_n\}$ is a disjoint sequence of sets in $\mathfrak{B}$, $G_n \subseteq E_n$ for each n and $\bigcup_{i=1}^{\infty} G_i = \bigcup_{i=1}^{\infty} E_i$

$\therefore \mu(G_n) \leq \mu(E_n)$ ... (i).

And $\mu(\bigcup_{i=1}^{\infty} E_i) = \mu(\bigcup_{i=1}^{\infty} G_i) = \sum_{i=1}^{\infty} \mu(G_i)$ by additivity

$\leq \sum_{i=1}^{\infty} \mu(E_i)$ by (i).

Hence $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

Exercise: Let (X, $\mathfrak{B}$, μ) be measure space. If $\{A_i, i \in \mathbb{Z}^+\}$ is a sequence of sets in $\mathfrak{B}$, then show that $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} A_i)$

Solution: Let $E_n = \bigcup_{i=1}^{n} A_i$ and $E = \bigcup_{i=1}^{\infty} A_i$. Then $E_n \in \mathfrak{B}$ and $E_n \subseteq E_{n+1}$

Then clearly $E = E_1 \cup (E_2 \sim E_1) \cup (E_3 \sim E_2) \cup (E_4 \sim E_3) \cup \ldots$

$= E_1 \cup \{\bigcup_{i=2}^{\infty}(E_i \sim E_{i-1})\}$ is a countable union of disjoint measurable sets.

$\therefore \mu(E) = \mu(E_1) + \sum_{i=2}^{\infty} \mu(E_i - E_{i-1}) = \mu(E_1) + \lim_{n\to\infty} \sum_{i=2}^{n} \mu(E_i - E_{i-1})$

$= \mu(E_1) + \lim_{n\to\infty} \sum_{i=2}^{n} [\mu(E_i) - \mu(E_{i-1})]$

$= \mu(E_1) + \lim_{n\to\infty} \{\mu(E_2) - \mu(E_1) + \mu(E_3) - \mu(E_2) + \cdots + \mu(E_n) - \mu(E_{n-1})\}$

$= \mu(E_1) + \lim_{n\to\infty} \{\mu(E_n) - \mu(E_1)\} = \mu(E_1) + \lim_{n\to\infty} \mu(E_n) - \mu(E_1).$

$\therefore \mu(E) = \lim_{n\to\infty} \mu(E_n)$

Hence $\mu(\cup_{i=1}^{\infty} A_i) \leq \lim_{n\to\infty} \mu(\cup_{i=1}^{n} A_i)$

Theorem: Let $(X, \mathfrak{B}, \mu)$ be measure space and $E_1, E_2 \in \mathfrak{B}$. Then prove that $\mu(E_1 \Delta E_2) = 0 \Rightarrow \mu(E_1) = \mu(E_2)$.

Proof: Since $E_1 \Delta E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$ is a disjoint union of measurable sets,
$0 = \mu(E_1 \Delta E_2) = \mu(E_1 \setminus E_2) + \mu(E_2 \setminus E_1)$.
But $\mu(E_1 \setminus E_2) \geq 0$ and $\mu(E_2 \setminus E_1) \geq 0$ since μ is non – negative.
$\therefore$ $\mu(E_1 \setminus E_2) = 0$ and $\mu(E_2 \setminus E_1) = 0$.
Since $E_1 = (E_1 \setminus E_2) \cup (E_1 \cap E_2)$ is a disjoint union of measurable sets,
$\mu(E_1) = \mu(E_1 \setminus E_2) + \mu(E_1 \cap E_2) = 0 + \mu(E_1 \cap E_2) = \mu(E_1 \cap E_2)$
Since $E_2 = (E_2 \setminus E_1) \cup (E_2 \cap E_1)$ is a disjoint union of measurable sets,
$\mu(E_2) = \mu(E_2 \setminus E_1) + \mu(E_2 \cap E_1) = 0 + \mu(E_2 \cap E_1) = \mu(E_1 \cap E_2)$
Thus $\mu(E_1) = \mu(E_1 \cap E_2) = \mu(E_2)$.

Definition: Let $(X, \mathfrak{B}, \mu)$ be measure space. μ is said to be **finite** if $\mu(X) < \infty$. μ is said to be **σ - finite** if there is a sequence $\{X_n\}$ in $\mathfrak{B}$ such that $X = \cup X_n$ and $\mu(X_n) < \infty$ for each n.

Definition: The measure space $(X, \mathfrak{B}, \mu)$ is said to be **complete** if $\mathfrak{B}$ contains all subsets of sets of measure zero. Ie. $A \in \mathfrak{B}$, $\mu(A) = 0$ and $B \subseteq A \Rightarrow B \in \mathfrak{B}$.

Example 1: If a coin is tossed either head or tail comes up when the coin falls.
Let us assume these are the only possibilities.
Let X = {H, T} where H stands for head and T for tail. Let $\mathfrak{B} = \{\phi, \{H\}, \{T\}, X\}$.
Define $\mu : \mathfrak{B} \to [0, 1]$ by $\mu(\phi) = 0$, $\mu(\{H\}) = \mu(\{T\}) = ½$ and $\mu(X) = 1$.
Then μ is a finite measure on $(X, \mathfrak{B})$.

Example 2: Let two coins be tossed. Let X = {HH, HT, TH, TT} where H stands for head and T for tail. Let $\mathfrak{B} = \wp(X)$. Define $\mu : \mathfrak{B} \to [0, 1]$ by $\mu(A)$ = probability of A where $A \subseteq X$. Then μ is a finite measure on $(X, \mathfrak{B})$.

Example 3: **Let X be an uncountable set.** $\mathfrak{B}$ = {A ⊆ X: A is countable or A′ is countable}. Define μ on $\mathfrak{B}$ by μ(A) = 0 if A ∈ $\mathfrak{B}$ is countable, μ(A) = 1 if A ∈ $\mathfrak{B}$ and A′ is countable. Then the measure μ is finite since μ(X) = 1 < ∞.

Example 4: Let X = ℝ, $\mathfrak{B}$ be the σ - algebra of Lebesgue measurable sets and m be the Lebesgue measure on $\mathfrak{B}$. Let $X_n = [-n, n]$, $n \in \mathbb{Z}^+$. Then $m(X_n) = 2n < \infty$ for all n and $\mathbb{R} = \cup X_n$. Hence m is σ - finite.

Proposition: Let (X, $\mathfrak{B}, \mu$) be measure space. Let Y ⊆ X, Y ∈ $\mathfrak{B}$.
Define $\mathfrak{B}_Y = \{A \in \mathfrak{B} : A \subseteq Y\}$ and $\mu_Y(A) = \mu(A)$.
Then (Y, $\mathfrak{B}_Y$, μ_Y)is a measure space. μ_Y is called restriction of μ to Y.

Hint: $\mathfrak{B}_Y$ is a σ - algebra of subsets of Y. $\mu_Y(\phi) = \mu(\phi) = 0$ and countable additivity of μ_Y is inherited from that of μ.

Definition: Let (X, $\mathfrak{B}$, μ) be measure space. A subset E of X is said to be of **finite measure** if E ∈ $\mathfrak{B}$ and μ(E) < ∞. A subset E of X is said to be of **σ - finite measure** if E is the union of a countable collection of measurable sets of finite measure.

Result: Prove that any measurable set contained in a set of σ - finite measure is itself a σ - finite measure.

Proof: Let A be a set of σ - finite measure of a measure space (X, $\mathfrak{B}, \mu$) and E be a measurable subset of A.
Then ∃ a sequence $\{A_n\}$ of measurable sets with $\mu(A_n) < \infty$ such that $A = \bigcup_{i=1}^{\infty} A_n$
Now $E = E \cap A = E \cap \bigcup_{i=1}^{\infty} A_n = \bigcup_{i=1}^{\infty}(E \cap A_n)$ and
since $E \cap A_n \subseteq A_n$, $\mu(E \cap A_n) \leq \mu(A_n) < \infty$.
∴ E is of σ - finite measure.

Result: Prove that union of countable collection of sets of σ - finite measure is again of σ - finite measure.

Proof: Let $\{E_n\}$ be a sequence of sets of σ - finite measure of a measure space (X, $\mathfrak{B}, \mu$).
Then ∃ a sequence $\{E_{n_i}\}$ of measurable sets ∋ $E_n = \bigcup_{i=1}^{\infty} E_{n_i}$ and $\mu(E_{n_i}) < \infty$ for each n.
Now $\bigcup_{i=1}^{\infty} E_n = \bigcup_{i=1}^{\infty}\left(\bigcup_{i=1}^{\infty} E_{n_i}\right)$ and $\mu(E_{n_i}) < \infty$.
Hence $\bigcup_{i=1}^{\infty} E_n$ is of σ - finite measure.

Proposition: If (X, $\mathfrak{B}$, μ) is a measure space, then we can find a complete measure space $(X, \mathfrak{B}_0, \mu_0)$ such that (i) $\mathfrak{B} \subseteq \mathfrak{B}_0$ (ii) $E \in \mathfrak{B} \Rightarrow \mu(E) = \mu_0(E)$.
(iii) $E \in \mathfrak{B}_0$ iff $E = A \cup B$ where $B \in \mathfrak{B}$ and $A \subseteq C$, $C \in \mathfrak{B}$, $\mu(C) = 0$.

Proof: Let (X, $\mathfrak{B}$, μ) be a measure space.
Now we have to construct a complete space $(X, \mathfrak{B}_0, \mu_0)$ satisfying (i), (ii) and (iii).
Define $\mathfrak{B}_0 = \{A \cup B: B \in \mathfrak{B}, \exists\, C \in \mathfrak{B} \ni A \subseteq C, \mu(C) = 0\}$,
Claim: $\mathfrak{B}_0$ is a σ - algebra.
Clearly $\phi \in \mathfrak{B}_0$.
Let $A \cup B \in \mathfrak{B}_0$. ∴ $B \in \mathfrak{B}, \exists\, C \in \mathfrak{B} \ni A \subseteq C, \mu(C) = 0$.
Then $\widetilde{(A \cup B)} = \tilde{A} \cap \tilde{B} = \tilde{A} \cap \tilde{B} \cap X = \tilde{A} \cap \tilde{B} \cap (C \cup \tilde{C})$
$= (\tilde{A} \cap \tilde{B} \cap C) \cup (\tilde{A} \cap \tilde{B} \cap \tilde{C}) = (\tilde{A} \cap \tilde{B} \cap C) \cup (\tilde{B} \cap \tilde{C}) \in \mathfrak{B}_0 \because A \subseteq C.$
Thus, $A \cup B \in \mathfrak{B}_0 \Rightarrow \widetilde{(A \cup B)} \in \mathfrak{B}_0$.
Let $\{A_i \cup B_i\}$ be a countable collection of members of $\mathfrak{B}_0$.
Then $\bigcup_{i=1}^{\infty}(A_i \cup B_i) = (\bigcup_{i=1}^{\infty} A_i) \cup (\bigcup_{i=1}^{\infty} B_i) \in \mathfrak{B}_0$ since $\bigcup_{i=1}^{\infty} B_i \in \mathfrak{B}$, $\exists \bigcup_{i=1}^{\infty} C_i \in \mathfrak{B}$ $\ni \bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} C_i$ and $\mu(\bigcup_{i=1}^{\infty} C_i) \leq \sum_{i=1}^{\infty} \mu(C_i) = 0$.
∴ $\mathfrak{B}_0$ is a σ - algebra.
Define $\mu_0: \mathfrak{B}_0 \to \mathbb{R} \cup \{-\infty, \infty\}$ by $\mu_0(A \cup B) = \mu(B)\ \forall\ A \cup B \in \mathfrak{B}_0$.
Claim: μ_0 is well defined
Let $A \cup B = A_1 \cup B_1 \in \mathfrak{B}_0$
$\Rightarrow B, B_1 \in \mathfrak{B}, \exists\, C, C_1 \in \mathfrak{B}, A \subseteq C, A_1 \subseteq C_1$ with $\mu C = \mu C_1 = 0$.
Now $B_1 \subseteq A_1 \cup B_1 = A \cup B \subseteq C \cup B$ ie. $B_1 \subseteq C \cup B$
∴ $\mu(B_1) \leq \mu C + \mu B = \mu B$. So $\mu B_1 \leq \mu B$. Similarly, $\mu B \leq \mu B_1$, so that $\mu B_1 = \mu B$
∴ $\mu_0(A \cup B) = \mu(B) = \mu B_1 = \mu_0(A_1 \cup B_1)$.
Also, $\mu_0(A \cup B) = \mu(B) \geq 0$ and $\mu_0(\phi) = \mu(\phi) = 0$
Let $\{A_i \cup B_i\}$ be a sequence of pairwise disjoint sets in $\mathfrak{B}_0$.
Then $\bigcup_{i=1}^{\infty}(A_i \cup B_i) = (\bigcup_{i=1}^{\infty} A_i) \cup (\bigcup_{i=1}^{\infty} B_i)$.
Here $\bigcup_{i=1}^{\infty} B_i \in \mathfrak{B}$, $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} C_i$ and $\mu(\bigcup_{i=1}^{\infty} C_i) \leq \sum_{i=1}^{\infty} \mu(C_i) = 0$.
∴ $\mu_0(\bigcup_{i=1}^{\infty}(A_i \cup B_i)) = \mu(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i) = \sum_{i=1}^{\infty} \mu_0(A_i \cup B_i)$
Hence μ_0 is a measure on $\mathfrak{B}_0$.
If $E \in \mathfrak{B}_0$ then, $\mu_0(E) = \mu_0(\phi \cup E) = \mu(E)$
Claim: $(X, \mathfrak{B}_0, \mu_0)$ is complete.
Let $A \cup B \in \mathfrak{B}_0$, $\mu_0(A \cup B) = 0$ and $A_1 \cup B_1 \subseteq A \cup B$.
$\Rightarrow B \in \mathfrak{B}, \exists\, C \in \mathfrak{B}, A \subseteq C$ with $\mu C = 0$.
∴ $\mu(C \cup B) \leq \mu C + \mu B = \mu C + \mu_0(A \cup B) = 0 + 0 = 0$ ... (1).

Now $A_1 \cup B_1 = A_1 \cup B_1 \cup \phi = \{(A_1 \cup B_1) \cap (A \cup B)\} \cup \phi$
$= \{(A_1 \cup B_1) \cap (C \cup B)\} \cup \phi \in \mathfrak{B}_0$ since $\phi \in \mathfrak{B}$, $\exists\, C \cup B \in \mathfrak{B} \ni (A_1 \cup B_1) \cap (C \cup B)$
$\subseteq C \cup B$ and $\mu(C \cup B) = 0$ by (1).

Definition: Let $(X, \mathfrak{B}, \mu)$ be a measure space. A subset E of X is said to be ***locally measurable*** if $E \cap B \in \mathfrak{B}$ for each $B \in \mathfrak{B}$ with $\mu(B) < \infty$.
Proposition: The collection $\mathcal{C}$ of all locally measurable sets is σ - algebra containing $\mathfrak{B}$.
Proof: Let $\mathcal{C}$ be a collection of all locally measurable sets.
Claim: $\mathcal{C}$ is non-empty: Let $B \in \mathfrak{B}$ with $\mu(B) < \infty$. Then $\phi \cap B = \phi \in \mathfrak{B}$. $\therefore \phi \in \mathcal{C}$ so that $\mathcal{C}$ is non – empty.
Claim: $\mathcal{C}$ is closed under countable unions:
Let $\{E_i\}$ be a sequence of sets in $\mathcal{C}$ and $B \in \mathfrak{B}$ with $\mu(B) < \infty$.
Now $(\bigcup_{i=1}^{\infty} E_i) \cap B = \bigcup_{i=1}^{\infty}(E_i \cap B) \in \mathfrak{B}$ since each $E_i \cap B \in \mathfrak{B}$ *and* $\mathfrak{B}$ is a σ - algebra. $\therefore \bigcup_{i=1}^{\infty} E_i \in \mathcal{C}$ so that $\mathcal{C}$ is closed under countable unions.
Claim: $\mathcal{C}$ is closed under complements:
Let $E \in \mathcal{C}$. Ie $E \cap B \in \mathfrak{B}$ for each $B \in \mathfrak{B}$ with $\mu(B) < \infty$.
Let $B \in \mathfrak{B}$ with $\mu(B) < \infty$.
Then $E' \cap B = (E' \cap B) \cup \phi = (E' \cap B) \cup (B' \cap B)$
$= (E' \cup B') \cap B = (E \cap B)' \cap B \in \mathfrak{B}$ since $\mathfrak{B}$ is closed under complements and intersection. Ie. $E \in \mathcal{C} \Rightarrow E' \in \mathcal{C}$ so that $\mathcal{C}$ is closed under complements.
Hence the collection $\mathcal{C}$ of all locally measurable sets is a σ - algebra
Claim: $\mathfrak{B} \subseteq \mathcal{C}$
Let $E \in \mathfrak{B}$ and $B \in \mathfrak{B}$ with $\mu(B) < \infty$.
Since $\mathfrak{B}$ is a σ - algebra $E \cap B \in \mathfrak{B}$. Ie. $E \cap B \in \mathfrak{B}$ for each $B \in \mathfrak{B}$ with $\mu(B) < \infty$.
$\Rightarrow E \in \mathcal{C}$. Hence $\mathfrak{B} \subseteq \mathcal{C}$.

Definition: The measure μ is called saturated if every locally measurable set is measurable (ie is in $\mathfrak{B}$).

Problem: Every σ - finite measure is saturated.
Solution: Let $(X, \mathfrak{B}, \mu)$ be a measure space and the measure μ be σ - finite.
$\therefore \exists$ a sequence $\{X_n\}$ of measurable sets in $\mathfrak{B}$ such that $X = \cup X_n$ and $\mu(X_n) < \infty$ for each n. Let E be locally measurable set in X.
Since $X_n \in \mathfrak{B}$ and $\mu(X_n) < \infty$ for each n and E is locally measurable, $X_n \cap A \in \mathfrak{B}$ for each n.
Now $E = X \cap E = (\cup X_n) \cap E = \cup (X_n \cap E) \in \mathfrak{B}$ since $\mathfrak{B}$ is σ - algebra.
Thus, E is measurable. Hence every σ - finite measure is saturated.

Proposition: Let $(X, \mathfrak{B})$ be a measurable space. μ, ν be two measures on $(X, \mathfrak{B})$. Let $\lambda = \mu + \nu$. Then, $(X, \mathfrak{B}, \lambda)$ is a measure space.

Proposition: Let $(X, \mathfrak{B})$ be a measurable space; μ, ν be two measures on $(X, \mathfrak{B})$ such that $\mu \geq \nu$. Then there is a measure on $(X, \mathfrak{B})$ such that $\lambda + \nu = \mu$. In addition if ν is σ - finite then λ is unique.

MEASURABLE FUNCTIONS ON ABSTRACT SPACES

Definition: Let $(X, \mathfrak{B})$ be a measurable space. Let f be an extended real-valued function defined on X. Then f is said to be measurable (w.r.t $\mathfrak{B}$) if $\forall\ \alpha$, $\{x: f(x) > \alpha\} \in \mathfrak{B}$.

Proposition: Let f be an extended real-valued function defined on X. Then the following statements are equivalent.

(i) $\{x: f(x) > \alpha\} \in \mathfrak{B}$ for each α.
(ii) $\{x: f(x) \geq \alpha\} \in \mathfrak{B}$ for each α.
(iii) $\{x: f(x) < \alpha\} \in \mathfrak{B}$ for each α.
(iv) $\{x: f(x) \leq \alpha\} \in \mathfrak{B}$ for each α.

Proof: Claim: (i) $\Rightarrow$ (ii). Assume (i). Let α be a real number.

Now $\{x: f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x: f(x) > \alpha - \frac{1}{n}\right\} \in \mathfrak{B}$ since $\left\{x: f(x) > \alpha - \frac{1}{n}\right\} \in \mathfrak{B}\ \forall \alpha - \frac{1}{n}$ *and* $\mathfrak{B}$ *is a* $\sigma -$ *algebra.*

$\therefore \{x: f(x) \geq \alpha\} \in \mathfrak{B}\ \forall\ \alpha$.

So, (i) $\Rightarrow$ (ii).

Claim: (ii) $\Rightarrow$ (iii).

Suppose $\{x: f(x) \geq \alpha\} \in \mathfrak{B}$ for each α.

Let α be a real number. Then $\{x: f(x) < \alpha\} = \overline{\{x: f(x) \geq \alpha\}} \in \mathfrak{B}$ as it is σ - algebra.

$\therefore \{x: f(x) < \alpha\} \in \mathfrak{B}\ \forall\ \alpha$.

So, (ii) $\Rightarrow$ (iii).

Claim: (iii) $\Rightarrow$ (iv). Assume (iii).

Let α be a real number. Then $\{x: f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x: f(x) < \alpha + \frac{1}{n}\right\} \in \mathfrak{B}$.

$\therefore\ \{x: f(x) \leq \alpha\} \in \mathfrak{B}$ for each α.

So, (iii) $\Rightarrow$ (iv).

Claim: (iv) $\Rightarrow$ (i), assume (iv). Let α be a real number.

Then $\{x: f(x) > \alpha\} = \overline{\{x: f(x) \leq \alpha\}} \in \mathfrak{B}$ since $\mathfrak{B}$ is a σ - algebra.

$\therefore \{x: f(x) > \alpha\} \in \mathfrak{B}$ for each α. So, (iv) $\Rightarrow$ (i).

Hence the proposition is proved.

Theorem: If c is a real number and f, g are measurable functions then f + c, cf, f + g, g – f and fg are also measurable, on an abstract measurable space.

Proof: Let c be a real number and f, g be measurable functions. Let α be a real number.

$\{x: (f + c)(x) > \alpha\} = \{x: f(x) + c > \alpha\} = \{x: f(x) > \alpha - c\}$ which is a measurable set since f is measurable. $\therefore$ f + c is measurable.

If c = 0, cf is measurable, as the set $\{x: cf(x) > \alpha\} = \phi$ or X according as $\alpha \geq 0$ or $\alpha < 0$, and ϕ and X both belong to $\mathfrak{B}$ as $\mathfrak{B}$ is a σ-algebra.

If c > 0, $\{x: cf(x) > \alpha\} = \{x: f(x) > c^{-1}\alpha\}$ and since $\{x: f(x) > c^{-1}\alpha\} \in \mathfrak{B}$ it follows that $\{x: f(x) > \alpha\} \in \mathfrak{B}\ \forall\ \alpha$.

Also, if c < 0, then $\{x: cf(x) > \alpha\} = \{x: f(x) < c^{-1}\alpha\}$ and since $\{x: f(x) < c^{-1}\alpha\} \in \mathfrak{B}$ it follows that $\{x: cf(x) > \alpha\} \in \mathfrak{B}\ \forall\ \alpha$. So, cf is measurable.

$f(x) + g(x) > \alpha$ iff $\exists$ a rational r_i such that $\alpha - g(x) < r_i < f(x)$ where $\langle r_i \rangle$, i = 1, 2, 3, … is an enumeration of the set of rationals.

$\therefore \{x: f(x) + g(x) > \alpha\} = \bigcup_{i=1}^{\infty}[\{x: f(x) > r_i\} \cap \{x: g(x) > \alpha - r_i\}]$.

Since $\{x: f(x) > r_i\} \in \mathfrak{B}\ \forall\ r_i$ and $\{x: g(x) > \alpha - r_i\} \in \mathfrak{B}\ \forall\ \alpha - r_i$,

$[\{x: f(x) > r_i\} \cap \{x: g(x) > \alpha - r_i\}] \in \mathfrak{B}$ and hance $\{x: f(x) + g(x) > \alpha\} \in \mathfrak{B}\ \forall\ \alpha$ since Q is countable.

Hence f + g is measurable. Now f – g = f + (– g).

Since f and – g are measurable, so, f + (– g) is measurable.

Hence f – g is measurable.

Finally, fg = $\frac{1}{4}\{(f + g)^2 - (f - g)^2\}$.

So, it is sufficient to show that f^2 is measurable whenever f is.

If $\alpha < 0$, $\{x: f^2(x) > \alpha\} = X \in \mathfrak{B}$.

If $\alpha \geq 0$, $\{x: f^2(x) > \alpha\} = \{x: f(x) > \sqrt{\alpha}\} \cup \{x: f(x) < -\sqrt{\alpha}\}$.

Since f is measurable so $\{x: f(x) > \sqrt{\alpha}\}$ and $\{x: f(x) < -\sqrt{\alpha}\} \in \mathfrak{B}$.

Hence their union belongs to $\mathfrak{B}$. Thus, $\{x: f^2(x) > \alpha\} \in \mathfrak{B}\ \forall\alpha$.

$\therefore$ f^2 is measurable.

It follows that $(f + g)^2$ and $(f - g)^2$ are measurable.

So, fg is measurable.

Theorem: If $\{f_n\}$ is a sequence of measurable functions then sup f_n, inf f_n, $\underline{\lim}_{n\to\infty} f_n$ and $\overline{\lim}_{n\to\infty} f_n$ are also measurable.

Proof: Let $f_1, f_2, \ldots, f_n$ be measurable.

Claim: Sup $\{f_1, f_2, \ldots f_n\}$ is measurable.

Note that Sup $\{f_1, f_2, \ldots, f_n\}(x) = $ Sup $\{f_1(x), f_2(x), \ldots, f_n(x)\}$.

Let α be a real number.

Now $\{x \in X: \text{Sup}\{f_1, f_2, \ldots, f_n\}(x) > \alpha\} = \cap_{i=1}^{n}\{x \in X: f_i(x) > \alpha\} \in \mathfrak{B}$.

$\therefore$ Sup $\{f_1, f_2, \ldots, f_n\}$ is measurable.

Thus, if $f_1, f_2, \ldots, f_n$ are measurable then Sup $\{f_1, f_2, \ldots, f_n\}$ and similarly inf $\{f_1, f_2, \ldots, f_n\}$ is measurable.

Since $\{f_n\}$ is a sequence of measurable functions on $\mathfrak{B}$, so, for all α and all n, $\{x: f_n(x) > \alpha\} \in \mathfrak{B}$.

$\therefore \{x: \sup_n f_n > \alpha\} = \cap_{n=1}^{\infty}\{x: f_n(x) > \alpha\} \in \mathfrak{B}$ as $\mathfrak{B}$ is σ - algebra.

So, Sup f_n is measurable.

Since $\inf_n f_n = -\sup(-f_n)$ and since $(-f_n)$ is measurable by above argument $\sup(-f_n)$ is and hence $-\sup(-f_n)$ is measurable. Ie $\inf_n f_n$ is measurable.

(iii) $\because$ $\{f_n\}$ is a sequence of measurable functions, as above, $g_n(x) = \sup_{i \geq n} f_i$ is measurable for each n.

$\therefore$ As above $\inf_n g_n$ is measurable. Since $\overline{\lim_{n\to\infty}} f_n = \inf_n\left\{\sup_{i\geq n} f_i\right\}$, $\overline{\lim_{n\to\infty}} f_n$ is measurable.

$\because$ $\{f_n\}$ is a sequence of measurable functions, as above, $h_n(x) = \inf_{i\geq n} f_i$ is measurable for each n.

$\therefore$ As above $\inf_n h_n$ is measurable. Since $\underline{lim}_{n\to\infty} f_n = sup_n\left\{\inf_{i\geq n} f_i\right\}$, it follows that $\underline{lim}_{n\to\infty} f_n$ is measurable.

Definition: A real valued function defined on X and which assumes at most a finite number of values is called a simple function.

$\varphi: X \to \mathbb{R}$ is a simple function iff $\varphi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$, $x \in X$ where $E_1, E_2, \ldots, E_n$ are pairwise disjoint subsets of X $\ni$ $X = E_1 \cup E_2 \cup \ldots \cup E_n$ and $c_1, c_2, \ldots, c_n$ are distinct numbers. $\{E_1, E_2, \ldots, E_n\}$ is called a finite partition of X. χ_{E_i} is the characteristic function of E_i. Clearly $\varphi(x) = c_i$ for $x \in E_i$, i = 1, 2, …, n.

Proposition: A simple function $\varphi = \sum_{i=1}^{n} c_i \chi_{E_i}$ is measurable iff each E_i is measurable.

Proof: If the simple function φ has another representation as $\varphi = \sum_{j=1}^{n} d_j \chi_{F_j}$ then, on $E_i \cap F_j$, φ must assume the values c_i, d_j which is not possible unless $c_i = d_j$, $E_i = F_j$.

Hence the representation is unique upto the addition of empty set.

φ is measurable implies $\{x \in X: \varphi(x) > \alpha\} = \{x \in X: \sum_{i=1}^{n} c_i \chi_{E_i}(x) > \alpha\} \in \mathfrak{B}$.

Let $\alpha < \min\{c_1, c_2, \ldots, c_n\}$, Then $\{x \in X: \sum_{i=1}^{n} c_i \chi_{E_i}(x) > \alpha\} = X \in \mathfrak{B}$.

Let $\alpha > \max\{c_1, c_2, \ldots, c_n\}$. Then $\{x \in X: \sum_{i=1}^{n} c_i \chi_{E_i}(x) > \alpha\} = \phi \in \mathfrak{B}$.

Let $\min\{c_1, c_2, \ldots, c_n\} \leq \alpha \leq \max\{c_1, c_2, \ldots, c_n\}$.

Then $\{x \in X: \sum_{i=1}^{n} c_i \chi_{E_i}(x) > \alpha\} = \biguplus_j E_j$ where $\biguplus_j$ indicates the union over all j (from 1 to n) ∋ $c_j > \alpha$. Such j's are finite in number and hence, $\biguplus_j E_j \in \mathfrak{B}$, for any α we must have $E_j \in \mathfrak{B}$. Hence the simple function φ is measurable iff each E_j is measurable.

Proposition: If f is measurable and μ is complete, then f = g almost everywhere implies g is measurable.
Proof: Let $E = \{x \in X: f(x) \neq g(x)\}$. By hypothesis $\mu(E) = 0$.
For any $\alpha \in \mathbb{R}$, $\{x \in X: g(x) > \alpha\} = \{x \in X: f(x) > \alpha\} \cup N$ where
$N = \{x \in X: g(x) > \alpha$ and $f(x) \neq g(x)\}$. Hence $N \subseteq E$. Since, f is measurable $\{x \in X: f(x) > \alpha\} \in \mathfrak{B}$. Since, μ is complete $N \in \mathfrak{B}$ and hence $\{x \in X: g(x) > \alpha\} \in \mathfrak{B}$. Hence g is measurable.
…

INTEGRATION.
Definition:
*If E is a measurable set, φ a nonnegative simple function and μ any measure, define
$\int_E \varphi d\mu = \sum_{i=1}^{n} C_i \mu(E_i \cap E)$ where $\varphi(x) = \sum_{i=1}^{n} C_i \chi_{E_i}(x)$.
Proposition: If a and b are positive numbers and φ and ψ are nonnegative simple functions, then $\int(a\varphi + b\psi) = a\int\varphi + b\int\psi$
Proof:

If a simple function φ takes the values $c_1, c_2, \ldots, c_n$ then $\varphi(x) = \sum_{i=1}^{n} C_i \chi_{A_i}$ where $A_i = \{x : \varphi(x) = c_i\}$.
Then the integral of φ with respect to μ is given by $\int_E \varphi d\mu = \sum_{i=1}^{n} C_i \mu(A_i)$.

Definition: Let f be a nonnegative extended real-valued measurable function on the measure space $(X, \mathfrak{B}, \mu)$. Then the integral of f is given by
$\int f d\mu = \sup\{\int \varphi d\mu: \varphi \leq f\}$ where φ is a simple function.

Definition: Let $(X, \mathfrak{B}, \mu)$ be a measure space. Let $E \in \mathfrak{B}$, and let f be a measurable function $f: E \to (0, \infty]$, then the integral of f over E is
$\int_E \varphi d\mu = \int f \chi_E \, d\mu$.

Fatou's lemma: Let $\{f_n\}$ be a sequence of nonnegative measurable function which converges almost everywhere on a set E to a function f then $\int_E f \leq \underline{\lim}_{n\to\infty} \int_E f_n$

Proof: Without loss of generality, we may assume that $f_n(x) \to f(x)$ for each $x \in E$.
From the definition of $\int f$, it suffices to show that φ is any nonnegative simple function with $\varphi \le f$ then $\int_E \varphi \le \underline{\lim}_{n\to\infty} \int_E f_n$

If $\int \varphi = \infty$, then there is a measurable set $A \subseteq E$ with $\mu A = \infty$ such that $f \ge r > 0$ on A.

Set $A_n = \{x \in E: f_k(x) > r \ \forall\ k \ge n\}$.

Then $\{A_n\}$ is an increasing sequence of measurable sets whose union contains A, since $\varphi \le \lim f_n$.

Thus, $\lim \mu\, A_n = \infty$.

Since $\int_E f_n \ge r\,\mu(A_n)$ we have $\underline{\lim}_{n\to\infty} \int_E f_n = \infty = \int_E \varphi$.

If $\int\varphi < \infty$, then the set $A = \{x \in E: \varphi(x) > 0\}$ is a measurable set of finite measure.

Let M be maximum of φ, ε be a given positive integer, and set $A_n = \{x \in E: f_k(x) > (1-\varepsilon)\,\varphi(x) \ \forall\ k \ge n\}$.

Then $\{A_n\}$ is an increasing sequence of measurable sets whose union contains A, and so, $\{A \sim A_n\}$ is a decreasing sequence of sets whose intersection is empty.

By a proposition, $\lim \mu(A \sim A_n) = 0$, and so, we can find an $n \ni \mu(A \sim A_k) < \varepsilon \ \forall\ k \ge n$.

Then for $k \ge n$ we have

$$\int_E f_k > \int_{A_k} f_k > (1-\varepsilon)\int_{A_k} \varphi \ \ge (1-\varepsilon)\int_E \varphi - \int_{A\sim A_k} \varphi \ge \int_E \varphi - \varepsilon\left[\int_E \varphi + M\right].$$

Hence $\underline{lim} \int_E f_n \ge \int_E \varphi - \varepsilon\left[\int_E \varphi + M\right]$.

Since ε is arbitrary, $\underline{lim} \int_E f_n \ge \int_E \varphi$

Monotone Convergence Theorem: Let $\{f_n\}$ be a sequence of nonnegative measurable function which converges almost everywhere to a function f and suppose that $f_n \le f$ for all n. Then $\int_E f = \underline{\lim}_{n\to\infty} \int_E f_n$

Proof: Since $f_n \le f$, we have $\int f_n \le \int f$.

Hence $\overline{lim}_{n\to\infty} \int f_n \le \int f$

By Fatou's lemma, $\int f \le \underline{lim}_{n\to\infty} \int f_n$

From (i) and (ii) we get $\int f \le \underline{lim}_{n\to\infty} \int f_n \le \overline{lim}_{n\to\infty} \int f_n \le \int f$

$\therefore \underline{lim}_{n\to\infty} \int f_n = \overline{lim}_{n\to\infty} \int f_n = \lim_{n\to\infty} \int f_n$

Thus, $\lim_{n\to\infty} \int f_n = \int f$

Hence the theorem.

Proposition: Suppose a and b are non negative numbers and f and g are nonnegative measurable functions, then (i) ∫(af + bg) = a∫f + b∫g (ii) ∫ f ≥ 0 with equality only if f = 0 a. e.

Proof: (i) Let $\{\varphi_n\}$ and $\{\psi_n\}$ be increasing sequences of simple functions which converge to f and g respectively.

Then $\{a\varphi_n + b\psi_n\}$ is an increasing sequence of simple functions which converge to af + bg.

By the Monotone Convergence Theorem, $\int (af + bg) = \lim \int (a\varphi_n + b\psi_n)$

$$= \lim (a\int\varphi_n + b\int\psi_n)$$

$$= a\int f + b\int g.$$

(ii) Obviously $\int f \geq 0$.

If $\int f = 0$, let $A_n = \{x: f(x) \geq \frac{1}{n}\}$.

Then $f \geq \frac{1}{n}\chi_{A_n}$ and so, $\mu(A_n) = \int \chi_{A_n} = 0$. Since the set where f > 0 is the union of the sets A_n, it has measure zero.

Hence f = 0 a. e.

Corollary: Let $\{f_n\}$ be a sequence of nonnegative measurable functions. Then $\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$.

Proof: Let $s_n = \sum_{k=1}^{\infty} f_k$ so that $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} f_n \ldots$(i)

Now s_n is a sequence of measurable functions such that $s_n \leq s_{n+1}$ and (i) holds.

∴ By monotone convergence theorem,

$$\int \sum_{n=1}^{\infty} f_n d\mu = \int \lim_{n\to\infty} s_n = \lim_{n\to\infty} \int s_n$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int f_k$$

$= \sum_{k=1}^{\infty} \int f_k$, proving corollary.

Definition: A nonnegative function f is said to be integrable over a measurable set E w. r. t any measure μ if it is measurable and $\int_E f\, d\mu < \infty$.

Any function f can be written as $f = f^+ - f^-$ where f^+ and f^- are positive and negative parts of f and that $|f| = f^+ + f^-$.

Definition: An arbitrary function f is said to be integrable if both f^+ and f^- are integrable. In this case we define $\int_E f\, d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$.

Proposition: Let $(X, \mathfrak{B}, \mu)$ be a measure space. If f and g are integrable functions over $E \in \mathfrak{B}$, then (i) $\int_E (af + bg)\, d\mu = a\int_E f\, d\mu + b\int_E g\, d\mu$.

(ii) If $|h| < |f|$ and h is measurable then h is integrable.

(iii) If f ≥ g a. e., then $\int_E f\, d\mu \geq \int_E g\, d\mu$.

Proof: (i)

(ii) Given $|h| < |f|$ we have $|h| = h^+ + h^-$.

Then $h^+ < |h|$ so that $\int h^+ d\mu \leq \int |f|\, d\mu$ since f is integrable function, so $|f|$ is integrable and $\int_E |f|\, d\mu < \infty$ hence h^+ is integrable.

Similarly, we can prove that h^- is integrable.

Hence h = $h^+ - h^-$ is integrable.

(iii) f ≥ g a. e. ⇒ f – g ≥ 0 a. e.

Hence ∫(f – g) dμ ≥ 0.

⇒ ∫ f dμ – ∫ g dμ ≥ 0 proving $\int_E f\, d\mu \geq \int_E g\, d\mu$.

Lebesgue Convergence Theorem: Let (X, 𝔅, μ) be a measure space. Let g be integrable function over E ∈ 𝔅 and suppose $\{f_n\}$ be a sequence of measurable functions such that on E, $|f_n(x)| \leq g(x)$, and such that almost everywhere on E, $f_n(x) \to f(x)$. Then $\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$.

Proof: Since, for each x, $|f_n(x)| \leq g(x)$ and $\lim f_n = f$ a. e. we have $|f| \leq g$ a. e. hence f_n and f are integrable. Also, since $-g \leq f_n \leq g$, $\{g + f_n\}$ is a sequence of nonnegative measurable functions.

Now by Fatou's lemma, $\int (g+f) d\mu \leq \underline{\lim}_{n\to\infty} \int (g+f_n) d\mu$.

So, $\int_E g\, d\mu + \underline{\lim}_{n\to\infty} \int_E f_n d\mu \geq \int_E g\, d\mu + \int_E f\, d\mu$.

Since g is integrable $\int_E g\, d\mu$ is finite.

$\therefore \int_E f d\mu \leq \underline{\lim}_{n\to\infty} \int_E f_n d\mu \ldots$ (i)

Again, *since $f_n \leq |f_n| \leq g$, $\{g - f_n\}$ is a sequence of nonnegative measurable functions.

Now by Fatou's lemma, $\int (g-f) d\mu \leq \underline{\lim}_{n\to\infty} \int (g-f_n) d\mu$.

So, $\int_E g\, d\mu - \underline{\lim}_{n\to\infty} \int_E f_n d\mu \geq \int_E g\, d\mu - \int_E f\, d\mu$. $\overline{\lim}_{n\to\infty} \int f_n \leq \int f$

Since g is integrable $\int_E g\, d\mu$ is finite.

$\therefore \overline{\lim}_{n\to\infty} \int f_n \leq \int f \ldots$ (ii).

From (i) and (ii) we get $\int f \leq \underline{\lim}_{n\to\infty} \int f_n \leq \overline{\lim}_{n\to\infty} \int f_n \leq \int f$

$\therefore \underline{\lim}_{n\to\infty} \int f_n = \overline{\lim}_{n\to\infty} \int f_n = \lim_{n\to\infty} \int f_n$

Thus, $\lim_{n\to\infty} \int f_n = \int f$

Hence the theorem.

Theorem: Let f be an integrable function on the measure space (X, $\mathfrak{B}$, μ). Then given ε > 0, there is a δ > 0, such that for each measurable set E with μ(E) < δ, $\left|\int_E f\right| < \varepsilon$.

Proof: The theorem is trivial if f is bounded function.
For any n, let $f_n(x) = f(x)$ if $f(x) \le n$ and $f_n(x) = n$ otherwise.
Then each f_n is bounded and $f_n(x) \to f(x)$ for each x.
By the monotone convergence theorem there is an integer N ∋ $\left|\int_E (f - f_N)\right| < \frac{\varepsilon}{2}$.
Choose δ > $\frac{\varepsilon}{2N}$.

If μ (E) < δ, $\left|\int_E f\right| = \left|\int_E (f - f_N + f_N)\right|$

$$\le \left|\int_E (f - f_N)\right| + \left|\int_E f_N\right|$$

$$\le \left|\int_E (f - f_N)\right| + \int_E |f_N|$$

$$< \frac{\varepsilon}{2} + N\frac{\varepsilon}{2N}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Ie. $\left|\int_E f\right| < \varepsilon$. Hence the theorem.

Theorem: Let (X, $\mathfrak{B}$, μ) be a measure space and g be a nonnegative measurable function on X. Set ν (E) = $\int_E g\,d\mu$. Then ν is a measure on $\mathfrak{B}$.

Proof: By the definition of ν, obviously, ν is non-negative, ν(ϕ) = 0.
Let $\{E_n\}$ be a sequence of pairwise disjoint sets.

$\nu(\cup_{n=1}^{\infty} E_n) = \int_{\cup_{n=1}^{\infty} E_n} g\,d\mu = \int_X g\,\chi_{\cup_{n=1}^{\infty} E_n} d\mu = \int_X \sum_{n=1}^{\infty} g.\chi_{E_n} d\mu$

$= \sum_{n=1}^{\infty} \int_X \chi_{E_n} g\,d\mu = \sum_{n=1}^{\infty} \int_{E_n} g\,d\mu = \sum_{n=1}^{\infty} \nu(E_n).$

Lemma: Suppose that to each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathfrak{B}$ such that $B_\alpha \subseteq B_\beta$ for α < β. Then there is a unique measurable extended real-valued function f on X such that f ≤ α on B_α and f ≥ α on X ~ B_α.

Proof: For each x ∈ X, define f(x) = inf {α∈D: x ∈ B_α} where, as usual, inf ϕ = ∞.
If x ∈ B_α, then f(x) ≤ α. If x ∉ B_α, then x ∉ B_β for each β < α, and so f(x) ≥ α.
To show that f is measurable, we take λ ∈ ℝ and choose a sequence $\{\alpha_n\}$, from D with $\alpha_n < \lambda$ and $\lambda = \lim \alpha_n$. Then {x: f(x) < λ} = $\cup_{n=1}^{\infty} B_{\alpha_n}$. For if f(x) < λ, then f(x) < α_n for some n, and so x ∈ B_{α_n}. If x ∈ B_{α_n} for any n, then f(x) < α_n < λ.
Thus, the sets {x: f(x) < λ} are all measurable, and so f is measurable.

To prove the unicity of f, let g be any extended real-valued function with $g \leq \alpha$ on B_α and $g \geq \alpha$ on $\widetilde{B_\alpha}$.
Then $x \in B_\alpha$ implies $g(x) \leq \alpha$, and so $\{\alpha \in D: x \in B_\alpha\} \subseteq \{\alpha \in D: \alpha \geq g\{x)\}$.
Since $g(x) < \alpha$ implies that $x \in B_\alpha$ we have $\{\alpha \in D: \alpha > g(x)\} \subseteq \{\alpha \in D: x \in B_\alpha\}$.
Because of the density of D we have $g(x) = \inf \{\alpha \in D: \alpha > g\{x)\}$
$= \inf \{\alpha \in D: \alpha \geq g(x)\} = \inf \{\alpha \in D: x \in B_\alpha\} = f(x)$.

Proposition: Suppose that for each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathfrak{B}$ such that $\mu(B_\alpha \sim B_\beta) = 0$ for $\alpha < \beta$. Then there is a measurable function f such that $f \leq \alpha$ a. e. on B_α and $f \geq \alpha$ a. e. on $X \sim B_\alpha$. If g is any other function with this property, then $g = f$ a. e.

Proof: Let C be a countable dense subset of D, and set $N = \cup (B_\alpha \sim B_\beta)$ for α and β in C with $\alpha < \beta$.
Then N is the countable union of sets of measure zero and so is itself a set of measure zero.
Let $B'_\alpha = B_\alpha \cup N$.
For α and β in C with $\alpha < \beta$ we have $B'_\alpha \sim B'_\beta = (B_\alpha \sim B_\beta) \sim N = \phi$.
Thus $B'_\alpha \subseteq B'_\beta$. By Lemma there is a measurable function f such that $f \leq \gamma$ on B'_γ and $f \geq \gamma$ on $X \sim B'_\gamma$.
Let $\alpha \in D$ and choose a sequence $\{\gamma n\}$ from C with $\alpha < \gamma_n$ and $\alpha = \lim \gamma_n$.
Then $B_\alpha \sim B'_{\gamma_n} \subseteq B_\alpha \sim B_{\gamma_n}$.
Thus, $P = \bigcup_n (B_\alpha \sim B'_{\gamma_n})$ is a countable union of null sets and so a n null set.
Let $A = \cap B'_{\gamma_n}$.
Then $f < \inf \gamma_n = \alpha$ on A, and $A \sim B_\alpha \subseteq P$.
Thus $f \leq \alpha$ almost everywhere on B_α.
A similar argument shows that $f \geq \alpha$ almost everywhere on $\tilde{B}_\alpha$
Let g be an extended real-valued function with $g \leq \gamma$ a. e. on B_γ and $g \geq \gamma$ on $\tilde{B}_\gamma$ for each $\gamma \in C$. Then $g \leq \gamma$ on $\tilde{B}_\gamma$ and $g \geq \gamma$ on $\tilde{B}'_\gamma$ except for x in a null set Q_γ.
Thus $Q = \cup Q_\gamma$ is a null set and we must have $f = g$ on $X \sim Q$.

GENERAL CONVERGENCE THEOREMS

Proposition: Let $(X, \mathfrak{B})$ be a measurable space, $\{\mu_n\}$ be a sequence of measures that converge setwise to a measure μ, and $\{f_n\}$ a sequence of non-negative measurable functions that converge pointwise to the function f. Then $\int f \, d\mu \leq \underline{lim} \int f_n \, d\mu_n$

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E – CONTENT

PAPER: M 401,

MEASURE THEORY

M. Sc. II YEAR, SEMESTER - IV

UNIT – II

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M 401, MEASURE THEORY
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UNIT II
SIGNED MEASURE

Definition: By a ***signed measure*** on the measurable space (X, $\mathfrak{B}$) we mean an extended real-valued set function ν defined for the sets of $\mathfrak{B}$ and satisfying the following conditions:
(i) ν assumes at most one of the values $+\infty$, $-\infty$.
(ii) $\nu(\phi) = 0$
(iii) $\nu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \nu(E_i)$, for any sequence $\{E_i\}$of disjoint measurable sets, the equality taken to mean that the series on the right converges absolutely if $\nu(\bigcup_{i=1}^{\infty} E_i)$ is finite and that it properly diverges otherwise.

Note: Thus, a measure is a special case of a signed measure, but a signed measure is not in general a measure.

Definition: A set A is a ***positive set*** with respect to a signed measure ν if A is measurable and for every measurable subset E of A we have $\nu E \geq 0$.
Ie. $A \in \mathfrak{B}$ is positive w.r.t. signed measure ν if $E \in \mathfrak{B}$, $E \subseteq A \Rightarrow \nu E \geq 0$.

Result; (i) ϕ set is a positive set.
(ii) If we take the restriction of ν to a positive set we obtain a measure.

Definition: A set B is called a ***negative set*** if it is measurable and every measurable subset of it has nonpositive ν measure.

Definition: A set that is both positive and negative with respect to ν is called a ***null set***.

Note: A measurable set is a null set if and only if every measurable subset of it has ν measure zero.

Note the distinction between a null set and a set of measure zero. While every null set must have measurable subsets of measure zero, a set of measure zero may well be a union of two sets whose measures are not zero but are negatives of each other. Similarly, a positive set is not to be confused with a set that merely has positive measure. Similar statements hold, of course, for negative sets.

Lemma: (i) Every measurable subset of a positive set is itself positive.
(ii) The union of a countable collection of positive sets is positive.
Proof: (i) Let (X, $\mathfrak{B}$) be a measurable space and ν be a signed measure.
Let A be a positive set of ν. Let B ⊆ A and B be measurable. Let E ⊆ B and E be measurable. Then νE ≥ 0 since E ⊆ B ⊆ A and A is positive.
∴ B is a positive set. Hence every measurable subset of a positive set is itself positive.
(ii) Let $\{A_n\}$ be a sequence of positive sets and A = $\bigcup_{n=1}^{\infty} A_n$.
Let E be any measurable subset of A.
Set $E_n = E \cap A_n \cap \tilde{A}_{n-1} \cap \tilde{A}_{n-2} \cap \ldots \cap \tilde{A}_1$
Then E_n is a measurable subset of A_n and so $\nu E_n \geq 0$.
Since the E_n are disjoint and E = ∪ E_n, we have νE = $\sum_{n=1}^{\infty} \nu E_n \geq 0$.
Thus, A is a positive set.

Lemma: Let (X, $\mathfrak{B}$) be a measurable space and ν be signed measure on $\mathfrak{B}$. Let E be a measurable set such that 0 < νE < ∞. Then there is a positive set A contained in E with νA > 0.

Proof: Case (i): Let E itself be a positive set, in which case the lemma is trivial. Case (ii): Suppose E contains measurable sets of negative measure. Let n_1 be the smallest positive integer such that there is a measurable set $E_1 \subseteq E$ with $\nu E_1 < -\frac{1}{n_1}$,
Proceeding inductively, if E ~ $\bigcup_{n=1}^{k-1} E_n$, is not already a positive set, let n_k, be the smallest positive integer for which there is a measurable set E_k such that
$E_k \subseteq E \sim \bigcup_{n=1}^{k-1} E_n$, and $\nu E_k < -\frac{1}{n_k}$.
If we set A = E ~ $\bigcup_{n=1}^{\infty} E_n$, then $E = A \cup [\bigcup_{n=1}^{\infty} E_n]$.
Since this is a disjoint union, we have νE = νA + $\sum_{n=1}^{\infty} \nu E_n$ with the series on the right absolutely convergent, as νE is finite.
Thus, $\sum \frac{1}{n_k}$ converges, and we have $n_k \to \infty$.
Since $\nu E_k \leq 0$ and νE > 0, we must have νA > 0.
To show that A is a positive set, let ε > 0 be given.
Since $n_k \to \infty$, we may choose k so large that $(n_k - 1)^{-1} < \varepsilon$.
Since A ⊆ E ~ $\bigcup_{n=1}^{k} E_n$, A can contain no measurable sets with measure less than $-(n_k - 1)^{-1}$, which is greater than – ε.
Thus, A contains no measurable sets of measure less than – ε.
Since ε is an arbitrarily positive number, it follows that A can contain no sets of negative measure and so must be a positive set.

Proposition (Hahn Decomposition Theorem): Let ν be a Signed measure on the measurable space (X, $\mathfrak{B}$). Then there is a positive set A and a negative set B such that X = A ∪ B and A ∩ B = ϕ.
Proof: Let (X, $\mathfrak{B}$) be a measurable space and ν be signed measure on (X, $\mathfrak{B}$). Without

loss of generality, we may assume that + ∞ is not assumed by ν. Ie. $E \in \mathfrak{B} \Rightarrow \nu E \neq \infty$.

Let $\lambda = \text{Sup}\,\{\nu E: E \in \mathfrak{B}, E \text{ is positive}\}$. Since ϕ is positive set, $\nu\phi = 0$ so that $\lambda \geq 0$.

For each n, $\lambda - \frac{1}{n}$ is not an upper bound of $\{\nu E: E \in \mathfrak{B}, E \text{ is positive}\}$.

$\therefore \exists$ a positive set $E_n \ni \nu E_n > \lambda - \frac{1}{n}$.

$\therefore\ \lambda - \frac{1}{n} < \nu E_n \leq \lambda$.

$\Rightarrow \lim\limits_{n\to\infty} (\lambda - \frac{1}{n}) \leq \lim\limits_{n\to\infty} \nu E_n \leq \lim\limits_{n\to\infty} \lambda. \Rightarrow \lambda \leq \lim\limits_{n\to\infty} \nu E_n \leq \lambda$.

$\therefore \exists$ a sequence of positive sets $\{E_n\} \ni \lim\limits_{n\to\infty} \nu E_n = \lambda$.

Set $A = \bigcup_{n=1}^{\infty} E_n$.

Then A is a positive set since each E_n is a positive set.

$\therefore$ by definition of λ, $\nu A \leq \lambda$… (i).

Since $A \sim E_n \subseteq A$ for any n, and A is positive $\nu(A \sim E_n) \geq 0$.

Since $A = E_n \cup (A \sim E_n)$, $\nu A = \nu E_n + \nu(A \sim E_n) \geq \nu E_n\ \forall$ n.

$\therefore\ \nu A \geq \lambda$ … (ii).

From (i) and (ii) $\nu A = \lambda < \infty$ … (iii).

Let B = X – A. Then $X = A \cup B$ and $A \cap B = \phi$.

Claim: B is a negative set.

If B contains a measurable subset D of positive measure then we have $0 < \nu D < \infty$. So, D contains a positive set $E \ni \nu E > 0$.

Then E and A are disjoint and $E \cup A$ is a positive set.

But then $\nu(E \cup A) = \nu E + \nu A > \lambda$ which is a contradiction.

Hence B is a negative set. Hence the theorem.

Result: Hahn decomposition is not unique.

Proof: Let ν be a Signed measure on the measurable space $(X, \mathfrak{B})$.

Then by Hahn decomposition there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \phi$.

Let $E\ (\neq \phi) \subseteq A$, $E \in \mathfrak{B}$ with $\nu(E) = 0$. Let $A' = A - E$ and $B' = B \cup E$.

Claim: A′, B′ is also a Hahn Decomposition for X.

$A' \cup B' = (A \cap \tilde{E}) \cup (B \cup E) = \{A \cup (B \cup E)\} \cap (\tilde{E} \cup B \cup E)$

$= \{(A \cup B) \cup E\} \cap \{(\tilde{E} \cup E) \cup B\} = X \cap X = X$.

$A' \cap B' = (A \cap \tilde{E}) \cap (B \cup E) = \{(A \cap \tilde{E}) \cap B\} \cup \{(A \cap \tilde{E}) \cap E\}$

$= \{(A \cap B) \cap \tilde{E}\} \cup \{A \cap (\tilde{E} \cap E)\} = \phi \cup \phi = \phi$.

A′ is positive: For $F \subseteq A' \Rightarrow F \subseteq A - E \Rightarrow F \subseteq A \Rightarrow \nu(F) \geq 0$ since A is positive.

B′ is negative: For $F \subseteq B' \Rightarrow F \subseteq B \cup E \Rightarrow F = F \cap (B \cup E) = (F \cap B) \cup (F \cap E)$

$\Rightarrow \nu(F) = \nu(F \cap B) + \nu(F \cap E)$.

But $\nu(F \cap E) \geq 0$ since $F \cap E \subseteq E \subseteq A$ and $\nu(F \cap E) \leq 0$ since $\nu(F \cap E) \leq \nu(E) = 0$

Ie $\nu(F \cap E) = 0$.

$\therefore\ \nu(F) = \nu(F \cap B) \leq \nu(B) \leq 0$.

Hence A′, B′ is also Hahn Decomposition for X.

Lemma: Any two Hahn decompositions differ by a null set.
Proof: Let (A, B) and (F, G) be two Hahn decompositions of ν.
Then, A and F are positive sets of ν, B and G are negative sets of ν.
$A \cap B = \phi$, $A \cup B = X$, $F \cap G = \phi$, $F \cup G = X$.
We can easily derive that $\nu(A\Delta F) = 0 = \nu(B\Delta G)$, $A \Delta F = (A \cap F') \cup (A' \cap F)$.
By additivity of ν, $0 = \nu(A \Delta F) = \nu(A \cap F') + \nu(A' \cap F)$.
Since, A and F are positive sets, $\nu(A \cap F') = 0 = \nu(A' \cap F)$. Thus, $\nu(A \cap F') = 0$, $\nu(A' \cap F) = 0$, $\nu(B \cap F') = 0$, $\nu(B' \cap F) = 0$.
Now, $A = (A \cap F') \cup (A \cap F) \Rightarrow \nu A = \nu(A \cap F)$
$F = (A' \cap F) \cup (A \cap F) \Rightarrow \nu F = \nu(A \cap F)$
Hence $\nu A = \nu F$. Similarly, $\nu B = \nu G$. Hence the lemma.

Definition: Let $(X, \mathfrak{B})$ be a fixed measurable space. If μ and ν are two measures defined on $(X, \mathfrak{B})$ we say that μ and ν are ***mutually singular***, denoted by $\mu \perp \nu$ if there are disjoint measurable sets A and B with $X = A \cup B$ such that $\mu(B) = \nu(A) = 0$. A measure ν is said to be ***absolutely continuous*** with respect to the measure μ (written as $\nu \ll \mu$) if $\nu A = 0$ for each set A for which $\mu A = 0$.

Jordan Decomposition theorem: Let ν be a signed measure on the measurable space $(X, \mathfrak{B})$. Then there are two mutually singular measures ν^+ and ν^- on $(X, \mathfrak{B})$ such that $\nu = \nu^+ - \nu^-$. Moreover, there is only one such pair of mutually singular measures.
Proof: Let (A, B) be a Hahn decomposition of X w.r.t ν.
Define ν^+ and ν^- on $\mathfrak{B}$ by $\nu^+(E) = \nu(E \cap A)$ and $\nu^-(E) = -\nu(E \cap B)$ … (i) for $E \in \mathfrak{B}$.
Since A is a positive set $\nu(E \cap A) \geq 0$ and for similar reasons
$\nu(E \cap B) \leq 0 \;\forall\, E \in \mathfrak{B}$.
So, $\nu^+(E) \geq 0$, $\nu^-(E) \geq 0$. $\nu^+(\phi) = \nu(\phi \cap A) = 0$, $\nu^-(\phi) = \nu(\phi \cap B) = 0$.
Let $\{E_i\}$ be a sequence of pairwise disjoint measurable subsets of X.
Then $\nu^+(\bigcup_{i=1}^{\infty} E_i) = \nu\{(\bigcup_{i=1}^{\infty} E_i) \cap A\} = \nu\{\bigcup_{i=1}^{\infty}(E_i \cap A)\} = \sum_{i=1}^{\infty} \nu(E_i \cap A) = \sum_{i=1}^{\infty} \nu^+(E_i)$.
Similarly, $\nu^-(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \nu^-(E_i)$.
Thus, ν^+ and ν^- are measures on $(X, \mathfrak{B})$.
Also, $\nu^+(B) = \nu(B \cap A) = \nu(\phi) = 0$ and $\nu^-(A) = -\nu(A \cap B) = -\nu(\phi) = 0$. Hence $\nu^+ \perp \nu^-$.
Further for $E \in \mathfrak{B}$, $E = (E \cap A) \cup (E \cap B)$ so that $\nu E = \nu(E \cap A) + \nu(E \cap B)$ which gives $\nu(E) = \nu^+(E) - \nu^-(E)$ ie. $\nu = \nu^+ - \nu^-$.
Claim: Decomposition is unique.
Let $\nu = \nu_1 - \nu_2$ be any other decomposition of ν into mutually singular measures.
Then we have disjoint measurable subsets A and B such that $X = A \cup B$ where
$B = A'$ and $\nu_1(B) = \nu_2(A) = 0$.
Let $D \in \mathfrak{B}$, and $D \subseteq A$. Then $B \cap D \subseteq B \cap A = \phi$ so that $\nu_2(B \cap D) = \nu_2(\phi) = 0$ But $D \cap B \subseteq D \subseteq A$. $\therefore 0 = \nu_2(D \cap B) \leq \nu_2(D) \leq \nu_2(A) = 0$.
$\Rightarrow \nu_2(D) = 0$ … (ii)

Then ν(D) = $\nu_1(D) - \nu_2(D) = \nu_1(D)$…(iii). Thus, νD ≥ 0 ∀ D ⊆ A.
∴ A is positive set w.r.t ν.
Similarly, B is a negative set.
Now for each E ∈ $\mathfrak{B}$, E ∩ A ⊆ A so ν_1(E ∩ A) = ν(E ∩ A) from (iii)… (iv)
Now $\nu_1(E)$ = ν_1{(E − B)∪(E − A) = ν_1(E − B) + ν_1(E − A) = ν_1(E∩B') since
ν_1(E ∩ B) ≤ ν_1(B) = 0.
Thus, $\nu_1(E) = \nu_1(E \cap A)$ = ν(E ∩ A) ∀ E ∈ $\mathfrak{B}$.
Ie. $\nu_1(E)$ = ν(E ∩ A) = $\nu^+(E)$ &
similarly $\nu_2(E)$ = –ν(E ∩ B) = $\nu^-(E)$ ∀ E ∈ $\mathfrak{B}$...(v) and so, every such decomposition of ν is obtained from a Hahn decomposition of X as
$\nu^+(E) = \nu(E \cap A)$ and $\nu^-(E) = -\nu(E \cap B)$.

[OR So, it is enough to show that if (A, B) and (F, G) are Hahn decompositions then measures obtained as in (v) are the same as ν^+ and ν^-.
Now ν(A ∪ F) = ν(A ∩ F) + ν(A Δ F) … (vi)
Note that A – F = A ∩ G ⊆ A and hence is a positive set.
Also, A ∩ G ⊆ G. So, A – F is also a negative set. Hence A – F is a null set. Similarly, F – A is a null set and so A Δ F is null set. Hence by (v), ν(A ∪ F) = ν(A ∩ F)...(vii).
For each E ∈ $\mathfrak{B}$, as A ∪ F is a positive set,
ν{E ∩ (A ∩ F)} ≤ ν(E ∩ A) ≤ ν{E ∩ (A ∪ F)}…(viii)
ν{E ∩ (A ∩ F)} ≤ ν(E ∩ F) ≤ ν{E ∩ (A ∪ F)}…(ix)
But the first and last terms in each of these inequalities are the same.
So, ν(E ∩ F) = ν(E ∩ A) ⇒ ν^+ defined in (i) is unique and $\nu^- = \nu^+ - \nu$.
Hence ν^- is also unique. Thus, the theorem is proved.]

<u>The Radon-Nikodym Theorem</u>:
Let (X, $\mathfrak{B}, \mu$) be a σ - finite measure space. Let ν be a measure defined on $\mathfrak{B}$ such that ν ≪ μ. Then there is a non – negative measurable function f such that ν(E) = $\int_E f\, d\mu$ for all E ∈ $\mathfrak{B}$. If g is also a non – negative measurable function with this property then, f = g a. e. [μ].

<u>Proof</u>: First assume that (X, $\mathfrak{B}, \mu$) is finite. For each α ∈ Q (rationals), ν – αμ is a signed measure. Let (A_α, B_α) be a Hahn decomposition of ν – αμ for each α ∈ Q. If α = 0 then, ν being non – negative, set A_0 = X, B_0 = ϕ.
For each α∈ Q, $B_\alpha \setminus B_\beta = B_\alpha \cap B_\beta' = B_\alpha \cap A_\beta$ so that (ν – αμ)($B_\alpha \setminus B_\beta$)
= (ν – αμ)($B_\alpha \cap A_\beta$) ≤ 0 and (ν – βμ)($B_\alpha \setminus B_\beta$) = (ν – βμ)($B_\alpha \cap A_\beta$) ≥ 0…(i).
If β > α then, these imply μ($B_\alpha \cap A_\beta$) ≤ 0. But μ($B_\alpha \cap A_\beta$) ≥ 0.
Thus, the family {B_α} is such that μ($B_\alpha \setminus B_\beta$) = 0 if β > α.
∴ ∃ a measurable function f ∋ f ≤ α a. e. on B_α and f ≥ α a. e. on B_β' ie. on A_β.

Since $B_0 = \phi$, which means $f \leq 0$ on ϕ and $f \geq 0$ a. e. on $A_0 = X$ we may take f to be non – negative.

Let E ∈ 𝔅. Define $E_j = E \cap \left(B_{\frac{j+1}{N}} \setminus B_{\frac{j}{N}}\right)$ where N is a fixed positive integer … (ii),

$E_\infty = E \setminus \cup_{j=1}^{\infty} B_{\frac{j}{N}}$…(iii).

Then, clearly E = $E_\infty \cup (\cup_{j=0}^{\infty} E_j)$, and this union is disjoint modulo null set.

Thus, $\nu(E) = \nu(E_\infty) + \sum_{j=0}^{\infty} \nu(E_j)$...(iv).

Since $E_j \subseteq B_{\frac{j+1}{N}} \setminus B_{\frac{j}{N}} = B_{\frac{j+1}{N}} \cap A_{\frac{j}{N}}$, we have $\frac{j}{N} \leq f(x) \leq \frac{j+1}{N}$ on E_j.

Hence on Integration, $\frac{j}{N}\mu(E_j) \leq \int_{E_j} f\, d\mu \leq \frac{j+1}{N}\, \mu(E_j)$…(v)

Since $E_j \subseteq A_{\frac{j}{N}}$ we have $\left(\nu - \frac{j}{N}\mu\right) E_j \geq 0$ ...(vi).

Also, $E_j \subseteq B_{\frac{j+1}{N}} \Rightarrow \left(\nu - \frac{j+1}{N}\mu\right) E_j \leq 0$…(vii).

From (vi) and (vii) $\frac{j}{N}\mu(E_j) \leq \nu(E_j) \leq \frac{j+1}{N}\, \mu(E_j)$…(viii).

From (v) and (viii) we get

$\nu(E_j) - \frac{1}{N}\mu(E_j) \leq \frac{j}{N}\mu(E_j) \leq \int_{E_j} f\, d\mu \leq \frac{j+1}{N}\, \mu(E_j) \leq \nu(E_j) + \frac{1}{N}\mu(E_j)$.

Ie. $\nu(E_j) - \frac{1}{N}\mu(E_j) \leq \int_{E_j} f\, d\mu \leq \nu(E_j) + \frac{1}{N}\mu(E_j)$… (ix).

Now taking the sum from j = 0 to ∞,

$\sum_{j=0}^{\infty} \nu(E_j) - \frac{1}{N}\sum_{j=0}^{\infty} \mu(E_j) \leq \sum_{j=0}^{\infty} \int_{E_j} f\, d\mu \leq \sum_{j=0}^{\infty} \nu(E_j) + \frac{1}{N}\sum_{j=0}^{\infty} \mu(E_j)$

Since $\sum_{j=0}^{\infty} \mu(E_j) = \mu\left(\cup_{j=0}^{\infty} E_j\right)$ and $\sum_{j=0}^{\infty} \int_{E_j} f\, d\mu = \int_{\cup_{j=0}^{\infty} E_j} f\, d\mu$, we get

$\sum_{j=0}^{\infty} \nu(E_j) - \frac{1}{N}\mu\left(\cup_{j=0}^{\infty} E_j\right) \leq \int_{\cup_{j=0}^{\infty} E_j} f\, d\mu \leq \sum_{j=0}^{\infty} \nu(E_j) + \frac{1}{N}\mu\left(\cup_{j=0}^{\infty} E_j\right)$...(x)

On E_∞, from the definition of f, f(x) = ∞ a. e. If $\mu(E_\infty) > 0$, then $\nu E_\infty = \infty$ as $(\nu - \alpha\mu)E_\infty > 0\ \forall\ \alpha \in Q$.

If $\mu(E_\infty) = 0$, then since $\nu \ll \mu$, $\nu E_\infty = 0$. If $\mu(E_\infty) = 0$, then $\int_{E_\infty} f\, d\mu = 0$.

Hence in either case $\nu(E_\infty) = \int_{E_\infty} f\, d\mu$ … (xi).

On adding (x) and (xi) we get,

$\nu(E_\infty) + \sum_{j=0}^{\infty} \nu(E_j) - \frac{1}{N}\mu\left(\cup_{j=0}^{\infty} E_j\right) \leq \int_{E_\infty} f\, d\mu \cup \int_{\cup_{j=0}^{\infty} E_j} f\, d\mu \leq \nu(E_\infty) + \sum_{j=0}^{\infty} \nu(E_j) + \frac{1}{N}\mu\left(\cup_{j=0}^{\infty} E_j\right)$

Hence $\nu(E) - \frac{1}{N}\mu(E) \leq \int_{E} f\, d\mu \leq \nu(E) + \frac{1}{N}\, \mu(E)$

$\Rightarrow \left|\nu(E) - \int_E f\, d\mu\right| \leq \frac{1}{N}\, \mu(E)\ \forall$ N.

Since μE is finite as $N \to \infty$, $\nu(E) = \int_E f\, d\mu$.

Let μ be σ - finite. Then $\exists$ measurable X_i, i = 1, 2, …∋ $X = \cup X_i$, $\mu X_i < \infty\ \forall$ i. Apply the above argument for each X_i to get the required function.

Uniqueness: Let g be any non – negative measurable function such that $\nu(E) = \int_E g\, d\mu\ \forall\ E \in \mathfrak{B}$.

Define $A_n = \{x \in X: f(x) - g(x) \geq \frac{1}{n}\} \in \mathfrak{B}$ and $B_n = \{x \in X: g(x) - f(x) \geq \frac{1}{n}\} \in \mathfrak{B}$. Since $f(x) - g(x) \geq \frac{1}{n}\ \forall\ x \in A_n$, $\int_{A_n} (f-g)\, d\mu \geq \frac{1}{n}\mu(A_n)$ by first mean value theorem.

$\Rightarrow \int_{A_n} f\, d\mu - \int_{A_n} g\, d\mu \geq \frac{1}{n}\mu(A_n)$

$\Rightarrow \nu(A_n) - \nu(A_n) \geq \frac{1}{n}\mu(A_n)$

$\Rightarrow \mu(A_n) \leq 0 \Rightarrow \mu(A_n) = 0$.

Let $C = \{x \in X: f(x) \neq g(x)\}$

$\quad = \cup (A_n \cup B_n)$

$\therefore \mu C = \Sigma\{\mu(A_n) + \mu(B_n)\} = \Sigma(0 + 0) = 0$. $\therefore \mu C = 0$. Hence f = g a. e.

Short Proof: First assume that $(X, \mathfrak{B}, \mu)$ is finite. For each rational α, $\nu - \alpha\mu$ is a signed measure. Let (A_α, B_α) be a Hahn decomposition of $\nu - \alpha\mu$ for each $\alpha \in Q$. and take $A_0 = X$, $B_0 = \phi$.

For each $\alpha \in Q$, $B_\alpha \setminus B_\beta = B_\alpha \cap A_\beta$ so that $(\nu - \alpha\mu)(B_\alpha \setminus B_\beta) \leq 0$ and hence $(\nu - \beta\mu)(B_\alpha \setminus B_\beta) \geq 0$…(i).

If $\beta > \alpha$ then, these imply $\mu(B_\alpha \cap A_\beta) = 0$. $\therefore \exists$ a measurable function f ∋ $f \leq \alpha$ a. e. on B_α and $f \geq \alpha$ a. e. on B_β' ie. on A_β.

Since $B_0 = \phi$, we may take f to be non – negative.

Let E ∈ B. Define $E_j = E \cap \left(B_{\frac{j+1}{N}} \setminus B_{\frac{j}{N}}\right)$ where N is a fixed positive integer … (ii),

$E_\infty = E \setminus \bigcup_{j=1}^{\infty} B_{\frac{j}{N}}$…(iii).

Then, clearly $E = E_\infty \cup (\bigcup_{j=0}^{\infty} E_j)$, and this union is disjoint modulo null set.

Thus, $\nu(E) = \nu(E_\infty) + \sum_{j=0}^{\infty} \nu(E_j)$...(iv).

Since $E_j \subseteq B_{\frac{j+1}{N}} \cap A_{\frac{j}{N}}$, we have $\frac{j}{N} \leq f(x) \leq \frac{j+1}{N}$ on E_j.

Hence on Integration, $\frac{j}{N}\mu(E_j) \leq \int_{E_j} f\, d\mu \leq \frac{j+1}{N}\, \mu(E_j)$…(v)

Since $\frac{j}{N}\mu(E_j) \leq \nu(E_j) \leq \frac{j+1}{N}\, \mu(E_j$, we get

$\nu(E_j) - \frac{1}{N}\mu(E_j) \leq \int_{E_j} f\, d\mu \leq \nu(E_j) + \frac{1}{N}\mu(E_j).$

Now taking the sum from j = 0 to ∞,

$\sum_{j=0}^{\infty} \nu(E_j) - \frac{1}{N}\sum_{j=0}^{\infty} \mu(E_j) \leq \sum_{j=0}^{\infty} \int_{E_j} f\, d\mu \leq \sum_{j=0}^{\infty} \nu(E_j) + \frac{1}{N}\sum_{j=0}^{\infty} \mu(E_j)$

$\Rightarrow \sum_{j=0}^{\infty} \nu(E_j) - \frac{1}{N}\mu(\cup_{j=0}^{\infty} E_j) \leq \int_{\cup_{j=0}^{\infty} E_j} f\, d\mu \leq \sum_{j=0}^{\infty} \nu(E_j) + \frac{1}{N}\mu(\cup_{j=0}^{\infty} E_j)$

On E_∞, we have, f(x) = ∞ a. e.

If $\mu(E_\infty) > 0$, then $\nu E_\infty = \infty$ as $(\nu - \alpha\mu)E_\infty > 0$ ∀ α ∈ Q.

If $\mu(E_\infty) = 0$, then since $\nu \ll \mu$, $\nu E_\infty = 0$. If $\mu(E_\infty) = 0$, then $\int_{E_\infty} f\, d\mu = 0$.

Hence in either case $\nu(E_\infty) = \int_{E_\infty} f\, d\mu$.

On adding this equality and our previous inequalities we get,

$\nu(E) - \frac{1}{N}\mu(E) \leq \int_E f\, d\mu \leq \nu(E) + \frac{1}{N}\mu(E)$

$\Rightarrow \left|\nu(E) - \int_E f\, d\mu\right| \leq \frac{1}{N}\mu(E)$ ∀ N.

Since μE is finite as N → ∞, $\nu(E) = \int_E f\, d\mu$.

Let μ be σ - finite. Then ∃ measurable X_i, i = 1, 2, …∍ X = ∪ X_i, $\mu X_i < \infty$ ∀ i. Apply the above argument for each X_i to get the required function.

Uniqueness: Let g be any measurable function such that $\nu(E) = \int_E g\, d\mu$ ∀ E ∈ 𝔅.

Define $A_n = \{x \in X: f(x) - g(x) \geq \frac{1}{n}\} \in$ 𝔅 and $B_n = \{x \in X: g(x) - f(x) \geq \frac{1}{n}\} \in$ 𝔅. Since $f(x) - g(x) \geq \frac{1}{n}$ ∀ x ∈ A_n, $\int_{A_n} (f - g)\, d\mu \geq \frac{1}{n}\mu(A_n)$ by first mean value theorem.

$\Rightarrow \int_{A_n} f\, d\mu - \int_{A_n} g\, d\mu \geq \frac{1}{n}\mu(A_n)$

$\Rightarrow \nu(A_n) - \nu(A_n) \geq \frac{1}{n}\mu(A_n)$

$\Rightarrow \mu(A_n) \leq 0 \Rightarrow \mu(A_n) = 0$.

Let C = {x ∈ X: f(x) ≠ g(x)}

$= \cup (A_n \cup B_n)$

$\therefore \mu C = \Sigma\{\mu(A_n) + \mu(B_n)\} = \Sigma(0 + 0) = 0$. ∴ μC = 0. Hence f = g a. e.

Note: The function f given by Radon–Nikodym Theorem is called the Radon–Nikodym derivative of ν with respect to μ. It is sometimes denoted by $\left[\frac{d\nu}{d\mu}\right]$.

Proposition (Lebesgue Decomposition): Let (X, 𝔅, μ) be a σ-finite measure space and ν a σ-finite measure defined on 𝔅. Then we can find a measure ν_0, singular with respect to μ, and a measure ν_1, absolutely continuous with respect to μ, such that $\nu = \nu_0 + \nu_1$. The measures ν_0 and ν_1 are unique.

Proof: Since μ and ν are σ - finite measures, so is the measure $\lambda = \mu + \nu$.
Let $E \in \mathfrak{B} \ni \lambda E = 0$.
Then $\mu E + \nu E = 0$ so that $\mu E = 0$ and $\nu E = 0$ since μ, ν are nonnegative.
$\therefore \nu \ll \lambda$ and $\mu \ll \lambda$.
By Radon – Nikodym theorem $\exists$ non negative measurable functions f and g such that
$\nu(E) = \int_E g \, d\lambda$ and $\mu(E) = \int_E f \, d\lambda$.
Define $A = \{x \in X: f(x) > 0\}$ and $B = \{x \in X: f(x) = 0\}$.
Then clearly $X = A \cup B$ and $A \cap B = \phi$.
Also $\mu B = \int_B f \, d\lambda = 0$.
Define $\nu_0: \mathfrak{B} \to [0, \infty) \cup \{\infty\}$ by $\nu_0(E) = \nu(E \cap B) \ \forall \ E \in \mathfrak{B}$.
and $\nu_1: \mathfrak{B} \to [0, \infty) \cup \{\infty\}$ by $\nu_1(E) = \nu(E \cap A) \ \forall \ E \in \mathfrak{B}$.
Then clearly ν_0, and ν_1 are measures.
Then $\nu_0(A) = \nu(A \cap B) = \nu(\phi) = 0$.
Thus $\exists$ disjoint measurable sets A and B with $X = A \cup B \ni \mu(B) = \nu_0(A) = 0$.
$\therefore \nu_0 \perp \mu$.
Also $\nu = \nu_0 + \nu_1$.
Let $E \in \mathfrak{B} \ni \mu E = 0$. $\therefore 0 = \mu E = \int_E f \, d\lambda. \Rightarrow f = 0$ a. e. on E w. r. t. λ.
$\therefore \lambda\{x \in X: f(x) > 0\} = 0$.
Then $\lambda(E \cap A) = 0$.
Since $\nu \ll \lambda$ it follows that $\nu_1(E) = \nu(E \cap A) = 0$.
Thus, $\mu(E) = 0 \Rightarrow \nu_1(E) = 0$.
$\therefore \nu_1 \ll \mu$.
Uniqueness: Suppose $\nu = \nu_0 + \nu_1$ and $\nu = \nu'_0 + \nu'_1$ where $\nu_0 \perp \mu$, $\nu'_0 \perp \mu$, $\nu_1 \ll \mu$, $\nu'_1 \ll \mu$.
So, $\exists$ A, B, A′, B′ $\ni X = A \cup B = A' \cup B'$, $A \cap B = A' \cap B' = \phi$ and $\nu_0(A) = \mu(B) = \nu'_0(A') = \mu(B') = 0$.
Let $E \in \mathfrak{B}$.
Then, $E = (E \cap A \cap A') \cup (E \cap A' \cap B) \cup (E \cap B \cap B') \cup (E \cap A \cap B')$.
Clearly μ is zero on the last three sets in this union and hence ν_1 and ν'_1 are zero by absolute continuity.
Since $\nu'_1 - \nu_1 = \nu_0 - \nu'_0$ we have $(\nu'_1 - \nu_1)(E) = (\nu'_1 - \nu_1)(E \cap A \cap A')$
$= (\nu_0 - \nu'_0)(E \cap A \cap A') = 0$ as $\nu_0(A) = \nu'_0(A') = 0$.
So, $\nu_1(E) = \nu'_1(E)$ which implies $\nu_0(E) = \nu'_0(E)$. Hence the theorem.

Problem: Show that if γ is a signed measure such that γ is mutually singular w. r. t μ and $\gamma \ll \mu$ then γ = 0.

Solution: Since $\gamma \perp \mu$, ∃ a measurable set A such that $\mu A = \gamma(\tilde{A}) = 0$.

Let E be any measurable set.

Then $E = (E \cap A) \cup (E \cap \tilde{A})$

$\Rightarrow \gamma E = \gamma(E \cap A) + \gamma(E \cap \tilde{A}) = \gamma(E \cap A) + 0 \because \gamma(\tilde{A}) = 0$.

$\Rightarrow \gamma E = \gamma(E \cap A)$ … (i).

Since $E \cap A \subseteq A$, $\mu(E \cap A) \le \mu(A) = 0$

$\Rightarrow \mu(E \cap A) = 0$.

Since $\gamma \ll \mu$, $\gamma(E \cap A) = 0$

$\Rightarrow \gamma E = 0$ from (i).

Thus, $\gamma E = 0\ \forall\ E \in \mathfrak{B}$.

$\Rightarrow \gamma = 0$.

The L^p Spaces.

Definition: Let p, $1 \le p < \infty$ be a real number. We define

$L^p = L^p(X,\mu) = \left\{ f \,/\, f: X \to \mathbb{R} \text{ is measurable and } \int_X |f|^p d\mu < \infty \right\}$.

Definition: For a function $f \in L^p(X,\mu)$, define $\|f\|_p = \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}$.

Definition: Two measurable functions f, g are said to be equivalent if there are equal almost everywhere. Ie. f ~ g iff f = g a. e.

Definition: A real number M is said to be an *essential bound* for the function f if $|f(x)| \le M\ a.e.$ on X. A function f defined on X is *essentially bounded* if it is bounded except possibly on a set of measure zero.

Essential supremum of f on X is defined as inf{M: m({x∈E: $|f(x)| > M$}) = 0}.

We denote the class of all measurable functions defined on X which are essentially bounded on X by $L^\infty(X,\mu)$. For f $\in L^\infty(X,\mu)$ we define $\|f\|_\infty$ = ess sup $|f|$.

THE MINKOWSKI AND HOLDER INEQUALITIES

Lemma: Let α, β be non – negative real numbers and 0 < λ < 1. Then $\alpha^\lambda \beta^{1-\lambda} \le \lambda\alpha + (1-\lambda)\beta$ with equality if α = β.

Proof: Define ϕ as $\phi(t) = (1 - \lambda) + \lambda t - t^\lambda$ for all real numbers t.

Then $\phi(1) = 1 - \lambda + \lambda - \mathbf{1} = \mathbf{0}$.

Also $\phi'(t) = \lambda - \lambda t^{\lambda - 1} = \lambda(1 - t^{\lambda - 1})$

$\phi''(t) = -\lambda(\lambda - 1)t^{\lambda - 2}$.

And $\phi'(t) = 0$ iff $t = 1$ and $\phi''(1) = -\lambda(\lambda - 1) > 0$.
∴ ϕ has local minimum at $t = 1$.
∴ $t < 1 \Rightarrow \phi$ is decreasing. Ie. $\phi(t) > \phi(1)$ ant $t > 1 \Rightarrow \phi$ is increasing ie. $\phi(t) > \phi(1)$.
Thus $t \neq 1 \Rightarrow \phi(t) > \phi(1) \Rightarrow (1 - \lambda) + \lambda t - t^{\lambda} > 0 \Rightarrow t^{\lambda} < (1 - \lambda) + \lambda t$
∴ we may say that $t^{\lambda} \leq (1 - \lambda) + \lambda t$ for all t and with equality if t = 1...(i)
If $\beta \neq 0$ put $t = \alpha / \beta$ in (i).
Then $\left(\frac{\alpha}{\beta}\right)^{\lambda} \leq 1 - \lambda + \lambda\left(\frac{\alpha}{\beta}\right)$
$\Rightarrow \frac{\alpha^{\lambda}}{\beta^{\lambda}} \leq (1 - \lambda) + \frac{\lambda\alpha}{\beta}$
$\Rightarrow \alpha^{\lambda}\beta^{1-\lambda} \leq \lambda\alpha + (1 - \lambda)\beta$ with equality if $\alpha = \beta$.

HOLDER'S INEQUALITY:

If p and q are non – negative extended real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and if $f \in L^p$, $g \in L^q$, then $fg \in L^1$ and $\int|fg| \leq \|f\|_p \|g\|_q$ equality holds iff for some non – zero constants α and β, we have $\alpha|f|^p = \beta|g|^q$ a.e.

Proof: If p = 1, q = ∞, then the in equality holds. So, assume that $1 < p < \infty$. First assume that $\|f\|_p = 1 = \|g\|_q$

Take $\alpha = |f(t)|^p, \beta = |g(t)|^q$ and $\lambda = \frac{1}{p}, 1 - \lambda = \frac{1}{q}$.

Then by Lemma we get $|f(t)||g(t)| \leq \frac{1}{p}|f(t)|^p + \frac{1}{q}|g(t)|^q$ and equality holds if $\alpha = \beta$ ie. $|f(t)|^p = |g(t)|^q$...(i)

$\Rightarrow \int|fg| \leq \frac{1}{p}\int|f(t)|^p + \frac{1}{q}\int|g(t)|^q = \frac{1}{p}\|f\|^p + \frac{1}{q}\|g\|^q = \frac{1}{p} + \frac{1}{q} = 1$

Ie. $\int|fg| \leq 1 = \|f\|_p\|g\|_q$

Let $f \in L^p$, $g \in L^q$. Now if $\|f\| = 0$ or $\|g\| = 0$ then the inequality is obvious. Assume that $\|f\| \neq 0$ and $\|g\| \neq 0$. Then $\frac{f}{\|f\|} \in L^p, \frac{g}{\|g\|} \in L^q$.

Also $\left\|\frac{f}{\|f\|}\right\| = 1$ and $\left\|\frac{g}{\|g\|}\right\| = 1$.

So by the above case $\int\left|\frac{f}{\|f\|}\frac{g}{\|g\|}\right| \leq 1$ and equality holds iff $\left|\frac{f}{\|f\|}\right|^p = \left|\frac{g}{\|g\|}\right|^q$ iff $\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q}$ iff $\|g\|_q^q|f|^p = \|f\|_p^p|g|^q$ a. e. ... (ii).

Now $\int\left|\frac{f}{\|f\|}\frac{g}{\|g\|}\right| \leq 1 \Rightarrow \frac{1}{\|f\|\|g\|}\int|fg| \leq 1 \Rightarrow \int|fg| \leq \|f\|_p\|g\|_q$

Also equality holds iff $\|g\|_q^q|f|^p = \|f\|_p^p|g|^q$ a. e. ie equality holds iff for some non – zero constants $\alpha = \|g\|_q^q$ and $\beta = \|f\|_p^p$, we have $\alpha|f|^p = \beta|g|^q$ a.e.

MINKOWSKI'S INEQUALITY:

If f, g ∈ L^p with $1 \leq p \leq \infty$, then f + g ∈ L^p and $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

Proof: Let f, g ∈ L^p with $1 \leq p \leq \infty$. Then f + g ∈ L^p since Lp is linear.

Now $\|f+g\|_1 = \int |f+g|dx$
$\leq \int (|f|+|g|)dx$
$= \int |f|\,dx + \int |g|dx$
$= \|f\|_1 + \|g\|_1$

Also $\|f+g\|_\infty$ = ess sup $|(f+g)(t)|$
$\leq$ ess sup $|f(t)|$ + ess sup $|g(t)|$
$= \|f\|_\infty + \|g\|_\infty$

So assume that 1 < p < ∞. Let q be the real number such that $\frac{1}{p}+\frac{1}{q}=1$.

Now $|f+g|^p = |f+g|^{p-1}\cdot|f+g|$
$\leq |f+g|^{p-1}\cdot(|f|+|g|)$
$= |f+g|^{p-1}\cdot|f| + |f+g|^{p-1}\cdot|g|$ … (i)

Claim: $|f+g|^{p-1} \in L^q$

Now $(|f+g|^{p-1})^q = |f+g|^{(p-1)q} = |f+g|^p$

Since f + g ∈ L^p, we have $\int |f+g|^p < \infty$.

Now $\int (|f+g|^{p-1})^q = \int |f+g|^p < \infty$.

So we have $|f+g|^{p-1} \in L^q$. Since f, g ∈ L^p and $|f+g|^{p-1} \in L^q$, we have by Holder's inequality, $\int |f||f+g|^{p-1} \leq \|f\|_p \|(f+g)^{p-1}\|_q$.

$$\int |g||f+g|^{p-1} \leq \|g\|_p \|(f+g)^{p-1}\|_q$$

But $\|(f+g)^{p-1}\|_q = \{\int (|f+g|^{p-1})^q\}^{\frac{1}{q}} = \left(\int |f+g|^{(p-1)q}\right)^{\frac{1}{q}} = \left(\int |f+g|^p\right)^{\frac{1}{q}} = \left\{\left(\int |f+g|^p\right)^{\frac{1}{p}}\right\}^{\frac{p}{q}} = \|f+g\|_p^{\frac{p}{q}}$

$\therefore \int |f||f+g|^{p-1} \leq \|f\|_p \|f+g\|_p^{\frac{p}{q}}$ and

$\int |g||f+g|^{p-1} \leq \|g\|_p \|f+g\|_p^{\frac{p}{q}}$…(ii).

From (i) and (ii), $\int |f+g|^p \leq \|f\|_p \|f+g\|_p^{\frac{p}{q}} + \|g\|_p \|f+g\|_p^{\frac{p}{q}}$

$\Rightarrow \int |f+g|^p \leq (\|f\|_p + \|g\|_p)\cdot\|f+g\|_p^{\frac{p}{q}}$

$\Rightarrow \|f+g\|_p^p \leq (\|f\|_p + \|g\|_p)\cdot\|f+g\|_p^{\frac{p}{q}}$

$\Rightarrow \|f+g\|_p^{p-\frac{p}{q}} \leq \|f\|_p + \|g\|_p$

$\Rightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p$ since $p - \frac{p}{q} = p\left(1-\frac{1}{q}\right) = p\left(\frac{1}{p}\right) = 1$.

<u>Definition</u>: A bounded linear functional on L^p is a linear map from L^p to R which is also continuous. $(L^p)^*$ = {x*: x* is a bounded linear functional on L^p}.

<u>Lemma</u>: Let f ∈ L^p (μ), 1 ≤ p < ∞. Then for any ε > 0, there is a simple function φ which vanishes outside a set of finite measure such that $\int |f-\varphi|^p d\mu$ < ε or $\|f-\varphi\|_p$ < ε.

Proof: W. l. g. suppose μX < ∞.
If possible suppose ∃ ε > 0 such that for any simple function,
$\varepsilon \le \|f-\varphi\|_p \le \|f-\varphi\|_p \frac{\mu(X)}{\mu(X)} \le \left\{\int_X |f-\varphi| d\mu\right\} \frac{1}{\mu(X)}$ … (i).
There exists a sequence of simple functions such that $\int f\, d\mu = \lim \int \varphi_n\, d\mu$. We can find φ_k such that $\int |f-\varphi_k|\, d\mu < \varepsilon\, \mu(X)$…(ii)
From (i) and (ii) we get ε < ε which is a contradiction. Hence $\|f-\varphi\|_p < \varepsilon$.

Proposition: Let g ∈ L^q, $\frac{1}{p}+\frac{1}{q}=1$, Define F(f) = $\int_X fg\, d\mu$ for all f ∈ L^p. Then F ∈ $(L^p)^*$.

Proposition: Let (X, 𝔅, μ) be a finite measure space and g be an integrable function such that for some constant M, $\left|\int g\varphi\, d\mu\right| \le M$ for all simple functions φ, Then g ∈ L^q.
Proof: Since $|g|^q$ is a non – negative measurable function, ∃ a sequence of non – negative measurable simple functions $\{\psi_n\} \ni \psi_n \uparrow |g|^q$.
Then the function $\varphi_n = \{\psi_n\}^{\frac{1}{p}}$(sgn g)… (i) is a well defined simple function for each n. Also, $\int_X |\varphi_n(x)|^p d\mu = \int \psi_n(x) d\mu$. Thus, $\|\varphi_n\|_p = \{\int \psi_n d\mu\}^{\frac{1}{p}}$. Since $\varphi_n g \ge |\varphi_n||\psi_n|^{\frac{1}{q}} = |\psi_n|^{\frac{1}{p}+\frac{1}{q}} = \psi_n$, we have
$\int \psi_n d\mu \le \int \varphi_n g\, d\mu \le M\|\varphi_n\|_p = M\{\int \psi_n d\mu\}^{\frac{1}{p}}$.
Since $1-\frac{1}{p}=\frac{1}{q}$, $\{\int \psi_n d\mu\}^{\frac{1}{p}} \le M$ or $\int \psi_n d\mu \le M^p$ and by Monotone convergence theorem, $\int |g|^q d\mu \le M^p$ ie. g ∈ L^q.

Proposition: Let $\{E_n\}$ be a sequence of disjoint measurable sets and for each n, let f_n be a function in L^p, 1 ≤ p < ∞, that f_n = 0 on E'_n.
Set f =$\sum_{n=1}^{\infty} f_n$. Then f ∈ L^p if and only if $\sum \|f_n\|_p^p < \infty$.
In this case, f = $\sum_n f_n$ in L^p that is $\|f-\sum_{i=1}^{n} f_i\|_p \to 0$, and $\|f\|^p = \sum_{n=1}^{\infty} \|f_n\|^p$.

Riesz Representation Theorem: Let f be a bounded linear functional on $L^p(\mu)$ with 1 ≤ p < ∞ and μ a σ - finite measure. Then there is a unique g in L^p, where $\frac{1}{p}+\frac{1}{q}=1$, such that F(f) = $\int fg\, d\mu$. Also, $\|F\| = \|g\|_q$.
Proof: Step (i): Let μ be finite measure.
Since f is bounded and measurable f ∈ L^p.
For any E ∈ 𝔅, E ⊆ X, define ν(E) = F(χ_E) where χ_E is the characteristic function of E.
Claim: ν is a signed measure.
Clearly ν(ϕ) = F(χ_ϕ) = 0.

Let $\{E_n\}$ be pairwise disjoint sequence of m'ble sets and E = $\bigcup_{n=1}^{\infty} E_n$
Then $|\nu(E_n)| = |F(\chi_{E_n})|$ = sgn $\{F(\chi_{E_n})\}\, F(\chi_{E_n})$
$\therefore \sum_{n=1}^{\infty}|\nu(E_n)| = \sum_{n=1}^{\infty} sgn[\{F(\chi_{E_n})\}\, F(\chi_{E_n})] = \sum_{n=1}^{\infty} F\{sgn\, F(\chi_{E_n})\}\, \chi_{E_n}$.
Write $f_n = sgn\, F(\chi_{E_n})\chi_{E_n}$.
Then $f_n = 0$ outside E_n. Set f = $\sum f_n$.
Then by proposition $f_n \in L^p$ and f $\in L^p$, $\|f\|^p = \sum_{n=1}^{\infty}\|f_n\|^p < \infty$.
Thus, $|F(f)| \leq \|F\|\|f\|_p < \infty$.
So, $\sum_{n=1}^{\infty}|\nu(E_n)| < \infty$.
Now $\sum_{n=1}^{\infty} \nu(E_n) = \sum_{n=1}^{\infty} F(\chi_{E_n}) = F\sum_{n=1}^{\infty} \chi_{E_n} = F(\chi_{\cup E_n}) = F(\chi_E) = \nu(E) = \nu(\bigcup_{n=1}^{\infty} E_n)$
Thus, ν is countably additive.
Hence ν is a signed measure.
Claim: ν ≪ μ.
Let μ(E) = 0.
Then 0 = μ(E) = $\int_E \chi_E \, d\mu$.
Hence $\chi_E = 0$ on E.
Thus, F(χ_E) = 0 and hence ν(E) = 0. ∴ ν ≪ μ.
∴ By Radon-Nikodym theorem, ∃ a measurable function g such that for any E ∈ 𝔅, $\int_E g\, d\mu$ = ν(E) < ∞. ∴ g is integrable.
For any f ∈ L^p, G(f) = $\int fg\, d\mu$.
Claim: G is a bounded linear functional on L^p.
For any simple function φ, F(φ) = $F\left(\sum_{i=1}^{k} c_i\chi_{E_i}\right) = \sum_{i=1}^{k} c_i F(\chi_{E_i})$
$= \sum_{i=1}^{k} c_i \nu(E_i) = \sum_{i=1}^{k} c_i \int_{E_i} g\, d\mu = \sum_{i=1}^{k} \int_{E_i} c_i g\, d\mu = \int \varphi g\, d\mu$ … (i).
Now, $\left|\int \varphi g\, d\mu\right| = |F(\varphi)| \leq \|F\|\|\varphi\|_p$ since F ∈ $(L^p)^*$.
Hence by a Proposition, g ∈ L^q.
Now $|G(f)| = \left|\int \varphi g\, d\mu\right| \leq \|F\|_p\|G\|_q$
Hence G ∈ $(L^p)^*$.
Next for any simple function φ ∈ L^p, G(φ) = $\int \varphi g\, d\mu$ = F(φ) from (i)
Hence F – G = 0 on the set of all simple functions. ∴ F = G on L^p.
∴ F(f) = G(f) = $\int fg\, d\mu$.
Uniqueness of g:
Let g_1, g_2 be ∍ $g_1 \in L^q$, $g_2 \in L^q$ and F(f) = $\int fg_1\, d\mu = \int fg_2\, d\mu$ ∀ f ∈ L^p.
⇒ $\int f(g_1 - g_2)\, d\mu = 0$ ie. $g_1 - g_2$ corresponds to zero functional. Hence $\|g_1 - g_2\| = 0$ which shows that $g_1 = g_2$ a.e.
Step (ii): Let μ be σ - finite, so that X = $\bigcup_{n=1}^{\infty} X_n$, $\mu(X_n) < \infty$ for each n.
∴ For each n ∃ $g_n \in L^q(X_n)$ ∍ $F_n(f) = \int_{X_n} fg_n\, d\mu,\ n\epsilon\mathbb{Z}, f \in L^p(X_n)$.
Take $g_n = 0$ outside X_n as also f.
Since each $F_n = F/X_nL^p(X_n)$, clearly $\|F_n\|_{X_n} = \|F\|$ ∀ n and hence $\|g_n\|_q \leq \|F\|$ ∀ n. … (ii). Further assume $g_{n+1} = g_n$ on X_n.

Define $g(x) = g_n(x)\ \forall\ x \in X_n$.
Take X_n to be increasing, so g is well defined and $|g_n| \uparrow\ |g|$.
By Monotone convergence theorem, $\int |g|^q\, d\mu = \lim \int |g_n|^q\, d\mu \leq \|F\|^q$ by (ii)
Hence $g \in L^q(X)$. We now obtain any $f \in L^p(X)$ as a limit of a sequence of function in $L^p(X_n)$. Define $f_n(x) = \begin{cases} f(x)\ if\ x\ \in\ X_n \\ \ 0\ if\ x\ \notin\ X_n \end{cases}$
Then $x \in X \Rightarrow x \in X_n$ for some n and $f(x) = f_k(x)$ for $k \geq n$.
$\therefore\ f_n \to f$ pointwise. Also, $\lim\limits_{n} \int\limits_{X} |f_n - f|^p\, d\mu = \int\limits_{X} \lim|f_n - f|\, d\mu = 0$
Hence $f_n \to f$ in $L^p(X)$. For any n, $|f_n g| \leq |fg|$ and $|fg|$ is integrable.
$\therefore\ f_n g \to fg$.
By Lebesgue convergence theorem, $\int fg\, d\mu = \lim \int f_n g_n\, d\mu = \lim \int f_n g\, d\mu$
$= \lim F(f_n) = F(f)$. Hence the theorem.

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E – CONTENT

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M 401: MEASURE THEORY: UNIT III
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MEASURE AND OUTER MEASURE
1. OUTER MEASURE AND MEASURABILITY

Definition: By an outer measure μ^* we mean a nonnegative extended real valued set function defined on all subsets of a space X and having the following properties

(i) $\mu^*(\phi) = 0$

(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

(iii) $E \subseteq \bigcup_{i=1}^{\infty} E_i \Rightarrow \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$

Here property (ii) is called the monotonicity of μ^* and the property (iii) is called the subadditivity of μ^*.

Result: Let μ^* be a nonnegative extended real valued set function defined on all subsets of a space X and having the following properties
(i) $\mu^*(\phi) = 0$
(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$
(iii)′ $E = \bigcup_{i=1}^{\infty} E_i$; E_i are disjoint $\Rightarrow \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$. Then μ^* is an outer measure.
Proof: Let $E \subseteq \bigcup_{i=1}^{\infty} E_i$ where $\{E_i\}$ is a sequence of subsets of X.
Then we can find pairwise disjoint sequence $\{E_i'\}$ such that each $E_i' \subseteq E_i$ and $\bigcup_{i=1}^{\infty} E_i' = \bigcup_{i=1}^{\infty} E_i$. ∴ by (iii)′, $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i') \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.
Hence μ^* is an outer measure.

ie. Property (iii) in the definition of an outer measure can be replaced by
(iii)′ $E = \bigcup_{i=1}^{\infty} E_i$; E_i are disjoint $\Rightarrow \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.

Definition: A set E is said to be measurable with respect to μ^* or μ* - measurable if for every set A we have $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \tilde{E})$.

Theorem: A set E is μ^* measurable if and only if $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap \tilde{E})$ for every set A.
Proof: If E is μ^* measurable, the inequality holds trivially.
Conversely suppose E is a set such that $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap \tilde{E})$ …(i) for every set A.
By sub additivity of μ*, we have $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap \tilde{E})$…(ii), since $A = (A \cap E) \cup (A \cap \tilde{E})$.
Hence from (i) and (ii) $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \tilde{E})$ for every set A showing E is μ*- measurable.
Remark: In view of above theorem, it is only necessary to show the inequality $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap \tilde{E})$ for every set A to prove the measurability of E.

Theorem: The class $\mathfrak{B}$ of μ* - measurable sets is a σ - algebra.

Proof : For any set A, we have $\mu^*(A \cap \phi) + \mu^*(A \cap \tilde{\phi}) = \mu^*(\phi) + \mu^*(A \cap X)$ $= 0 + \mu^*(A) = \mu^*(A)$ so that ϕ is μ* - measurable…(i).
Suppose E is μ* measurable. Then for any set A, $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \tilde{E}) \Rightarrow$ for any set A, $\mu^*(A) = \mu^*\left(A \cap (\widetilde{\tilde{E}})\right) + \mu^*(A \cap \tilde{E}) \Rightarrow \tilde{E}$ is μ* - measurable…(ii)
Suppose E_1, and E_2, are μ*-measurable sets. Then for any set A, we get, by the measurability of E_2 that $\mu^*(A) = \mu^*(A \cap E_2) + \mu^*(A \cap \widetilde{E_2})$…(iii)
If we take $A \cap \widetilde{E_2}$ for A, then measurability of E_1, gives
$\mu^*(A \cap \widetilde{E_2}) = \mu^*(A \cap \widetilde{E_2} \cap E_1) + \mu^*(A \cap \widetilde{E_2} \cap \widetilde{E_1})$.
Substituting this value of $\mu^*(A \cap \widetilde{E_2})$ in (iii),
$\mu^*(A) = \mu^*(A \cap E_2) + \mu^*(A \cap \widetilde{E_2} \cap E_1) + \mu^*(A \cap \widetilde{E_2} \cap \widetilde{E_1})$… (iv)
Since $A \cap (E_1 \cup E_2) = (A \cap E_2) \cup (A \cap \widetilde{E_2} \cap E_1)$ is a disjoint union, $\mu^*\{A \cap (E_1 \cup E_2)\} \leq \mu^*(A \cap E_2) + \mu^*(A \cap \widetilde{E_2} \cap E_1)$… (v).
∴ From (iv) and (v) $\mu^*(A) \geq \mu^*\{A \cap (E_1 \cup E_2)\} + \mu^*(A \cap \widetilde{E_2} \cap \widetilde{E_1})$
$= \mu^*\{A \cap (E_1 \cup E_2)\} + \mu^*(A \cap \widetilde{E_1 \cup E_2})$.
Thus, for any set A, $\mu^*(A) \geq \mu^*\{A \cap (E_1 \cup E_2)\} + \mu^*\{A \cap (\widetilde{E_1 \cup E_2})\}$.
Therefore $E_1 \cup E_2$ is μ* - measurable.
Thus, $E_1 \cup E_2 \in \mathfrak{B}$ if E_1, and $E_2 \in \mathfrak{B}$ ...(v).
By induction this can be extended to any finite number of sets.
From (1) and (2) we get that B is an algebra of sets.

To prove $\mathfrak{B}$ is a σ - algebra let $\{E_i\}$ be a sequence of pairwise disjoint sets in $\mathfrak{B}$ and E $= \bigcup_{i=1}^{\infty} E_i$. Write $G_n = \bigcup_{i=1}^{n} E_i$.
Then for any n, G_n is μ*-measurable set.
Also $G_n \subseteq$ E gives $\tilde{E} \subseteq \widetilde{G_n}$ so that $\mu^*(A) = \mu^*(A \cap G_n) + \mu^*(A \cap \widetilde{G_n}) \geq \mu^*(A \cap G_n) + \mu^*(A \cap \tilde{E})$.
Since $G_n \cap E_n = E_n$ and $G_n \cap \widetilde{E_n} = G_{n-1}$ we get, by the measurability of E_n that $\mu^*(A \cap G_n) = \mu^*(A \cap E_n) + \mu^*(A \cap G_{n-1})$.
Thus, by induction, $\mu^*(A \cap G_n) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$.
∴ for every n, $\mu^*(A) \geq \mu^*(A \cap \tilde{E}) + \sum_{i=1}^{n} \mu^*(A \cap E_i)$.
Since this is true for every n, we have $\mu^*(A) \geq \mu^*(A \cap \tilde{E}) + \sum_{i=1}^{\infty} \mu^*(A \cap E_i)$
$\geq \mu^*(A \cap \tilde{E}) + \mu^*(A \cap E)$ since $A \cap E \subseteq \bigcup_{i=1}^{\infty}(A \cap E_i)$.
Thus, E is measurable.
∴ $\mathfrak{B}$ *is a* σ - algebra.

Theorem: Suppose μ* is an outer measure and $\mathfrak{B}$ is the class of all μ*measurable sets. If $\bar{\mu}$ is the restriction of μ* to $\mathfrak{B}$ (that is, $\bar{\mu}: \mathfrak{B} \rightarrow$ IR is such that $\bar{\mu}(E) = \mu^*(E)$ for $E \in \mathfrak{B}$, then $\bar{\mu}$ is a complete measure on $\mathfrak{B}$.

Proof: Let $\bar{\mu}$ be the restriction of μ* to $\mathfrak{B}$
Clearly $\bar{\mu}$ is nonnegative set function.
Also, $\bar{\mu}(\phi) = \mu^*(\phi) = 0$.

Let E_1 and E_2 be disjoint μ*-measurable sets. Then by the measurability of E_2,
$\bar{\mu}(E_1 \cup E_2) = \mu^*(E_1 \cup E_2) = \mu^*[(E_1 \cup E_2) \cap E_2] + \mu^*[(E_1 \cup E_2) \cap \widetilde{E_2}]$
$= \mu^*(E_2) + \mu^*(E_1) = \bar{\mu}(E_2) + \bar{\mu}(E_1)$.
Thus, by induction, $\bar{\mu}$ is finitely additive set function.
Let $\{E_i\}$ be a sequence of pairwise disjoint measurable sets and $E = \bigcup_{i=1}^{\infty} E_i$.
Then, $\bar{\mu}(E) \geq \bar{\mu}(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \bar{\mu}(E_i)$ for all n.
$\therefore \bar{\mu}(E) \geq \sum_{i=1}^{\infty} \bar{\mu}(E_i)$.
But $\bar{\mu}(E) \leq \sum_{i=1}^{\infty} \bar{\mu}(E_i)$ by the subadditivity of μ*. Hence $\bar{\mu}$ is countably additive and thus $\bar{\mu}$ is a measure.
Let $E \in \mathfrak{B}$ with $\bar{\mu}(E) = 0$ and $A \subseteq E$. Then $0 \leq \mu^*(A) \leq \mu^*(E) = \bar{\mu}(E) = 0$.
$\Rightarrow \mu^*(A) = 0$. $\therefore \mu^*(A \cap E) = 0$. So, $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap \tilde{A})$. Thus, A is μ* - measurable. Ie. $A \in \mathfrak{B}$. Hence $\bar{\mu}$ is complete.

2 THE EXTENSION THEOREM:

<u>Definition</u>: By a **measure on an algebra** we mean a nonnegative extended real valued set function μ defined on an algebra $\mathcal{A}$ of sets such that
(i) μ(ϕ) = 0
(ii) If $\{A_i\}$ is a disjoint sequence of sets in $\mathcal{A}$ whose union is also in $\mathcal{A}$, then
$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

<u>Example</u>: The Lebesgue measure defined on the class of all intervals is a measure on the algebra.

<u>Definition</u>: Suppose μ is a measure on an algebra $\mathcal{A}$.
For any E, define $\mu^*(E) = \inf \{\sum_{i=1}^{\infty} \mu(A_i) : \{A_i\}$ is a sequence of sets in $\mathcal{A}$ such that $E \subseteq \bigcup_{i=1}^{\infty} A_i\}$.

<u>Theorem</u>: Suppose μ is a measure on an algebra $\mathcal{A}$ and $\mu^*(E) = \inf \{\sum_{i=1}^{\infty} \mu(A_i) : E \subseteq \bigcup_{i=1}^{\infty} A_i\}$. Then (i) if $A \in \mathcal{A}$ and $\{A_i\}$ is any sequence of sets in $\mathcal{A}$ such that $A \subseteq \bigcup_{i=1}^{\infty} A_i$ then $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$. (ii) $A \in \mathcal{A} \Rightarrow \mu^*(A) = \mu(A)$.
(iii) μ^* is an outer measure. (iv) Each $A \in \mathcal{A}$ is μ^*- measurable.
<u>Proof</u>: (i) Let $A \in \mathcal{A}$ and $\{A_i\}$ be any sequence of sets in $\mathcal{A}$ such that $A \subseteq \bigcup_{i=1}^{\infty} A_i$
Let $B_i = A \cap A_i \cap \tilde{A}_{i-1} \cap \ldots \cap \tilde{A}_1$. Then $B_i \in \mathcal{A}$ and $B_i \subseteq A_i$. Also $A = \bigcup_{i=1}^{\infty} B_i$.
$\therefore$By the countable additivity of μ, $\mu A = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$
Ie. $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$.
(ii) Let $A \in \mathcal{A}$. If $\{A_i\}$ is any sequence of sets in $\mathcal{A} \ni A \subseteq \bigcup_{i=1}^{\infty} A_i$ then by (i), $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$. $\Rightarrow \mu(A)$ is a lower bound of $\{\sum_{i=1}^{\infty} \mu(A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i\}$. $\therefore \mu(A) \leq \mu^*(A)$
Since $A \in \mathcal{A}$, A is a cover for A, so $\mu(A) \geq \mu^*(A)$. Hence $\mu(A) = \mu^*(A)$.
(iii) Since μ is nonnegative extended real valued set function so is μ^*.
Since $\phi \in \mathcal{A}$, by (ii) $\mu^*(\phi) = \mu(\phi) = 0$.

Let E ⊆ F. Then for every sequence $\{A_i\}$ in $\mathcal{A}$ with F ⊆ $\bigcup_{i=1}^{\infty} A_i$ we have E ⊆ $\bigcup_{i=1}^{\infty} A_i$.
Then $\{\sum_{i=1}^{\infty} \mu(A_i): E \subseteq \bigcup_{i=1}^{\infty} A_i\} \supseteq \{\sum_{i=1}^{\infty} \mu(A_i): F \subseteq \bigcup_{i=1}^{\infty} A_i\}$ so that
Inf $\{\sum_{i=1}^{\infty} \mu(A_i): E \subseteq \bigcup_{i=1}^{\infty} A_i\} \leq$ Inf $\{\sum_{i=1}^{\infty} \mu(A_i): F \subseteq \bigcup_{i=1}^{\infty} A_i\}$.
∴ E ⊆ F ⇒ μ^*(E) ≤ μ^*(F).
Let $E \subseteq \bigcup_{i=1}^{\infty} E_i$. If $\mu^*(E_i) = \infty$ for at least one I, then we are through.
∴ Assume $\mu^*(E_i) < \infty$ for each i.
Let ε > 0. Then ∃ a sequence $\{A_j{}^{(i)}\}$ of sets in $\mathcal{A}$ ∍ $E_i \subseteq \bigcup_{j=1}^{\infty} A_j{}^{(i)}$ and $\sum_{j=1}^{\infty} \mu(A_j{}^{(i)}) < \mu^*(E_i) + \frac{\varepsilon}{2^i}$.
Now $E \subseteq \bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} A_j{}^{(i)}$ so that $\mu^*(E) < \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu(A_j{}^{(i)}) < \sum_{i=1}^{\infty} \left\{\mu^*(E_i) + \frac{\varepsilon}{2^i}.\right\} = \sum_{i=1}^{\infty} \mu^*(E_i) + \varepsilon$.
Since ε is arbitrary we get $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.
Hence μ^* is an outer measure called outer measure induced by μ.
(iv) Let A ∈ $\mathcal{A}$, E be an arbitrary set of finite measure and ε > 0.
Then ∃ a sequence $\{A_i\}$ of sets in $\mathcal{A}$ ∍ $E \subseteq \bigcup_{i=1}^{\infty} A_i$ and $\sum_{i=1}^{\infty} \mu(A_i) < \mu^*(E) + \varepsilon$.
By the additivity of μ on $\mathcal{A}$, $\mu(A_i) = \mu(A_i \cap A) + \mu(A_i \cap \tilde{A})$.
∴ $\mu^*(E) + \varepsilon > \sum_{i=1}^{\infty}\{\mu(A_i \cap A) + \mu(A_i \cap \tilde{A})\} = \sum_{i=1}^{\infty} \mu(A_i \cap A) + \sum_{i=1}^{\infty} \mu(A_i \cap \tilde{A}) \geq \mu^*(E \cap A) + \mu^*(E \cap \tilde{A})$ since $E \cap A \subseteq \bigcup_{i=1}^{\infty}(A_i \cap A)$ and $E \cap \tilde{A} \subseteq \bigcup_{i=1}^{\infty}(A_i \cap \tilde{A})$.
Since ε is arbitrary, $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap \tilde{A})$ for every set E.
Hence A is μ^*-measurable.

Notation: For a given Algebra $\mathcal{A}$ of sets, $\mathcal{A}_\sigma$ denotes those sets that are countable unions of sets of $\mathcal{A}$ and $\mathcal{A}_{\sigma\delta}$ denotes those sets that are countable intersections of sets of $\mathcal{A}_\sigma$.

Theorem: Let μ be a measure on an algebra $\mathcal{A}$, μ^* be the outer measure induced by μ and E be any set. Then (i) for each ε > 0, there is a set A ∈ $\mathcal{A}_\sigma$ with E ⊆ A and $\mu^*(A) \leq \mu^*(E) + \varepsilon$. (ii) There is a B ∈ $\mathcal{A}_{\sigma\delta}$ with E ⊆ B and $\mu^*(E) = \mu^*(B)$.

Proof: Let ε > 0. Then ∃ a sequence $\{A_i\}$ of sets in $\mathcal{A}$ ∍ $E \subseteq \bigcup_{i=1}^{\infty} A_i$ and $\sum_{i=1}^{\infty} \mu(A_i) < \mu^*(E) + \varepsilon$.
Writing $A = \bigcup_{i=1}^{\infty} A_i$, we have A ∈ $\mathcal{A}_\sigma$, E ⊆ A and
$\mu^*(A) = \mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i) = \sum_{i=1}^{\infty} \mu(A_i) < \mu^*(E) + \varepsilon$.
Thus, for each ε > 0, there is a set A ∈ $\mathcal{A}_\sigma$ with E ⊆ A and $\mu^*(A) \leq \mu^*(E) + \varepsilon$.
(ii) By (i) for each n ≥ 1, ∃ a set $A_n \in \mathcal{A}_\sigma$ with $E \subseteq A_n$ and $\mu^*(A_n) \leq \mu^*(E) + \frac{1}{n}$.
Write $B = \bigcap_{n=1}^{\infty} A_n$ so that B ∈ $\mathcal{A}_{\sigma\delta}$ and E ⊆ B.
Also, since $B \subseteq A_n$ for every n, $\mu^*(B) \leq \mu^*(A_n) \leq \mu^*(E) + \frac{1}{n}$ and
hence $\mu^*(B) \leq \mu^*(E)$ …(1)
Since E ⊆ B, $\mu^*(E) \leq \mu^*(B)$ …(2).
From (1) and (2) $\mu^*(B) = \mu^*(E)$.

Note: If we apply this proposition in the case that E is a measurable set of finite measure, we see that E is difference of a set B ∈ $\mathcal{A}_{\sigma\delta}$ and a set of measure zero. This gives the structure of measurable sets of finite measure.

The next theorem extends this to the σ-finite case which can be considered as generalization of the first principle of Littlewood.

Theorem: Let μ be a σ-finite measure on an algebra $\mathcal{A}$ and let μ* be the outer measure induced by μ. A set E is μ*-measurable if and only if E is the proper difference A – B of a set A ∈ $\mathcal{A}_{\sigma\delta}$, and a set B with μ*(B) = 0.
Proof : Suppose E is the proper difference A – B of a set A ∈ $\mathcal{A}_{\sigma\delta}$, and a set B with μ*(B) = 0. Since the class of all μ*-measurable sets is a σ-algebra, we get A is measurable. Since $\bar{\mu}$ is complete, each set of μ*-measure zero must be measurable. ∴ B is measurable and hence E = A – B is μ* measurable.
Conversely suppose E is measurable. Since μ is σ - finite, ∃ a countable sequence of pairwise disjoint sets {X_i} in $\mathcal{A}$ with μ(X_i) < ∞ and X = $\bigcup_{i=1}^{\infty} X_i$.
Then E = X ∩ E = $\bigcup_{i=1}^{\infty} X_i$ ∩ E = $\bigcup_{i=1}^{\infty}(X_i \cap E) = \bigcup_{i=1}^{\infty} E_i$ where E_i = X_i ∩ E is a disjoint union of the measurable sets. Also, for each positive integer n, there exists a set $A_{ni} \in \mathcal{A}_\sigma$ such that $E_i \subseteq A_{ni}$ and $\bar{\mu}(A_{ni}) \le \bar{\mu}(E_i) + \frac{1}{n2^i}$.
Setting $A_n = \bigcup_{i=1}^{\infty} A_{n\,i}$, we find E ⊆ A_n and $A_n - E \subseteq \bigcup_{i=1}^{\infty}(A_{n\,i} - E_i)$.
Hence $\bar{\mu}(A_n - E) \le \sum_{i=1}^{\infty} \bar{\mu}(A_{n\,i} - E_i) \le \sum_{i=1}^{\infty} \frac{1}{n2^i} = \frac{1}{n}$.
Since $A_n \in \mathcal{A}_\sigma$ the set A = $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}_{\sigma\delta}$ and for each n, A – E ⊆ $A_n - E$.
Hence $\bar{\mu}$ (A – E) ≤ $\bar{\mu}$ $(A_n - E) \le \frac{1}{n}$.
Since this holds for each positive integer n, we get $\bar{\mu}$ (A – E) = 0.
Writing B = A – E we find E = A – B, where A ∈ $\mathcal{A}_{\sigma\delta}$,, and μ*(B) = 0.

Theorem (Caratheodory): Let μ be a measure on an algebra $\mathcal{A}$, and μ* the outer measure induced by μ. Then the restriction $\bar{\mu}$ of μ* to the μ*-measurable sets is an extension of μ to a σ-algebra containing $\mathcal{A}$. If μ is finite (or σ-finite) so is $\bar{\mu}$. If μ is σ-finite, then $\bar{\mu}$ is the only measure on the smallest σ-algebra containing $\mathcal{A}$ which is an extension of μ.

Proof: The fact that $\bar{\mu}$ is an extension of μ from $\mathcal{A}$ to be a measure on a σ-algebra containing $\mathcal{A}$ follows directly from the facts A ∈ $\mathcal{A} \Rightarrow \mu^*(A) = \mu(A)$, each A ∈ $\mathcal{A}$ is μ^*- measurable and the class of μ^*- measurable sets is a σ-algebra. Already verified if μ is finite (or σ-finite) so is $\bar{\mu}$.
To show the unicity of $\bar{\mu}$ when μ is σ-finite, we let $\mathfrak{B}$ be the smallest σ-algebra containing $\mathcal{A}$ and $\tilde{\mu}$ some measure on $\mathfrak{B}$ that agrees with μ on $\mathcal{A}$.
Since each set in $\mathcal{A}_\sigma$, can be expressed as disjoint countable union of sets in $\mathcal{A}$, the measure $\tilde{\mu}$ must agree with $\bar{\mu}$ on $\mathcal{A}_\sigma$. Let B be any set in $\mathfrak{B}$ with finite outer measure.
Then by Proposition, there is an A in $\mathcal{A}_\sigma$, such that B ⊆ A and
$\mu^*(A) \le \mu^*(B) + \varepsilon$
Since B ⊆ A, $\tilde{\mu}(B) \le \tilde{\mu}(A) = \mu^*(A) \le \mu^*(B) + \varepsilon$.
Since ε is an arbitrary positive number, we have $\tilde{\mu}(B) \le \mu^*(B)$ for each B ∈ $\mathfrak{B}$.
Since the class of sets measurable with respect to μ* is a σ-algebra containing $\mathcal{A}$, each

B in $\mathfrak{B}$ must be measurable. If B is measurable and A is in $\mathcal{A}_\sigma$ with B ⊆ A and μ*A < μ*B + ε, then $\mu^*(A) = \mu^*(B) + \mu^*(A{\sim}B)$, and so,
$\tilde{\mu}(A{\sim}B) \leq \mu^*(A{\sim}B) \leq \varepsilon$, if $\mu^*(B) < \infty$.
Hence $\mu^*(B) \leq \mu^*(A) = \tilde{\mu}(A) = \tilde{\mu}(B) + \tilde{\mu}(A{\sim}B) \leq \tilde{\mu}(B) + \varepsilon$.
Since ε is arbitrary, $\mu^*(B) \leq \tilde{\mu}(B)$ and so, $\mu^*(B) = \tilde{\mu}(B)$.
If μ is σ-finite measure, let $\{X_i\}$ be a countable disjoint collection of sets in $\mathcal{A}$ with X = $\bigcup_{i=1}^{\infty} X_i$ and $\mu(X_i)$ finite.
If B is any set in $\mathfrak{B}$, then B = $\bigcup_{i=1}^{\infty}(X_i \cap B)$ and this is a countable disjoint union of sets in $\mathfrak{B}$, and so, we have $\tilde{\mu}(B) = \sum_{i=1}^{\infty} \tilde{\mu}(X_i \cap B)$ and $\bar{\mu}(B) = \sum_{i=1}^{\infty} \bar{\mu}(X_i \cap B)$.
Since $\mu^*(X_i \cap B) < \infty$, $\bar{\mu}(X_i \cap B) = \tilde{\mu}(X_i \cap B)$.

Definition: A collection $\mathcal{C}$ of subsets of X is **a semi-algebra of sets** if the intersection of any two sets in $\mathcal{C}$ is again in $\mathcal{C}$ and the complement of any set in $\mathcal{C}$ is a finite disjoint union of sets in $\mathcal{C}$.

Definition: If $\mathcal{C}$ is any semialgebra of sets, then the collection $\mathcal{A}$ consisting of the empty set and all finite disjoint unions of sets in $\mathcal{C}$ is an algebra of sets which is called the **algebra generated by** $\mathcal{C}$.

If μ is a set function defined on $\mathcal{C}$, it is natural to attempt to define a finitely additive set function on $\mathcal{A}$ by setting μA = $\sum_{i=1}^{n} \mu(E_i)$, whenever A is the disjoint union of the set E_i in $\mathcal{C}$. Since a set A in $\mathcal{A}$ may possibly be represented in several ways as a disjoint union of sets in $\mathcal{C}$, we must be certain that such a procedure leads to a unique value for μA. The following proposition gives conditions under which this procedure can be carried out and will give a measure on the algebra $\mathcal{A}$.

Proposition: Let $\mathcal{C}$ be a semialgebra of sets and μ a nonnegative set function defined on $\mathcal{C}$ with μϕ = 0 (if ϕ ∈ $\mathcal{C}$). Then μ has a unique extension to a measure on the algebra $\mathcal{A}$ generated by $\mathcal{C}$ if the following conditions are satisfied:
1. If a set C in $\mathcal{C}$ is the union of a finite disjoint collection $\{C_i\}$ of sets in, then μC = $\sum_{i=1}^{n} \mu(C_i)$.
2. If a set C in $\mathcal{C}$ is the union of a countable disjoint collection $\{C_i\}$ of sets in $\mathcal{C}$, then μC ≤ $\sum_{i=1}^{\infty} \mu(C_i)$.

3. THE LEBESGUE-STIELTJES INTEGRAL

Definition: Let X be the set of real numbers and $\mathfrak{B}$ the class of all Borel sets, A measure μ defined on $\mathfrak{B}$ and finite for bounded sets is called a **Baire measure** (on the real line). To each finite Baire measure we associate a function F by setting F(x) = μ(–∞, x]. The function F is called the **cumulative distribution function** of μ and is real-valued and monotone increasing.

Lemma: If μ is a finite Baire measure on the real line, then its cumulative distribution function F is a monotone increasing bounded function which is continuous on the right. Moreover, $\lim_{x\to-\infty} F(\text{x}) = 0$.

Proof: We have μ(a, b] = F(b) – F(a),

Since (a, b] is the intersection of the sets $\left]\text{a}, \text{b} + \frac{1}{n}\right]$, by a Proposition

μ(a, b] = $\lim_{n\to\infty} \mu \left]\text{a}, \text{b} + \frac{1}{n}\right] \Rightarrow$ F(b) – F(a) = $\lim_{n\to\infty} \left\{F\left(\text{b} + \frac{1}{n}\right) - F(a)\right\}$

and so F(b) = $\lim_{n\to\infty} \text{F}\left(\text{b} + \frac{1}{n}\right)$ = F(b+).

Thus a cumulative distribution function is continuous on the right. Similarly,

μ{b} = $\lim_{n\to\infty} \mu \left]\text{b} - \frac{1}{n}, b\right] = \lim_{n\to\infty} \left\{F(\text{b}) - F\left(b - \frac{1}{n}\right)\right\}$ = F(b) – F(b-).

Hence F is continuous at b if and only if the set {b} consisting of b alone has measure zero. Since ϕ = $\bigcap_{n=1}^{\infty}(-\infty, n]$, we have $\lim_{n\to-\infty} F(\text{n}) = 0$, and hence $\lim_{x\to-\infty} F(\text{x}) = 0$ because of the monotonicity of F.

Lemma: Let F be a monotone increasing function continuous on the right.
If (a, b] ⊆ $\bigcup_{i=1}^{\infty}(a_i, b_i]$, then F(b) – F(a) ≤ $\sum_{i=1}^{\infty}\{F(b_i) - F(a_i)\}$

Proposition: Let F be a monotone increasing function which is continuous on the right. Then there is a unique Baire measure μ such that for all a and b we have
μ(a, b] = F(b) – F(a).

Proof: If we let $\mathcal{C}$ be the semialgebra consisting of all intervals of the form (a, b] or (a, ∞) and set μ(a, b] = F(b) – F(a), then μ. is easily seen to satisfy condition (1) of a Proposition, and since Lemma is precisely the second condition, we see that μ admits a unique extension to a measure on the algebra generated by 𝔅. By Theorem 8 this μ can be extended to a σ-algebra containing $\mathcal{C}$. Since the class 𝔅 of Borel sets is the smallest σ-algebra containing $\mathcal{C}$, we have an extension of to a Baire measure. The measure μ is σ-finite, since X is the union of the intervals (n, n + 1] and each has finite measure. Thus, the extension of μ to 𝔅 is unique.

Corollary: Each bounded monotone function which is continuous on the right is the cumulative distribution function of a unique finite Baire measure provided F(∞) = 0.

Definition: If φ is a nonnegative Borel measurable function and F is a monotone increasing function which is continuous on the right, we define the **Lebesgue-Stieltjes integral** of φ with respect to F to be $\int \varphi\, dF = \int \varphi\, d\mu$ where μ is the Baire measure having F as its cumulative distribution function. If φ is both positive and negative, we say that it is integrable with respect to F if it is integrable with respect to μ.

Definition: If F is any monotone increasing function, then there is a unique function F* which is monotone increasing, continuous on the right, and agrees with F wherever F is continuous on the right, and we define the Lebesgue-Stieltjes integral of φ with respect to F by $\int \varphi\, dF = \int \varphi\, dF^*$. If F is a monotone function, continuous on the right, then $\int_a^b \varphi\ dF$ agrees with the Riemann-Stieltjes integral whenever the latter is defined. The Lebesgue-Stieltjes integral is only defined when F is monotone (or more generally of bounded variation), while the Riemann-Stieltjes integral can exist when F is not of bounded variation, say when F is continuous and φ is of bounded variation.

4. PRODUCT MEASURES

Let (X, 𝔄, μ) and (Y, 𝔅, ν) be two complete measure spaces, and consider the direct product X × Y of X and Y. If A ⊆ X and B ⊆ Y, we call A × B a rectangle. If A ∈ 𝔄 and B ∈ 𝔅 we call A × B a measurable rectangle.
The collection ℜ of measurable rectangles is a semi-algebra, since
$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ and
$\sim(A \times B) = (\bar{A} \times B) \cup (A \times \bar{B}) \cup (\bar{A} \times \bar{B})$.
If A × B is a measurable rectangle, we set $\lambda(A \times B) = \mu A \cdot \upsilon B$

14. **Lemma**: Let $\{(A_i \times B_i)\}$ be a countable disjoint collection of measurable rectangles whose union is a measurable rectangle A × B. Then
$\lambda(A \times B) = \Sigma\, \lambda(A_i \times B_i)$.
Proof: Fix a point x ∈ A. Then for each y ∈ B, the point $\langle x, y\rangle$ belongs to exactly one rectangle Ai × Bi. Thus B is the disjoint union of those Bi such that x is in the corresponding Ai,. Hence

$$\sum \upsilon B_i \cdot \chi_{A_i}(x) = \upsilon B \cdot \chi_A(x)$$

since ν is countably additive. Thus, by the corollary of the Monotone Convergence Theorem (11.14), we have

$$\sum \int \upsilon B_i \cdot \chi_{A_i} d\mu = \int \upsilon(B) \cdot \chi_A d\mu$$

or $\Sigma \nu B_i \cdot \mu A_i = \nu B \cdot \mu A$.♦

The lemma implies that λ satisfies the conditions of Proposition 9 and hence has a unique extension to a measure on the algebra ℜ′ consisting of all finite disjoint unions of sets in ℜ. Theorem 8 allows us to extend λ to be a complete measure on a σ-algebra

$\mathcal{S}$ containing $\mathfrak{R}$. This extended measure is called the product measure of μ and ν and is denoted by μ × ν. If μ and ν are finite (or σ-finite), so is μ × ν. If X and Y are the real line and μ and ν are both Lebesgue measure, then μ × ν is called two-dimensional Lebesgue measure for the plane.
If E is any subset of X × Y and x a point of X, we define the x cross section E_x by E_x = {y: <x, y> ∈ E}, and similarly for the y cross section for y in Y. The characteristic function of E, is related to that of E by $\chi_{E_x}(y) = \chi_E(x,y)$
We also have $(\bar{E})_x$ = ~ (E_x) and (∪ E_α)$_x$ = ∪(E_α)$_x$ for any collection {E_α}.

15. **Lemma**: Let x be a point of X and E a set in $\mathfrak{R}_{\sigma\delta}$. Then E_x is a measurable subset of Y.
Proof: The lemma is trivially true if E is in the class $\mathfrak{R}$ of measurable rectangles. We next show it to be true for E in $\mathfrak{R}_\sigma$ Let E = $\bigcup_{i=1}^{\infty} E_i$ where each E_i is a measurable rectangle. Then $\chi_{E_x}(y) = \chi_E(x,y) = \sup_i \chi_{E_i}(x,y) = \sup_i \chi_{(E_i)_x}(y)$. Since each E_i is a measurable rectangle, $\chi_{(E_i)_x}(y)$ is a measurable function of y, and so χ_{E_x} must also be measurable, whence E_x is measurable.
Suppose now that E = $\bigcap_{i=1}^{\infty} E_i$ with $E_i \in \mathfrak{R}_\sigma$. Then $\chi_{E_x} = \chi_E(x,y) = \inf_i \chi_{E_i}(x,y)$ $= \inf_i \chi_{(E_i)_x}(y)$ and we see that χ_{E_x} is measurable. Thus, E_x is measurable for any E in $\mathfrak{R}_{\sigma\delta}$.♦

16. **Lemma**: Let E be a set in $\mathfrak{R}_{\sigma\delta}$ with μ × ν (E) < ∞. Then the function g defined by g(x) = νE_x is a measurable function of x and $\int g\, d\mu$ = μ × ν(E)
Proof: The lemma is trivially true if E is a measurable rectangle. We first note that any set in $\mathfrak{R}_\sigma$, is a disjoint union of measurable rectangles. Let {E_i} be a disjoint sequence of measurable rectangles, and let E = ∪ E_i. Set $g_i(x) = \upsilon[(E_i)_x]$
Then each g_i is a nonnegative measurable function, and g = $\sum g_i$
thus, g is measurable, and by the corollary of the Monotone Convergence Theorem we have $\int g\, d\mu = \sum \int g_i d\mu = \sum \mu \times \upsilon(E_i)$
Consequently, the lemma holds for E $\in \mathfrak{R}_\sigma$. Let E be a set of finite measure in $\mathfrak{R}_{\sigma\delta}$ Then there is a sequence {E_i} of sets in $\mathfrak{R}_\sigma$such that $E_{i+1} \subseteq E_i$ and E = ∩ E_i. It follows from Proposition 6 that we may take μ × ν (E_i) < ∞. Let $g_i(x) = \nu[(E_i)_x]$.
Since $\int g_1 d\mu$ = μ × ν(E1) < ∞, we have $g_1(x) < \infty$ for almost all x. For an x with $g_1(x) < \infty$, we have $\langle (E_i)_x \rangle$ a decreasing sequence of measurable sets of finite measure whose intersection is E_x. Thus, by Proposition 11.2 we have g(x) = ν(E_x) = lim ν[(E_i)$_x$) = lim $g_i(x)$.
Hence $g_i \to g$ a.e., and so g is measurable. Since $0 \le g_i \le g_1$, the Lebesgue Convergence Theorem implies that $\int g\, d\mu = \lim \int g_i d\mu$ = lim μ × ν(E_i). the last

equality following from Proposition 11.2. ♦

17. **Lemma**: Let E be a set for which μ × ν(E) = 0. Then for almost all x we have $\nu(E_x)$ = 0.

Proof: By Proposition 6 there is a set F in $\Re_{\sigma\delta}$ such that E ⊆ F and μ × ν (F) = 0. It follows from Lemma 16 that for almost all x we have $\nu(F_x)$ = 0. But $E_x \subseteq F_x$ and so νE_x = 0 for almost all x since ν js complete. ♦

18. **Proposition**: Let E be a measurable subset of X × Y such that μ × ν(E) is finite. Then for almost all x the set E_x is a measurable subset of Y. The function g defined by g(x) = $\nu(E_x)$ is a measurable function defined for almost all x and $\int g\, d\mu$ = μ × ν(E). ″
Proof: By Proposition 6 there is a set F in $\Re_{\sigma\delta}$, such that E ⊆ F and μ × ν (F) = μ × ν (E). Let G = F ~ E. Since E and F are measurable, so is G, and μ × ν(F) = μ × ν (E) + μ × ν(G).
Since μ × ν(E) is finite and equal to μ × ν(F), we have μ × ν(G) = 0. Thus by Lemma 17 we have $\nu(G_x)$, = 0 for almost all x. Hence g(x) = νE_x = νF_x a.e.; so g is a measurable function by Lemma 16. Again by Lemma 16 $\int g\, d\mu$ = μ × ν(F) = μ × ν(E). ♦

19. **Theorem (Fubini)**: Let (X, 𝔄, μ) and (Y, 𝔅, ν) be two complete measure spaces and f an integrable function on X x Y. Then
1). For almost all x the function f_x defined by $f_x(y) = f(x, y)$ is an integrable function on Y.
2). For almost all y the function f^y defined by $f^y(x) = f(x, y)$ is an integrable function on X.
3), $\int_Y$f(x,y) d$\upsilon(y)$ is an integrable function on X,
4). $\int_X$f(x,y) d$\mu(x)$ is an Integrable function on Y.
5). $\int_X[\int_Y f\, dv]d\mu = \int_{X\times Y} f\, d(\mu\times v)$ = $\int_Y[\int_X f\, d\mu]dv$
Proof: Because of the symmetry between x and y it suffices to prove (1), (3), and the first half of (5). If the conclusion of the theorem holds for each of two functions, it also holds for their difference, and hence it is sufficient to consider the case when f is nonnegative. Proposition 18 asserts that the theorem is true if f is the characteristic function of a measurable set of finite measure, and hence the theorem must be true if f is a simple function which vanishes outside a set of finite measure. Proposition 11.7 asserts that each nonnegative integrable function f is the limit of an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions, and, since each φ_n is integrable and simple, it must vanish outside a set of finite measure. Thus f_x is the limit of the increasing sequence $\{(\varphi_n)_x\}$ and is measurable. By the Monotone Convergence Theorem $\int_Y$f(x, y) d$\upsilon(y)$ = lim $\int_Y \varphi_n$(x, y) d$\upsilon(y)$
and so this integral is a measurable function of x.

Again, by the Monotone Convergence Theorem

$\int_X [\int_Y f dv] d\mu$ = lim $\int_X [\int_Y \varphi_n dv] d\mu$

= $\int_{X\times Y} \varphi_n d(\mu \times v)$

= $\int_{X\times Y} f d(\mu \times v)$ ♦

20. **Theorem (Tonelli)**: Let (X, 𝔄, μ) and (Y, 𝔅, ν) be two σ-finite measure spaces and f be a non- negative measurable function on X × Y. Then

1). For almost all x the function f_x defined by $f_x(y) = f(x, y)$ is a measurable function on Y.

2). For almost all y the function f^y defined by $f^y(x) = f(x, y)$ is a measurable function on X.

3), $\int_Y$f(x, y) d$v(y)$ is a measurable function on X,

4). $\int_X$f(x, y) d$\mu(x)$ is a measurable function on Y.

5). $\int_X [\int_Y f dv] d\mu = \int_{X\times Y} f d(\mu \times v)$ = $\int_Y [\int_X f d\mu] dv$

Proof: Because of the symmetry between x and y it suffices to prove (1), (3), and the first half of (5).

Case (i): Suppose that f is a characteristic function of measurable set with finite measure.

Let f = χ_E where E is a measurable set.

Since f is integrable, $\int_{X\times Y} f\ d(\mu \times v)$ < ∞.

We have $f_x = \chi_{E_x}$. Clearly f_x is measurable function on Y

$\int_{X\times Y} f_x\ d(\mu \times v)$ = $\int_{X\times Y} \chi_{E_x} d(\mu \times v)$ < ∞.

∴ f_x is integrable ⇒ f_x is measurable.

Consider g(x) = $\int_Y$f dv.

But we know that g(x) = νE_x is measurable with μ × ν E < ∞ and

$\int_X$g(x) dμ = μ × νE < ∞ we have g is integrable function.

∴ $\int_Y$f dv is measurable function.

Now $\int_X$g(x) dμ = $\int_X [\int_Y f dv] d\mu$ = μ × νE.

$\int_{X\times Y} f\ d(\mu \times v)$ = $\int_{X\times Y} \chi_E d(\mu \times v)$ = μ × ν (E) = $\int_X$g(x) dμ = $\int_X [\int_Y f dv] d\mu$

Case (ii): Suppose that f is a simple function which vanishes outside a set of finite measure. Since a simple function is a linear combination of characteristic function by case (i) the theorem is true.

Case (iii): Let f be an integrable function on X × Y.

Choose an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions which converges to f.

Since $\varphi_n$is measurable and simple, it must vanish outside a set of finite measure.

and f = lim φ_n.

$\therefore f_x$ = lim $(\varphi_n)_x$

Since $f_x$is measurable and hence integrable. Then by monotone convergence theorem

$\int_Y f_x$ dv = lim $\int \varphi_n(x, y)\ d\ v$

By case (ii) the RHS of the above equation is measurable and hence

$\int_Y f_x$ dυ is measurable.

Again, by Monotone convergence theorem

$\int_X [\int_Y f d\upsilon] d\mu$ = lim $\int_X [\int_Y \varphi_n(x, y)\, d\upsilon] d\mu$ = lim $\int_{X\times Y} \varphi_n(x, y)\, d(\mu \times \upsilon)$

= $\int_{X\times Y} f d(\mu \times \upsilon)$ ♦

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E – CONTENT

PAPER: M 401,

MEASURE THEORY

M. Sc. II YEAR, SEMESTER - IV

UNIT – IV

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M 401: MEASURE THEORY
UNIT IV
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6 INNER MEASURE

Let μ be a measure on an algebra $\mathcal{A}$ and μ* the induced outer measure. Then μ*E may be thought of as the largest possible measure for E compatible with μ. We can also define an inner measure μ_* which assigns to a given set E the smallest measure compatible with μ:

Definition: Let μ be a measure on an algebra $\mathcal{A}$ and μ* the induced outer measure. We define the inner measure μ_* induced by μ by setting
$\mu_*(E)$ = sup {μA – μ*(A ~ E): A ∈ $\mathcal{A}$, μ*(A ~ E) < ∞}.

[Inner measure was important historically because the measurability of a set was originally characterized using both inner ang outer measure. In the historical context inner measure was first defined for bounded subsets of R. For such sets the definition above is equivalent to the historical one: $\mu_*(E)$ = l(I) – μ(I ~ E) where I is a finite interval containing E. A bounded set E was then defined to be measurable if $\mu_*(E)$ = μ*(E), and the measurability of unbounded sets was defined in terms of their intersections with finite intervals. Even in the case of a bounded set this procedure is more cumbersome than the elegant approach of Carathéodory, which we have followed in this chapter. Apart from this historical importance, inner measure is useful for the extension of μ from $\mathcal{A}$ to an algebra containing $\mathcal{A}$ and a given set E (which need not be measurable) and for determining the freedom we have in extending μ to a σ-algebra containing $\mathcal{A}$.]

Lemma: Prove that $\mu_*(E) \le \mu^*(E)$. If E ∈ $\mathcal{A}$, then show that $\mu_*(E)$ = μ(E) = $\mu^*(E)$.

Proof: Let A ∈ $\mathcal{A}$ and μ*(A ~ E) < ∞.
Since $A = (A \cap E) \cup \left(A \cap \tilde{E}\right)$, by finite sub additivity of μ, we have $\mu(A) \le \mu^*(A \cap E) + \mu^*(A \cap \tilde{E})$.
$\therefore \mu(A) - \mu^*(A \cap \tilde{E}) \le \mu^*(A \cap E)$
$\le \mu^*(E)$ since $A \cap E \subseteq E$.
Ie. $\mu(A) - \mu^*(A \cap \tilde{E}) \le \mu^*(E)$ ∀ A ∈ $\mathcal{A}$ and μ*(A ~ E) < ∞. $\therefore \mu^*(E)$ is an upper bound of $\{\mu(A) - \mu^*\left(A \cap \tilde{E}\right)$: A ∈ $\mathcal{A}$, μ*(A ~ E) < ∞
Consequently, by definition of μ_*, $\mu_*(E) \le \mu^*(E)$…(i)
Let E ∈ $\mathcal{A}$. Then μ*(E) = μ(E) so that $\mu_*(E) \le$ μ(E)…(ii)
Now put A = E so that A ∈ $\mathcal{A}$ and μ*(A ~ E) = 0 < ∞. $\therefore \mu_*(E)$ = Sup{μ(E)} ≥ μ(E)
Ie. $\mu_*(E) \ge$ μ(E)…(iii)
From (ii) and (iii) $\mu_*(E) = \mu^*(E)$.

Lemma: If $E \subseteq F$ then $\mu_*(E) \le \mu_*(F)$.
Proof: Let $E \subseteq F$.
Let $A \in \mathcal{A}$ and $\mu^*(A \sim E) < \infty$.
Then $\sim F \subseteq \sim E$ and so, $A \sim F \subseteq A \sim E$. $\therefore \mu^*(A \sim F) \le \mu^*(A \sim E) < \infty$.
Also, $\mu^*(A \sim F) \le \mu^*(A \sim E) \Rightarrow \mu(A) - \mu^*(A \sim F) \ge \mu(A) - \mu^*(A \sim E)$.
$\therefore \sup \{\mu A - \mu^*(A \sim E): A \in \mathcal{A}, \mu^*(A \sim E) < \infty\} \le \sup \{\mu A - \mu^*(A \sim F): A \in \mathcal{A}, \mu^*(A \sim F) < \infty\}$.
Ie $\mu_*(E) \le \mu_*(F)$.

[One of the difficulties of using the definition of inner measure is that we must take supremum of $\mu(A) - \mu^*(A \sim E)$ overall $A \in \mathcal{A}$ with $\mu^*(A \sim E) < \infty$. The next lemma shows that this expression is monotone in A and enables us to calculate $\mu_*(E)$ more easily.]

Lemma: Let A and B be two sets in $\mathcal{A}$ with $\mu^*(A \sim E) < \infty$ and $\mu^*(B \sim E) < \infty$.
If $A \subseteq B$, we have $\mu A - \mu^*(A \sim E) \le \mu B - \mu^*(B \sim E)$. If also $E \subseteq A$, we have equality, and hence $\mu_*(E) = \mu(A) - \mu^*(A \sim E)$.
Proof: Let $A \subseteq B$. Then $B = A \cup (B \sim A)$, is a disjoint union.
$\therefore$ By additivity of μ, $\mu(B) = \mu(A) + \mu(B \sim A)$.
$\Rightarrow \mu(B \sim A) = \mu(B) - \mu(A)$ … (i)
Observe that $B \sim E \subseteq (B \sim A) \cup (A \sim E)$.
$\therefore$ By sub additivity of μ^* we have $\mu^*(B \sim E) \le \mu^*(B \sim A) + \mu^*(A \sim E)$.
Since $B \sim A \in \mathcal{A}$, $\mu^*(B \sim A) = \mu(B \sim A)$.
$\therefore \mu^*(B \sim E) \le \mu(B \sim A) + \mu^*(A \sim E)$.
$\Rightarrow \mu^*(B \sim E) \le \mu(B) - \mu(A) + \mu^*(A \sim E)$ from (i). –
$\Rightarrow \mu(A) - \mu^*(A \sim E) \le \mu(B) - \mu^*(B \sim E)$
Let $E \subseteq A$. Then $B \sim E = (B \sim A) \cup (A \sim E)$ is a disjoint union and so proceeding as above we get $\mu^*(B \sim E) = \mu(B) - \mu(A) + \mu^*(A \sim E)$
ie. $\mu A - \mu^*(A \sim E) = \mu B - \mu^*(B \sim E)$.
Now taking supremum over all sets $B \in \mathcal{A}$ with $\mu^*(B \sim E) < \infty$ we have
$\mu A - \mu^*(A \sim E) = \sup \{\mu B - \mu^*(B \sim E): B \in \mathcal{A}, \mu^*(B \sim E) < \infty\} = \mu_*(E)$. Hence $\mu_*(E) = \mu A - \mu^*(A \sim E)$

[This lemma and its corollary show that if μ is a finite measure, then $\mu_*(E) = \mu X - \mu^*(\tilde{E})$. In this case the development of the theory and properties of inner measure are relatively straightforward. The complexity of the treatment of inner measure in this section is caused by having the concept apply to measures that are not σ-finite.

Corollary: If A $\in \mathcal{A}$, then μA = $\mu_*(A \cap E)$ + $\mu^*(A \cap \tilde{E})$.

Proof: Let $\mu^*(A \cap \tilde{E}) = \infty$.

Then μA = ∞ and there is nothing to prove.

Let $\mu^*(A \cap \tilde{E}) < \infty$. Set F = A ∩ E.

Then F ⊆ E so that $\tilde{E} \subseteq \tilde{F}$ and so, $A \cap \tilde{E} \subseteq A \cap \tilde{F}$ ie. $A \cap \tilde{E} \subseteq A \sim F$ …(i)

Let x ∈ A ~ F ⇒ x ∈ A and x ∉ F.

⇒ x ∈ A and x ∉ A ∩ E

⇒ x ∈ A and x ∉ E ⇒ x ∈ A and x ∈ $\tilde{E}$

⇒ x ∈ $A \cap \tilde{E}$, ie A ~ F ⊆ $A \cap \tilde{E}$ …(ii)

∴ from (i) and (ii) A ~ F = A ∩ $\tilde{E}$

But by above lemma $\mu_*(F)$ = μ(A) – μ*(A ~ F) since F ⊆ A

⇒ $\mu_*(F)$ = μ(A) – μ*(A ∩ $\tilde{E}$).

⇒ μA = $\mu_*(A \cap E)$ + $\mu^*(A \cap \tilde{E})$.

Lemma: Let B be a μ*-measurable set with μ*B < ∞. Then $\mu_*(B)$ = μ*B.

Proof: Let ε > 0. Since μ*B < ∞, there is a set A ∈ $\mathcal{A}$ with μ*(B ~ A) < ε. [Let μ be a finite measure on an algebra $\mathcal{A}$ and μ* the induced outer measure. A set E is measurable iff for each ε > 0 there is a set A in $\mathcal{A}_\delta$, A ⊆ E ∋ μ*(E ~ A) < ε]

Since A is measurable, μ*B = μ*(B ∩ A) + μ*(B ∩ $\tilde{A}$) = μ*(B ∩ A) + μ*(B ~ A) and so μ*B < μ*(B ∩ A) + ε.

∴ Now μ*(A ∩ B) > μ*B – ε…(i)

Since $\mu_*(B)$ = Sup {μA – μ*(A ~ B): A ∈ $\mathcal{A}$, μ*(A ~ B) < ∞},

$\mu_*(B)$ ≥ μA – μ*(A ~ B) = μ*(A) – μ*(A ∩ $\tilde{B}$) = μ*(A ∩ B) since B is measurable

> μ*B – ε.

Ie. $\mu_*(B)$ > μ*B – ε ∀ ε > 0.

⇒ $\mu_*(B)$ ≥ μ*B.

But $\mu_*(B)$ ≤ μ*B.

Hence $\mu_*(B)$ = μ*B.

Proposition: Let E be a set with $\mu_*(E) < \infty$. Then there is a set H ∈ $\mathcal{A}_{\delta\sigma}$ such that H ⊆ E and $\bar{\mu}(H) = \mu_*(E)$.

Proof: Since $\mu_*(E)$ = Sup {μA – μ*(A ~ E): A ∈ $\mathcal{A}$, μ*(A ~ E) < ∞}, for each n ∃ a set A_n in $\mathcal{A}$ with μ*(A_n ~ E) < ∞ and μA_n – μ*(A_n ~ E) > $\mu_*(E)$ - $\frac{1}{n}$.

By a Proposition, [Let μ be a measure on an algebra $\mathcal{A}$, μ^* be the outer measure induced by μ and E be any set. Then (i) for each ε > 0, there is a set A ∈ $\mathcal{A}_\sigma$ with E ⊆ A and $\mu^*(A) \le \mu^*(E) + \varepsilon$. (ii) There is a B ∈ $\mathcal{A}_{\sigma\delta}$ with E ⊆ B and $\mu^*(E) = \mu^*(B)$.] ∃

$G_n \in \mathcal{A}_{\sigma\delta} \ni G_n \supseteq A_n \sim E$ and $\bar{\mu}(G_n) = \mu^*(A_n \sim E)$. Let $H_n = A_n \sim G_n$. Then $H_n \in \mathcal{A}_{\delta\sigma}$ and $H_n \subseteq E$. [$x \in H_n \Rightarrow x \in A_n \sim G_n \Rightarrow x \in A_n, x \notin G_n \Rightarrow x \in A_n$ and $x \notin A_n \cap \tilde{E} \Rightarrow x \in A_n$ and $x \in E$]

Moreover, $\bar{\mu}(H_n) = \mu(A_n) - \bar{\mu}(G_n) > \mu_*(E) - \frac{1}{n}$.
Let $H = \cup H_n$. Then $H \subseteq E$, $H \in \mathcal{A}_{\delta\sigma}$ and $\bar{\mu}(H) \geq \mu_*(E)$

Hence $\bar{\mu}(H) = \mu_*(E)$.
[Moreover, since $A_n \sim E \subseteq G_n$, $\bar{\mu}(A_n \sim E) \leq \bar{\mu}(G_n)$ … (i).
Again, since $A_n = H_n \cup G_n$, $\bar{\mu}(A_n) = \bar{\mu}(H_n) + \bar{\mu}(G_n)$.

$$\begin{aligned}\therefore \bar{\mu}(H_n) &= \bar{\mu}(A_n) - \bar{\mu}(G_n)\\ &\geq \bar{\mu}(A_n) - \bar{\mu}(A_n - E)\\ &\geq \mu(A_n) - \bar{\mu}(A_n - E)\\ &> \mu_*(E) - \frac{1}{n}\end{aligned}$$

Since as $n \to \infty$, $1/n \to 0$ so that $\bar{\mu}(H_n) \geq \mu_*(E)$

Let $H = \cup H_n$. Then $H \subseteq E$, $H \in \mathcal{A}_{\delta\sigma}$ and $\bar{\mu}(H_n) \geq \mu_*(E)$
$H \subseteq E \Rightarrow \mu^*(H) \leq \mu^*(E)$
$\Rightarrow \bar{\mu}(H) \leq \mu^*(E) = \mu_*(E) \Rightarrow \bar{\mu}(H) \leq \mu_*(E)$
$\therefore \bar{\mu}(H) = \mu_*(E)$]

Corollary: If $\mu_*(E) < \infty$, $\mu_*(E) = \sup\{\bar{\mu}(B): B \subseteq E, B \text{ measurable}, \bar{\mu}(B) < \infty\}$.

Proposition: Suppose $\mu^*E < \infty$. Then E is measurable if and only if $\mu_*(E) = \mu^*E$.
Proof: Suppose $\mu^*E < \infty$ and E is measurable.
By the lemma [Let B be a μ^*-measurable set with $\mu^*B < \infty$. Then $\mu_*(B) = \mu^*B$]
we have $\mu_*(E) = \mu^*E$.
Conversely suppose $\mu_*(E) = \mu^*E < \infty$. Then by a Proposition [Let $\mu \in$ be a measure on an algebra $\mathcal{A}$, μ^* be the outer measure induced by μ and E be any set. There is a B $\in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(E) = \mu^*(B)$.] $\exists$ measurable set $G \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq G$ and $\mu^*(E) = \mu^*(G)$ and by above proposition [Let E be a set with $\mu_*(E) < \infty$. Then there is a set $H \in \mathcal{A}_{\delta\sigma}$ such that $H \subseteq E$ and $\bar{\mu}(H) = \mu_*(E)$] $\exists$ measurable set $H \in \mathcal{A}_{\sigma\delta}$ with $H \subseteq E$, $\bar{\mu}(H) = \mu_*(E)$.
$\therefore \bar{\mu}(H) = \mu_*(E) = \mu^*(E) = \mu^*(G)$.
ie. $\bar{\mu}(H) = \mu^*(G)$.
$\Rightarrow \bar{\mu}(H) = \bar{\mu}(G)$
Thus, we have E differs from a measurable set by a set of measure zero and hence E is measurable. [Hint $G - E \subseteq G - H$. Put $B = G - E$. Then $E = G - B$ where $\bar{\mu}(B) \leq \bar{\mu}(G -$

H) = 0].

Theorem: Let E and F be two disjoint sets. Then
$\mu_*(E) + \mu_*(F) \leq \mu_*(E \cup F) \leq \mu_*(E) + \mu^*(F) \leq \mu^*(E \cup F) \leq \mu^*(E) + \mu^*(F)$
Proof: Suppose $\mu_*(E)$ or $\mu_*(F)$ is infinite.
Then the first inequality follows from the monotonicity of μ_*.
Suppose $\mu_*(E)$ and $\mu_*(F)$ are both finite. By a proposition [Let E be a set with $\mu_*(E) <$ ∞. Then there is a set H ∈ $\mathcal{A}_{\delta\sigma}$ such that H ⊆ E and $\bar{\mu}(H) = \mu_*(E)$]
∃ G and H, measurable sets, with G ⊆ E and H ⊆ F such that $\bar{\mu}(G) = \mu_*(E)$ and $\bar{\mu}(H)$ = $\mu_*(F)$.
Then G ∪ H is a measurable set of finite outer measure contained in E ∪ F.
Thus, $\mu_*(E \cup F) \geq \mu_*(G \cup H) = \bar{\mu}(G \cup H) = \bar{\mu}(G) + \bar{\mu}(H) = \mu_*(E) + \mu_*(F)$ proving the first inequality.
Suppose $\mu^*(F) = \infty$, Then the second inequality is trivial.
Suppose $\mu^*(F) < \infty$. Let A ∈ $\mathcal{A}$ with $\mu^*(A \sim (E \cup F) < \infty$.
Since A ~ E ⊆ {A ~ (E ∪ F)} ∪ F we have μ^*(A ~ E) ≤ μ^*{A ~ (E ∪ F)} + μ^*(F).
Thus, μ^*(A ~ E) < ∞ and
μA - μ^*{A ~ (E ∪ F)} ≤ μA - μ^*(A ~ E) + μ^*F ≤ $\mu_*(E) + \mu^*(F)$
Taking the supremum over A, we get $\mu_*(E \cup F) \leq \mu_*(E) + \mu^*(F)$.
To prove the 3rd inequality, we choose a measurable set G ⊆ E with $\bar{\mu}(G) = \mu_*(E)$.
Then the measurability of G implies that $\mu_*(E) + \mu^*(F) = \bar{\mu}(G) + \mu^*(F) = \mu^*(G \cup F)$ $\leq \mu^*(E \cup F)$. Thus, $\mu_*(E) + \mu^*(F) \leq \mu^*(E \cup F)$.
By the sub additivity of outer measure $\mu^*(E \cup F) \leq \mu^*(E) + \mu^*(F)$

Corollary: If {E$_i$} is any disjoint sequence of sets, then $\sum_{i=1}^{\infty} \mu_* E_i \leq \mu_*(\bigcup_{i=1}^{\infty} E_i)$
Proof: Set E = $\bigcup_{i=1}^{\infty} E_i$. Repeated application of the first inequality in a Theorem gives us $\sum_{i=1}^{n} \mu_* E_i \leq \mu_*(\bigcup_{i=1}^{n} E_i) \leq \mu_*(E)$. The corollary follows by letting n tend to ∞.
Lemma: Let {A$_i$} be a disjoint sequence of sets in $\mathcal{A}$.
Then $\mu_*(E \cap \bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu_*(E \cap A_i)$
Proof: Since we may replace E by $E \cap \bigcup_{i=1}^{\infty} A_i$ we may suppose E ⊆ $\bigcup_{i=1}^{\infty} A_i$ = C.
Let B ∈ $\mathcal{A}$ with μ*(B ~ E) < ∞.
Since C is μ*-measurable, μB = μ*(B ∩ C) + μ*(B∩ $\tilde{C}$) and
y*(B ∩ $\tilde{E}$) = μ*(B ∩ C ∩ $\tilde{E}$) + μ*(B ∩ $\tilde{E}$ ∩ $\tilde{C}$) = μ*(B ∩ C ∩ $\tilde{E}$) + μ*(B ∩ $\tilde{C}$), since $\tilde{C}$ ⊆ $\tilde{E}$.
Thus, μ*(B ∩ $\tilde{C}$) ≤ μ*(B ∩ $\tilde{E}$) < ∞, and so
μB - μ*(B ~ E) = μ*(B ∩ C) - μ*(B ∩ $\tilde{E}$ ∩ C) = $\sum_{i=1}^{\infty} \mu(A_i \cap B)$ - μ*(B ∩ $\tilde{E}$ ∩ C).
Hence μB - μ*(B ~ E) ≤ $\sum_{i=1}^{\infty} \mu_*(A_i \cap E)$.
Taking the supremum over B gives $\mu_* E \leq \sum_{i=1}^{\infty} \mu_*(A_i \cap E)$.

The opposite inequality follows from Corollary.

Theorem: Let μ be a measure on an algebra $\mathcal{A}$ of subsets of X and E any subset of X. If $\mathfrak{B}$ is the algebra generated by $\mathcal{A}$ and E and if $\tilde{\mu}$ is any extension of μ to $\mathfrak{B}$, then $\mu_*(E) \le \tilde{\mu}(E) \le \mu^*(E)$.
Moreover, there are extensions $\overline{\mu}$ and $\underline{\mu}$ of μ to $\mathfrak{B}$ (and hence also to the σ-algebra generated by $\mathfrak{B}$ such that $\overline{\mu}(E) = \mu^*(E)$ and $\underline{\mu}(E) = \mu_*(E)$.

Proof: Let $\{A_i\}$ be any disjoint sequence of sets from $\mathcal{A}$ with $E \subseteq \bigcup_{i=1}^{\infty} A_i$. then $E = \bigcup_{i=1}^{\infty}(A_i \cap E)$, and so $\tilde{\mu}(E) = \sum_{i=1}^{\infty} \tilde{\mu}(A_i \cap E) \le \sum_{i=1}^{\infty} \mu(A_i)$.
Thus, $\tilde{\mu}(E) \le \mu^*(E)$.
If A is any set in $\mathcal{A}$ with $\mu^*(A \sim E) < \infty$, then $\tilde{\mu}(A \sim E) \le \mu^*(A \sim E)$, and $\mu(A) - \mu^*(A \sim E) \le \mu(A) - \tilde{\mu}(A \sim E) = \tilde{\mu}(E \cap A) \le \tilde{\mu}(E)$.
Thus, $\mu_*(E) \le \tilde{\mu}(E)$.
Hence $\mu_*(E) \le \tilde{\mu}(E) \le \mu^*(E)$.
The sets B in $\mathfrak{B}$ are the sets of the form $B = (A \cap E) \cup (A' \cap \tilde{E})$ with A and A′ in $\mathcal{A}$, since the collection of all sets of this form is an algebra contained in $\mathfrak{B}$ and containing $\mathcal{A}$ and E.
For each $B \in \mathfrak{B}$ define $\overline{\mu}$ and $\underline{\mu}$ by $\overline{\mu}(B) = \mu^*(B \cap E) + \mu_*(B \cap \tilde{E})$ and $\underline{\mu}(B) = \mu_*(B \cap E) + \mu^*(B \cap \tilde{E})$.
Then $\overline{\mu}$ and $\underline{\mu}$ are monotone, nonnegative functions defined on $\mathfrak{B}$, and since for $A \in \mathcal{A}$, we have $\mu A = \mu_*(A \cap E) + \mu^*(A \cap \tilde{E})$ it follows that $\overline{\mu}A = \underline{\mu}A = \mu A$ for $A \in \mathcal{A}$. For any A we have $\mu^*(A \cap E) = \sum_{i=1}^{\infty} \mu^*(A \cap A_i)$. By a lemma we have Then $\mu_*(E \cap \bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu_*(E \cap A_i)$. So, $\overline{\mu}$ and $\underline{\mu}$ are countably additive on $\mathfrak{B}$. Hence the theorem follows.♦

7. **Extension by Sets of Measure Zero**

The results of Section 2 allow us to extend a measure μ on an algebra $\mathcal{A}$ to a σ-algebra containing $\mathcal{A}$ and those of Section 6 provide for the extension from $\mathcal{A}$ to a o-algebra containing $\mathcal{A}$ and one additional set. It is sometimes useful to be able to extend to a σ-algebra containing $\mathcal{A}$ and some collection $\mathfrak{M}$ of subsets of X and to extend in such a way that each of the sets in $\mathfrak{M}$ has measure zero. A necessary condition for this to be possible is that whenever we have a set $A \in \mathcal{A}$ such that $A \subseteq M \in \mathfrak{M}$, then $\mu A = 0$. This condition is not in general sufficient, since a countable union of sets in $\mathfrak{M}$ may contain an A with positive measure, but, if we assume that $\mathfrak{M}$ is closed under countable unions, then the condition is sufficient.

39. **Proposition**: Let μ be a measure on a σ-algebra $\mathcal{A}$ of subsets of X, and let 𝔐 be a collection of subsets of X which is closed under countable unions and which has the property that for each A ∈ 𝔐 with A ⊆ M ∈ 𝔐 we have μA = 0. Then there is an extension $\bar{\mu}$ of μ to the smallest σ-algebra 𝔅 containing $\mathcal{A}$ and 𝔐 such that $\bar{\mu}$M = 0 for each M ∈ 𝔐.

Proof:
Since the collection of sets which are subsets of a set in 𝔐 satisfies the same hypothesis as 𝔐, we may assume that each subset of a set in 𝔐 is itself in 𝔐. With this assumption the collection 𝔅 = {B: B = A Δ M, A ∈$\mathcal{A}$, M ∈ 𝔐} is a σ-algebra containing $\mathcal{A}$ and 𝔐, and since each σ-algebra containing $\mathcal{A}$ and 𝔐 contains 𝔅, 𝔅 is the smallest σ-algebra containing $\mathcal{A}$ and 𝔐.
If B = A_1 Δ M_1 = A_2 Δ M_2, then A_1 Δ A_2 = M_1 Δ M_2, and so μ(A_1 Δ A_2) = 0. Thus μA_1 = μA_2, and, if we define $\bar{\mu}$B to be μA_1, then $\bar{\mu}$ is well defined on 𝔅 and is an extension of μ. It remains only to show that $\bar{\mu}$ is countably additive.
Let B = ∪ B_i, B_i ∩ B_j = ϕ. If B_i = A_i Δ M_i, then A_i Δ A_j ∈ 𝔐. Setting $A_n' = A_n \cap \tilde{A}_1 \cap \ldots \cap \tilde{A}_{n-1}$ we have $A_i' \cap A_j' = \phi$, and $A_n \cap A_n'$∈ 𝔐 Thus B_i = $A_i' \Delta M_i'$ and B = A Δ M, where A = ∪A_i'; and M ⊆ ∪M_i';. Thus $\bar{\mu}$B = μA = ΣμA_i'; = Σ $\bar{\mu}$$B_i$. ♦

We observe that the condition that μA = 0 for each A ∈ $\mathcal{A}$ with A ⊆ M simply states that μ_*M = 0. Thus, the proposition states that we can extend the domain of μ to include any collection 𝔐 of sets of inner measure zero provided that 𝔐 is closed under countable unions. Note that on the σ-algebra generated by $\mathcal{A}$ and 𝔐 we have $\bar{\mu}$= μ_*. Thus, this proposition gives a generalization of the process of completion which extends the domain of a measure by adding sets of outer measure zero.

8. **Caratheodory Outer measure**.

Suppose X is a set of points and Γ is a set of real – valued functions on X. Now *we find a sufficient condition under which* an outer measure μ* will have the property that every function in Γ is μ* measurable.

Definition: Two sets are said to be ***separated by the function φ*** if there are numbers a and b with a > b such that φ is greater than a on one and less than b on the other.

Definition: An outer measure μ* is called a ***Caratheodory outer measure*** with respect to Γ if it satisfies the following axiom: (iv) If A and B are two sets which are separated by some function in Γ, then μ*(A ∪B) = μ*(A) + μ*(B).S

Proposition: If μ* is a Caratheodory outer measure with respect to Γ, then every function in Γ is μ* measurable.

Proof: Let μ* be a Caratheodory outer measure w.r.t Γ. Given the real number a and the function φ ∈ Γ, we must show that the set E = {x: φ(x) > a} is μ* measurable or equivalently, that given any set A, μ*(A) ≥ μ*(A ∩E) + μ*(A ∩ $\tilde{E}$).

Since this inequality is trivial if μ*(A) = ∞, let μ*(A) < ∞.
We begin by setting B = E ∩ A, C = $\tilde{E}$∩ A, and
$B_n = \left\{x: (x \in B \;\&\; \left(\varphi(x) > a + \frac{1}{n}\right)\right\}$.
Defining $R_n = B_n \sim B_{n-1}$, we have $B = B_n \cup [\bigcup_{k=n+1}^{\infty} R_k]$.
Now on B_{n-2} we have $\varphi > a + \frac{1}{n-2}$, while on R_n we have $\varphi \leq a + \frac{1}{n-1}$.
Thus φ separates R_n and B_{n-2} and hence separates R_{2k} and $\bigcup_{j=1}^{k-1} R_{2j}$, since the latter set is contained in B_{2k-2}.
Consequently $\mu^*\left[\bigcup_{j=1}^{k} R_{2j}\right] = \mu^* R_{2k} + \left[\bigcup_{j=1}^{k-1} R_{2j}\right] = \sum_{j=1}^{k} \mu^* R_{2j}$ by induction. Since $\sum_{j=1}^{k} R_{2j}$ ⊆ B ⊆ A we have $\sum_{j=1}^{k} \mu^* R_{2j} \leq \mu^* A$, and so the series $\sum_{j=1}^{\infty} \mu^* R_{2j}$ converges. Similarly, the series $\sum_{j=1}^{\infty} \mu^* R_{2j+1}$ converges, and therefore also the series $\sum_{k=1}^{\infty} \mu^* R_k$.
⇒ given ε > 0, we can choose n so large $\sum_{k=1}^{\infty} \mu^* R_k < \varepsilon$.
Then by subadditivity of μ*, μ*B ≤ $\mu^* B_n + \sum_{k=n+1}^{\infty} \mu^* R_k < \mu^* B_n + \varepsilon$. S
Now $\mu^* A \geq \mu^*(B_n \cup C) = \mu^*(B_n) + \mu^*(C)$ since φ separates B_n and C. Consequently, $\mu^* A \geq \mu^*(B) + \mu^*(C) - \varepsilon$.
Since ε is arbitrary positive quantity, $\mu^* A \geq \mu^*(B) + \mu^*(C)$
ie. $\mu^* A \geq \mu^*(A \cap E) + \mu^*(A \cap \bar{E})$ ♦.

9 **Hausdorff Measures**

Definition: By a Borel measure on a metric space X we mean a measure that is defined on some σ-algebra containing the σ-algebra of Borel sets in X. For each positive real number α we will define a particular Borel measure m_α, called the Hausdorff measure on X of dimensions α. These measures are particularly important for the Euclidean spaces R^n, but much of their theory goes through just as easily for an arbitrary metric space X. To define m_α, we take ε > 0 and set $\lambda_\alpha^{(\varepsilon)}$= inf $\sum_{i=1}^{\infty} r_i^\alpha$. where the <$r_i$> are the radii of a sequence of balls, <B_i> that cover E and for which r_i < ε.
Observe that $\lambda_\alpha^{(\varepsilon)}$ increases as ε decreases. Set $m_\alpha^*(E)$ = sup $\sum_{i=1}^{\infty} \lambda_\alpha^\varepsilon(E)$ as ε → 0. Then we have $m_\alpha^*(E) = \lim_{\varepsilon\to 0} \sum_{i=1}^{\infty} \lambda_\alpha^\varepsilon(E)$. It is readily verified that m_α^* is countably subadditive and thus an outer measure.
If E and F are two subsets of X with ρ(E, F) > δ, then
$\lambda_\alpha^{(\varepsilon)}(E \cup F) = \lambda_\alpha^{(\varepsilon)}(E) + \lambda_\alpha^{(\varepsilon)}(F)$ as soon as ε < δ:
For if <B_i> is a sequence of balls of radii less than ε covering E U F, no ball can meet both E and F. Taking limits as ε → 0, we have $m_\alpha^*(E \cup F) \geq m_\alpha^*(E) + m_\alpha^*(F)$. Thus m_α^* induces a Borel measure m_α, on X by Proposition 41.
The measure m_α is called Hausdorff α-dimensional measure.
It is customary to normalize mα by dividing by the quantity $\pi_\alpha = \frac{2\pi^{\alpha/2}}{\alpha\Gamma\left(\frac{\alpha}{2}\right)}$.

Thus π_1 = 2, π_2= π, π_3= 4π/3, and π_n is the volume of the unit ball in R^n.
We refer to this measure as normalized Hausdorff measure.
In R^n the normalized Hausdorff measure m_n is equal to Lebesgue measure.